Linearized Active Circuits: Transfer Functions and Stability

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Abstract

We develop a theoretical framework for local stability analysis of active microwave circuits around an equilibrium. We first characterize linearized (partial) transfer functions of ideal circuits around such an equilibrium, and show they can be unstable even though there are no unstable poles. Next, we consider linearized transfer functions of realistic circuits, comprising components which are passive at sufficiently high frequency. We establish that such realistic transfer functions are stable if and only if they have no poles in the closed right half-plane, and that there are at most finitely many such poles. This suggests that anti-analytic projection-based methods and meromorphic approximation techniques in Hardy spaces should be helpful to check for stability.

Key words: Circuits, transfer function, active components, transistor, diode, transmission line, negative resistor, stability, Hardy spaces.

1 Introduction

This work is motivated by the stability analysis of circuits containing active (non-linear) components, such as transistors and diodes, as well as distributed elements like transmission lines. A typical example is that of amplifiers. The design of such devices relies today on powerful simulation tools in the frequency domain, like the so-called harmonic-balance techniques or DC simulations. These can compute only steady-state solutions (no transient) constrained to a predefined set of frequencies. The obtained solutions may either be stable, hence physically observable, or unstable, thus physically immaterial. Testing stability is thus mandatory before implementing the circuit. To this effect, a wide variety of methods has been proposed and we refer the reader to [27, 26] and their bibliography for a small sample of literature on the subject.

In this paper we focus on local stability analysis around a fixed operating point, which is an equilibrium of the autonomous circuit (*i.e.* a dc solution thereof). Local stability can be studied by computing the response of the circuit to small signal perturbations at its nodes, using AC analysis around a dc solution computed *via* a DC simulation, see [15]. For instance, a small sinusoidal current source can input the circuit in parallel at a particular node, and the voltage at this node is taken as the output of the system. Then, a sweep of the frequency range is performed and the corresponding transfer function is estimated, pointwise in a bandwidth, as being the impedance seen by the current probe at the node under consideration. This approach, which requires no internal knowledge of the circuit, aroused considerable interest in the microwave community [26, 13]. A common methodology in practice is to approximate the simulated frequency response of the linearized system by a rational function, the poles of which are used to assess the stability of the equilibrium. Namely, the poles of a rational approximant are used to indicate the location of the poles of the true system, poles lying in the closed right half-plane indicate instability (such poles are called unstable).

The development of such techniques raises several interesting questions from the point of view of system and function theory. First, it seems like the class of transfer functions involved in the linearization process has not been characterized up to now. Second, though it is not difficult to show that singularities at finite distance are indeed poles, one expects that their number may be infinite as is the case for most delay systems [19] (models for transmission lines contain delays). Third, the speed of approximation by rational functions to transfer functions of delay systems is rather low [16, 29, 4], hence high order models are typically needed to reach good accuracy on a broad frequency interval. However, when the degree goes large, rational approximation techniques based on interpolation which are often favored by electronics engineers are known to generate spurious poles whose physical interpretation is uneasy [10, 22]; in fact, the extent to which the singularities of a rational approximant indicate those of the approximated function is a longstanding issue in approximation theory that cannot be answered independently of the approximation method one is using, by Runge's theorem [25] [1]. Fourth, it is unclear whether the absence of unstable poles does guarantee stability, as this fails to hold for some retarded systems, see *e.g.* [20]. Altogether, one might anticipate that constructing rational approximants is nontrivial in this context, and interpreting the information they contain even more so.

Below, we aim at contributing to build a rigorous setup to analyze transfer functions of linearized circuits comprising active elements (diodes and transistors) as well as standard passive components (resistors, inductors, capacitors and transmission lines). We describe in Section 2 ideal models in the frequency domain for these elements, and we proceed in Section 3 with preliminaries on partial transfer functions. Through Sections 3, 5 and 6, we establish the nature of linearized transfer functions of such ideal circuits. In section 7, we discuss stability and show with an example that ideal circuits may be unstable even though their transfer function has no pole in the closed right half-plane. We also introduce more realistic models of active components, in which small inductive and capacitive effects make them passive at infinite frequency (as opposed to ideal components). Finally, in Sections 8 and 9, we study the stability of realistic circuits and show that instability without unstable pole can no longer happen: our main result (Section 9) is that a realistic circuit is unstable if and only if it has poles in the closed right half-plane. Moreover, these must be finite in number so that the unstable part of the linearized transfer function of a realistic circuit is demonstrably rational. In retrospect, this justifies to look for unstable poles thereof to check for stability at an operating point.

Hereafter, given a matrix M, we let M^t denote its transpose. The identity matrix is written $\mathbf{Id}$, irrespective of its size which will be understood from the context.

2 Description of ideal linear circuits

2.1 Electronic components under consideration

The circuits that we consider are made of classical passive RLC components, active components and transmission lines. We only study small perturbations around a operating point, hence it is legitimate to linearize the active components. In this section, we give a detailed account of these elementary ideal models, along with equations satisfied by currents and voltages at their terminals. These involve complex impedances and admittances [5, 7], *i.e.* they express the relations between Laplace transforms of currents and voltages. We denote Laplace transforms with uppercase symbols, *e.g.* $V = V(s)$ is a function of a complex variable s which stands for the Laplace transform of the voltage $v = v(t)$ which is a function of the time t. Variables of interest to linearized models are incremental rather than absolute quantities, but we denote them like ordinary intensity or voltage for notational convenience. By convention, currents are oriented so as to *enter* electronic components.

2.1.1 Dipoles

A dipole, also called a branch [7], is an electronic box with 2 terminals, labeled 1, 2, such that the current I_1 through the box, oriented from 1 to 2, is related to the potentials V_1, V_2 at the terminals by a linear equation of the form

$$V_1 - V_2 = Z\, I_1, \tag{1}$$

where $Z = Z(s)$ is the impedance. The reciprocal $1/Z$ is called the admittance. The dipole is passive if $\Re Z(s) \geq 0$ when $\Re s \geq 0$ [1]. Elementary passive dipoles considered in this paper are the following (see Figure 1):

- Ideal resistor, with a real and positive impedance R.
- Ideal inductor, with impedance of the form $Z(s) = Ls$, $L > 0$.
- Ideal capacitor, with impedance of the form $Z(s) = 1/(Cs)$, $C > 0$.

These typically arise as linearized versions of more realistic non-linear components.

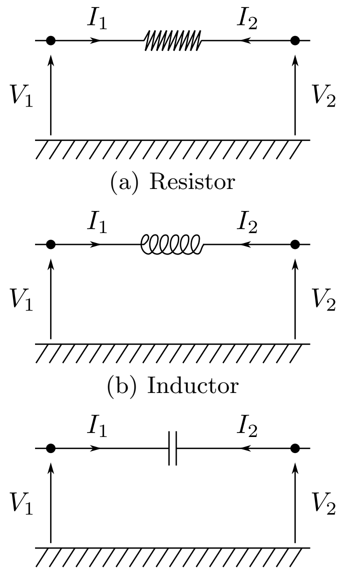
2.1.2 Diodes.

A commonly accepted model says that the current through a diode is a non-linear function of the voltage (one assumes the diode has no inductive nor capacitive effect). Supposing this function is differentiable, we may linearize the behavior around an operating point. Taking Laplace transforms, we get $I = g\,U$, so the (linearized) diode appears as a standard linear dipole with admittance $g \in \mathbb{R}$ (see Figure 2). Typical in our context are tunnel diodes which behave in the frequency range of concern (once correctly biased) as ideal negative resistors: $g < 0$, and satisfy (1) with $Z(s) = 1/g$.

2.1.3 Transmission lines.

A transmission line (see Figure 4) is commonly modeled as a concatenation of infinitesimal capacitors, resistors and inductors with the same impedance [23] (see Figure 3, in which G denotes the conductance of the resistor and the hatched region is the ground).

[1] Runge's theorem entails that a continuous function on a segment can be approximated uniformly arbitrary well by a rational function with prescribed pole location.



