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Jean-François Jabir*, Denis Talay† and Milica Tomašević †‡

Abstract: In this work, we prove the well–posedness of a singularly interacting stochastic particle system and we establish propagation of chaos result towards the one-dimensional parabolic-parabolic Keller-Segel model.
Key words: Interacting particle system; Singular McKean-Vlasov dynamic; Keller-Segel model; .
Classification: 39A50, 60H30, 80C22.

1 Introduction

The standard d-dimensional parabolic–parabolic Keller–Segel model for chemotaxis describes the time evolution of the density ρ_t of a cell population and of the concentration c_t of a chemical attractant:

$$\begin{cases} \partial_t \rho(t,x) = \nabla \cdot (\frac{1}{2}\nabla\rho - \chi\rho\nabla c), \quad t>0,\ x\in\mathbb{R}^d, \\ \alpha\partial_t c(t,x) = \frac{1}{2}\triangle c - \lambda c + \rho, \quad t>0,\ x\in\mathbb{R}^d, \\ \rho(0,x)=\rho_0(x), c(0,x)=c_0(x), \end{cases} \tag{1}$$

for some parameters $\chi>0$, $\lambda\geq 0$ and $\alpha\geq 0$. See e.g. Corrias *et al.* [4], Perthame [9] and references therein for theoretical results on this system of PDEs and applications to biology. When $\alpha=0$, the system (1) is parabolic–elliptic, and when $\alpha=1$ (or more generally, when $0<\alpha\leq 1$), the system is parabolic–parabolic.

For the parabolic–elliptic version of the model with $d=2$, the first stochastic interpretation of this system is due to Haškovec and Schmeiser [6] who analyze a particle system with McKean–Vlasov interactions and Brownian noise. More precisely, as the ideal interaction kernel should be strongly singular, they introduce a kernel with a cut-off parameter and obtain the tightness of the particle probability distributions w.r.t. the cut-off parameter and the number of particles. They also obtain partial results in the direction of the propagation of chaos. More recently, in the subcritical case, that is, when the parameter χ of the parabolic–elliptic model is small enough, Fournier and Jourdain [5] obtain the well–posedness of a particle system without cut-off. In addition, they obtain a consistency property which is weaker than the propagation of chaos. They also describe complex behaviors of the particle system in the sub and super critical cases. Cattiaux and Pédèches [3] obtain the well-posedness of this particle system without cut-off by using Dirichlet forms rather than pathwise approximation techniques.

For a parabolic–parabolic version of the model with a smooth coupling between ρ_t and c_t, Budhiraja and Fan [2] study a particle system with a smooth time integrated kernel and prove it propagates chaos. Moreover, adding a forcing potential term to the model, under a suitable convexity assumption, they obtain uniform in time concentration inequalities for the particle system and uniform in time error estimates for a numerical approximation of the limit non-linear process.

For the pure parabolic–parabolic model without cut-off or smoothing, in the one-dimensional case with $\alpha=1$, Talay and Tomašević [12] have proved the well-posedness of PDE (1) and of the following non-linear SDE:

$$\begin{cases} dX_t = b(t,X_t)dt + \left\{\chi\int_0^t (K_{t-s}\star\rho_s)(X_t)ds\right\}dt + dW_t, \quad t>0, \\ \rho_s(y)dy := \mathcal{L}(X_s), \quad X_0\sim\rho_0(x)dx, \end{cases} \tag{2}$$

where $K_t(x) := e^{-\lambda t}\frac{\partial}{\partial x}(\frac{1}{(2\pi t)^{1/2}}e^{-\frac{x^2}{2t}})$ and $b(t,x)=e^{-\lambda t}\frac{\partial}{\partial x}\mathbb{E}[c_0(x+W_t)]$.

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Under the sole condition that the initial probability law $\mathcal{L}(X_0)$ has a density, it is shown that the law $\mathcal{L}(X)$ uniquely solves a non-linear martingale problem and its time marginals have densities. These densities coupled with a suitable transformation of them uniquely solve the one–dimensional parabolic–parabolic Keller–Segel system without cut-off. In Tomašević [13], additional techniques are being developed for the two-dimensional version of (2).

The objective of this note is to analyze the particle system related to (2). It inherits from the limit equation that at each time $t > 0$ each particle interacts in a singular way with the past of all the other particles. We prove that the particle system is well–posed and propagates chaos to the unique weak solution of (2). Compared to the stochastic particle systems introduced for the parabolic–elliptic model, an interesting fact occurs: the difficulties arising from the singular interaction can now be resolved by using purely Brownian techniques rather than by using Bessel processes. Due to the singular nature of the kernel K, we need to introduce a partial Girsanov transform of the N-particle system in order to obtain uniform in N bounds for moments of the corresponding exponential martingale. Our calculation is based on the fact that the kernel K is in $L^1(0,T;L^2(\mathbb{R}))$. We aim to address in the close future the multi-dimensional Keller–Segel particle system where the $L^1(0,T;L^2(\mathbb{R}^d))$-norm of the kernel is infinite.

The paper is organized as follows. In Section 2 we state our two main results and comment our methodology. In Section 3 and Appendix we prove technical lemmas. In Section 4 we prove our main results.

In all the paper we denote by C any positive real number independent of N.

2 Main results

Our main results concern the well–posedness and propagation of chaos of

$$\begin{cases} dX_t^{i,N} = \left\{ \frac{1}{N}\sum_{j=1,j\neq i}^N \int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N})ds \; \mathbb{1}_{\{X_t^{i,N}\neq X_t^{j,N}\}}\right\} dt + dW_t^i, \\ X_0^{i,N} \text{ i.i.d. and independent of } W := (W^i, 1\leq i\leq N), \end{cases} \tag{3}$$

where $K_t(x) = \frac{-x}{\sqrt{2\pi}t^{3/2}}e^{-\frac{x^2}{2t}}$ and the W^i's are N independent standard Brownian motions. It corresponds to $\alpha = 1$, $\lambda = 0$, $\chi = 1$, and $c_0' \equiv 0$. It is easy to extend our methodology to (2) under the hypotheses made in [12].

***Theorem* 1.** *Given $0 < T < \infty$ and $N \in \mathbb{N}$, there exists a weak solution $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, 0\leq t\leq T), \mathbb{Q}^N, W, X^N)$ to the N-interacting particle system (3) that satisfies, for any $1 \leq i \leq N$,*

$$\mathbb{Q}^N\left(\int_0^T \left(\frac{1}{N}\sum_{j=1,j\neq i}^N \int_0^t K_{t-s}(X_t^{i,N} - X_s^{j,N})ds\mathbb{1}_{\{X_t^{i,N}\neq X_t^{j,N}\}}\right)^2 dt < \infty\right) = 1. \tag{4}$$

In view of Karatzas and Shreve [7, Chapter 5, Proposition 3.10], one has the following uniqueness result:

Corollary 1. *Weak uniqueness holds in the class of weak solutions satisfying (4).*

The construction of a weak solution to (3) involves arguments used by Krylov and Röckner [8, Section 3] to construct a weak solution to SDEs with singular drifts. It relies on the Girsanov transform which removes all the drifts of (3).

