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Jean-François Jabir[*], Denis Talay[†] and Milica Tomašević [†‡]

Abstract: In this work, we prove the well–posedness of a singularly interacting stochastic particle system and we establish propagation of chaos result towards the one-dimensional parabolic-parabolic Keller-Segel model.
Key words: Chemotaxis model; Interacting particle system; Singular McKean-Vlasov dynamic.
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1 Introduction

The standard d-dimensional parabolic–parabolic Keller–Segel model for chemotaxis describes the time evolution of the density ρ_t of a cell population and of the concentration c_t of a chemical attractant:

$$\begin{cases} \partial_t \rho(t,x) = \nabla \cdot (\frac{1}{2}\nabla\rho - \chi\rho\nabla c), \quad t>0,\ x\in\mathbb{R}^d, \\ \alpha\partial_t c(t,x) = \triangle c - \lambda c + \rho, \quad t>0,\ x\in\mathbb{R}^d. \\ \rho(0,x)=\rho_0(x), c(0,x)=c_0(x). \end{cases} \tag{1}$$

See e.g. Corrias *et al.* [3], Perthame [8] and references therein for theoretical results on this system of PDEs and applications to Biology.

For the parabolic–elliptic version of the model, that is, when $\alpha = 0$, the first stochastic interpretation of this system is due to Haškovec and Schmeiser [5] who analyzed a particle system with McKean–Vlasov interactions and Brownian noise. More precisely, as the ideal interaction kernel should be strongly singular, they introduce a kernel with a cut-off parameter and obtain the tightness of the particle probability distributions w.r.t. the cut-off parameter and the number of particles. They also obtain partial results in the direction of the propagation of chaos. More recently, in the subcritical case, that is, when the parameter χ of the parabolic–elliptic model is small enough, Fournier and Jourdain [4] obtained the well–posedness of a particle system without cut-off. In addition, they obtain a consistency property which is weaker than the propagation of chaos. They also describe complex behaviors of the particle system in the sub and super critical cases. Cattiaux and Pédèches [2] also obtain the well-posedness of this particle system by using Dirichlet forms rather than pathwise approximation techniques.

W.l.o.g. one may suppose that $\|\rho_0\|_{L^1(\mathbb{R}^d)} = 1$. In Talay and Tomašević [11], the authors propose a different type of stochastic representation of the parabolic–parabolic ($\alpha = 1$) Keller–Segel model: we construct a McKean–Vlasov non-linear stochastic differential system with a singular interaction which is non classical in the sense that it involves all the past time marginals of the probability distribution of the solution. The system reads

$$\begin{cases} dX_t = b(t,X_t)dt + \left\{\chi \int_0^t (K_{t-s}\star\rho_s)(X_t)ds\right\}dt + dW_t, \quad t>0, \\ p_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0(x)dx, \end{cases} \tag{2}$$

where $K_t(x) := e^{-\lambda t}\nabla(\frac{1}{(2\pi t)^{d/2}}e^{-\frac{|x|^2}{2t}})$ and $b(t,x) = e^{-\lambda t}\nabla\mathbb{E}[c_0(x+\sqrt{2}W_t)]$. Without any other condition than the absolute continuity w.r.t. Lebesgue's measure of the initial distribution of X_0 we show that this system is well-posed in the one-dimensional case and that $\mathcal{L}(X_t)$ has a density which uniquely solves a non-linear martingale problem. This density coupled with a suitable transformation of it are shown to uniquely

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solve the one–dimensional Keller–Segel system. In the work in progress Tomašević [12], additional techniques are developed to handle the two-dimensional system.

In this note we consider the stochastic particle system which naturally approximates the solution to (2) in the one-dimensional case. It inherits from the limit equation the non classical property that at each time t each particle interacts in a singular way with the past of all the other particles. We prove that the particle system is well–posed and propagates chaos to the unique weak solution of (2). Compared to the stochastic particle systems introduced for the parabolic–elliptic model, an interesting phenomenon occurs: the interaction with the past of the other particles can be handled by purely Brownian techniques (up to Girsanov transforms) rather than using Bessel processes. However, we emphasize that our technique cannot be extended to the multi-dimensional Keller–Segel situation without deep additional techniques because we here implicitly use that the particles interact through a functional defined on the path space by means of a kernel $K \in L^1(0,T;L^2(\mathbb{R}))$. In higher dimensions, the $L^1(0,T;L^2(\mathbb{R}^d))$-norm of the kernel is infinite.

The paper is organized as follows. In Section 2 we state our two main results and comment our techniques to prove them. In Section 3 we prove some technical lemmas. In Section 4 we prove that the particle system is well–posed and propagates chaos to the solution of (2).

2 Main results

Our main results concern the well–posedness and propagation of chaos of the following system of one–dimensional particles:

$$\begin{cases} dX_t^{i,N} = b(t,X_t^{i,N})dt + \left\{\frac{\chi}{N}\sum_{j=1,j\neq i}^N \int_0^t K_{t-s}(X_t^{i,N}-X_s^{j,N})ds\mathbb{1}_{\{X_t^i \neq X_t^j\}}\right\}dt + dW_t^i, \quad t\leq T, \\ X_0^i, \text{ i.i.d. independent of } (W^i, 1\leq i\leq N), \end{cases}$$

where T is an arbitrary time horizon, and the $(W_t^i;\, 0\leq t\leq T)$ are N independent standard Brownian motions. For simplicity, we here assume that $\lambda = 0, \chi = 1$ and $b \equiv 0$. Under the hypothesis in [11], the dynamics (5) for the Keller–Segel equation involves a bounded and Lipschitz drift $b(t,x)$. It is easy to extend the methodology below to this case. We thus consider

$$\begin{cases} dX_t^{i,N} = \left\{\frac{1}{N}\sum_{j=1,j\neq i}^N \int_0^t K_{t-s}(X_t^{i,N}-X_s^{j,N})ds\mathbb{1}_{\{X_t^{i,N} \neq X_t^{j,N}\}}\right\}dt + dW_t^i, \quad t\leq T, \\ X_0^i, \text{ i.i.d. independent of } (W^i, 1\leq i\leq N), \end{cases} \tag{3}$$

where K_t is now $K_t(x) = \frac{-x}{\sqrt{2\pi}t^{3/2}}e^{-\frac{x^2}{2t}}$. Throughout the rest of the paper, for each $N \geq 1$, a weak solution to the N-interacting particle system (3) will be understood in the standard sense (see e.g. Karatzas and Shreve [6]). The existence of a weak solution to (3) is given by the following theorem:

***Theorem* 1.** *Given $0 < T < \infty$ and $N \in \mathbb{N}$, for any initial distribution of $(X_0^i, i\leq N)$, there exists a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, 0\leq t\leq T), \mathbb{Q}^N, W, X)$ to the N-interacting particle system (3) that satisfies*

$$\forall 1\leq i\leq N, \quad \mathbb{Q}^N\left(\int_0^T \left(\frac{1}{N}\sum_{j=1,j\neq i}^N \int_0^t K_{t-s}(X_t^{i,N}-X_s^{j,N})ds\mathbb{1}_{\{X_t^i\neq X_t^j\}}\right)^2 dt < \infty\right) = 1. \tag{4}$$

In view of [6, Chapter 5, Proposition 3.10], one has the following uniqueness result:

Corollary 1. *Weak uniqueness holds in the class of weak solutions satisfying (4).*

The construction of a weak solution to (3) involves arguments used by Krylov and Röckner [7, Section 3] to construct a weak solution to SDEs with singular drifts. It relies on the Girsanov transformation which removes all the drifts of (3).