(a) Resistor

(b) Inductor

(c) Capacitor

Fig. 1. Symbols for linear dipoles

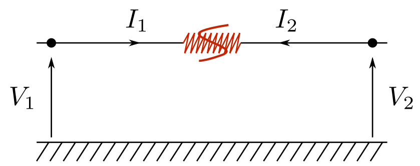


Fig. 2. Symbol for the linearized diode

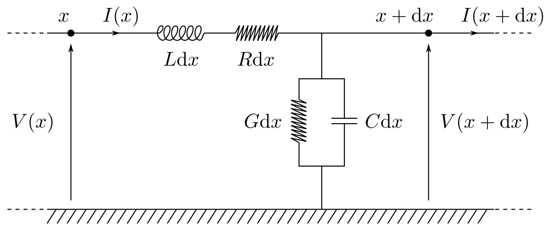


Fig. 3. Model of a transmission line

This model leads to the so-called telegrapher's equation [17, sec. 9.7.3]:

$$\begin{cases} \dfrac{\partial V}{\partial x} = -(R + Ls)I \\[2ex] \dfrac{\partial I}{\partial x} = -(G + Cs)V \end{cases} \implies \begin{cases} \dfrac{\partial^2 V}{\partial x^2} = \gamma^2 V \\[2ex] \dfrac{\partial^2 I}{\partial x^2} = \gamma^2 I, \end{cases} \tag{2}$$

where γ is one of the two complex square roots of $(R + Ls)(G + Cs)$. This γ is sometimes called the propagation coefficient (note that it is frequency-dependent) while $z_0 = (R + Ls)/\gamma$, is the so-called characteristic impedance of the line. Whichever determination of the square root we choose will be irrelevant. Note that $(R + Ls)(G + Cs)$ cannot vanish in the half-plane $\{s : \Re s > \max\{-R/L, -G/C\}\}$, therefore γ may be chosen analytic there [25], thus a fortiori analytic in $\{\Re s > 0\}$.

Multiplying R, G, L and C by some constant α does not change the value of z_0, but has the effect of multiplying γ by α. Therefore, the relations between the currents and voltages at the terminals of a line of length ℓ with characteristics R, G, L and C are the same as the relations at the terminals of a line of length 1 with characteristics R/ℓ, G/ℓ, L/ℓ and C/ℓ. Thus, from a theoretical viewpoint, the length ℓ of the line can be chosen at will. Hereafter we set $\ell = 1$, and we want to express the relations between $I_1 = I(0), V_1 = V(0), I_2 = -I(1)$ and $V_2 = V(1)$. Solving the telegrapher's equations, we see that the behavior of a transmission line is characterized by the linear relations:

$$\begin{cases} V_2 = \cosh(\gamma)\, V_1 - z_0 \sinh(\gamma)\, I_1, \\[1ex] I_2 = \dfrac{\sinh(\gamma)}{z_0}\, V_1 - \cosh(\gamma)\, I_1. \end{cases} \tag{3}$$

All transmission lines will be assumed to share a common ground. In Section 6, it will be convenient to materialize the current loss between terminals of a line as resulting from a current occurring in a wire (which does not actually exist) connected to the ground. This virtual wire is drawn with a dotted segment on Figure 4. The transmission line can be viewed as a quadripole, with two poles connected to the ground.

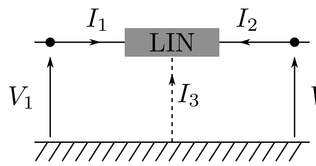


Fig. 4. Symbol for transmission lines

This circuit satisfies the relations

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} z_0 \coth(\gamma) & \dfrac{z_0}{\sinh(\gamma)} \\[2ex] \dfrac{z_0}{\sinh(\gamma)} & z_0 \coth(\gamma) \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \tag{4}$$

and $I_1 + I_2 + I_3 = 0$. In particular, the choice of the square root defining γ is irrelevant, for coth and sinh are odd functions. Equation (4) looks like (1) except that we deal with vector currents and potentials. Accordingly, the impedance Z is a non-diagonal 2×2 matrix. Thus, the transmission line is not a dipole but a pair of coupled dipoles (ground-end1, ground-end2) each of which has one terminal which is grounded. Positive realness is then defined as a matrix inequality: $Z(s) + Z^*(s) \geq 0$, where the star means transpose-conjugate.

Lemma 1 *For a transmission line, the impedance matrix $Z(s)$ as well as the admittance matrix $Y(s)$ are positive real.*

Proof. Let us write

$$P := V_1\,\bar I_1 + V_2\,\bar I_2 = V(0)\,\overline{I(0)} - V(\ell)\,\overline{I(\ell)}$$
$$= -\int_0^\ell \left(\frac{\partial V}{\partial x}(\xi)\,\overline{I(\xi)} + V(\xi)\,\frac{\partial \bar I}{\partial x}(\xi)\right)\,\mathrm{d}\xi.$$

Replacing $\partial V/\partial x$ and $\partial I/\partial x$ by their values in terms of I and V deduced from Equation (2), we obtain:

$$P = -\int_0^\ell \left(-(R+Ls)\,|I(\xi)|^2 - \overline{(G+Cs)}\,|V(\xi)|^2\right)\,\mathrm{d}\xi,$$

therefore

$$\Re(P) \qquad (5)$$
$$= \int_0^\ell (R + L\,\Re(s))\;|I(\xi)|^2 + (G + C\,\Re(\bar s))\;|V(\xi)|^2\,\mathrm{d}\xi.$$

Clearly the integrand is positive for $\Re(s) \geq 0$, hence $\Re(V_1\,\bar I_1 + V_2\,\bar I_2)$ is positive when $\Re(s) \geq 0$. This in turn implies that both $Y(s)$ and $Z(s)$ are positive real. □

2.1.4 Transistors.

A transistor is typically modeled by a controlled current source, usually combined with some resistors and (non-linear) capacitors. After linearization the latter become ordinary capacitors, so we are left to describe the current sources and their linearization. A controlled current source has 3 terminals. When the transistor if a Field Effect Transistor (FET), these terminals are called gate, source, and drain, denoted respectively by G, S and D (see Figure 5). Their behavior is described by a relation

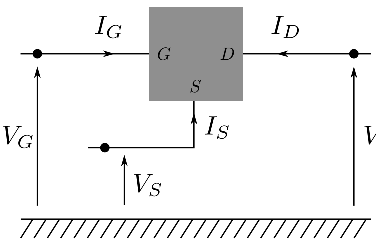


Fig. 5. Controlled current source.

of the form $i_D = f(v_{GS}, v_{DS})$ where f is a non-linear real-valued function and $v_{GS} = v_G - v_S$, $v_{DS} = v_D - v_S$. As in the case of diodes, this simple model assumes no inductive nor capacitive effect, as f only depends on v_{GS}, v_{DS} and not on their time derivatives, nor on the derivative of i_D. Moreover the function f is increasing in both variables. Also, no current enters the gate: $i_G = 0$. Around a operating point, we make use of the linear approximation:

$$i_D = g_m\,v_{GS} + g_d\,v_{DS}, \quad g_m > 0, \quad g_d > 0, \qquad (6)$$

where g_m and g_d are the partial derivatives of f at this point, assuming they exist (see Figure 6). Thus, a transistor is again a pair of coupled dipoles (gate-source and drain-source) with a common terminal (source).

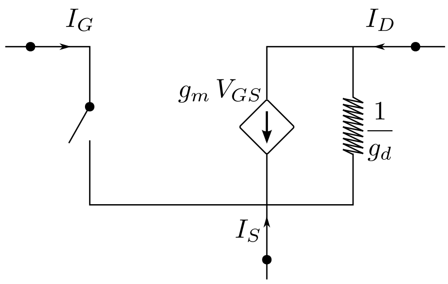


Fig. 6. Model for the linearized controlled current source.

3 Nodal analysis and partial transfer functions

Formally speaking, a circuit is a directed graph with labeled vertices (called junction nodes), and edges (called branches). Branches correspond to dipoles and nodes to terminals thereof. We restrict to circuits built from elements listed in Section 2.1. To form a graph representing a given electric device, we take these elements as branches and connect them according to the device. Coupled branches with a common node may occur if transmission lines or transistors are used. Of course many different circuits may represent to the same device.