***Remark* 1.** *Our construction shows that the law of the particle system is equivalent to Wiener's measure. Thus, a.s. the set $\{t \leq T, X_t^{i,N} = X_t^{j,N}\}$ has Lebesgue measure zero.*

Our second main theorem concerns the propagation of chaos of the system (3). Before we proceed to its statement, we need to define the non-linear martingale problem (MPKS) associated to the non-linear SDE:

$$\begin{cases} dX_t = \left\{\int_0^t (K_{t-s} \star \rho_s)(X_t)ds\right\}dt + dW_t, \quad t\leq T, \\ \rho_s(y)dy := \mathcal{L}(X_s), \quad X_0 \sim \rho_0(x)dx. \end{cases} \tag{5}$$

For any measurable space E we denote by $\mathcal{P}(E)$ the set of probability measures on E.

***Definition* 1.** *$\mathbb{Q} \in \mathcal{P}(C[0,T];\mathbb{R})$ is a solution to (MPKS) if:*

(i) $\mathbb{Q}_0(dx) = \rho_0(x)\ dx$;

(ii) *For any* $t \in (0,T]$, *the one dimensional time marginal* $\mathbb{Q}_t$ *of* $\mathbb{Q}$ *has a density* ρ_t *w.r.t. Lebesgue measure on* $\mathbb{R}$ *which belongs to* $L^2(\mathbb{R})$ *and satisfies*

$$\exists C_T, \quad \forall\, 0 < t \le T, \quad \|\rho_t\|_{L^2(\mathbb{R})} \le \frac{C_T}{t^{1/4}};$$

(iii) *Denoting by* $(x(t);\, t \le T)$ *the canonical process of* $C([0,T];\mathbb{R})$, *we have: For any* $f \in C_b^2(\mathbb{R})$, *the process defined by*

$$M_t := f(x(t)) - f(x(0)) - \int_0^t \left(\Big(\int_0^s \int K_{s-r}(x(s)-y)\rho_r(y)dydr \Big) f'(x(s)) + \frac{1}{2} f''(x(s)) \right) ds$$

is a $\mathbb{Q}$*-martingale w.r.t. the canonical filtration.*

In [12], the authors prove that (MPKS) admits a unique solution and that a suitable notion of weak solution to (5) is equivalent to the notion of solution to (MPKS).

***Theorem* 2.** *Assume that the* $X_0^{i,N}$*'s are i.i.d. and that the initial distribution of* $X_0^{1,N}$ *has a density* ρ_0*. The empirical measure* $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ *of* (3) *converges in the weak sense, when* $N \to \infty$*, to the unique weak solution of* (5).

To prove the tightness and weak convergence of μ^N, we use a Girsanov transform which removes a fixed small number of the drifts of (3) rather than all the drifts. This trick, which may be useful for other singular interactions, allows us to get uniform w.r.t. N bounds for the needed Girsanov exponential martingales.

3 Preliminaries

On the path space define the functional $F_t(x,\widehat{x}) = \left(\int_0^t K_{t-s}(x_t - \widehat{x}_s)ds\ \mathbb{1}_{\{x_t \neq \widehat{x}_t\}} \right)^2$, where $(x,\widehat{x}) \in C([0,T];\mathbb{R}) \times C([0,T];\mathbb{R})$. The objective of this section is to show that $\int_0^T F_t(w,Y)dt$ has finite exponential moments when w is a Brownian motion and Y is a process independent of w. The following key property of the kernel K_t will be used:

$$\|K_t\|_{L^p(\mathbb{R})} = \left(C \int_0^\infty \frac{z^p}{t^{p-1/2}} e^{-\frac{pz^2}{2}} dz \right)^{1/p} = \frac{C_p}{t^{1-1/2p}},\ 1 \le p < \infty. \tag{6}$$

We will proceed as in the proof of the local Novikov Condition (see [7, Chapter 3, Corollary 5.14]) by localizing on small intervals of time.

***Lemma* 1.** *Let* $w := (w_t)$ *be a* $(\mathcal{G}_t)$*-Brownian motion with an arbitrary initial distribution* μ_0 *on some probability space equipped with a probability measure* $\mathbb{P}$ *and a filtration* $(\mathcal{G}_t)$*. There exists a universal real number* $C_0 > 0$ *such that*

$$\forall x \in C([0,T];\mathbb{R}), \quad \forall 0 \le t_1 \le t_2 \le T, \quad \int_{t_1}^{t_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_1}} F_t(w,x)dt \le C_0\sqrt{T}\sqrt{t_2 - t_1}.$$

Proof. By the definition of F,

$$\int_{t_1}^{t_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_1}} F_t(w,x)dt \le \int_{t_1}^{t_2}\int_0^t\int_0^t \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_1}} |K_{t-s}(w_t - x_s)K_{t-u}(w_t - x_u)|dsdudt. \tag{7}$$

Let $g_t(x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$. In view of (6), one has

$$\sqrt{\int K_{t-s}^2(y + w_{t_1} - x_s))g_{t-t_1}(y)dy} \le C\frac{\|K_{t-s}\|_{L^2(\mathbb{R})}}{(t-t_1)^{1/4}} \le \frac{C}{(t-s)^{3/4}(t-t_1)^{1/4}}.$$

Here we used that the density of $w_t - w_{t_1}$ is bounded by $\frac{C}{\sqrt{t-t_1}}$. Coming back to (7),

$$\int_{t_1}^{t_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_1}} F_t(w,x)dt \le \int_{t_1}^{t_2} \frac{C}{\sqrt{t-t_1}} \int_0^t\int_0^t \frac{1}{(t-s)^{3/4}(t-u)^{3/4}}dsdudt = \int_{t_1}^{t_2} \frac{C\sqrt{t}}{\sqrt{t-t_1}}dt.$$

□

Lemma 2. *Same assumptions as in Lemma 1. Let C_0 be as in Lemma 1. For any $\kappa > 0$, there exists $C(T,\kappa)$ independent of μ_0 such that, for any $0 \leq T_1 \leq T_2 \leq T$ satisfying $T_2 - T_1 < \frac{1}{C_0^2 T \kappa^2}$,*

$$\forall x \in C([0,T];\mathbb{R}), \quad \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left[\exp\left\{\kappa \int_{T_1}^{T_2} F_t(w,x)dt\right\}\right] \leq C(T,\kappa).$$

Proof. We adapt the proof of Khasminskii's lemma in Simon [10]. Admit for a while we have shown that there exists a constant $C(\kappa,T)$ such that for any $M \in \mathbb{N}$

$$\sum_{k=1}^{M} \frac{\kappa^k}{k!} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}} \left(\int_{T_1}^{T_2} F_t(w,x)dt\right)^k \leq C(T,\kappa), \tag{8}$$

provided that $T_2 - T_1 < \frac{1}{C_0^2 T \kappa^2}$. The desired result then follows from Fatou's lemma.

