The Unicycle in Presence of a Single Disturbance: Observability Properties

Agostino Martinelli

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Abstract

This paper investigates the observability properties of a mobile robot that moves on a planar surface by satisfying the unicycle dynamics and that is equipped with exteroceptive sensors (visual or range sensors). In accordance with the unicycle dynamics, the motion is powered by two independent controls, which are the linear and the angular speed, respectively. We assume that both these speeds are known. We consider the case when the robot motion is affected by a disturbance (or unknown input) that produces an additional (unknown and time dependent) robot speed along a fixed direction. The goal of the paper is to obtain the observability properties of the state that characterizes the robot configuration. The novelty of this observability analysis is that it takes into account the presence of an unknown and time dependent disturbance. Previous works that analyzed similar localization problems, either did not consider the presence of disturbances, or assumed disturbances constant in time. In order to deal with an unknown and time dependent disturbance, the paper adopts a new analytic tool [18]. This analytic tool is the solution of a fundamental open problem in control theory (the Unknown Input Observability problem in the general nonlinear case). We show that the application of this analytic tool is very simple and can be implemented automatically. Additionally, we simulate the aforementioned system and we show that a simple estimator based on an Extended Kalman Filter provides results that fully agree with what we could expect from the observability analysis.

1 INTRODUCTION

In many mobile robotics applications one fundamental requirement for a mobile robot is its ability to estimate its own configuration during the motion (localization problem).

It has become praxis in robotics to provide an observability analysis prior to solving an estimation problem. Usually this is a proof that the system is locally weakly observable [6] according to nonlinear observability theory. In the last decade, a great effort has been devoted to analyze the observability properties in several estimation robotics problems (e.g., in SLAM [4, 8, 12, 20, 21], in visual-inertial sensor fusion [7, 9, 10, 11, 13, 15, 16, 19]). Note that in all these works, the dynamics of the considered systems are assumed either unaffected by disturbances or, in presence of disturbances, these are assumed to be constant in time.

Very recently, also the case when some of the inputs are unknown and time dependent has been considered in the framework of mobile robotics [2, 16]. In [2] the authors investigate the observability properties of a fundamental problem in the framework of mobile robotics (the bearing SLAM). In [16] the author derives the observability properties in the case of robots equipped with visual and inertial sensors, when some of the inertial sensors are missing. In both these cases, the authors introduced new methods to solve a fundamental problem in control theory that is the Unknown Input Observability problem. The problem of state observability for systems driven by unknown inputs (UI) is a fundamental problem in control theory. This problem was introduced and firstly investigated in the seventies [1, 3, 5, 22]. A huge effort has then been devoted to design observers for both linear and nonlinear systems in presence of UI. On the other hand, the problem of finding a simple analytic condition that allows us to check the weak local observability of the state is still open in the nonlinear case and for any number of UI. The methods introduced in [2, 16] only provide sufficient conditions for the state observability. They are based on a suitable state extension.

Very recently, this fundamental problem has been solved in the general case [18]. In other words, the analytic criterion that allows us to check the weak local observability of the state in the general nonlinear case and characterized by any number of unknown inputs has been introduced. This analytic tool is based on a simple recursive algorithm that computes the observable codistribution for a nonlinear system whose dynamics are driven by multiple known inputs, a nonlinear drift and multiple unknown inputs (note that the case of a single unknown input was previously dealt in [17]). In this paper, in section 3, we provide a concise description of the analytic criterion. Then, the paper adopts the analytic criterion in order to investigate the observability properties of a mobile robot that moves

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on a planar surface by satisfying the unicycle dynamics and that is equipped with exteroceptive sensors. The robot dynamics and observation are defined in section 2. In accordance with the unicycle dynamics, the motion is powered by two independent controls, which are the linear and the angular speed, respectively. We assume that both these speeds are known. In other words, we consider these two speeds as two known inputs of our system. We consider the case when the robot motion is affected by a disturbance that produces an additional (unknown and time-dependent) robot speed along a fixed direction. We consider both the case when this fixed direction is known and unknown. The goal of the paper is to obtain the observability properties of the state that characterizes the robot configuration. Note that the disturbance is an unknown and time dependent input of our system. Hence, our system is characterized by two known inputs and one unknown input. In section 4 we derive the observability properties of the systems defined in section 2, by using the analytic tool provided in section 3. In section 5, we simulate the aforementioned system and we show that, a simple estimator based on an Extended Kalman Filter, provides results that agree with what we could expect from the observability analysis. Finally, conclusions are provided in section 6