***Remark* 1.** *In (3), the indicator of $\{X_t^{i,N} \neq X_t^{j,N}\}$ makes it clear that the drift coefficient is well defined. Our construction shows that the law of the particle system is equivalent to the Wiener measure. Consequently, almost surely the set $\{t\leq T, X_t^i = X_t^j\}$ has Lebesgue measure zero.*

Our second main theorem concerns the propagation of chaos of the system (3).

***Theorem* 2.** *Assume that the initial distribution of X_0^i has a density ρ_0 for all $i \leq N$. The empirical measure $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ of (3) converges, when $N \to \infty$, in distribution to the unique weak solution of*

$$\begin{cases} dX_t = \Big\{ \int_0^t (K_{t-s} \star \rho_s)(X_t) ds \Big\} dt + dW_t, \quad t \leq T, \\ p_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0(x)dx. \end{cases} \tag{5}$$

To prove the tightness and weak convergence of μ^N, we do not use the Girsanov transformation which removes all the drifts of (3). Actually, it seems difficult to get uniform bounds w.r.t. N for the exponential moment appearing in the Novikov criterion. We thus work with a new probability measure under which only a fixed number of particles are independent Brownian motions. This trick might be useful when dealing with other particle systems with singular interaction.

3 Preliminaries

Let F_t be a functional on the path space,

$$F_t(\omega, \widehat{\omega}) = \left(\int_0^t K_{t-s}(w_t - \widehat{w}_s) ds \mathbb{1}_{\{w_t \neq \widehat{w}_t\}} \right)^2, \tag{6}$$

where $(\omega, \widehat{\omega}) \in C([0,T];\mathbb{R}) \times C([0,T];\mathbb{R})$. The objective of this section is to show that $\int_0^T F_t(W,X)dt$ has finite exponential moments under some conditions on W and X. The following key property of the kernel K_t will be used

$$\|K_t\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} \frac{x^2}{t^2} g_t^2(x)\, dx \right)^{1/2} = \left(2 \int_0^\infty \frac{z^2}{t^{3/2}} e^{-\frac{z^2}{2}} dz \right)^{1/2} \leq \frac{C}{t^{3/4}}. \tag{7}$$

We will proceed as in the proof of the local Novikov Condition (see Karatzas and Shreve [6, Chapter 3, Corollary 5.14]) by localizing on small intervals of time. Thus, we start with the following lemmas. In all the sequel $C > 0$ denotes any universal constant that may change from line to line.

***Lemma* 1.** *Let $T > 0$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, 0 \leq t \leq T), \mathbb{P})$ be a filtered probability space equipped with a Brownian motion W. There exists $C_0 > 0$ such that for any path $x \in C([0,T];\mathbb{R})$ and any $0 \leq t_1 \leq t_2 \leq T$ one has*

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{t_1}^W} \left(\int_{t_1}^{t_2} F_t(W,x) dt \right) \leq C_0 \sqrt{T} \sqrt{t_2 - t_1}.$$

Proof. By Fubini-Tonelli theorem and the definition of F,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{t_1}^W} \left(\int_{t_1}^{t_2} F_t(W,x) dt \right) \leq \int_{t_1}^{t_2} \int_0^t \int_0^t \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{t_1}^W} |K_{t-s}(W_t - x_s) K_{t-u}(W_t - x_u)| ds du dt. \tag{8}$$

Notice that

$$\sqrt{\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{t_1}^W} (K_{t-s}^2(W_t - x_s))} = \sqrt{\int K_{t-s}^2(y + W_{t_1} - x_s)) g_{t-t_1}(y) dy} \leq \frac{C \|K_{t-s}\|_{L^2(\mathbb{R})}}{(t-t_1)^{1/4}} \leq \frac{C}{(t-s)^{3/4}(t-t_1)^{1/4}}.$$

Here we used that the density of $W_t - W_{t_1}$ is bounded by $\frac{C}{\sqrt{t-t_1}}$. Coming back to (8), Cauchy-Schwarz inequality leads to

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{t_1}^W} \left(\int_{t_1}^{t_2} F_t(W,x) dt \right) \leq C \int_{t_1}^{t_2} \frac{1}{\sqrt{t-t_1}} \int_0^t \int_0^t \frac{1}{(t-s)^{3/4}(t-u)^{3/4}} ds du dt = C \int_{t_1}^{t_2} \frac{\sqrt{t}}{\sqrt{t-t_1}} dt,$$

which ends the proof. □

***Lemma* 2.** *Same assumptions as in Lemma 1. Let C_0 be as in Lemma 1. For any $\kappa > 0$, there exists a constant $C(T,\kappa)$ such that for any $0 \leq T_1 \leq T_2 \leq T$, satisfying $T_2 - T_1 < \frac{1}{C_0^2 T \kappa^2}$, one has*

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{T_1}^W} \left[\exp\left\{ \kappa \int_{T_1}^{T_2} F_t(W,x) dt \right\} \right] \leq C(T,\kappa).$$

Proof. We adapt the proof of Khasminskii's lemma in Simon [9]. Admit for a while we have shown that there exists a constant $C(\kappa, T)$ such that for any $M \in \mathbb{N}$

$$\sum_{k=1}^{M} \frac{\kappa^k}{k!} \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{T_1}^W} \left(\int_{T_1}^{T_2} F_t(W^1, W^2) dt \right)^k \leq C(\kappa, T), \tag{9}$$

provided that $T_2 - T_1 < \frac{1}{C_0^2 T \kappa^2}$. To obtain the desired result, it remains to apply Fatou's lemma.

We now prove (9). Start with the equality

$$\left(\int_{T_1}^{T_2} F_t(W,x) dt \right)^k = k! \int_{T_1}^{T_2} F_{t_1}(W,x) \int_{t_1}^{T_2} F_{t_2}(W,x) \int_{t_2}^{T_2} \cdots \int_{t_{k-2}}^{T_2} F_{t_{k-1}}(W,x) \int_{t_{k-1}}^{T_2} F_{t_k}(W,x)\, dt_k\, dt_{k-1} \cdots dt_2\, dt_1.$$

By the tower property of conditional expectation,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{T_1}^W} \left[\left(\int_{T_1}^{T_2} F_t(W,x) dt \right)^k \right] = k! \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{T_1}^W} \Big[\int_{T_1}^{T_2} F_{t_1}(W,x) \int_{t_1}^{T_2} F_{t_2}(W,x) \int_{t_2}^{T_2} \cdots$$
$$\cdots \int_{t_{k-2}}^{T_2} F_{t_{k-1}}(W,x) \left(\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{t_{k-1}}^W} \int_{t_{k-1}}^{T_2} F_{t_k}(W,x)\, dt_k \right) dt_{k-1} \cdots dt_2\, dt_1 \Big].$$

In view of Lemma 1,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{t_{k-1}}^W} \int_{t_{k-1}}^{T_2} F_{t_k}(W,x)\, dt_k \leq C_0 \sqrt{T} \sqrt{T_2 - t_{k-1}} \leq C_0 \sqrt{T} \sqrt{T_2 - T_1}.$$