To each junction node j is associated a potential V_j, and to each edge k an electric current I_k. One of the junction nodes, say V_n, is the ground (its potential is 0 by convention). *We always assume that the graph associated with a circuit is connected.*

In order to check stability of a circuit, the following simulation experiment is made. An ideal current source I_{in} is plugged in between the ground and some junction node k of the linearized circuit. To express the effect of I_{in} on

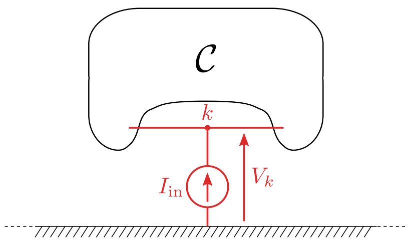


Fig. 7. Partial transfer function at node k

the potential V_k, we use nodal analysis which is a classical method to derive voltages at the nodes of a circuit in terms of the branch currents [7, sect.2.9]. Specifically, we denote by $V = (V_1,\,\ldots,\,V_{n-1})^t$ the vector of all node voltages (except V_n, the reference ground voltage) and by $I = (I_1,\,\ldots,\,I_p)^t$ the vector of all currents in the branches. The *(node-branch) incidence matrix* of the circuit, say $A = (A_{ij})$, has $n-1$ rows corresponding to the nodes (except the ground) and p columns corresponding the branches. It is defined by the rule:

$$\begin{cases} A_{ij} = 1 \text{ if edge } e_j \text{ is incident away from node } i, \\ A_{ij} = -1 \text{ if edge } e_j \text{ is incident towards node } i, \\ A_{ij} = 0 \text{ otherwise.} \end{cases}$$

Since the graph is connected, A has full row rank $(n-1)$ [9, Th. 2.1]. Now, because we plug in I_{in} at node k, Kirchhoff's law gives us

$$A\,I = \begin{pmatrix} 0 \ \dots \ I_{\text{in}} \ \dots \ 0 \end{pmatrix}^t. \tag{7}$$

Next, we substitute currents with voltages using relations (1), (4) and (6). For this, we form the *branch admittance matrix*, a block diagonal matrix $\mathbf{Y}_b = \operatorname{diag}(Y_1, Y_2, \dots, Y_h)$ where the Y_j are the admittances of the branches (or admittance matrices of pairs of coupled branches in case of transistors and lines). With a convenient ordering of nodes and edges, it holds that $I = \mathbf{Y}_b A^t\, V$, and (7) yields

$$\mathbf{Y}\,V = \begin{pmatrix} 0 \ \dots \ I_{\text{in}} \ \dots \ 0 \end{pmatrix}^t, \tag{8}$$

where $\mathbf{Y}(s)$ is a $(n-1)\times(n-1)$ matrix, called *nodal admittance matrix*, which is related to the branch admittance matrix through ([7] eq. (2.9.8))

$$\mathbf{Y} = A\mathbf{Y}_b A^t. \tag{9}$$

The presence of active components like diodes and transistors may result in $\mathbf{Y}$ being singular [3, 8]. We will return to this point in section 8, but for the time being *we suppose that the matrix* $\mathbf{Y}$ *in Equation* (8) *is invertible.* By Cramer's rule, we then have:

$$V_j = (-1)^{k+j}\,\frac{\mathbf{Y}_{j,k}}{\det \mathbf{Y}}\,I_{\text{in}}. \tag{10}$$

where $\mathbf{Y}_{j,k}$ denotes the minor of $\mathbf{Y}$ obtained by deleting row j and column k. Thus, the voltage at node j depends linearly on I_{in} injected at node k, and the ratio V_j/I_{in} is the *partial transfer function* or *partial frequency response* of the circuit at node j from node k.

Obviously, from (10), each partial frequency response of a circuit made of positive resistors, negative resistors, along with standard (i.e. positive) capacitors and inductors, belongs to the field $\mathbb{R}(s)$ of rational functions in the variable s with real coefficients. The converse statement that every real rational function occurs in this manner is a little harder, but can be established along the lines of classical Foster synthesis [7, thm. 5.2.1] by relaxing sign conditions therein, see also [8]. In Section 5, we produce a different proof as a step towards characterizing partial transfer functions of circuits involving also lines and transistors, which is a more general result established in Section 6.

More precisely, let $\mathcal{E}$ be the smallest field containing $\mathbb{R}(s)$ as well as all functions of the form $\gamma(s)\,\sinh(\gamma(s))$ and $\cosh(\gamma(s))$, where $\gamma(s) = \sqrt{(a+bs)(c+ds)}$ for some real numbers $a, b, c, d \geq 0$ Again, it is easy to see that each partial frequency response of a circuit built from elements listed in Section 2.1 lies in $\mathcal{E}$. Indeed, all entries of the matrix Y given in Equation (8) belong to $\mathcal{E}$, which is a field. Thus, as each partial frequency response is the ratio of a minor of Y and its determinant, it belongs to $\mathcal{E}$ as well. The converse statement lies deeper, and is established in Theorem 15.

We shall deal with particular circuits consisting of a single dipole connected to the ground at one of its two terminals. Such an object we call a *grounded branch.* The impedance of the dipole is called the impedance of the grounded branch. It is but the ratio of the voltage at the ungrounded terminal and the current through the grounded branch, *i.e.* it holds in Figure 8 that $V = ZI$.

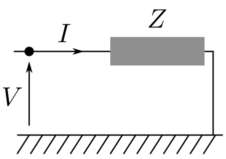


Fig. 8. Representation of a grounded branch

Remark 2 *A circuit with a single node (besides ground) and partial frequency response* $R(s)$ *can be synthesized with a one-port circuit of impedance* $R(s)/(1-R(s))$ *and a resistor of resistance* 1*, as is easily checked on Figure 9.*

By Remark 2, the smallest field containing all partial frequency responses of circuits built from positive or negative resistors, inductors, capacitors and transmission lines, is also the smallest field containing all impedances of grounded branches made of such elements. This elementary observation is technically quite useful. Indeed, whereas dipoles are easy to compose in series or in parallel, which results in nice algebraic combinations of their impedance functions, it is not so for transmission lines. Their behavior, recalled in Equation (4), cannot be described by a single scalar relation as one fundamentally needs two linear relations to express it. Thus, when composing lines with other elements in a circuit, one has to multiply 2×2 matrices which makes it difficult to keep track of the algebraic structure of the resulting elements. In contrast, transmission lines can be used in a fairly simple manner to design impedances of grounded branches, as we will see in Section 6.1.

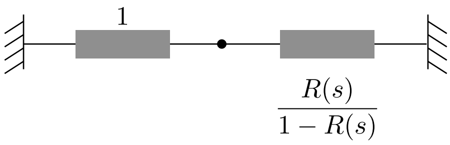


Fig. 9. A circuit with partial frequency response $R(s)$

4 Combining grounded branches

4.1 Parallel composition of grounded branches

Lemma 3 *Given two grounded branches with impedance X and Y respectively, it is possible to build a grounded branch with impedance $XY/(X+Y)$.*

Proof. This is clear from Figure 10. □

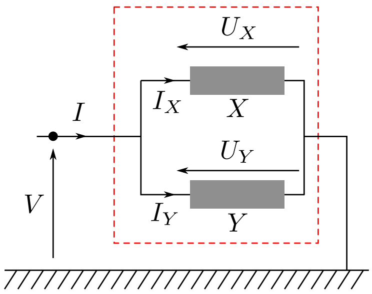


Fig. 10. Composition of one-port circuits in parallel

4.2 Series of a grounded branch and a dipole

Lemma 4 *Given a grounded branch with impedance X and a dipole with impedance Y, it is possible to build a grounded branch with impedance $X+Y$.*

Proof. This is clear from Figure 11. □

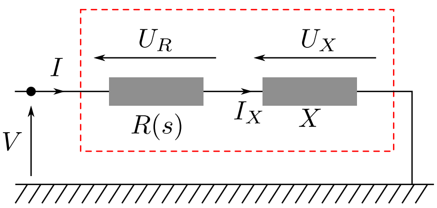


Fig. 11. Composition of a grounded branch and a dipole in series

Obviously one cannot compose grounded branches in series.