We now prove (8). By the tower property of conditional expectation,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left[\left(\int_{T_1}^{T_2} F_t(w,x)dt\right)^k\right] = k! \int_{T_1}^{T_2}\int_{t_1}^{T_2}\int_{t_2}^{T_2}\cdots\int_{t_{k-2}}^{T_2}\int_{t_{k-1}}^{T_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\Big[F_{t_1}(w,x)F_{t_2}(w,x)$$
$$\times \cdots \times F_{t_{k-1}}(w,x)\left(\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_{k-1}}} F_{t_k}(w,x)\right)\Big]\, dt_k\, dt_{k-1}\cdots dt_2\, dt_1.$$

In view of Lemma 1,

$$\int_{t_{k-1}}^{T_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{t_{k-1}}} F_{t_k}(w,x)\, dt_k \leq C_0\sqrt{T}\sqrt{T_2 - t_{k-1}} \leq C_0\sqrt{T}\sqrt{T_2 - T_1}.$$

Therefore, by Fubini's theorem,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left[\left(\int_{T_1}^{T_2} F_t(w,x)dt\right)^k\right] \leq k! C_0\sqrt{T}\sqrt{T_2 - T_1}\int_{T_1}^{T_2}\int_{t_1}^{T_2}\int_{t_2}^{T_2}\cdots\int_{t_{k-2}}^{T_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\Big[F_{t_1}(w,x)F_{t_2}(w,x)$$
$$\times\cdots\times F_{t_{k-1}}(w,x)\Big] dt_{k-1}\cdots dt_2\, dt_1.$$

Now we repeatedly condition with respect to $\mathcal{G}_{t_{k-i}}$ $(i \geq 2)$ and combine Lemma 1 with Fubini's theorem. It comes:

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}}\left(\int_{T_1}^{T_2} F_t(w,x)dt\right)^k \leq k!(C_0\sqrt{T}\sqrt{T_2 - T_1})^{k-1}\int_{T_1}^{T_2} \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{T_1}} F_{t_1}(w,x)dt_1 \leq k!(C_0\sqrt{T}\sqrt{T_2 - T_1})^k.$$

Thus, (8) is satisfied provided that $T_2 - T_1 < \frac{1}{C_0^2 T\kappa^2}$. □

Proposition 1. *Let $T > 0$. Same assumptions as in Lemma 1. Suppose that the filtered probability space is rich enough to support a continuous process Y independent of (w_t). For any $\alpha > 0$,*

$$\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\alpha\int_0^T F_t(w,Y)dt\right\}\right] \leq C(T,\alpha),$$

where $C(T,\alpha)$ depends only on T and α, but does neither depend on the law $\mathcal{L}(Y)$ nor of μ_0.

Proof. Observe that

$$\mathbb{E}_{\mathbb{P}} \exp\left\{\alpha\int_0^T F_t(w,Y)dt\right\} = \int_{C([0,T];\mathbb{R})} \mathbb{E}_{\mathbb{P}} \exp\left\{\alpha\int_0^T F_t(w,x)dt\right\}\mathbb{P}^Y(dx). \tag{9}$$

Set $\delta := \frac{1}{2C_0^2 T\alpha^2} \wedge T$, where C_0 is as in Lemma 1. Set $n := \left[\frac{T}{\delta}\right]$. Then,

$$\exp\left\{\alpha\int_0^T F_t(w,x)dt\right\} = \prod_{m=0}^{n} \exp\left\{\alpha\int_{(T-(m+1)\delta)\vee 0}^{T-m\delta} F_t(w,x)\, dt\right\}.$$

Condition the right-hand side by $\mathcal{G}_{(T-\delta)\vee 0}$. Notice that δ is small enough to be in the setting of Lemma 2. Thus,

$$\mathbb{E}_{\mathbb{P}} \exp\left\{\alpha \int_0^T F_t(w,x)dt\right\} \leq C(T,\alpha)\mathbb{E}_{\mathbb{P}} \prod_{m=1}^{n} \exp\left\{\kappa N \int_{(T-(m+1)\delta)\vee 0}^{T-m\delta} F_t(w,x)\, dt\right\}.$$

Successively, conditioning by $\mathcal{G}_{(T-(m+1))\vee 0}$ for $m = 1, 2, \dots n$ and using Lemma 2,

$$\mathbb{E}_{\mathbb{P}} \exp\left\{\alpha \int_0^T F_t(w,x)dt\right\} \leq C^n(T,\alpha)\mathbb{E}_{\mathbb{P}} \exp\left\{\int_0^{(T-n\delta)\vee 0} F_t(w,x)dt\right\} \leq C(T,\alpha).$$

The proof is completed by plugging the preceding estimate into (9). □

4 Existence of the particle system and propagation of chaos

4.1 Existence: Proof of Theorem 1

We start from a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, 0 \leq t \leq T), \mathbb{W})$ on which are defined an N - dimensional Brownian motion $W = (W^1, \dots, W^N)$ and the random variables $X_0^{i,N}$(see (3)). Set $\bar{X}_t^{i,N} := X_0^{i,N} + W_t^i$ ($t \leq T$) and $\bar{X} := (\bar{X}^{i,N}, 1 \leq i \leq N)$. Denote the drift terms in (3) by $b_t^{i,N}(x)$, $x \in C([0,T];\mathbb{R})^N$, and the vector of all the drifts as $B_t^N(x) = (b_t^{1,N}(x), \dots, b_t^{N,N}(x))$. For a fixed $N \in \mathbb{N}$, consider

$$Z_T^N := \exp\left\{\int_0^T B_t^N(\bar{X}) \cdot dW_t - \frac{1}{2}\int_0^T \left|B_t^N(\bar{X})\right|^2 dt\right\}.$$

To prove Theorem 1, it suffices to prove the following Novikov condition holds true (see e.g. [7, Chapter 3, Proposition 5.13]):

***Proposition* 2.** *For any $T > 0$, $N \geq 1$, $\kappa > 0$, there exists $C(T,N,\kappa)$ such that*

$$\mathbb{E}_{\mathbb{W}}\left(\exp\left\{\kappa \int_0^T |B_t^N(\bar{X})|^2 dt\right\}\right) \leq C(T,N,\kappa). \tag{10}$$

Proof. Drop the index N for simplicity. Using the definition of (B_t^N) and Jensen's inequality one has

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa \int_0^T \left|B_t^N(\bar{X})\right|^2 dt\right\}\right] \leq \mathbb{E}_{\mathbb{W}}\left[\exp\left\{\frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \int_0^T \kappa N F_t(\bar{X}^i, \bar{X}^j)\, dt\right\}\right],$$

from which we deduce

$$\mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa \int_0^T \left|B_t^N(\bar{X})\right|^2 dt\right\}\right] \leq \frac{1}{N}\sum_{i=1}^N \frac{1}{N}\sum_{j=1}^N \mathbb{E}_{\mathbb{W}}\left[\exp\left\{\kappa N \int_0^T F_t(\bar{X}^i, \bar{X}^j)\, dt\right\}\right].$$

As the $\bar{X}^i$'s are independent Brownian motions, we are in a position to use Proposition 1. This concludes the proof.