Therefore,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{T_1}^W} \left[\left(\int_{T_1}^{T_2} F_t(W,x) dt \right)^k \right] \leq k! C_0 \sqrt{T} \sqrt{T_2 - T_1} \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{T_1}^W} \Big[\int_{T_1}^{T_2} F_{t_1}(W,x) \int_{t_1}^{T_2} F_{t_2}(W,x) \int_{t_2}^{T_2} \cdots$$
$$\cdots \int_{t_{k-2}}^{T_2} F_{t_{k-1}}(W,x) dt_{k-1} \cdots dt_2\, dt_1 \Big].$$

Now we condition with respect to $\mathcal{F}_{t_{k-2}}$ and use Lemma 1. Repeating this procedure $k-1$ times,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{T_1}^W} \left[\left(\int_{T_1}^{T_2} F_t(W,x) dt \right)^k \right] \leq k! (C_0 \sqrt{T} \sqrt{T_2 - T_1})^{k-1} \mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{T_1}^W} \int_{T_1}^{T_2} F_{t_1}(W,x) dt_1.$$

Again, by Lemma 1,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{F}_{T_1}^W} \left[\left(\int_{T_1}^{T_2} F_t(W,x) dt \right)^k \right] \leq k! (C_0 \sqrt{T} \sqrt{T_2 - T_1})^k$$

Thus, (9) is satisfied provided that $T_2 - T_1 < \frac{1}{C_0^2 T \kappa^2}$. □

***Proposition* 1.** *Let $T > 0$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, 0 \leq t \leq T), \mathbb{P})$ be a filtered probability space on which are defined two independent one dimensional stochastic processes, a Brownian motion W and a process X. For any $\alpha > 0$,*

$$\mathbb{E}_{\mathbb{P}} \left[\exp \left\{ \alpha \int_0^T F_t(W,X) dt \right\} \right] \leq C(T, \alpha),$$

where the constant $C(T,\alpha)$ depends only on T and α, but does not depend of the law of the process X.

Proof of Proposition 1. Observe that

$$\mathbb{E}_{\mathbb{P}} \left[\exp \left\{ \alpha \int_0^T F_t(W,X) dt \right\} \right] = \int_{C([0,T];\mathbb{R})} \mathbb{E}_{\mathbb{P}} \left[\exp \left\{ \alpha \int_0^T F_t(W,x) dt \right\} \right] \mathbb{P}^X(dx). \tag{10}$$

Set $\delta := \frac{1}{2C^2T\alpha^2} \wedge T$, where C_0 is fixed in Lemma 1. Set $n := \left[\frac{T}{\delta}\right]$. Then,

$$\exp\left\{\alpha \int_0^T F_t(W,x)dt\right\} = \prod_{m=0}^{n} \exp\left\{\alpha \int_{(T-(m+1)\delta)\vee 0}^{T-m\delta} F_t(W,x)\,dt\right\}.$$

Condition the quantity of interest by $\mathcal{F}^W_{(T-\delta)\vee 0}$,

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\alpha \int_0^T F_t(W,x)dt\right\}\right] \le \mathbb{E}_{\mathbb{P}}\left[\prod_{m=1}^{n} \exp\left\{\alpha \int_{(T-(m+1)\delta)\vee 0}^{T-m\delta} F_t(W,x)\,dt\right\} \mathbb{E}_{\mathbb{P}}^{\mathcal{F}^W_{T-\delta}}\left(\exp\{\alpha \int_{(T-\delta)\vee 0}^{T} F_t(W,x)dt\}\right)\right].$$

As δ is small enough, we are in the setting of Lemma 2. Thus,

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\alpha \int_0^T F_t(W,x)dt\right\}\right] \le C(T,\alpha)\mathbb{E}_{\mathbb{P}}\left[\prod_{m=1}^{n} \exp\left\{\kappa\chi^2 N \int_{(T-(m+1)\delta)\vee 0}^{T-m\delta} F_t(W,x)\,dt\right\}\right].$$

Successively, conditioning by $\mathcal{F}^W_{(T-(m+1))\vee 0}$ for $m = 1, 2, \dots n-1$ and using Lemma 2,

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\alpha \int_0^T F_t(W,x)dt\right\}\right] \le C^n(T,\alpha)\mathbb{E}_{\mathbb{Q}}\left[\exp\left\{\int_0^{(T-n\delta)\vee 0} F_t(W,x)dt\right\}\right].$$

As the length of the last integral is less than δ, apply one more time Lemma 2,

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\alpha \int_0^T F_t(W,x)dt\right\}\right] \le C(T,\alpha).$$

Finally, plugging the preceding estimate in (10), the proof is complete. $\square$

4 Existence of the particle system and propagation of chaos

4.1 Existence: Proof of Theorem 1

We start from the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t; 0 \le t \le T), \mathbb{W})$ on which is defined a N - dimensional Brownian motion $W = (W^1, \dots, W^N)$. Define the drift-less system

$$\begin{cases} \bar{X}^{i,N}_t = \bar{X}^i_0 + W^i_t, \quad t \le T, \\ \bar{X}^i_0 \text{ i.i.d. and independent of } (W). \end{cases} \tag{11}$$

Denote the drift terms in (3) by $b^{i,N}_t(x)$, $x \in C([0,T];\mathbb{R})^N$, and the vector of all the drifts as $B^N_t(x) = (b^{1,N}_t(x), \dots b^{N,N}_t(x))$. For a fixed $N \in \mathbb{N}$, consider

$$Z^N_T := \exp\left\{\int_0^T B^N_t(\bar{X}) \cdot dW_t - \frac{1}{2}\int_0^T \left|B^N_t(\bar{X})\right|^2 dt\right\}.$$

To prove Theorem 1, it suffices to prove the following Novikov condition holds true (see e.g. [6, Chapter 3, Proposition 5.13])

Proposition 2. *For any $T > 0$, $N \ge 1$, $\kappa > 0$, there exists $C(T,N,\kappa)$ such that*

$$\mathbb{E}_{\mathbb{W}}\left(\exp\{\kappa \int_0^T |B^N_t(\bar{X})|^2 dt\}\right) \le C(T,N,\kappa).$$

Proof. We drop the index N for notation convenience. By the definition of (B_t),

$$\begin{aligned}\mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa \int_0^T \left|B^N_t(\bar{X})\right|^2 dt\right\}\right] &= \mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa \sum_{i=1}^N \int_0^T \left|\frac{1}{N}\sum_{j=1}^N \int_0^t K_{t-s}(\bar{X}^i_t - \bar{X}^j_s)\,ds \mathbb{1}_{\{\bar{X}^i_t \neq \bar{X}^j_t\}}\right|^2 dt\right\}\right] \\ &= \mathbb{E}_{\mathbb{W}}\left[\exp\left\{\frac{1}{N}\sum_{i=1}^N \int_0^T \left(\kappa N \left|\frac{1}{N}\sum_{j=1}^N \int_0^t K_{t-s}(\bar{X}^i_t - \bar{X}^j_s)\,ds \mathbb{1}_{\{\bar{X}^i_t \neq \bar{X}^j_t\}}\right|^2\right) dt\right\}\right].\end{aligned}$$