4.3 What the inverter makes possible.

Hereafter we make extensive use of the widget depicted in Figure 12, called an inverter. Using Lemmas 3 and 4, one easily sees that it is equivalent to a dipole whose impedance is

$$X+\frac{-X\,(X+Z)}{-X+(X+Z)}=\frac{-X^2}{Z}. \tag{11}$$

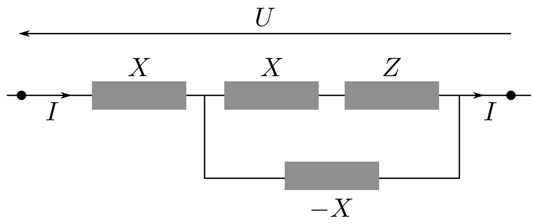


Fig. 12. The inverter network

Since negative resistors are allowed (recall they are linearized diodes), the invertor can actually be realized with $X=1$, hence:

Lemma 5 *Given a grounded branch with impedance Z, it is possible to built a grounded branch with impedance $-1/Z$.*

Remark 6 *By Lemma* (5) *it is possible to emulate negative capacitors and inductors.*

4.4 Series-like composition of grounded branches

The invertor also allows to emulate composition in series:

Lemma 7 *Given grounded branches with impedances X and Y respectively, it is possible to build a grounded branch with impedance $X+Y$.*

Proof. By Lemma 5 it is possible to build grounded branches with impedances $-1/X$ and $-1/Y$, respectively. Hence, using Lemma 3 we get a ground branch with impedance

$$\frac{(-1/X)(-1/Y)}{(-1/X)+(-1/Y)}=\frac{-1}{Y+X}$$

From Lemma 5 again, we now get a grounded branch with impedance $X+Y$. □

As a consequence, we obtain the following lemma.

Lemma 8 *(a) Given grounded branches with impedances X and $-X$ respectively, it is possible to build grounded branches with impedances $\pm X^2$.*
(b) Given grounded branches with impedances X, $-X$, Y and $-Y$ respectively, it is possible to build grounded branches with impedances $\pm XY$.

Proof. (a) We use the inverter (11) again, this time with Z a resistor of resistance ± 1. This way we obtain a grounded branch with impedance $\pm X^2$.
(b) From Lemma 7 we get grounded branches with impedances $\pm(X+Y)$ and $\pm(X-Y)$. Thus, by (a), we can built grounded branches with impedances $\pm(X+Y)^2$ and $\pm(X-Y)^2$. Using Lemma 7 again, we get grounded branches with impedances $\pm 2XY$.
To recap, we just proved that given grounded branches with impedances $\pm X$ and $\pm Y$, one can build grounded branches with impedances $\pm 2XY$. Applying this result to $\pm 2XY$ and a resistor of resistance $\pm 1/4$, we get a grounded branch with impedance $\pm XY$. □

5 Frequency responses of R-L-C circuits with negative resistors

Theorem 9 *Any $R(s) \in \mathbb{R}(s)$ can be realized as the impedance of a grounded branch comprised only of positive and negative resistors, capacitors and inductors.*

Proof. Let $R(s) = P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are real polynomials. Using Remark 6 and Lemma 8, we get by induction grounded branches with impedances $\pm\alpha s^k$ for any $\alpha > 0$ and $k \in \mathbb{Z}$. Appealing to Lemma 7 allows us to obtain grounded branches with impedances $\pm P(s)$ and $\pm Q(s)$. Now, by Lemma 5, we obtain grounded branches with impedances $\pm 1/Q(s)$ and thus, using Lemma 8 again, grounded branches of impedances $\pm P(s)/Q(s)$. □

Corollary 10 *The class of partial frequency responses of circuits made of positive and negative resistors, capacitors and inductors is exactly $\mathbb{R}(s)$.*

Proof. If $R(s)$ lies in $\mathbb{R}(s)$, so does $R(s)/(1-R(s))$ and we can use Remark 2 to conclude the proof. □

Remark 11 *Note that Corollary 10 is concerned with a single partial transfer function and says nothing about synthesizing an arbitrary rational matrix as the transfer matrix of a circuit of the prescribed type. Whether this is possible or not is still an open issue, see [8] where transformers and gyrators are added to the list of admissible elements to get a positive answer.*

Remark 12 *Circuits made of positive and negative resistors, capacitors, inductors and linearized transistors have exactly the same class of partial frequency responses as those without transistors. Indeed, elements in the branch admittance matrix $\mathbf{Y}_b$ corresponding to the behavior of transistors (see Equation (6)) also belong to $\mathbb{R}(s)$, so frequency responses of circuits with transistors in turn lies in $\mathbb{R}(s)$. Since all functions from $\mathbb{R}(s)$ are already realizable without transistors, this remains true* a fortiori *if transistors are allowed.*

6 Frequency responses of ideal linear circuits

6.1 Using transmission lines as grounded branches

A transmission line may be construed as a grounded branch by forcing either the current or the voltage at one end. To see this, recall from Section 2.1.3 that a transmission line can be viewed as a pair of coupled dipoles, one at each end, each of which has one terminal which is grounded and the other free for connection, the coupling between them being described by Equation (3) where we set $\gamma(s) = \sqrt{(a+bs)(c+ds)}$ for some $a, b, c, d \geq 0$ and $z_0 = (a+bs)/\gamma(s)$. Consider first the line depicted in Figure 13 where the end number 2 is open, not connected to anything. This amounts to impose $I_2 = 0$ in Equation (3) whence, by construction, we get a grounded branch with impedance $\coth(\gamma(s))(a+bs)/\gamma(s)$.

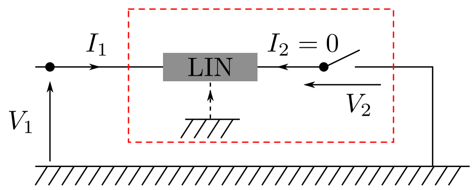


Fig. 13. Line in open circuit: grounded branch with impedance $\coth(\gamma(s))(a+bs)/\gamma(s)$

Likewise, Figure 14 shows a line with end number 2 in short circuit (*i.e.* connected to the ground), which amounts to impose $V_2 = 0$ in Equation (3). This way, we get a grounded branch with impedance $\tanh(\gamma(s))\gamma(s)/(c+ds)$.

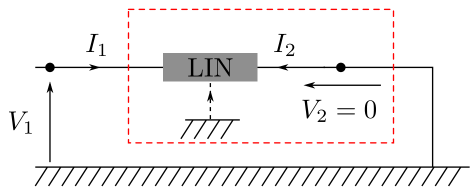


Fig. 14. Line in short-circuit: grounded branch with impedance $\tanh(\gamma(s))\gamma(s)/(c+ds)$.

Lemma 13 *Let a, b, c, and d be non negative numbers. Set $\gamma(s) = \sqrt{(a+bs)(c+ds)}$, where the determination of the square root is arbitrary. Then, one can construct grounded branches with impedances $\pm\tanh(\gamma(s))\gamma(s)/(a+bs)$.*

Proof. Applying Lemma 5 to the circuit in Figure 13, we get a grounded branch with impedance $-\tanh(\gamma(s))\gamma(s)/(a+bs)$. Renaming (a, b, c, d) as (c, d, a, b), the circuit in Figure 14 yields a grounded branch with impedance $\tanh(\gamma(s))\gamma(s)/(a+bs)$. □

Lemma 14 *Notation being as in Lemma 13, one can realize grounded branches with impedances $\pm\cosh^2(\gamma(s))$.*

Proof. According to Lemma 13, there are grounded branches with impedances $\pm\tanh(\gamma(s))\gamma(s)/(a+bs)$. Hence using part (a) of Lemma 8, we can build grounded branches with impedances

$$\pm\frac{\gamma(s)^2}{(a+bs)^2}\ \tanh^2(\gamma(s)) = \pm\frac{c+ds}{a+bs}\ \tanh^2(\gamma(s)).$$

Now, by Theorem 9, there are grounded branches with impedances $\pm(a+bs)/(c+ds)$. Thus, in view of Lemma 8 (b), we can build grounded branches with impedances $\pm\tanh^2(\gamma(s))$. Thanks to Lemma 7, we then obtain grounded branches with impedances $\pm(\tanh^2(\gamma(s))-1)$ and finally, using Lemma 5, we get grounded branches with impedances

$$\pm\frac{1}{1-\tanh^2(\gamma(s))} = \pm\cosh^2(\gamma(s)).$$

□

6.2 Circuits with transmission lines

By definition, every element of the field $\mathcal{E}$ introduced in Section 3 can be written as the quotient of two real linear combinations of functions of the form

$$s^k \prod_{i=1}^{l}\gamma_i(s)\ \sinh(\gamma_i(s)) \prod_{i=l+1}^{m}\cosh(\gamma_i(s)), \qquad (12)$$

with $k\in\mathbb{N}$ and $\gamma_i(s)=\sqrt{(a_i+b_is)(c_i+d_is)}$ where a_i, b_i, c_i, d_i are non negative real numbers.