2 The considered system

We consider a vehicle that moves on a $2D$-environment. The configuration of the vehicle in a global reference frame can be characterized through the vector $[x_v,\ y_v,\ \theta_v]^T$, where x_v and y_v are the Cartesian vehicle coordinates, and θ_v is the vehicle orientation. We assume that the vehicle motion satisfies the unicycle dynamics and we also consider the case when the robot motion is affected by a disturbance that produces an additional (and unknown) robot speed (denoted by w) along a fixed direction (denoted by γ). Hence, the dynamics are characterized by the following differential equations:

$$\begin{bmatrix} \dot{x}_v & = v\cos\theta_v + w\cos\gamma \\ \dot{y}_v & = v\sin\theta_v + w\sin\gamma \\ \dot{\theta}_v & = \omega_v \end{bmatrix} \tag{2.1}$$

where v and ω_v are the linear and the rotational speed, respectively, in absence of disturbance ($w = 0$). We assume that these two speeds are known (we refer to them as to the known inputs), w is unknown (we refer to it as to the unknown input or disturbance) and γ is constant in time and we consider both the case when it is known and unknown.

We consider the following three cases of output (see also figure 1 for an illustration):

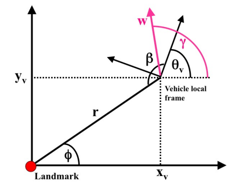

Figure 1: The vehicle state together with the three considered outputs.

1. the distance from the origin, denoted by r (e.g., a landmark is at the origin and its distance is measured by a range sensor);
2. the bearing of the origin in the local frame, denoted by β (e.g., a landmark is at the origin and its bearing angle is measured by an on-board camera);
3. the bearing of the vehicle in the global frame, denoted by ϕ (e.g., a camera is placed at the origin).

3 Extended Observability Rank Condition

This section provides the analytic criterion to investigate the observability properties of a nonlinear system whose dynamics are driven by multiple known inputs a nonlinear drift and multiple unknown inputs (or disturbances). In absence of unknown inputs, the analytic criterion is the observability rank condition introduced by Hermann and Krener [6]. We call the new criterion, able to account for the presence of unknown inputs, the *extended observability rank condition*. The derivation of this criterion is available on the book in [18]. In the driftless case and with a single unknown input, the criterion was introduced in [17].

We will refer to a nonlinear control system with m_u known inputs ($u \triangleq [u_1, \cdots, u_{m_u}]^T$) and m_w unknown inputs or disturbances ($w \triangleq [w_1, \cdots, w_{m_w}]^T$). The state is the vector $x \in M$, with M an open set of $\mathbb{R}^n$. We assume that the dynamics are nonlinear with respect to the state and affine with respect to the inputs (both known and unknown). Finally, for the sake of simplicity, we will refer to the case of a single output y (the extension to the case of multiple outputs is trivial and is given in [18]). Our system is characterized by the following equations:

$$\begin{cases} \dot{x} = g^0(x) + \sum_{i=1}^{m_u} f^i(x)u_i + \sum_{j=1}^{m_w} g^j(x)w_j \\ y = h(x) \end{cases} \tag{3.2}$$

where $g^0(x)$, $f^i(x)$, $i = 1, \cdots, m_u$, and $g^j(x)$, $j = 1, \cdots, m_w$, are vector fields in M and the function $h(x)$ is a scalar function defined on the open set M. Finally, we assume that the unknown inputs $w_1, \cdots, w_{m_w}$ are analytic functions of time.

The system in (2.1) is characterized by $m_w = 1$ and dynamics linear (and not affine) in the inputs (i.e., without the term g^0). In other words, the system belongs to the following class of systems:

$$\begin{cases} \dot{x} = \sum_{i=1}^{m_u} f^i(x) u_i + g(x) w \\ y = h(x) \end{cases} \tag{3.3}$$

These systems are precisely the ones dealt in [17].
We now outline all the steps to investigate the weak local observability at a given point x_0 of a nonlinear system characterized by (3.3). The reader is addressed to [18] (sections 3.3 and 4.2) for the general case described by (3.2). Basically, these steps are the steps necessary to compute the observable codistribution (i.e., the steps of algorithms 2 below) and to prove that the differential of a given state component belongs to this codistribution.

In the sequel, we will denote with the symbol $\mathcal{D}$ the differential with respect to the state x. For instance, if $x = [x_1, \ x_2]^T$ and $h = x_1 + x_2^2$, we have[1]: $\mathcal{D}h = \mathcal{D}x_1 + 2x_2 \mathcal{D}x_2$.