By Jensen's inequality and the definition of F_t,

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa\int_0^T \left|B_t^N(\bar{X})\right|^2\,dt\right\}\right] \leq \mathbb{E}_{\mathbb{W}}\left[\exp\left\{\frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \int_0^T \kappa N F_t(\bar{X}^i,\bar{X}^j)\,dt\right\}\right],$$

from which we deduce

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa\int_0^T \left|B_t^N(\bar{X})\right|^2\,dt\right\}\right] \leq \frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa N\int_0^T F_t(\bar{X}^i,\bar{X}^j)\,dt\right\}\right].$$

By exchangeability of the particle system,

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa\int_0^T \left|B_t^N(\bar{X})\right|^2\,dt\right\}\right] \leq \mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa N\int_0^T F_t(\bar{X}^1,\bar{X}^2)\,dt\right\}\right]. \tag{12}$$

Now we are in the position to use Proposition 1 which ends the proof. $\square$

4.2 Girsanov transformation with $1 \leq k < N$ particles

In the proof of Theorem 1, we used Property (7) of kernel K and a Girsanov transform reducing the particle system to (11). However, the estimate for the exponential martingale in (12) goes to infinity as $N \to \infty$. Thus, one could not use that Girsanov transform to prove the tightness and propagation of chaos for the particle system.

Using Novikov's condition as in Proposition 1, one gets that there exists a weak solution to

$$\begin{cases} d\widehat{X}_t^{l,N} = dW_t^l, \quad 1 \leq l \leq k, \quad t \leq T, \\ d\widehat{X}_t^{i,N} = \left\{\frac{1}{N}\sum_{j=k+1}^N \int_0^t K_{t-s}(\widehat{X}_t^{i,N} - \widehat{X}_s^{j,N})ds\mathbb{1}_{\{\widehat{X}_t^{i,N}\neq \widehat{X}_t^{j,N}\}}\right\}dt + dW_t^i, \quad k+1 \leq i \leq N, \quad t \leq T, \\ \widehat{X}_0^i \text{ i.i.d. and independent of } (W). \end{cases} \tag{13}$$

Denote the constructed probability measure by $\mathbb{Q}^{k,N}$. Notice that $(\widehat{X}^{l,N}, 1 \leq l \leq k)$ is independent of $(\widehat{X}^{i,N}, k+1 \leq i \leq N)$. We now study the exponential local martingale associated to the change of drift between (3) and (13). For $x \in C([0,T];\mathbb{R})^N$ define the vector,

$$\beta_t^{(k)}(x) := \Big(b_t^{1,N}(x), \ldots, b_t^{k,N}(x), \frac{1}{N}\sum_{i=1}^k \int_0^t K_{t-s}(x_t^{k+1,N} - x_s^{i,N})ds\mathbb{1}_{\{x_t^{k+1,N}\neq x_t^{i,N}\}}, \ldots,$$
$$\frac{1}{N}\sum_{i=1}^k \int_0^t K_{t-s}(x_t^{N,N} - x_s^{i,N})ds\mathbb{1}_{\{x_t^{N,N}\neq x_t^{i,N}\}}\Big).$$

Consider

$$Z_T^{(k)} := \exp\left\{-\int_0^T \beta_t^{(k)}(\widehat{X})\cdot dW_t - \frac{1}{2}\int_0^T |\beta_t^{(k)}(\widehat{X})|^2 dt\right\}.$$

We prove the following proposition,

Proposition 3. *For any $T > 0$, $\gamma > 0$ and $k \geq 1$ there exists $N_0 \geq k$ and $C(T,\gamma,k)$ such that*

$$\forall N \geq N_0, \quad \mathbb{E}_{\mathbb{Q}^{k,N}} \exp\left\{\gamma\int_0^T |\beta_t^{(k)}(\widehat{X})|^2 dt\right\} \leq C(T,\gamma,k).$$

Proof. For $x \in C([0,T];\mathbb{R})^N$, one has

$$|\beta_t^{(k)}(x)|^2 = \sum_{i=1}^k \left(\frac{1}{N}\sum_{j=1}^N \int_0^t K_{t-s}(x_t^i - x_s^j)ds\right)^2 + \frac{1}{N^2}\sum_{j=1}^{N-k}\left(\sum_{i=1}^k \int_0^t K_{t-s}(x_t^{k+j} - x_s^i)ds\right)^2$$

By Jensen's inequality,

$$|\beta_t^{(k)}|^2 \leq \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{N} F_t(x^i,x^j) + \frac{k}{N^2}\sum_{j=1}^{N-k}\sum_{i=1}^{k} F_t(x^{k+j},x^i).$$

For notation convenience, we drop the index N. Applying Cauchy-Schwarz inequality and after applying Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^{k,N}} \exp\left\{\gamma\int_0^T |\beta_t^{(k)}(\widehat{X})|^2 dt\right\}\\
&\leq \left(\mathbb{E}_{\mathbb{Q}^{k,N}} \exp\left\{\sum_{i=1}^{k}\frac{2\gamma}{N}\sum_{j=1}^{N}\int_0^T F_t(\widehat{X}^i,\widehat{X}^j)dt\right\}\right)^{1/2}\left(\mathbb{E}_{\mathbb{Q}^{k,N}} \exp\left\{\frac{2\gamma k}{N^2}\sum_{j=1}^{N-k}\sum_{i=1}^{k}\int_0^T F_t(\widehat{X}^{k+j},\widehat{X}^i)dt\right\}\right)^{1/2}\\
&\leq \left(\prod_{i=1}^{k}\mathbb{E}_{\mathbb{Q}^{k,N}} \exp\left\{\frac{2\gamma k}{N}\sum_{j=1}^{N}\int_0^T F_t(\widehat{X}^i,\widehat{X}^j)dt\right\}\right)^{\frac{1}{2k}}\times\left(\prod_{j=1}^{N-k}\mathbb{E}_{\mathbb{Q}^{k,N}} \exp\left\{\frac{2\gamma k(N-k)}{N^2}\sum_{i=1}^{k}\int_0^T F_t(\widehat{X}^{k+j},\widehat{X}^i)dt\right\}\right)^{\frac{1}{2(N-k)}}.
\end{aligned}$$

Again by Jensen's inequality,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}^{k,N}} \exp\left\{\gamma\int_0^T |\beta_t^{(k)}(\widehat{X})|^2 dt\right\}\\
&\leq \left(\prod_{i=1}^{k}\frac{1}{N}\sum_{j=1}^{N}\mathbb{E}_{\mathbb{Q}^{k,N}}\left\{2\gamma k\int_0^T F_t(\widehat{X}^i,\widehat{X}^j)dt\right\}\right)^{\frac{1}{2k}}\times\left(\prod_{j=1}^{N-k}\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}_{\mathbb{Q}^{k,N}} \exp\left\{\frac{2\gamma k^2}{N}\int_0^T F_t(\widehat{X}^{k+j},\widehat{X}^i)dt\right\}\right)^{\frac{1}{2(N-k)}}.
\end{aligned}$$

In view of Proposition 1, it now remains to prove that there exists $N_0 \in \mathbb{N}$ such that

$$\sup_{N\geq N_0}\mathbb{E}_{\mathbb{Q}^{k,N}}\left[\exp\left\{\frac{2\gamma k^2}{N}\int_0^T F_t(\bar{X}^{k+j},\bar{W}^i)dt\right\}\right] \leq C(T,k,\gamma).$$

We postpone the proof of this inequality to the Appendix A. Finally, for $N \geq N_0$

$$\mathbb{E}_{\mathbb{Q}^{k,N}} \leq C(T,k,\gamma)^{k\frac{1}{2k}}C(T,k,\gamma)^{(N-k)\frac{1}{2(N-k)}} = C(T,k,\gamma).$$

□

4.3 Propagation of chaos : Proof of Theorem 2

In all the sequel, for any measurable space E we denote by $\mathcal{P}(E)$ the set of probability measures on E.