□

4.2 Girsanov transform for $1 \leq r < N$ particles

In the proof of Theorem 1 we used (6) and a Girsanov transform. However, the right-hand side of (10) goes to infinity with N. Thus, Proposition 2 cannot be used to prove the tightness and propagation of chaos of the particle system. We instead define an intermediate particle system. For any integer $1 \leq r < N$, proceeding as in the proof of Theorem 1 one gets the existence of a weak solution on $[0,T]$ to

$$\begin{cases} d\widehat{X}_t^{l,N} = dW_t^l, \quad 1 \leq l \leq r, \\ d\widehat{X}_t^{i,N} = \left\{\frac{1}{N}\sum_{j=r+1}^N \int_0^t K_{t-s}(\widehat{X}_t^{i,N} - \widehat{X}_s^{j,N})ds\; \mathbb{1}_{\{\widehat{X}_t^{i,N} \neq \widehat{X}_t^{j,N}\}}\right\} dt + dW_t^i, \quad r+1 \leq i \leq N, \\ \widehat{X}_0^{i,N} \text{ i.i.d. and independent of } (W) := (W^i, 1 \leq i \leq N). \end{cases} \tag{11}$$

Below we set $\hat{X} := (\hat{X}^{i,N}, 1 \leq i \leq N)$ and we denote by $\mathbb{Q}^{r,N}$ the probability measure under which $\hat{X}$ is well defined. Notice that $(\widehat{X}^{l,N}, 1 \leq l \leq r)$ is independent of $(\widehat{X}^{i,N}, r+1 \leq i \leq N)$. We now study the exponential local martingale associated to the change of drift between (3) and (11). For $x \in C([0,T];\mathbb{R})^N$ set

$$\beta_t^{(r)}(x) := \Big(b_t^{1,N}(x), \dots, b_t^{r,N}(x), \frac{1}{N}\sum_{i=1}^r \int_0^t K_{t-s}(x_t^{r+1} - x_s^i)ds\; \mathbb{1}_{\{x_t^{r+1} \neq x_t^i\}}, \dots,$$
$$\frac{1}{N}\sum_{i=1}^r \int_0^t K_{t-s}(x_t^N - x_s^i)ds\; \mathbb{1}_{\{x_t^N \neq x_t^i\}}\Big).$$

In the sequel we will need uniform w.r.t N bounds for moments of

$$Z_T^{(r)} := \exp\left\{-\int_0^T \beta_t^{(r)}(\widehat{X}) \cdot dW_t - \frac{1}{2}\int_0^T |\beta_t^{(r)}(\widehat{X})|^2 dt\right\}. \tag{12}$$

Proposition 3. *For any $T > 0$, $\gamma > 0$ and $r \geq 1$ there exists $N_0 \geq r$ and $C(T,\gamma,r)$ s.t.*

$$\forall N \geq N_0, \quad \mathbb{E}_{\mathbb{Q}^{r,N}} \exp\left\{\gamma \int_0^T |\beta_t^{(r)}(\widehat{X})|^2 dt\right\} \leq C(T,\gamma,r).$$

Proof. For $x \in C([0,T];\mathbb{R})^N$, one has

$$|\beta_t^{(r)}(x)|^2 = \sum_{i=1}^r \left(\frac{1}{N}\sum_{j=1}^N \int_0^t K_{t-s}(x_t^i - x_s^j)ds \mathbb{1}_{\{x_t^j \neq x_t^i\}}\right)^2$$
$$+ \frac{1}{N^2}\sum_{j=1}^{N-r}\left(\sum_{i=1}^r \int_0^t K_{t-s}(x_t^{r+j} - x_s^i)ds \mathbb{1}_{\{x_t^{r+j} \neq x_t^i\}}\right)^2.$$

By Jensen's inequality,

$$|\beta_t^{(r)}|^2 \leq \frac{1}{N}\sum_{i=1}^r\sum_{j=1}^N F_t(x^i, x^j) + \frac{r}{N^2}\sum_{j=1}^{N-r}\sum_{i=1}^r F_t(x^{r+j}, x^i).$$

For simplicity we below write $\mathbb{E}$ (respectively, $\hat{X}^i$) instead of $\mathbb{E}_{\mathbb{Q}^{r,N}}$ (respectively, $\hat{X}^{i,N}$). Observe that

$$\mathbb{E}\exp\left\{\gamma\int_0^T |\beta_t^{(r)}(\widehat{X})|^2 dt\right\}$$
$$\leq \Big(\mathbb{E}\exp\Big\{\sum_{i=1}^r \frac{2\gamma}{N}\sum_{j=1}^N \int_0^T F_t(\widehat{X}^i, \widehat{X}^j)dt\Big\}\Big)^{1/2}\Big(\mathbb{E}\exp\Big\{\frac{2\gamma r}{N^2}\sum_{j=1}^{N-r}\sum_{i=1}^r \int_0^T F_t(\widehat{X}^{r+j}, \widehat{X}^i)dt\Big\}\Big)^{1/2}$$
$$\leq \Big(\prod_{i=1}^r \frac{1}{N}\sum_{j=1}^N \mathbb{E}\exp\Big\{2\gamma r\int_0^T F_t(\widehat{X}^i, \widehat{X}^j)dt\Big\}\Big)^{\frac{1}{2r}}\Big(\prod_{j=1}^{N-r}\frac{1}{r}\sum_{i=1}^r \mathbb{E}\exp\Big\{\frac{2\gamma r^2}{N}\int_0^T F_t(\widehat{X}^{r+j}, \widehat{X}^i)dt\Big\}\Big)^{\frac{1}{2(N-r)}}.$$

In view of Proposition 1, it now remains to prove that there exists $N_0 \in \mathbb{N}$ such that

$$\sup_{N \geq N_0} \mathbb{E}\left[\exp\left\{\frac{2\gamma r^2}{N}\int_0^T F_t(\hat{X}^{r+j}, \hat{X}^i)dt\right\}\right] \leq C(T,r,\gamma).$$

We postpone the proof of this inequality to the Appendix (see Proposition 4). ☐

4.3 Propagation of chaos : Proof of Theorem 2

4.3.1 Tightness

We start with showing the tightness of $\{\mu^N\}$ and of an auxiliary empirical measure which is needed in the sequel.