For a given codistribution Ω and a given vector field $f = f(x)$ (both defined on the open set M), we denote by $\mathcal{L}_f \Omega$ the codistribution whose covectors are the Lie derivatives along f of the covectors in Ω. We remind the reader that the Lie derivative of a scalar function $h(x)$ along the vector field $f(x)$ is defined as follows:

$$\mathcal{L}_f h \triangleq \frac{\partial h}{\partial x} f$$

which is the product of the row vector $\frac{\partial h}{\partial x}$ with the column vector f. Hence, it is a scalar function. Additionally, by definition of Lie derivative of covectors, we have: $\mathcal{L}_f \mathcal{D}h = \mathcal{D}\mathcal{L}_f h$. Finally, given two vector spaces V_1 and V_2, we denote by $V_1 + V_2$ their sum, i.e., the span of all the generators of both V_1 and V_2.
To start we need to introduce the following scalar:

[1]The span of the differentials of a set of scalar functions is a codistribution. The reader non familiar with the theory of distributions can simply consider the differential as the gradient operator. The gradient of a scalar function is a line vector. For instance, if $x = [x_1, \ x_2]^T$ and $h = x_1 + x_2^2$, we obtain for its gradient the line vector function $[1, \ 2x_2]$. Later, we adopt this representation. According to this, a codistribuion will be the span of a set of line vectors and a covector (i.e., an element of a codistribution) will be a line vector.

$$L_g^1 \triangleq \mathcal{L}_g h \tag{3.4}$$

The analytic computation is based on the assumption that $L_g^1 \neq 0$ on a given neighbourhood of x_0 (the reader is addressed to [18] to see how to proceed in the case when this assumption is not satisfied). Finally, we need to introduce a new set of vector fields ${}^i\phi_m \in \mathbb{R}^n$ ($i = 1, \cdots, m_u$ and for any integer m). They are obtained recursively by the following algorithm:

Algorithm 1

1. ${}^i\phi_0 = f^i$;
2. ${}^i\phi_m = \frac{[{}^i\phi_{m-1}, \ g]}{L_g^1}$

where the parenthesis $[\cdot, \cdot]$ denote the Lie bracket of vector fields, defined as follows:

$$[a, \ b] \triangleq \frac{\partial b}{\partial x} a(x) - \frac{\partial a}{\partial x} b(x)$$

In other words, for each $i = 1, \cdots, m_u$, we have one new vector field at every step of the algorithm. In the case when $m_u = 1$, we denote by ϕ_m the vector field ${}^1\phi_m$.

The algorithm that generates the entire observable codistribution for the system in (3.3) is the following:

Algorithm 2 Observable codistribution in the case $m_w = 1$ and $g^0 = 0$

1. $\Omega_0 = span\{\mathcal{D}h\}$;
2. $\Omega_m = \Omega_{m-1} + \sum_{i=1}^{m_u} \mathcal{L}_{f^i} \Omega_{m-1} + \mathcal{L}_{\frac{g}{L_g^1}} \Omega_{m-1} + \sum_{i=1}^{m_u} span\left\{\mathcal{L}_{{}^i\phi_{m-1}} \mathcal{D}h\right\}$

It is possible to prove that this algorithm converges in less than $n + 2$ steps (n is the dimension of the state). The convergence criterion is trivial in the case dealt by the following lemma (the proof can be found in [18]).

Lemma 1 *Let us denote by Λ_j^i the distribution generated by ${}^i\phi_0, {}^i\phi_1, \cdots, {}^i\phi_j$ (i.e., the vectors obtained by running algorithm 1 for a given $i = 1, \cdots, m_u$) and by $m_i (\leq n-1)$ the smallest integer for which $\Lambda_{m_i+1}^i = \Lambda_{m_i}^i$ (n is the dimension of the state x). If $\mathcal{L}_{{}^i\phi_j} L_g^1 = 0$, $\forall i = 1, \cdots, m_u$ and $\forall j = 0, \cdots, m_i$, the convergence of algorithm 2 occurs in at most $n-1$ steps and it occurs at the smallest integer j such that $\Omega_{j+1} = \Omega_j$.*

The steps to investigate the weak local observability at a given point x_0 of a nonlinear system characterized by (3.3) are reported below. Note that, in the trivial case analyzed by the previous lemma, the criterion provided below simplifies, since we do not need to compute the quantity $\tau \left(= \frac{\mathcal{L}_g^2 h}{(\mathcal{L}_g^1 h)^2}\right)$, and we do not need to check that its differential belongs to the codistribution computed at every step of algorithm 2. In practice, we skip the steps 4 and 5 in the procedure below.