Theorem 15 *Any element of the field $\mathcal{E}$ can be realized as the impedance of a grounded branch made of elements listed in Section 2.1, namely positive and negative resistors, capacitors, inductors, linearized transistors and transmission lines.*

Proof. It is enough to prove that, for any non negative numbers a, b, c, and d, one can realize $\pm\gamma(s)\ \sinh(\gamma(s))$ and $\pm\cosh(\gamma(s))$, where $\gamma(s)$ is defined as in Lemma 13. Indeed, by Lemma 8 (b) and Theorem 9, this will imply that any element of the form (12) can be obtained as the transfer function of a grounded branch. Then, Lemma 7 allows us to realize sums of such elements, and appealing to Lemmas 5 and 8 (b) will achieve the proof.

Let a, b, c,d be arbitrary nonnegative numbers, and define as before $\gamma(s)=\sqrt{(a+bs)(c+ds)}$. Set $a'=a/2$, $b'=b/2$, $c'=c/2$, $d'=d/2$, and put $\gamma'(s)=\sqrt{(a'+b's)(c'+d's)}$. Clearly $\gamma(s)=\pm2\gamma'(s)$ (the determinations of the square roots in the definitions of γ and γ' are not necessarily the same). According to Lemma 14, there are grounded branches with impedances $\pm\cosh^2(\gamma'(s))$. Now, using the identity $\cosh(2x) = 2\cosh^2(x)-1$, we see that $\cosh^2(\gamma'(s)) = \frac{1}{2}\cosh(\pm\gamma(s))+\frac{1}{2}$. By Lemma 7, and since cosh is an even function, we otain this way ground branches with impedances $\pm\cosh(\gamma(s))$. Thus, it remains to show that impedances $\pm\gamma(s)\ \sinh(\gamma(s))$ can also be realized.

Since we can realize ground branches with impedances $\pm\frac{1}{2}\cosh(\gamma(s))$ and $\pm\frac{\gamma(s)}{a+bs}\ \tanh(\gamma(s))$ (*cf.* Lemma 13), it follows from Lemma 8 that we can also get impedances

$$\pm\frac{\gamma(s)}{2(a+bs)}\sinh(\gamma(s)).$$

In another connection, by Theorem 9, there are grounded branches with impedances $\pm2(a+bs)$. Applying Lemma 8 therefore yields grounded branches with impedances $\pm\gamma(s)\sinh\gamma(s)$, as desired. □

The next corollary immediately follows from Remark 2.

Corollary 16 *The class of functions realizable as partial transfer functions of circuits made of positive and negative resistors, capacitors, inductors, linearized transistors and transmission lines is exactly $\mathcal{E}$.*

Remark 17 *The partial transfer functions of circuits made of positive and negative resistors, capacitors, inductors, linearized transistors and transmission lines are meromorphic functions, that is to say, the only possible singularities are poles. Indeed, since $\gamma_i(s)$ is a function with a branchpoint of order 2 at each of its zeros, while* sinh *is entire and odd, we see that $\gamma_i\sinh\gamma_i$ is an entire function. Also, since* cosh *is an even entire function,* $\cosh\gamma_i$ *is also entire. Altogether, the functions in* (12) *are thus entire as well.*

Remark 18 *If in Corollary 16 we restrict ourselves to lossless transmission lines (for which $R=G=0$ in Figure 3), then the class of partial transfer functions obtained by combining such lines with the other elements consists exactly of quotients of quasi-polynomials*

$$\frac{\sum_{j=1}^{l}P_j(s)e^{\alpha_j s}}{\sum_{j=1}^{m}Q_j(s)e^{\beta_j s}},$$

where P_j and Q_j are arbitrary real polynomials and α_j and β_j arbitrary real numbers.

7 On stability and passivity

A standard definition of stability is: a linear stationary control system is stable when its transfer function belongs to $\mathcal{H}^\infty$, the space of bounded analytic functions in the open right half-plane. This is equivalent to require that the system maps inputs of finite energy (*i.e.* L^2 signals) to outputs of finite energy [19]. It should be noted that this definition considers as unstable certain ideal passive components, such as pure inductors and capacitors, since their impedance is unbounded on the imaginary axis. More generally, for a rational transfer function to be stable, it is necessary and sufficient that it has no pole in the closed right half-plane including at infinity. Such poles will be called unstable. Thus, by Corollary 10, partial frequency responses of circuits involving positive resistors, capacitors and inductors, as well as ideal linearized diodes (which are negative capacitors, see Section 2.1.2), are unstable if and only if they have at least one unstable pole. It is natural to ask whether a similar statement holds for functions from the class $\mathcal{E}$. This is actually false. Indeed, it is known there are quotient of quasi-polynomials (*cf.* Remark 18) which are unstable though they have no unstable pole (these are neutral systems [20]). By Corollary 16, such functions can be realized as partial frequency responses of circuits comprised of positive and negative, resistors, capacitors, inductors, transmission lines and linearized transistors. In [3] such a circuit was synthesized using these components to have partial transfer function $Z(s) = \frac{2f(s)}{f(s)+2}$, where

$$f(s) = s\,\tanh(s) - \frac{1}{s+1}. \tag{13}$$

It can be shown that $Z(s)$ lies in $\mathcal{E}$, has no poles in the closed right half-plane nor at infinity (where it has an essential singularity), and does not belong to $\mathcal{H}^\infty$.

At first, example (13) casts doubt on whether assessing the stability of an equilibrium, for active electronic devices, can be achieved upon checking if the linearized circuit has unstable poles. Still, one may argue that such ideal models are somewhat unrealistic. Indeed, even if it does not show at working frequencies, no passive component is truly lossless and no active component has gain at all frequencies, for there are always small resistive, capacitive and inductive effects in a physical device. In fact, advanced simulators usually provide one with a library of more accurate models, and we now review how the components listed in Section 2 can be modified to make them more realistic. We will see in the next section that using only such modified components restricts considerably the class of partial frequency responses of corresponding circuits.

To rule out unrealistic behaviors of passive components, it is natural to select multiports that involve short or open circuits at no frequency. This is the object of the following definition.

Definition 19 *A passive and square multiport is said to be realistic if its complex impedance matrix Z and its admittance $Y = Z^{-1}$ satisfy, for $\Re(s) \geq 0$:*
(i) $Y(s) + Y^*(s) \geq \alpha\,\mathbf{Id}$ for some $\alpha > 0$,
(ii) $Z(s) + Z^*(s) \geq \beta\,\mathbf{Id}$ for some $\beta > 0$.

Note that (i) entails that Z is stable, when viewed as a transfer function. Indeed, since $Y + Y^* = Y(Z + Z^*)Y^*$, we have for $\Re(s) \geq 0$ that

$$Z(s) + Z^*(s) \geq \alpha Z(s)Z^*(s), \tag{14}$$

and letting $u(s)$ be a maximizing vector of $Z^*(s)$ with unit norm we get

$$\begin{aligned} 2\|Z(s)\| &\geq u^*(s)\Big(Z(s) + Z^*(s)\Big)u(s) \\ &\geq \alpha u^*(s)Z(s)Z^*(s)u(s) = \alpha\|Z(s)\|^2, \end{aligned} \tag{15}$$

hence $\|Z(s)\| \leq 1/(2\alpha)$ which implies $Z \in H^\infty$.