Lemma 3. *Let $\mathbb{Q}^N$ be as above. The sequence $\{\mu^N\}$ is tight under $\mathbb{Q}^N$. In addition, let $\nu^N := \frac{1}{N^4}\sum_{i,j,k,l=1}^N \delta_{X_\cdot^{i,N},X_\cdot^{j,N},X_\cdot^{k,N},X_\cdot^{l,}}$ The sequence $\{\nu^N\}$ is tight under $\mathbb{Q}^N$.*

Proof. The tightness of $\{\mu^N\}$, respectively $\{\nu^N\}$, results from the tightness of the intensity measure $\{\mathbb{E}_{\mathbb{Q}^N}\mu^N(\cdot)\}$, respectively$\{\mathbb{E}_{\mathbb{Q}^N}\nu^N(\cdot)\}$: See Sznitman [11, Prop. 2.2-ii]. By symmetry, in both cases it suffices to check the tightness of $\{\text{Law}(X^{1,N})\}$. We aim to prove

$$\exists C>0, \forall N\geq N_0, \quad \mathbb{E}_{\mathbb{Q}^N}[|X_t^{1,N}-X_s^{1,N}|^4]\leq C_T|t-s|^2, \quad 0\leq s,t\leq T, \tag{13}$$

where N_0 is as in Proposition 3. Let $Z_T^{(1)}$ be as in (12). One has

$$\mathbb{E}_{\mathbb{Q}^N}[|X_t^{1,N}-X_s^{1,N}|^4]=\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-1}|\widehat{X}_t^{1,N}-\widehat{X}_s^{1,N}|^4].$$

As $\widehat{X}^{1,N}$ is a one dimensional Brownian motion under $\mathbb{Q}^{1,N}$,

$$\mathbb{E}_{\mathbb{Q}^N}[|X_t^{1,N}-X_s^{1,N}|^4]\leq(\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}])^{1/2}(\mathbb{E}_{\mathbb{Q}^{1,N}}[|\widehat{X}_t^{1,N}-\widehat{X}_s^{1,N}|^8])^{1/2}\leq(\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}])^{1/2}C|t-s|^2.$$

Observe that, for a Brownian motion $(W^\sharp)$ under $\mathbb{Q}^{1,N}$,

$$\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}]=\mathbb{E}_{\mathbb{Q}^{1,N}}\exp\left\{2\int_0^T\beta_t^{(1)}(\hat{X})\cdot dW_t^\sharp-\int_0^T|\beta_t^{(1)}(\hat{X})|^2dt\right\}.$$

Adding and subtracting $3\int_0^T|\beta_t^{(1)}|^2dt$ and applying again the Cauchy-Schwarz inequality,

$$\mathbb{E}_{\mathbb{Q}^{1,N}}[(Z_T^{(1)})^{-2}]\leq\left(\mathbb{E}_{\mathbb{Q}^{1,N}}\exp\left\{6\int_0^T|\beta_t^{(1)}(\hat{X})|^2dt\right\}\right)^{1/2}.$$

Applying Proposition 3 with $k=1$ and $\gamma=6$, we obtain the desired result. □

4.3.2 Convergence

To prove Theorem 2 we have to show that any limit point of $\{\text{Law}(\mu^N)\}$ is $\delta_{\mathbb{Q}}$, where $\mathbb{Q}$ is the unique solution to (MPKS). Since the particles interact through an unbounded singular functional, we adapt the arguments in Bossy and Talay [1, Thm. 3.2].

Let $\phi\in C_b(\mathbb{R}^p)$, $f\in C_b^2(\mathbb{R})$, $0<t_1<\cdots<t_p\leq s<t\leq T$ and $m\in\mathcal{P}(C[0,T];\mathbb{R})$. Set

$$G(m):=\int_{(C[0,T];\mathbb{R})^2}\phi(x_{t_1}^1,\ldots,x_{t_p}^1)\Big(f(x_t^1)-f(x_s^1)-\frac{1}{2}\int_s^t f''(x_u^1)du-\int_s^t f'(x_u^1)\mathbb{1}_{\{x_u^1\neq x_u^2\}}\int_0^u K_{u-\theta}(x_u^1-x_\theta^2)d\theta du\Big)dm(x^1)\otimes dm(x^2).$$

We start with showing that

$$\lim_{N\to\infty}\mathbb{E}[\big(G(\mu^N)\big)^2]=0. \tag{14}$$

Observe that

$$G(\mu^N)=\frac{1}{N}\sum_{i=1}^N\phi(X_{t_1}^{i,N},\ldots,X_{t_p}^{i,N})\Big(f(X_t^{i,N})-f(X_s^{i,N})-\frac{1}{2}\int_s^t f''(X_u^{i,N})du-\frac{1}{N}\sum_{j=1}^N\int_s^t f'(X_u^{i,N})\mathbb{1}_{\{X_u^{i,N}\neq X_u^{j,N}\}}\int_0^u K_{u-\theta}(X_u^{i,N}-X_\theta^{j,N})d\theta\, du\Big).$$

Apply Itô's formula to $\frac{1}{N}\sum_{i=1}^N(f(X_t^{i,N})-f(X_s^{i,N}))$. It comes:

$$\mathbb{E}[\big(G(\mu^N)\big)^2]\leq\frac{C}{N^2}\mathbb{E}\Big(\sum_{i=1}^N\int_s^t f'(X_u^{i,N})dW_u^i\Big)^2\leq\frac{C}{N}.$$

Thus, (14) holds true.

Suppose for a while we have proven the following lemma:

Lemma 4. *Let $\Pi^\infty \in \mathcal{P}(\mathcal{P}(C([0,T];\mathbb{R})^4))$ be a limit point of $\{law(\nu^N)\}$. Then*

$$\lim_{N\to\infty} \mathbb{E}[\big(G(\mu^N)\big)^2] = \int_{\mathcal{P}(C([0,T];\mathbb{R})^4)} \Bigg\{ \int_{C([0,T];\mathbb{R})^4} \Big[f(x_t^1) - f(x_s^1) - \frac{1}{2}\int_s^t f^{''}(x_u^1)du \\ - \int_s^t f^{'}(x_u^1)\mathbb{1}_{\{x_u^1\neq x_u^2\}} \int_0^u K_{u-\theta}(x_u^1 - x_\theta^2)d\theta du \Big] \times \phi(x_{t_1}^1,\dots,x_{t_p}^1)d\nu(x^1\dots,x^4) \Bigg\}^2 d\Pi^\infty(\nu), \tag{15}$$

and

i) Any $\nu \in \mathcal{P}(C([0,T];\mathbb{R})^4)$ belonging to the support of Π^∞ is a product measure: $\nu = \nu^1\otimes\nu^1\otimes\nu^1\otimes\nu^1$.

ii) For any $t\in(0,T]$, the time marginal ν_t^1 of ν^1 has a density ρ_t^1 which satisfies

$$\exists C_T,\ \forall 0<t\le T,\quad \|\rho_t^1\|_{L^2(\mathbb{R})} \le \frac{C_T}{t^{\frac{1}{4}}}.$$

Then, combining (14) with the above result, we get

$$\int_{C([0,T];\mathbb{R})} \phi(x_{t_1}^1,\dots,x_{t_p}^1)\Big[f(x_t^1) - f(x_s^1) - \frac{1}{2}\int_s^t f''(x_u)du \\ - \int_s^t f'(x_u^1)\int_0^u\int K_{u-\theta}(x_u^1-y)\rho_\theta^1(y)dyd\theta du\Big]d\nu^1(x^1) = 0.$$

We deduce that ν^1 solves (MPKS) and thus that $\nu^1 = \mathbb{Q}$. As by definition Π^∞ is a limit point of Law(ν^N), it follows that any limit point of Law(μ^N) is $\delta_{\mathbb{Q}}$, which ends the proof.