1. For the chosen x_0, compute $L_g^1 (= \mathcal{L}_g^1 h)$. In the case when $L_g^1 = 0$, choose another function in the space of functions $\mathcal{F}$ (defined as the space that contains h and its Lie derivative up to any order along the vector fields $f^1, \cdots, f^{m_u}$) such that its Lie derivative along g does not vanish[2].

2. Compute the codistribution Ω_0 and Ω_1 (at x_0) by using algorithm 2.

3. Compute the vector fields ${}^i\phi_m$ ($i = 1, \cdots, m_u$) by using algorithm 1, starting from $m = 0$, to check if the considered system is in the special case dealt by lemma 1. In this trivial case, set $m' = 0$, use the recursive step of algorithm 2 to build the codistribution Ω_m for $m \geq 2$, and skip to step 6.

4. Compute $\tau \left(= \frac{\mathcal{L}_g^2 h}{(L_g^1)^2}\right)$ and $\mathcal{D}\tau$.

5. Use the recursive step of algorithm 2 to build the codistribution Ω_m for $m \geq 2$, and, for each m, check if $\mathcal{D}\tau \in \Omega_m$. Denote by m' the smallest m such that $\mathcal{D}\tau \in \Omega_m$.

6. For each $m \geq m'$, check if $\Omega_{m+1} = \Omega_m$ and denote by $\Omega^* = \Omega_{m^*}$, where m^* is the smallest integer such that $m^* \geq m'$ and $\Omega_{m^*+1} = \Omega_{m^*}$ (note that $m^* \leq n + 2$).

7. If the differential of a given state component (x_j, $j = 1, \cdots, n$) belongs to Ω^* (namely if $\mathcal{D}x_j \in \Omega^*$) on a given neighbourhood of x_0, then x_j is weakly locally observable at x_0. If this holds for all the state components, the state x is weakly locally observable at x_0. Finally, if the dimension of Ω^* is smaller than n on a given neighbourhood of x_0, then the state is not weakly locally observable at x_0.

4 Observability properties of the system defined in section 2

In this section we derive the observability properties of the system defined in section 2 by using the analytic criterion presented in the previous section. We deal with both the cases when γ is known and unknown (in section 4.1 and 4.2, respectively). For both cases, we consider the three outputs defined in section 2, separately. For simplicity sake, we actually consider the following three outputs: $y = r^2 = x_v^2 + y_v^2$ instead of $y = r$, $y = \tan\beta = \frac{y_v - x_v \tan\theta_v}{x_v + y_v \tan\theta_v}$ instead of $y = \beta$ and $y = \tan\phi = \frac{y_v}{x_v}$ instead of $y = \phi$. Obviously, the result of the observability analysis does not change. As it will be seen, in all the cases, the systems always meet the assumptions of lemma 1. Hence, we will skip the fourth and the fifth step in the procedure given in section 3. Note that the most of systems investigated in [18] do not satisfy these assumptions and for them we had to compute the scalar τ and verify that its differential belongs to the codistribution returned by algorithm 2.

4.1 Observability properties when γ is known

The state is $[x_v,\ y_v,\ \theta_v]^T$ and its dynamics are provided by the three equations in (2.1), where γ is a known parameter. From (2.1) and (3.3) we easily obtain: $m_u = 2$, $u_1 = v$, $u_2 = \omega_v$,

$$f^1 = \begin{bmatrix} \cos\theta_v \\ \sin\theta_v \\ 0 \end{bmatrix}, \quad f^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad g = \begin{bmatrix} \cos\gamma \\ \sin\gamma \\ 0 \end{bmatrix}$$

We follow the seven steps mentioned in section 3.

$y = r^2$

First Step We compute L_g^1 in (3.4). We obtain:

$$L_g^1 = 2(x_v \cos\gamma + y_v \sin\gamma)$$

Second Step We have $\Omega_0 = span\{[x_v, y_v, 0]\}$ and $\Omega_1 = span\{[x_v, y_v, 0], [\cos\theta_v, \sin\theta_v, y_v\cos\theta_v - x_v\sin\theta_v]\ \}$.

Third Step We compute ${}^1\phi_1$ and ${}^2\phi_1$ through algorithm 1. We obtain: ${}^1\phi_1 = {}^2\phi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. As a result, all the subsequent steps of algorithm 1 provide null vectors. Therefore, the assumptions of lemma 1 are trivially met. We set $m' = 0$ and we skip to the sixth step.

Sixth Step We use algorithm 2 to compute Ω_2. We obtain $\Omega_2 = \Omega_1 + d\mathcal{L}_{\frac{g}{L_g^1}} \mathcal{L}_{f^1} h$. Hence, we obtain that Ω_2 has dimension equal to 3, which is the dimension of the state. Hence, algorithm 2 has converged.