Conditions (ii) is a frequency domain version of so-called *strict input passivity* of the system with transfer function Z, while (14) expresses its *strict output passivity*, see [14]. In the literature, a rational function satisfying (i) is called strongly strictly positive real [6, Def 2.54]. It should be noticed that, contrary to the rational case, functions in $\mathcal{E}$ may have no limit at infinity.

For instance, a realistic inductor can be obtained by connecting an ideal inductor in parallel with a large resistance R, and then composing the resulting dipole in series with a small resistance r. We may think of R, *e.g.* as being the resistance of the air around the inductor, and of r as the resistance of the conductor constitutive of the element. Similar considerations apply to capacitors.

As for transmission lines, we have the following result:

Lemma 20 *A transmission line for which R, L, G, C are strictly positive is realistic (see Figure 3).*

Proof. Let $Z(s)$ be the impedance of the line shown in (4) and set $\lambda = \max\{-R/L, -G/C\} < 0$. As $\gamma(s)$ and $z_0(s)$ are analytic and non-vanishing in the half-plane $\Pi_\lambda = \{s : \Re s > \lambda\}$, and since moreover γ is never pure imaginary in Π_λ, the matrix Z is analytic there. As $\det Z(s) = z_0^2$, we get that $Y = Z^{-1}$ is likewise analytic. From (5) and the symmetry of Z, Y, we observe that $Z(s) + Z^*(s) = 2\Re Z(s)$ and $Y(s) + Y^*(s) = 2\Re Y(s)$ are positive symmetric matrices in Π_λ whose entries, being real parts of analytic functions, are harmonic functions of s. They are in fact positive definite at each $s \in \Pi_\lambda$, as (5) implies if $I_1 = x_1 + iy_1$ and $I_2 = x_2 + iy_2$ are not both zero that

$$(x_1, x_2)\Re(Z(s))(x_1, x_2)^t) + (y_1, y_2)\Re(Z(s))(y_1, y_2)^t > 0$$

(for in this case $I(\xi)$ is not identically zero). We claim that

$$\Re(Y(i\omega)) \geq \alpha\mathbf{Id} \;\text{ for some } \alpha > 0, \tag{16}$$

and this will imply that (i) of definition 19 is met. Indeed, if (16) holds and $u \in \mathbb{R}^2$ is a unit vector, the function $-u^t\Re(Y)u$ is a nonpositive harmonic function in Π_0 whose limit at every point of the imaginary axis exists and is at most $-\alpha$, therefore $-u^t\Re(Y)u$ is at most $-\alpha$ everywhere in Π_0 by the extended maximum principle[24, thm. 3.6.9]. To prove (16), we may restrict to large $|\omega|$ because on any compact interval of the imaginary axis it certainly holds by strict positivity of $\Re(Y(i\omega))$ and continuity of the latter with respect to ω. Without loss of generality, we choose the branch of the square root which is positive for positive arguments. In view of (4), we can write:

$$Y = z_0^{-1}\begin{pmatrix} \frac{1+e^{-2\gamma}}{1-e^{-2\gamma}} & -\frac{2e^{-\gamma}}{1-e^{-2\gamma}} \\ -\frac{2e^{-\gamma}}{1-e^{-2\gamma}} & \frac{1+e^{-2\gamma}}{1-e^{-2\gamma}} \end{pmatrix} = z_0^{-1}M. \quad (17)$$

Since $z_0(i\omega) \to \sqrt{L/C} > 0$ when $\omega \to \pm\infty$ and the matrix M in (17) is symmetric, it is enough to show that $M(i\omega)$ is bounded and that the eigenvalues of $\Re(M(i\omega))$ are greater than some $\delta > 0$ for $|\omega|$ large enough. Now, using the Taylor expansion of $(1+x)^{1/2}$ at $x \sim 0$, we get

$$\Re\gamma(i\omega) = \sqrt{LC}\Big(i\omega + \frac{R+G}{2} + O\big(|\omega|^{-1}\big)\Big)$$

so that $\Re\gamma(i\omega) \geq \kappa = \sqrt{LC}(R+G)/3 > 0$, say, for $|\omega|$ large enough. Then $|e^{-2\gamma(i\omega)}| \leq e^{-\kappa} < 1$, hence $M(i\omega)$ is bounded. To prove that the eigenvalues of $\Re(M(i\omega))$ are greater than $\delta > 0$, we check that its trace and determinant are positive with $\det\Re(M(i\omega)) \geq \eta > 0$ for $|\omega|$ large. Since the trace is bounded (recall $M(i\omega)$ is bounded), we will be done. First, it is readily seen that

$$\operatorname{tr}\Re(M(i\omega)) = 2\frac{1-e^{-4\Re(\gamma(i\omega))}}{|1-e^{-2\gamma}|^2}$$

is indeed positive. Next, a short computation yields that

$$\begin{aligned}\det\Re(M(i\omega)) \quad &= \frac{(1-e^{-4\Re(\gamma(i\omega))})^2}{|1-e^{-2\gamma}|^4} \\ &\quad -4\frac{(\Re(e^{-\gamma(i\omega)} - e^{-\gamma(i\omega)-2\bar\gamma(i\omega)}))^2}{|1-e^{-2\gamma}|^4} \\ = \Big(1-e^{-4\Re\gamma(i\omega)}&+2\cos(\Im\gamma(i\omega))e^{-\Re\gamma(i\omega)}(1-e^{-2\Re\gamma(i\omega)})\Big) \\ \times\Big(1-e^{-4\Re\gamma(i\omega)}&-2\cos(\Im\gamma(i\omega))e^{-\Re\gamma(i\omega)}(1-e^{-2\Re\gamma(i\omega)})\Big) \\ &\times|1-e^{-2\gamma}|^{-4}.\end{aligned}$$

So, we are left to prove that

$$1-e^{-4\Re\gamma(i\omega)} \pm 2\cos(\Im\gamma(i\omega))e^{-\Re\gamma(i\omega)}(1-e^{-2\Re\gamma(i\omega)})$$

is positive and bounded away from 0 for $|\omega|$ large enough. As $0 < e^{-\Re\gamma(i\omega)} < e^{-\kappa} < 1$, this comes from the fact that

$$1-2x+2x^3-x^4 = (1-x)^3(1+x)$$

is strictly positive for $x \in (0,1)$. The same argument with z_0^{-1} replaced by z_0 in (17) gives us (ii) of definition 19. □

Until now, we considered linearized diodes as pure negative resistors and linearized transistors as pure current sources controlled by voltages. Such ideal models are usually good at working frequencies, but they are not realistic in that they still show gain at very high frequencies, whereas physical devices will not. Below we describe more realistic versions thereof.

7.1 Realistic models of linearized active components

We adopt the paradigm that "what happens at very high frequencies is unimportant beyond passivity", and we set up a somewhat general definition of "realistic" to accommodate various models used in practice. The bottom line is that a realistic model for an active linearized component can be obtained by embedding an ideal model of that component in a circuit comprised of passive elements, whose characteristics make them negligible at working frequencies (*e.g.* a very small resistor in series, or a very small capacitor in parallel) but turn the whole circuit into a passive device at sufficiently high frequencies.

Definition 21 *A linearized square multiport is said to be Y-realistic if its complex impedance matrix Y is in $\mathcal{E}$ and there exists $K > 0$ such that for any s satisfying $\Re(s) \geq 0$ and $|s| > K$:*
(i) $Y(s) + Y^*(s) \geq \alpha\,\mathbf{Id}$ for some $\alpha > 0$.
Similarly a linearized square multiport is said to be Z-realistic if its complex admittance matrix Z is in $\mathcal{E}$ and there exists $K' > 0$ such that for any s satisfying $\Re(s) \geq 0$ and $|s| > K'$:
(ii) $Z(s) + Z^*(s) \geq \beta\,\mathbf{Id}$ for some $\beta > 0$.
A square multiport is said to be realistic if it is Z and Y-realistic.