4.3.3 Proof of Lemma 4

Proof of (15): Step 1. Notice that

$$\mathbb{E}[\big(G(\mu^N)\big)^2] = \frac{1}{N^2}\mathbb{E}\sum_{i,k=1}^N \Phi_2(X^{i,N},X^{k,N}) + \frac{1}{N^3}\mathbb{E}\sum_{i,k,l=1}^N \Phi_3(X^{i,N},X^{k,N},X^{l,N}) \\ + \frac{1}{N^3}\mathbb{E}\sum_{i,j,k=1}^N \Phi_3(X^{k,N},X^{i,N},X^{j,N}) + \frac{1}{N^4}\mathbb{E}\sum_{i,j,k,l=1}^N \Phi_4(X^{i,N},X^{j,N},X^{k,N},X^{l,N}), \tag{16}$$

where

$$\Phi_2(X^{i,N},X^{k,N}) := \phi(X_{t_1}^{i,N},\dots,X_{t_p}^{i,N})\ \phi(X_{t_1}^{k,N},\dots,X_{t_p}^{k,N}) \\ \times\Big(f(X_t^{i,N}) - f(X_s^{i,N}) - \frac{1}{2}\int_s^t f''(X_u^{i,N})du\Big)\Big(f(X_t^{k,N}) - f(X_s^{k,N}) - \frac{1}{2}\int_s^t f''(X_u^{k,N})du\Big),$$

$$\Phi_3(X^{i,N},X^{k,N},X^{l,N}) := -\phi(X_{t_1}^{i,N},\dots,X_{t_p}^{i,N})\ \phi(X_{t_1}^{k,N},\dots,X_{t_p}^{k,N}) \\ \times\Big(f(X_t^{i,N}) - f(X_s^{i,N}) - \frac{1}{2}\int_s^t f''(X_{u_1}^{i,N})du_1\Big)\int_s^t f'(X_u^{k,N})\mathbb{1}_{\{X_u^{k,N}\neq X_u^{l,N}\}}\int_0^u K_{u-\theta}(X_u^{k,N}-X_\theta^{l,N})\ d\theta\ du,$$

$$\Phi_4(X^{i,N},X^{j,N},X^{k,N},X^{l,N}) := \phi(X_{t_1}^{i,N},\dots,X_{t_p}^{i,N})\phi(X_{t_1}^{k,N},\dots,X_{t_p}^{k,N})\int_s^t\int_s^t\int_0^{u_1}\int_0^{u_2} f'(X_{u_1}^{i,N})\ f'(X_{u_2}^{k,N}) \\ \times K_{u_1-\theta_1}(X_{u_1}^{i,N}-X_{\theta_1}^{j,N})K_{u_2-\theta_2}(X_{u_2}^{k,N}-X_{\theta_2}^{l,N})\mathbb{1}_{\{X_{u_1}^{i,N}\neq X_{u_1}^{j,N}\}}\mathbb{1}_{\{X_{u_2}^{k,N}\neq X_{u_2}^{l,N}\}}d\theta_1\ d\theta_2\ du_1\ du_2.$$

Let C_N be the last term in the r.h.s. of (16). In Steps 2-4 below we prove that C_N converges as $N\to\infty$ and we identify its limit. Define the function F on $\mathbb{R}^{2p+6}$ as

$$F(x^1,\dots,x^{2p+6}) := \phi(x^7,\dots,x^{p+6})\ \phi(x^{p+7},\dots,x^{2p+6})\ f'(x^1)\ f'(x^3) \\ \times K_{u_1-\theta_1}(x^1-x^2)K_{u_2-\theta_2}(x^3-x^4)\mathbb{1}_{\{x^1\neq x^5\}}\mathbb{1}_{\{x^3\neq x^6\}}\mathbb{1}_{\{\theta_1<u_1\}}\mathbb{1}_{\{\theta_2<u_2\}}. \tag{17}$$

We set $C_N = \int_s^t \int_s^t \int_0^{u_1} \int_0^{u_2} A_N \; d\theta_1 \; d\theta_2 \; du_1 \; du_2$ with

$$A_N := \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \mathbb{E}(F(X_{u_1}^{i,N}, X_{\theta_1}^{j,N}, X_{u_2}^{k,N}, X_{\theta_2}^{l,N}, X_{u_1}^{j,N}, X_{u_2}^{l,N}, X_{t_1}^{i,N}, \dots, X_{t_p}^{i,N}, X_{t_1}^{k,N}, \dots, X_{t_p}^{k,N})).$$

We now aim to show that A_N converges pointwise (Step 2), that $|A_N|$ is bounded from above by an integrable function w.r.t. $d\theta_1 \; d\theta_2 \; du_1 \; du_2$ (Step 3), and finally to identify the limit of C_N (Step 4).

Proof of (15): Step 2. Fix $u_1, u_2 \in [s,t]$ and $\theta_1 \in [0, u_1)$ and $\theta_2 \in [0, u_2)$. Define τ^N as

$$\tau^N := \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \delta_{X_{u_1}^{i,N}, X_{\theta_1}^{j,N}, X_{u_2}^{k,N}, X_{\theta_2}^{l,N}, X_{u_1}^{j,N}, X_{u_2}^{l,N}, X_{t_1}^{i,N}, \dots, X_{t_p}^{i,N}, X_{t_1}^{k,N}, \dots, X_{t_p}^{k,N}}.$$

Define the measure $\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ on $\mathbb{R}^{2p+6}$ as $\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}(A) = \mathbb{E}(\tau^N(A))$. The convergence of $\{\text{law}(\nu^N)\}$ implies the weak convergence of $\mathbb{Q}^N_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ to the measure on $\mathbb{R}^{2p+6}$ defined by

$$\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}(A) := \int_{\mathcal{P}(C([0,T];\mathbb{R})^4)} \int_{C([0,T];\mathbb{R})^4} \mathbb{1}_A(x^1_{u_1}, x^2_{\theta_1}, x^3_{u_2}, x^4_{\theta_2}, x^2_{u_1}, x^4_{u_2}, x^1_{t_1}, \dots, x^1_{t_p}, x^3_{t_1}, \dots, x^3_{t_p}) d\nu(x^1, x^2, x^3, x^4) d\Pi^\infty(\nu).$$