The following theorem comes from [11].

Theorem 3. *For any initial density ρ_0 there exists a unique solution $\mathbb{Q} \in \mathcal{P}(C[0,T];\mathbb{R})$ to the following non-linear martingale problem (MPKS):*

(i) $\mathbb{Q}_0(dx) = \rho_0(x)\ dx$;

(ii) For any $t \in (0,T]$, the one dimensional time marginals $\mathbb{Q}_t$ of $\mathbb{Q}$ have densities ρ_t w.r.t. Lebesgue measure on $\mathbb{R}$ which belong to $L^2(\mathbb{R})$ for any $0 < t \leq T$ and satisfy

$$\forall\, 0 < t \leq T, \quad \|\rho_t\|_{L^2(\mathbb{R})} \leq \frac{C_T}{t^{1/4}},$$

for some $C_T > 0$ depending only on T;

(iii) Denoting by $(x(t);\, t \leq T)$ the canonical process of $C([0,T];\mathbb{R})$, we have: For any $f \in C_b^2(\mathbb{R})$, the process defined by:

$$M_t := f(x(t)) - f(x(0)) - \int_0^t \left(\left(\int_0^s\int K_{s-r}(x(s)-y)\rho_r(y)dydr\right)\frac{\partial f}{\partial x}(x(s)) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x(s))\right) ds$$

is a $\mathbb{Q}$-martingale w.r.t. the canonical filtration.

Let us start with showing that the sequence $\{\mu^N\}$ is tight. We apply [10, Proposition 2.2-ii] and show that the family $\{\text{law}(X^{1,N})\}$ is tight. As the (X_0^i)'s share the same probability law it suffices to show

Proposition 4. *Let $T > 0$ and let N_0 be as in Proposition 3.*

$$\exists C > 0, \forall N \geq N_0, \quad \sup_{N\geq 1} \mathbb{E}_{\mathbb{Q}^N}[|X_t^{1,N} - X_s^{1,N}|^4] \leq C_T|t-s|^2, \quad 0 \leq s,t \leq T. \tag{14}$$

Proof. Let $Z_T^{(1)}$ and $Q^{1,N}$ be as in the previous section. One has,

$$\mathbb{E}_{\mathbb{Q}^N}[|X_t^{1,N} - X_s^{1,N}|^4] = \mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-1}|\widehat{X}_t^1 - \widehat{X}_s^1|^4],$$

where $\widehat{X}^1$ is a one dimensional Brownian motion under $\mathbb{Q}^{1,N}$. Applying Cauchy-Schwarz inequality,

$$\mathbb{E}_{\mathbb{Q}^N}[|X_t^{1,N} - X_s^{1,N}|^4] \leq (\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}])^{1/2}(\mathbb{E}_{\mathbb{Q}^{1,N}}[|\widehat{X}_t^1 - \widehat{X}_s^1|^8])^{1/2} \leq (\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}])^{1/2}C|t-s|^2.$$

Observe that

$$\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}] = \mathbb{E}_{\mathbb{Q}^{1,N}}[\exp\left\{2\int_0^T \beta_t^{(1)} \cdot dW_t - \int_0^T \|\beta_t^{(1)}\|^2 dt\right\}].$$

Adding and subtracting $8\int_0^T |\beta_t^{(1)}|^2 dt$ and applying again the Cauchy-Schwarz inequality,

$$\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}] \leq (\mathbb{E}_{\mathbb{Q}^{1,N}}[\exp\left\{14\int_0^T |\beta_t^{(1)}|^2 dt\right\}])^{1/2}.$$

Applying Proposition 3 with $k = 1$ and $\gamma = 14$, we obtain the desired result. □

We now proceed to the proof of Theorem 2.

Let $\phi \in C_b(\mathbb{R}^k)$, $f \in C_b^2(\mathbb{R})$, $0 < t_1 < \cdots < t_k \leq s < t \leq T$ and $m \in \mathcal{P}(C[0,T];\mathbb{R})$. We define

$$G(m) := \int_{(C[0,T];\mathbb{R})^2} \phi(x_{t_1}^1, \ldots, x_{t_k}^1)\times$$
$$\left(f(x_t^1) - f(x_s^1) - \int_s^t f''(x_u^1)du - \chi\int_s^t f'(x_u^1)\mathbb{1}_{\{x_u^1 \neq x_u^2\}}\int_0^u K_{u-\theta}(x_u^1 - x_\theta^2)d\theta du\right) dm(x^1)\otimes dm(x^2).$$

Consider a subsequence of $\{\mu_N\}$, still denoted by $\{\mu_N\}$, which converges in law to some probability distribution Π. Let us show that the support of Π is the set of solutions to (MPKS). As (MPKS) has a unique solution this will imply that μ_N converges in law towards $\mathbb{Q}$.

We start with showing that

$$\lim_{N\to\infty} \mathbb{E}[G(\mu^N)]^2 = 0. \tag{15}$$

Observe that

$$G(\mu^N) = \frac{1}{N}\sum_{i=1}^N f(X_{t_1}^{i,N}, \ldots, X_{t_k}^{i,N})\times$$
$$\left(f(X_t^{i,N}) - f(X_s^{i,N}) - \int_s^t f''(X_u^{i,N})du - \chi\int_s^t f^{'}(X_u^{i,N})\frac{1}{N}\sum_{j\neq i}\mathbb{1}_{\{X_u^{i,N}\neq X_u^{j,N}\}}\int_0^u K_{u-\theta}(X_u^{i,N} - X_\theta^{j,N})d\theta\, du\right).$$

By Itô's formula,

$$\mathbb{E}[G(\mu^N)]^2 \leq \frac{C}{N^2}\mathbb{E}\left(\sum_{i=1}^N\int_s^t f'(X_u^{i,N})\sqrt{2}dW_u^i\right)^2 \leq \frac{C}{N^2}\sum_{i=1}^N \mathbb{E}\left(W_t^i - W_s^i\right)^2 \leq \frac{C}{N},$$

which implies (15). Our next goal is the following equality:

$$\lim_{N\to\infty}\mathbb{E}[G(\mu^N)]^2 =$$
$$\int_{\mathcal{P}(C([0,T];\mathbb{R})^4)}\left\{\int_{C([0,T];\mathbb{R})^4}\left[f(x_t^1) - f(x_s^1) - \int_s^t f^{''}(x_u^1)du - \chi\int_s^t f^{'}(x_u^1)\mathbb{1}_{\{x_u^1\neq x_u^2\}}\int_0^u K_{u-\theta}(x_u^1 - x_\theta^2)d\theta du\right]\right.$$
$$\left.\times\phi(x_{t_1}^1,\ldots,x_{t_k}^1)dv(x^1,x^2,x^3,x^4)\right\}^2 d\Pi^\infty(\nu). \tag{16}$$

Since the particles do not interact through a bounded kernel, but through an unbounded singular functional defined on the path space, we need to adapt arguments in Bossy and Talay [1, Theorem 3.2]. Define

$$\nu^N := \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \delta_{X^{i,N}_{\cdot}, X^{j,N}_{\cdot}, X^{k,N}_{\cdot}, X^{l,N}_{\cdot}}.$$

As shown in Sznitman [10], the tightness of $\{\nu^N\}$ results from the tightness of the intensity measures $< I^N, f >:= \mathbb{E} < \nu^N, f >$. By symmetry, the later reduces to the tightness of the laws of $X^{1,N}$'s which is implied by Proposition 4. Let us still denote by $\{\nu^N\}$ a convergent subsequence, and by $\Pi^\infty \in \mathcal{P}(\mathcal{P}(C([0,T];\mathbb{R})^4))$ its limit. We take a non relabeled convergent subsequence of $\{\text{law}(\nu^N)\}$. Denote it's limit by $\Pi^\infty \in \mathcal{P}(\mathcal{P}(C([0,T];\mathbb{R})^4))$.