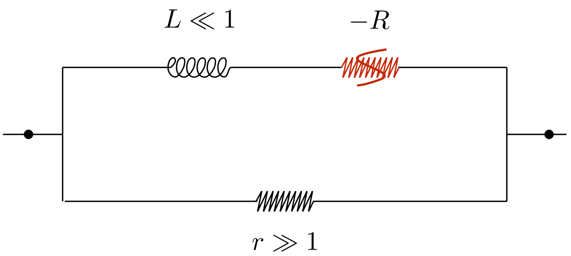
In the remaining of this section, we will exhibit and discuss some realistic models of linearized active components. In the propositions to come, the words "negligible with respect to other quantities in the circuit" are not given a precise definition. They simply mean that, over the working range of frequencies, the realistic model and the ideal model agree up to insignificant order, so that the ideal model could as well be used which corresponds to standard practice with simulators. It must be stressed, however, that the asserted negligibility is not used in the proof of forthcoming Theorem 27 and Corollary 28. There, only passivity at high frequency matters.

Proposition 22 *A possible model of realistic linearized diode is a dipole whose complex impedance Z and complex admittance Y have the following properties.*

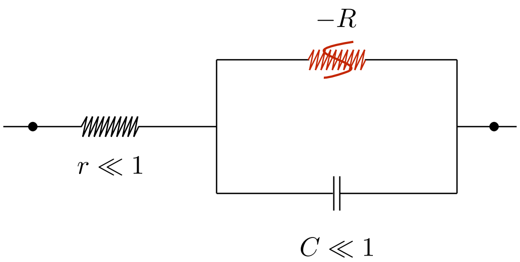
- *$Z(s)$ is a rational function with real coefficients.*

- *There is $\omega_0 > 0$ such that, whenever $|s| < \omega_0$, then $Z(s) = -R + \epsilon(s)$ where $R < 0$ and ϵ is negligible compared to other quantities in the circuit..*
- *There is $\omega_1 > \omega_0$ such that, whenever $|s| > \omega_1$ and $\Re(s) \geq 0$, then $\Re(Y(s)) \geq \alpha$ and $\Re(Z(s)) \geq \beta$, for some $\alpha, \beta > 0$.*

Both circuits proposed in Figure 15 are valid according to Proposition 22, but many other, more complex models could also be used.



(a) With inductive effect and high resistance (e.g., of the air around the diode)



(b) With capacitive effect and small resistance (e.g., of the wire)

Fig. 15. Two realistic models of a linearized diode

Transistors can also be modeled in a realistic way, to account for the fact that actual devices have no gain anymore at very high frequencies. For instance, the model presented in Figure 16 corresponds to what is called the *intrinsic model of the linearized transistor* [28]. It involves capacitive effects at the junctions between semiconductors, and it only satisfies condition (i) of definition 21. As in the case of a diode, more complex and accurate models, incorporating both inductive and capacitive effects, could be given, that we do not touch upon as they are not significant to our discussion.

Proposition 23 *A possible Y-realistic linearized field effect transistor (FET) circuit is given on Figure 16. We call $Y(s)$ the admittance matrix $Y(s)$ of this circuit, relating the tension vector $(V_G - V_S, V_D - V_S)^t$ to the corresponding currents $(I_G, I_D)^t$.*

- *The admittance matrix admits following asymptotic expansion at infinity,*

$$\begin{cases} Y_{1,1}(s) &= \frac{1}{r} + O(1/s) \\ Y_{2,1}(s) &= \frac{-C_{gd}}{r_g(C_{gd}+C_g)} + O(1/s) \\ Y_{1,2}(s) &= \frac{-C_{gd}}{r_g(C_{gd}+C_g)} + O(1/s) \\ Y_{2,2}(s) &= C_{gd}(1 - \frac{C_{gd}}{C_{gd}+C_g})s + 2g_d + 2g_m \frac{C_{gd}}{C_{gd}+C_g} \\ & \quad + \frac{(C_{gd})^2}{r_g(C_{gd}+C_g)^2} + O(1/s) \end{cases} \tag{18}$$

- *Under the assumption that $g_d > 0$, there exists ω_0 such that for $|s| > \omega_0$ and $\Re(s) \geq 0$ the matricial inequality $Y(s) + Y^*(s) \geq \gamma \mathbf{Id}$ holds for some $\gamma > 0$.*

Proof. Applying Ohm's law to the circuit of Figure 16 we obtain

$$\begin{pmatrix} I_G \\ I_D \end{pmatrix} = \underbrace{\begin{pmatrix} (C_g + C_{gd})s & -C_{gd}s \\ -C_{gd}s + g_m & C_{gd}s + g_d + C_d \end{pmatrix}}_{y(s)} \begin{pmatrix} U_i \\ V_D - V_S \end{pmatrix}.$$

In terms of $y(s)$, the admittance matrix $Y(s)$ express as:

$$Y(s) = \begin{pmatrix} \frac{y_{1,1}}{1+r_g y_{1,1}(s)} & \frac{y_{1,2}(s)}{1+r_g y_{1,1}(s)} \\ y_{2,1} - \frac{r_g y_{2,1} y_{1,1}(s)}{1+r_g y_{1,1}(s)} & y_{2,2}(s) - \frac{r_g y_{2,1}(s) y_{1,2}(s)}{1+r_g y_{1,1}(s)} \end{pmatrix} \tag{19}$$

Asymptotic expansions based on Taylor expansions yield (18). Eventually, checking that the principal minors of the asymptotic expansion of $Y(s)+Y^*(s)$ deduced from (18) are bounded away from zero for $\Re(s) \geq 0$ and $g_d > 0$, yields the second assertion of the proposition.

Note that this model could be rendered Z-realistic, and therefor completely realistic, by adding a resistor in series at port 2 of the device.

□

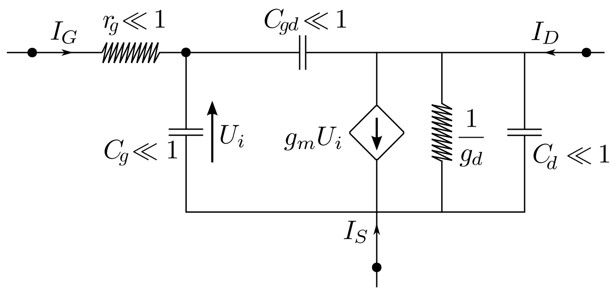


Fig. 16. Intrinsic model of a linearized transistor

8 Realistic partial transfer functions

We consider a circuit whose linearization comprises only realistic components in the sense of Definitions 19 and

21, namely, realistic passive dipoles, transmission lines with $R, L, G, C > 0$, realistic linearized diodes and transistors. The partial frequency responses of such a circuit belong to the class $\mathcal{E}$ introduced in Section 6, just like partial frequency responses of ideal circuits. This comes from the fact that a partial frequency response is still obtained by inverting a matrix with entries in $\mathcal{E}$. However, unlike ideal circuits, not all functions in $\mathcal{E}$ can be realized as partial frequency responses of circuits made of realistic components. In particular, the function f defined in (13) is not realizable this way. This is a consequence of the following proposition.

Proposition 24 *Assume that a linearized circuit made of Y-realistic components is excited at node k by a small current source as in Figure 7.*
Then, there exists $K > 0$ such that, for any s satisfying $\Re(s) \geq 0$ and $|s| \geq K$, the following properties hold.

(1) The nodal admittance matrix $\mathbf{Y}(s)$ of the linearized circuit (cf. (8)) satisfies $\mathbf{Y}(s) + \mathbf{Y}^*(s) \geq \alpha\, \mathbf{Id}$, for some $\alpha > 0$.
(2) $\mathbf{Y}(s)$ is invertible, therefore each partial frequency response $Z_{k,j}(s)$ of the circuit to the current source at node k is well-defined, by (10).
(3) the matrix $\mathbf{Z}(s) = \mathbf{Y}(s)^{-1}$ is bounded, uniformly with respect to s such that $\Re(s) \geq 0$ and $|s| \geq K$.

Proof. With the notation of Section 3, we consider the branch admittance matrix

$$\mathbf{Y}_b = \mathrm{diag}(Y_1, Y_2, \ldots, Y_h),$$

where Y_i is the impedance matrix associated with component i of the circuit.