Let us show that this probability measure has an L^2-density w.r.t. the Lebesgue measure on $\mathbb{R}^{2p+6}$. Let $h \in C_c(\mathbb{R}^{2p+6})$. By weak convergence,

$$\begin{aligned} &\left| < \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, h > \right| \\ &= \left| \lim_{N\to\infty} \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \mathbb{E} h(X_{u_1}^{i,N}, X_{\theta_1}^{j,N}, X_{u_2}^{k,N}, X_{\theta_2}^{l,N}, X_{u_1}^{j,N}, X_{u_2}^{l,N}, X_{t_1}^{i,N}, \dots, X_{t_p}^{i,N}, X_{t_1}^{k,N}, \dots, X_{t_p}^{k,N}) \right|. \end{aligned}$$

When, in the preceding sum, at least two indices are equal, we bound the expectation by $\|h\|_\infty$. When $i \neq j \neq k \neq l$, we apply Girsanov's transform in Section 4.2 with four particles and Proposition 3. This procedure leads to

$$\begin{aligned} \left| < \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, h > \right| \leq \lim_{N\to\infty} \Big(\|h\|_\infty \frac{C}{N} \\ + \frac{C_T}{N^4} \sum_{i\neq j\neq k\neq l} \Big(\mathbb{E} h^2(\hat{X}_{u_1}^{i,N}, \hat{X}_{\theta_1}^{j,N}, \hat{X}_{u_2}^{k,N}, \hat{X}_{\theta_2}^{l,N}, \hat{X}_{u_1}^{j,N}, \hat{X}_{u_2}^{l,N}, \hat{X}_{t_1}^{i,N}, \dots, \hat{X}_{t_p}^{i,N}, \hat{X}_{t_1}^{k,N}, \dots, \hat{X}_{t_p}^{k,N}) \Big)^{1/2} \Big). \end{aligned}$$

All the processes $\hat{X}^{i,N}, \dots, \hat{X}^{l,N}$ being independent Brownian motions we deduce that

$$\left| < \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, h > \right| \leq C_{u_1,u_2,\theta_1,\theta_2,t_1,\dots,t_p} \|h\|_{L^2(\mathbb{R}^{2p+6})}.$$

It follows from Riesz's representation theorem that $\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ has a density w.r.t. Lebesgue's measure in $L^2(\mathbb{R}^{2p+6})$. Therefore, the functional F is continuous $\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ - a.e. Since for any fixed $u_1, u_2 \in [s,t]$ and $\theta_1 \in [0, u_1)$, $\theta_2 \in [0, u_2)$, F is also bounded we have

$$\lim_{N\to\infty} A_N = < \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, F > .$$

Proof of (15): Step 3. In view of the definition (17) of F we may restrict ourselves to the case $i \neq j$ and $k \neq l$. Use the Girsanov transforms from Section 4.2 with $r_{i,j,k,l} \in \{2,3,4\}$ according to the respective cases $(i = k, j = l)$, $(i = k, j \neq l)$, $(i \neq k, j \neq l)$, etc. Below we write r instead of $r_{i,j,k,l}$. By exchangeability it comes:

$$A_N = \Big| \frac{1}{N^4} \sum_{i\neq j, k\neq l} \mathbb{E}_{\mathbb{Q}^{r,N}} (Z_T^{(r)} F(\cdots)) \Big| \leq \frac{1}{N^4} \sum_{i\neq j, k\neq l} \Big(\mathbb{E}_{\mathbb{Q}^{r,N}} (Z_T^{(r)})^2 \Big)^{1/2} \Big(\mathbb{E}_{\mathbb{Q}^{r,N}} (F^2(\cdots)) \Big)^{1/2}.$$

By Proposition 3, $\mathbb{E}_{\mathbb{Q}^{r,N}}(Z_T^{(r)})^2$ can be bounded uniformly w.r.t. N. As the functions f and ϕ are bounded we deduce

$$\sqrt{\mathbb{E}_{\mathbb{Q}^{r,N}}(F^2(\cdots))} \leq C\mathbb{1}_{\{\theta_1<u_1\}}\mathbb{1}_{\{\theta_2<u_2\}}\left(\mathbb{E}_{\mathbb{Q}^{r,N}}(K^2_{u_1-\theta_1}(W^i_{u_1}-W^j_{\theta_1})K^2_{u_1-\theta_1}(W^k_{u_2}-W^l_{\theta_2}))\right)^{1/2},$$

for $i \neq j$, $k \neq l$ and $r \equiv r_{i,j,k,l}$. In view of (6), for any $0 < \theta < u < T$ we have

$$\left(\mathbb{E}_{\mathbb{Q}^{r,N}}(K^4_{u-\theta}(W^i_u - W^j_\theta)\right)^{1/4} \leq \frac{C}{u^{1/8}}\|K_{u-\theta}\|_{L^4(\mathbb{R})} \leq \frac{C}{u^{1/8}(u-\theta)^{7/8}}.$$

Therefore,

$$\left(\mathbb{E}_{\mathbb{Q}^{r,N}}(F^2(\cdots))\right)^{1/2} \leq C\frac{\mathbb{1}_{\{\theta_1<u_1\}}\mathbb{1}_{\{\theta_2<u_2\}}}{u_1^{1/8}(u_1-\theta_1)^{7/8}u_2^{1/8}(u_2-\theta_2)^{7/8}}.$$

We thus have obtained:

$$A_N \leq C\frac{\mathbb{1}_{\{\theta_1<u_1\}}\mathbb{1}_{\{\theta_2<u_2\}}}{u_1^{1/8}(u_1-\theta_1)^{7/8}u_2^{1/8}(u_2-\theta_2)^{7/8}}.$$

We remark that the r.h.s. belongs to $L^1((0,T)^4)$.

Proof of (15): Step 4. Steps 2 and 3 allow us to conclude that

$$\lim_{N\to\infty} C_N = \int_s^t\int_s^t\int_s^t\int_s^t < \mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}, F > \ d\theta_1 d\theta_2 du_1 du_2.$$

By definition of $\mathbb{Q}_{u_1,\theta_1,u_2,\theta_2,t_1,\dots,t_p}$ and F we thus have obtained that

$$\begin{aligned}\lim_{N\to\infty} C_N = &\int_{P(C([0,T];\mathbb{R})^4)}\int_s^t\int_s^t\int_{C([0,T];\mathbb{R})^4} f'(x^1_{u_1})f'(x^3_{u_2})\phi(x^1_{t_1},\dots,x^1_{t_p})\phi(x^3_{t_1},\dots,x^3_{t_p})\\ &\times\int_0^{u_1}\int_0^{u_2} K_{u_1-\theta_1}(x^1_{u_1}-x^2_{\theta_1})K_{u_2-\theta_2}(x^3_{u_2}-x^4_{\theta_2})\mathbb{1}_{\{x^1_{u_1}\neq x^2_{u_1}\}}\mathbb{1}_{\{x^3_{u_2}\neq x^4_{u_2}\}}\\ &d\nu(x^1,x^2,x^3,x^4)\ d\theta_1\ d\theta_2\ du_1\ du_2\ d\Pi^\infty(\nu).\end{aligned}$$

A similar procedure is applied to the three other terms in the r.h.s. of (16). Together with the preceding, we obtain (15).