For some functionals φ and Φ which we do not explicit here, we have

$$\mathbb{E}[G(\mu^N)]^2 = \frac{1}{N^2} \sum_{i,j=1}^{N} \varphi(X^{i,N}, X^{j,N}) + \frac{1}{N^3} \sum_{i,j,k=1}^{N} \Phi(X^{i,N}, X^{j,N}, X^{k,N}) + C_N, \tag{17}$$

where

$$C_N := \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \int_s^t \int_s^t \int_0^{u_1} \int_0^{u_2} \int_{C([0,T];\mathbb{R})^4} f'(x^1_{u_1}) f'(x^3_{u_2}) \phi(x^1_{t_1}, \dots, x^1_{t_k}) \phi(x^3_{t_1}, \dots, x^3_{t_k})$$
$$\times K_{u_1-\theta_1}(x^1_{u_1} - x^2_{\theta_1}) K_{u_2-\theta_2}(x^3_{u_2} - x^4_{\theta_2}) \mathbb{1}_{\{x^1_{u_1} \neq x^2_{u_1}\}} \mathbb{1}_{\{x^2_{u_2} \neq x^4_{u_2}\}} d\mathbb{P}_{X^i_{\cdot}, X^j_{\cdot}, X^k_{\cdot}, X^l_{\cdot}}(x^1, x^2, x^3, x^4) d_{\theta_1} d_{\theta_2} d_{u_1} d_{u_2}.$$

Let us study C_N. Define the function F on $\mathbb{R}^{2k+6}$ as

$$F(x^1, \dots, x^{2k+6}) := f'(x^1) f'(x^3) \phi(x^7, \dots, x^{k+6}) \phi(x^{k+7}, \dots, x^{2k+6})$$
$$\times K_{u_1-\theta_1}(x^1 - x^2) K_{u_2-\theta_2}(x^3 - x^4) \mathbb{1}_{\{x^1 \neq x^5\}} \mathbb{1}_{\{x^2 \neq x^6\}} \mathbb{1}_{\{\theta_1 < u_1\}} \mathbb{1}_{\{\theta_2 < u_2\}}.$$

Set

$$A_N := \left| \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \mathbb{E}(F(X^{i,N}_{u_1}, X^{j,N}_{\theta_1}, X^{k,N}_{u_2}, X^{l,N}_{\theta_2}, X^{j,N}_{u_1}, X^{l,N}_{u_2}, X^{i,N}_{t_1}, \dots, X^{i,N}_{t_k}, X^{k,N}_{t_1}, \dots, X^{k,N}_{t_k})) \right|. \tag{18}$$

To study the convergence of C_N we aim to apply the Dominated Convergence Theorem and thus (Step 1) show that A_N converges pointwise and (Step 2) that A_N is bounded from above by an integrable function w.r.t. $d_{\theta_1} d_{\theta_2} d_{u_1} d_{u_2}$.

Step 1. Fix $u_1, u_2 \in [s,t]$ and $\theta_1 \in [0, u_1)$ and $\theta_2 \in [0, u_2)$. Define τ^N as

$$\tau^N := \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \delta_{X^{i,N}_{u_1}, X^{j,N}_{\theta_1}, X^{k,N}_{u_2}, X^{l,N}_{\theta_2}, X^{j,N}_{u_1}, X^{l,N}_{u_2}, X^{i,N}_{t_1}, \dots, X^{i,N}_{t_k}, X^{k,N}_{t_1}, \dots, X^{k,N}_{t_k}}.$$

Define the measure $\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}$ on $\mathbb{R}^{2k+6}$ as

$$\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}(A) = \mathbb{E}(\tau^N(A)).$$

The convergence of $\{\text{law}(\nu^N)\}$ implies the weak convergence of $\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}$ to a measure on $\mathbb{R}^{2k+6}$ defined by

$$\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}(A)$$
$$:= \int_{\mathcal{P}(C([0,T];\mathbb{R})^4)} \int_{C([0,T];\mathbb{R})^4} \mathbb{1}_A(x^1_{u_1}, x^2_{\theta 1}, x^3_{u_2}, x^4_{\theta_2}, x^2_{u_1}, x^4_{u_2}, x^1_{t_1}, \dots, x^1_{t_k}, x^3_{t_1}, \dots, x^3_{t_k}) d\nu(x^1, x^2, x^3, x^4) d\Pi^\infty(\nu).$$

Let us show that this probability measure has a density on $\mathbb{R}^{2k+6}$. Let $h \in C_c^\infty(\mathbb{R}^{2k+6})$. By weak convergence,

$$
\begin{aligned}
&|< \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}, h >| \\
&= \left| \lim_{N\to\infty} \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \mathbb{E} h(X^{i,N}_{u_1}, X^{j,N}_{\theta 1}, X^{k,N}_{u_2}, X^{l,N}_{\theta_2}, X^{j,N}_{u_1}, X^{l,N}_{u_2}, X^{i,N}_{t_1}, \dots, X^{i,N}_{t_k}, X^{k,N}_{t_1}, \dots, X^{k,N}_{t_k}) \right|.
\end{aligned}
$$

When $i \neq j \neq k \neq l$, use the Girsanov transformation from Section 4.2 for $k = 4$. By Proposition 3 and Cauchy-Schwarz inequality,

$$
\begin{aligned}
|< \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}, h >| &\leq \left(\mathbb{E} h^2(W^{i,N}_{u_1}, W^{j,N}_{\theta 1}, W^{k,N}_{u_2}, W^{l,N}_{\theta_2}, W^{j,N}_{u_1}, W^{l,N}_{u_2}, W^{i,N}_{t_1}, \dots, W^{i,N}_{t_k}, W^{k,N}_{t_1}, \dots, W^{k,N}_{t_k}) \right)^{1/2} \\
&\leq C_{u_1,u_2,\theta_1,\theta_2,t_1,\dots,t_k} \|h\|_{L^2(\mathbb{R}^{2k+6})}.
\end{aligned}
$$

Thus

$$
|< \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}, h >| \leq \lim_{N\to\infty} (C_{u_1,u_2,\theta_1,\theta_2,t_1,\dots,t_k} \|h\|_{L^2(\mathbb{R}^{2k+6})} + \|h\|_\infty (\frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3})).
$$