(i) According to Definition 21, the fact that i is a Y-realistic square multiport implies that there exists α_i such that $\Re(Y_i(s)) \geq \alpha_i\, \mathbf{Id}$ whenever $\Re(s) \geq 0$ and $|s|$ is large enough. Let $\alpha_{\min} > 0$ be the infimum of the α_i. Then $\alpha_{\min} > 0$ and we have that $\mathbf{Y}_b(s) + \mathbf{Y}_b^*(s) \geq \alpha_{\min}\, \mathbf{Id}$ for all s with $\Re(s) \geq 0$ and $|s| \geq K$, for some K.

Since the incident matrix A has full rank, by the assumed connectivity of the graph of a circuit, the symmetric matrix $A\,A^t$ is nonsingular so there exist $\lambda > 0$ such that $A\,A^t \geq \lambda\, \mathbf{Id}$. Letting $\alpha = \alpha_{\min}\, \lambda$, we deduce from (9) and the realness of A that assertion (1) holds.

(ii) Assume that for some s satisfying $\Re(s) \geq 0$ and $|s| \geq K$, the matrix $Y(s)$ fails to be invertible. Then, there is a non zero complex vector v such that $\mathbf{Y}(s)v = 0$, thus also $v^*\mathbf{Y}(s)v = 0$. Taking the real part, we get $v^*(\mathbf{Y}(s) + \mathbf{Y}^*(s))v = 0$ which contradicts the first item. Therefore assertion (2) holds.

(iii) That $\|Z(s)\| \leq 1/(2\alpha)$ when $\mathbf{Y}(s) + \mathbf{Y}^*(s) \geq \alpha\, \mathbf{Id}$ was proven earlier, see (14) and (15). Assertion (3) now follows from assertion (1). □

Remark 25 *We have proven that the branch admittance $\mathbf{Y_b}$ matrix of the circuit is Y-realistic. Working with the branch impedance matrix $\mathbf{Z}_b$ instead, it can be similarly proved that $\mathbf{Z}_b$ is Z-realistic. In fact, a linearized circuit made of active and passive realistic components is realistic in the sense of Definition 21.*

Remark 26 *The dual of proposition 24 can be proven, changing Z in Y and Y-realistic into Z-realistic.*

9 Application to stability analysis.

We derive in this section some consequences of Proposition 24, which point at a remarkable difference between ideal linearized circuits, described in Section 6, and realistic ones introduced in Section 8.

Theorem 27 *Let $Z(s)$ be a partial frequency response of a realistic linearized circuit. Then, for $|s| > K$ and $\Re(s) \geq 0$, $Z(s)$ is bounded.*

Proof. This is an immediate consequence of Proposition 24, assertion (3). □

Recall the space H^∞ of bounded analytic functions in the right half-plane Π_0, and the function field $\mathcal{E}$ described in Section 6.2. Recall also that a rational function is said to be strictly proper if it vanishes at infinity, *i.e.* if either the degree of the numerator is strictly less than the degree of the denominator or else it is the zero function.

Corollary 28 *With the assumption and notation of Theorem 27, $Z(s)$ has only finitely many unstable poles. Specifically, there is a function $h \in H^\infty \cap \mathcal{E}$ and a strictly proper rational function $\mathfrak{r}$ having poles in the closed right half-plane only, such that $Z(i\omega) = h(i\omega) + \mathfrak{r}(i\omega)$.*

Proof. Let $\overline{D(0,K)}$ denote the closed disk centered at 0 of radius K. By the previous theorem, the unstable poles of $Z(s)$ (if any) lie in the compact set $\overline{\Pi}_0 \cap \overline{D(0,R)}$. But since $Z(s)$ is an element of $\mathcal{E}$, it is a meromorphic function on $\mathbb{C}$ and so it can only have finitely many poles on a compact set. Hence Z has at most finitely many unstable poles, as announced.

Number these unstable poles as $s_1 \ldots s_N$, with respective multiplicities $\nu_1, \ldots \nu_N$. Let $\mathfrak{r}_j = p_{\nu_j - 1}(s)/(s - s_j)^{\nu_j}$ be the principal part of $Z(s)$ at s_j, where $p_{\nu_j-1}(s)$ is a polynomial of degree $\nu_j - 1$. It is a strictly proper rational function. Set $\mathfrak{r} = \sum_{j=1}^{N} \mathfrak{r}_j$; if $N = 0$ the sum is empty and we have that $\mathfrak{r} = 0$. By construction, $Z - \mathfrak{r}$ is a meromorphic function with no poles in the closed right half-plane. Moreover it is bounded there for large $|s|$, because so is Z by assertion (3) of Theorem 27 and so is the strictly proper rational function $\mathfrak{r}$. Hence $Z - \mathfrak{r} \in H^\infty$, as was to be shown. □

In the notation of Corollary 28, checking stability of an equilibrium in a circuit to a small current perturbation at node k means finding out whether $\mathfrak{r} = 0$ or not for each partial frequency response Z from node k. Of course, such a clear-cut answer is hard to make from simulations of $Z(i\omega)$ at finitely many points of the imaginary axis. Still, Corollary 28 suggests that identification methods should favor in this case a model class consisting of meromorphic functions with prescribed number n of poles in the right half-plane, because the theoretical response is of this type with $n = N$. This seems better suited than trying to fit a rational approximant with free poles to the non-rational function Z.

Two approximation techniques appear to be of special interest in this connection. The first is the half-plane version of the Adamjan-Arov-Krein theory on meromorphic approximation with n poles in the uniform norm, also known as Hankel norm approximation, which is of standard use today in control and order reduction [11, 21]. The second is best meromorphic approximation with n unstable poles in L^2 of the line, which is equivalent to H^2-best rational approximation on the disk [2] for which efficient algorithms exist [18]. One would typically use this kind of approximation for increasing values of n: the case $n = 0$ gives an estimate of the size of the unstable part, while the case $n = N$ (of course N is unknown) would in principle allow one to recover $\mathfrak{r}$. Note that both algorithms work in the matrix-valued setting, which should be helpful to improve the estimation of $\mathfrak{r}$ by jointly approximating several partial frequency responses from the same node using a common denominator (*cf.* (10)) or even a block of partial frequency responses from a set of nodes to another set of nodes, using a matrix fractional representation for the block (*cf.* (9)). It is worth stressing that, at the functional level, computing $\mathfrak{r}$ is a linear operation. Assume indeed that the partial frequency response function $Z(s)$ belongs to $L^2(i\mathbb{R})$. This is a realistic assumption in that it is fulfilled as soon as the response rolls off like $1/|\omega|$ at infinity, which is typical of capacitive effects. Then, $Z(s)$ will be stable if and only if it belongs to the *Hardy space* $\mathcal{H}^2$ of the right half plane [2]. Similarly, let $\mathcal{H}^2_-$ denote the Hardy space of the left half-plane. Using the orthogonal decomposition

$$L^2(i\mathbb{R}) = \mathcal{H}^2 \oplus \mathcal{H}^2_-,$$

we see that the orthogonal projection of $Z(s)$ onto $\mathcal{H}^2_-$ is precisely $\mathfrak{r}$. In particular, the L^2-norm of the latter, as compared to the expected numerical error, provides one with an initial cheap test for instability. When the response is not square summable but merely bounded on the imaginary axis, similar considerations are still valid in a non Hilbertian context.

[2] This space consists of holomorphic functions in the right half plane such that $\sup_{x>0} \|F(x+\,.)\|_{L^2(i\mathbb{R})} < \infty$ [12, Ch. 8].

Of course several basic issues remain to be addressed in practice. First of all, one has to extrapolate finitely many pointwise data for $Z(i\omega)$ on a limited range of frequencies into a function given at all frequencies. In this connection, the behavior at infinity is an important question that requires special care, for instance working with weights. This is essential to estimate the unstable part, which is not a trivial task. Also, one may have to study the quantitative behavior of the response in greater detail to decide how significant this unstable part with respect to numerical errors. Such questions are left here for further research, but the fact that $Z(i\omega)$ may in principle be computed with good precision at a great many frequencies, unlike in most identification problems, leads the authors to the belief that the problem is indeed amenable to function-theoretic techniques.

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