Proof of i) and ii). Now, we prove the claims i) and ii) of Lemma 4.

i) For any measure $\nu \in \mathcal{P}(C([0,T];\mathbb{R})^4)$, denote its first marginal by ν^1. One easily gets Π^∞ a.e., $\nu = \nu^1\otimes\nu^1\otimes\nu^1\otimes\nu^1$ (see [1, Lemma 3.3]).

ii) Take $h \in C_c(\mathbb{R})$. Using similar arguments as in the above Step 1, for any $0 < t \leq T$ one has $\Pi^\infty(d\nu)$ a.e.,

$$\begin{aligned}&<\nu^1_t, h> = \lim_{N\to\infty}\mathbb{E}_{\mathbb{Q}^N} <\mu^N_t, h> = \lim_{N\to\infty}\mathbb{E}_{\mathbb{Q}^N}(h(X^{1,N}_t)) = \lim_{N\to\infty}\mathbb{E}_{\mathbb{Q}^{1,N}}(Z^{(1)}_T h(W^{1,N}_t))\\ &\leq \frac{C}{t^{1/4}}\|h\|_{L^2(\mathbb{R})}.\end{aligned}$$

5 Appendix

***Proposition* 4.** *Same assumptions as in Proposition 1. There exists $N_0 \in \mathbb{N}$ depending only on T and α, such that*

$$\sup_{N\geq N_0}\mathbb{E}_{\mathbb{P}}\left[\exp\left\{\frac{\alpha}{N}\int_0^T\left(\int_0^t K_{t-s}(Y_t - w_s)ds\mathbb{1}_{\{w_t\neq Y_t\}}\right)^2 dt\right\}\right] \leq C(T,\alpha).$$

Compared to the proof of Proposition 3, as w and Y exchanged places in the left-hand side, it is not so obvious to use the independence of Brownian increments. However, the weight $\frac{1}{N}$ enables us to skip the localization part (see Lemmas 1 and 2).

Proof. Fix $N \in \mathbb{N}$. Set $I := \int_0^T \left(\int_0^t K_{t-s}(Y_t - w_s)ds\right)^2 dt$. One has

$$I^k \leq C\left(\int_0^T \int_0^t \frac{ds}{(t-s)^{3/4}} \int_0^t \frac{(Y_t - w_s)^2}{(t-s)^{9/4}} e^{-\frac{(Y_t-w_s)^2}{t-s}} dsdt\right)^k$$
$$\leq CT^{k/4}\left(\int_0^T \int_0^t \frac{(Y_t - w_s)^2}{(t-s)^{9/4}} e^{-\frac{(Y_t-w_s)^2}{t-s}} dsdt\right)^k.$$

For $0 \leq s < T$ and for $(\omega, \widehat{\omega}) \in C([0,T];\mathbb{R}) \times C([0,T];\mathbb{R})$, define the functional H_s as

$$H_s(\omega, \widehat{\omega}) = \int_s^T \frac{(\omega_t - \widehat{\omega}_s)^2}{(t-s)^{9/4}} e^{-\frac{(\omega_t - \widehat{\omega}_s)^2}{t-s}} dt.$$

As the processes Y and w are independent,

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T H_s(Y,w)ds\right)^k = \int_{C([0,T];\mathbb{R})} \mathbb{E}_{\mathbb{P}}\left(\int_0^T H_s(x,w)ds\right)^k \mathbb{P}^Y(dx).$$

As before we observe that, for any $x \in C([0,T];\mathbb{R})$,

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T H_s(x,w)ds\right)^k = k!\mathbb{E}_{\mathbb{P}} \int_0^T H_{s_1}(x,w) \int_{s_1}^T \dots \mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{s_{k-1}}}\left(\int_{s_{k-1}}^T H_{s_k}(x,w)ds_k\right)\dots ds_1.$$

Therefore,

$$\mathbb{E}_{\mathbb{P}}^{\mathcal{G}_{s_{k-1}}}\left(\int_{s_{k-1}}^T H_{s_k}(x,w)ds_k\right) = \int_{s_{k-1}}^T \int_{s_k}^T \int \frac{(x_t - z - w_{s_{k-1}})^2}{(t-s_k)^{9/4}} e^{-\frac{(x_t-z-w_{s_{k-1}})^2}{t-s_k}} g_{s_k - s_{k-1}}(z)dzdtds_k$$
$$\leq \int_{s_{k-1}}^T \frac{C}{\sqrt{s_k - s_{k-1}}} \int_{s_k}^T \frac{1}{(t-s_k)^{3/4}} \int z^2 e^{-z^2} dzdtds_k \leq CT^{1/4}\sqrt{T - s_{k-1}} \leq CT^{3/4}.$$

Finally,

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T H_s(x,w)ds\right)^k \leq k!CT^{3/4}\mathbb{E}_{\mathbb{P}}\left(\int_0^T H_{s_1}(x,w) \int_{s_1}^T \cdots \int_{s_{k-2}}^T H_{s_{k-1}}(x,w)ds_{k-1}\dots ds_1\right).$$

Repeat the previous procedure $k-2$ times and use that the density of w_{s_1} is bounded by $\frac{C}{\sqrt{s_1}}$. It comes:

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^T H_s(x,w)ds\right)^k \leq k!C^{k-1}T^{3(k-1)/4}\mathbb{E}_{\mathbb{P}}\left(\int_0^T H_{s_1}(x,w)ds_1\right)$$
$$\leq k!C^{k-1}T^{3(k-1)/4}\mathbb{E}_{\mathbb{P}} \int_0^T \frac{C}{\sqrt{s_1}} \int_{s_1}^T \int \frac{(x_t - w_{s_1})^2}{(t-s_1)^{9/4}} e^{-\frac{(x_t-w_{s_1})^2}{t-s_1}} dxdtds_1 \leq k!C^kT^{\frac{3k}{4}}.$$

This implies that for any $M \geq 1$,

$$\mathbb{E}_{\mathbb{P}} \sum_{k=1}^M \frac{\alpha^k I^k}{N^k k!} \leq \sum_{k=1}^M \frac{\alpha^k C^k T^k}{N^k}.$$

Choose N_0 large enough to have $\frac{\alpha}{N_0}CT < 1$. To conclude, we apply Fatou's lemma.

□

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