It follows that $\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}$ is absolutely continuous with respect to Lebesgue's measure. Therefore, the functional F is continuous $\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}$ - a.e. This implies that, for $u_1, u_2 \in [s,t]$ and $\theta_1 \in [0, u_1)$, $\theta_2 \in [0, u_2)$,

$$
\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \int_{C([0,T];\mathbb{R})^4} f'(x^1_{u_1}) f'(x^3_{u_2}) \phi(x^1_{t_1}, \dots, x^1_{t_k}) \phi(x^3_{t_1}, \dots, x^3_{t_k}) \\
&\times K_{u_1-\theta_1}(x^1_{u_1} - x^2_{\theta_1}) K_{u_2-\theta_2}(x^3_{u_2} - x^4_{\theta_2}) \mathbb{1}_{\{x^1_{u_1} \neq x^2_{u_1}\}} \mathbb{1}_{\{x^2_{u_2} \neq x^4_{u_2}\}} d\mathbb{P}_{X^i_\cdot, X^j_\cdot, X^k_\cdot, X^l_\cdot}(x^1, x^2, x^3, x^4) \\
&=< \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}, F > .
\end{aligned}
$$

Step 2. By definition of F we may and do restrict to the case $i \neq j$ and $k \neq l$. Use the Girsanov transformations from Section 4.2 with the number of particles $m \in \{2, 3, 4\}$. It comes:

$$
A_N = \left| \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \mathbb{E}_{\mathbb{Q}^{(m)}} (Z_T^{(m)} F(\cdots)) \right| \leq \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \left(\mathbb{E}_{\mathbb{Q}^{(m)}} (Z_T^{(m)})^2 \right)^{1/2} \left(\mathbb{E}_{\mathbb{Q}^{(m)}} (F^2(\cdots)) \right)^{1/2}.
$$

By Proposition 3, $\mathbb{E}_{\mathbb{Q}^{(m)}} (Z_T^{(m)})^2$ can be bounded uniformly w.r.t. N. As the functions f and ϕ are bounded we deduce

$$
\left(\mathbb{E}_{\mathbb{Q}^{(m)}} (F^2(\cdots)) \right)^{1/2} \leq C \mathbb{1}\{\theta_1 < u_1\} \mathbb{1}\{\theta_2 < u_2\} \left(\mathbb{E}_{\mathbb{Q}^{(m)}} (K^2_{u_1-\theta_1}(W^i_{u_1} - W^j_{\theta_1}) K^2_{u_1-\theta_1}(W^k_{u_2} - W^l_{\theta_2})) \right)^{1/2},
$$

for $i \neq j$ and $k \neq l$. For any $0 < \theta < u < T$ we have

$$
\left(\mathbb{E}_{\mathbb{Q}^{(m)}} (K^4_{u-\theta}(W^i_u - W^j_\theta)) \right)^{1/4} \leq \frac{C}{u^{1/8}} \|K_{u-\theta}\|_{L^4(\mathbb{R})} \leq \frac{C}{u^{1/8}(u-\theta)^{7/8}}.
$$

Therefore,

$$
\left(\mathbb{E}_{\mathbb{Q}^{(m)}} (F^2(\cdots)) \right)^{1/2} \leq C \frac{\mathbb{1}_{\{\theta_1 < u_1\}} \mathbb{1}_{\{\theta_2 < u_2\}}}{u_1^{1/8}(u_1 - \theta_1)^{7/8} u_2^{1/8}(u_2 - \theta_2)^{7/8}}.
$$

We thus have obtained:

$$
A_N \leq C \frac{\mathbb{1}_{\{\theta_1 < u_1\}} \mathbb{1}_{\{\theta_2 < u_2\}}}{u_1^{1/8}(u_1 - \theta_1)^{7/8} u_2^{1/8}(u_2 - \theta_2)^{7/8}}.
$$

We remark that the r.h.s. belongs to $L^1(0,T)^4$.

Conclusion. Steps 1 and 2 allow us to conclude that

$$\lim_{N\to\infty}\int_s^t\int_s^t\int_s^t\int_s^t\frac{1}{N^4}\sum_{i,j,k,l=1}^N\int_{C([0,T];\mathbb{R})^4}f'(x^1_{u_1})f'(x^3_{u_2})\phi(x^1_{t_1},\dots,x^1_{t_k})\phi(x^3_{t_1},\dots,x^3_{t_k})\mathbb{1}_{\{\theta_1<u_1\}}\mathbb{1}_{\{\theta_2<u_2\}}$$
$$\times K_{u_1-\theta_1}(x^1_{u_1}-x^2_{\theta_1})K_{u_2-\theta_2}(x^3_{u_2}-x^4_{\theta_2})\mathbb{1}_{\{x^1_{u_1}\neq x^2_{u_1}\}}\mathbb{1}_{\{x^2_{u_2}\neq x^4_{u_2}\}}d\mathbb{P}_{X^{i,N}_\cdot,X^{j,N}_\cdot,X^{k,N}_\cdot,X^{l,N}_\cdot}(x^1,x^2,x^3,x^4)d\theta_1d\theta_2du_1du_2$$
$$=\int_s^t\int_s^t\int_s^t\int_s^t<\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k},F>d\theta_1d\theta_2du_1du_2.$$

By definition of $\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_k}$

$$\lim_{N\to\infty}C_N=\int_{P(C([0,T];\mathbb{R})^4)}\int_s^t\int_s^t\int_{C([0,T];\mathbb{R})^4}\chi^2f'(x^1_{u_1})f'(x^3_{u_2})\phi(x^1_{t_1},\dots,x^1_{t_k})\phi(x^3_{t_1},\dots,x^3_{t_k})$$
$$\times\int_0^{u_1}\int_0^{u_2}K_{u_1-\theta_1}(x^1_{u_1}-x^2_{\theta_1})K_{u_2-\theta_2}(x^3_{u_2}-x^4_{\theta_2})\mathbb{1}_{\{x^1_{u_1}\neq x^2_{u_1}\}}\mathbb{1}_{\{x^3_{u_2}\neq x^4_{u_2}\}}$$
$$d\nu(x^1,x^2,x^3,x^4)d\theta_1d\theta_2du_1du_2d\Pi^\infty(\nu).$$

For any measure $\nu\in P(C([0,T];\mathbb{R})^4)$, denote its first marginal by ν^1. It is easy to get that

$$\Pi^\infty\quad\text{a.e.},\quad\nu=\nu^1\otimes\nu^1\otimes\nu^1\otimes\nu^1$$

(see [1, Lemma 3.3]). Therefore

$$\lim_{N\to\infty}C_N=\int_{\mathcal{P}(C([0,T];\mathbb{R})^4)}\Big[\int_{C([0,T];\mathbb{R})^2}\int_s^t\chi^2f'(x^1_u)\phi(x^1_{t_1},\dots,x^1_{t_k})$$
$$\times\int_0^u K_{u-\theta}(x^1_u-x^2_\theta)\mathbb{1}_{\{x^1_u\neq x^2_u\}}d\theta du\ d\nu^1(x^1)\otimes d\nu^1(x^2)\Big]^2d\Pi^\infty(\nu).$$

The two first terms in the r.h.s. of (17) can be treated similarly and thus (16) holds true. Combining this with (15) we have that Π^∞ a.e.,

$$\int_{C([0,T];\mathbb{R})^2}\phi(x^1_{t_1},\dots,x^1_{t_k})\Big[f(x^1_t)-f(x^1_s)-\int_s^t f''(x_u)du$$
$$-\int_s^t f'(x^1_u)\int_0^u K_{u-\theta}(x^1_u-x^2_\theta)\mathbb{1}_{\{x^1_u\neq x^2_u\}}d\theta du\Big]d\nu^1(x^1)\otimes d\nu^1(x^2)=0.$$

In addition, notice that $\nu^1=\lim_{N\to\infty}\mu^N$. Take $h\in L^2(\mathbb{R})$. Using similar arguments as in the above Step 1, for any $0<t\leq T$ one has

$$<\nu^1_t,h>=\lim_{N\to\infty}<\mu^N_t,h>=\lim_{N\to\infty}\mathbb{E}_{\mathbb{Q}^N}(h(X^{1,N}_t))=\lim_{N\to\infty}\mathbb{E}_{\mathbb{Q}^{1,N}}(Z^{(1)}_T h(W^{1,N}_t))\leq\frac{C}{t^{1/4}}\|h\|_{L^2(\mathbb{R})}.$$

Thus, ν^1 solves (MPKS). As (MPKS) admits a unique solution, Theorem 2 is proved.

Appendix A

Proposition 5. *Same assumptions as in Proposition 1. Then, there exists $N_0\in\mathbb{N}$ depending only on T and α, such that*

$$\sup_{N\geq N_0}\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\frac{\alpha}{N}\int_0^T\left(\int_0^t K_{t-s}(X_t-W_s)ds\mathbb{1}_{\{W_t\neq X_t\}}\right)^2dt\right\}\right]\leq C(T,\alpha).$$

The only difficulty to overcome is the fact that now the Brownian motion and X changed places. Thus, it is not obvious how to use the Brownian increments. However, the presence of the factor $\frac{1}{N}$ enables us to skip the localization part (see Lemmas 1 and 2).

Proof of Proposition 5 . Fix $N \in \mathbb{N}$. Set $II_k := \left(\int_0^T \left(\int_0^t K_{t-s}(X_t - W_s)ds\right)^2 dt\right)^k$. By Cauchy-Schwarz inequality,

$$II_k \leq \left(\int_0^T \int_0^t \frac{ds}{(t-s)^{3/4}} \int_0^t \frac{(X_t - W_s)^2}{(t-s)^{9/4}} e^{-\frac{(X_t-W_s)^2}{t-s}} dsdt\right)^k \leq 4^k T^{k/4} \left(\int_0^T \int_0^t \frac{(X_t - W_s)^2}{(t-s)^{9/4}} e^{-\frac{(X_t-W_s)^2}{t-s}} dsdt\right)^k.$$

By Fubini-Tonneli's theorem,

$$II_k \leq 4^k T^{k/4} \left(\int_0^T \int_s^T \frac{(X_t - W_s)^2}{(t-s)^{9/4}} e^{-\frac{(X_t-W_s)^2}{t-s}} dtds\right)^k.$$

For $0 \leq s < T$ and for $(\omega, \widehat{\omega}) \in C([0,T];\mathbb{R}) \times C([0,T];\mathbb{R})$, define the functional H_s,

$$H_s(\omega, \widehat{\omega}) = \int_s^T \frac{(w_t - \bar{w}_s)^2}{(t-s)^{9/4}} e^{-\frac{(w_t-\bar{w}_s)^2}{t-s}} dt.$$

As the processes X and W are independent,

$$\mathbb{E}_{\mathbb{P}} \left(\int_0^T H_s(X,W)ds\right)^k = \int_{C([0,T];\mathbb{R})} \mathbb{E}_{\mathbb{P}} \left(\int_0^T H_s(x,W)ds\right)^k \mathbb{P}^X(dx).$$

As above we observe that

$$\left(\int_0^T H_s(x,W)ds\right)^k = k! \int_0^T H_{s_1}(x,W) \int_{s_1}^T \cdots \int_{s_{k-1}}^T H_{s_k}(x,W) ds_k \ldots ds_1,$$

from which

$$\mathbb{E}_{\mathbb{P}} \left(\int_0^T H_s(x,W)ds\right)^k = k! \mathbb{E}_{\mathbb{P}} \int_0^T H_{s_1}(x,W) \int_{s_1}^T \cdots \mathbb{E}_{\mathbb{P}}^{\mathcal{F}^W_{s_{k-1}}} \left(\int_{s_{k-1}}^T H_{s_k}(x,W) ds_k\right) \ldots ds_1.$$

Therefore,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{F}^W_{s_{k-1}}} \left(\int_{s_{k-1}}^T H_{s_k}(x,W)ds_k\right) = \int_{s_{k-1}}^T \int_{s_k}^T \int \frac{(x_t - z - W_{s_{k-1}})^2}{(t-s_k)^{9/4}} e^{-\frac{(x_t-z-W_{s_{k-1}})^2}{t-s_k}} g_{s_k - s_{k-1}}(z) dzdtds_k$$

$$\leq \int_{s_{k-1}}^T \frac{C}{\sqrt{s_k - s_{k-1}}} \int_{s_k}^T \frac{1}{(t-s_k)^{3/4}} \int z^2 e^{-z^2} dzdtds_k \leq CT^{1/4}\sqrt{T - s_{k-1}} \leq CT^{3/4}.$$

Finally,

$$\mathbb{E}_{\mathbb{P}} \left(\int_0^T H_s(x,W)ds\right)^k \leq k! CT^{3/4} \mathbb{E}_{\mathbb{P}} \left(\int_0^T H_{s_1}(x,W) \int_{s_1}^T \cdots \int_{s_{k-2}}^T H_{s_{k-1}}(x,W) ds_{k-1} \ldots ds_1\right).$$

Repeat the previous procedure $k-2$ times. It comes:

$$\mathbb{E}_{\mathbb{P}} \left(\int_0^T H_s(x,W)ds\right)^k \leq k! C^{k-1} T^{3(k-1)/4} \mathbb{E}_{\mathbb{P}} \left(\int_0^T H_{s_1}(x,W)ds_1\right).$$

Using that the density of W_{s_1} is bounded by $\frac{C}{\sqrt{s_1}}$ we deduce that

$$\mathbb{E}_{\mathbb{P}} \left(\int_0^T H_{s_1}(x,W)ds_1\right) \leq \int_0^T \frac{C}{\sqrt{s_1}} \int_{s_1}^T \int \frac{(x_t - W_{s_1})^2}{(t-s_1)^{9/4}} e^{-\frac{(x_t-W_{s_1})^2}{t-s_1}} dxdtds_1 \leq \int_0^T \frac{1}{\sqrt{s_1}} \int_{s_1}^T \frac{dt}{(t-s_1)^{3/4}} ds_1 \leq CT^{3/4},$$

and thus $\mathbb{E}_{\mathbb{P}}\left(\int_0^T H_s(X,W)ds\right)^k \leq k!C^kT^k$. This implies that for any $M \geq 1$,

$$\mathbb{E}_{\mathbb{P}}\sum_{k=1}^{M}\frac{\alpha^k II_k}{N^k k!} \leq \sum_{k=1}^{M}\frac{\alpha^k C^k T^k}{N^k}.$$

Choose N_0 large enough to have $\frac{\alpha}{N_0}CT < 1$. We end the proof by applying Fatou's lemma to $\sum_{k=1}^{M}\frac{\alpha^k II_k}{N^k k!}$. $\square$

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