Seventh Step We conclude that the state is weakly locally observable.

$y = \tan\beta$

First Step We compute L_g^1 in (3.4). We obtain:

[2]If the Lie derivative of any function in $\mathcal{F}$ vanishes, it means that the unknown input can be ignored to obtain the observability properties (the system is not canonic with respect to the unknown input, as shown in [18]).

$$L_g^1 = -\frac{y_v \cos\gamma - x_v \sin\gamma}{x_v^2 \cos^2\theta_v + 2\sin\theta_v \cos\theta_v x_v y_v - y_v^2 \cos^2\theta_v + y_v^2},$$

Second Step By an explicit computation (by using algorithm 2) we obtain that the dimension of Ω_0 is 1 and the dimension of Ω_1 is 2.

Third Step We compute ${}^1\phi_1$ and ${}^2\phi_1$ through algorithm 1. They vanish as in the previous case. As a result, all the subsequent steps of algorithm 1 provide null vectors. Therefore, the assumptions of lemma 1 are trivially met. We set $m' = 0$ and we skip to the sixth step.

Sixth Step By using algorithm 2 we obtain that Ω_2 has dimension equal to 3, which is the dimension of the state. Hence, algorithm 2 has converged.

Seventh Step We conclude that the state is weakly locally observable.

$y = \tan\phi$

First Step We compute L_g^1 in (3.4). We obtain:

$$L_g^1 = -\frac{y_v \cos\gamma - x_v \sin\gamma}{x_v^2}$$

Second Step We have $\Omega_0 = span\{[-y_v, x_v, 0]\}$. Additionally, we obtain that the dimension of Ω_1 is 2.

Third Step We compute ${}^1\phi_1$ and ${}^2\phi_1$ through algorithm 1. They vanish as in the previous case. As a result, all the subsequent steps of algorithm 1 provide null vectors. Therefore, the assumptions of lemma 1 are trivially met. We set $m' = 0$ and we skip to the sixth step.

Sixth Step By using algorithm 2 we obtain that Ω_2 has dimension equal to 3, which is the dimension of the state. Hence, algorithm 2 has converged.

Seventh Step We conclude that the state is weakly locally observable.

4.2 Observability properties when γ is unknown The state is $[x_v,\ y_v,\ \theta_v,\ \gamma]^T$ and its dynamics are provided by the following four equations:

$$\begin{cases} \dot{x}_v &= v\cos\theta_v + w\cos\gamma \\ \dot{y}_v &= v\sin\theta_v + w\sin\gamma \\ \dot{\theta}_v &= \omega_v \\ \dot{\gamma} &= 0 \end{cases} \quad (4.5)$$

From (4.5) and (3.3) we easily obtain: $m_u = 2$, $u_1 = v$, $u_2 = \omega_v$,

$$f^1 = \begin{bmatrix} \cos\theta_v \\ \sin\theta_v \\ 0 \\ 0 \end{bmatrix}, \quad f^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad g = \begin{bmatrix} \cos\gamma \\ \sin\gamma \\ 0 \\ 0 \end{bmatrix}$$

As in section 4.1, we follow the seven steps mentioned in section 3 and we consider the same expressions for the outputs.

$y = r^2$

First Step We obviously obtain the same expression for L_g^1 as in the case $y = r^2$ of section 4.1.

Second Step We have $\Omega_0 = span\{[x_v, y_v, 0, 0]\}$. Additionally, $\Omega_1 = span\{[x_v, y_v, 0, 0], [\cos\theta_v, \sin\theta_v, y_v\cos\theta_v - x_v\sin\theta_v, 0]\}$.

Third Step We compute ${}^1\phi_1$ and ${}^2\phi_1$ through algorithm 1. We obtain: ${}^1\phi_1 = {}^2\phi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. As a result, all the subsequent steps of algorithm 1 provide null vectors. Therefore, the assumptions of lemma 1 are trivially met. We set $m' = 0$ and we skip to the sixth step.

Sixth Step By using algorithm 2 we obtain that Ω_2 has dimension equal to 3. Additionally, by a direct computation, it is possible to check that $\Omega_3 = \Omega_2$ meaning that $m^* = 2$ and $\Omega^* = \Omega_2$.

Seventh Step The dimension of the observable codistribution is $3 < 4$. We conclude that the state is not weakly locally observable. In particular, since the differential of every state component does not belong to Ω_2, we conclude that no state component is observable.

$y = \tan\beta$

First Step We obviously obtain the same expression for L_g^1 as in the case $y = \tan\beta$ of section 4.1.

Second Step We compute Ω_0 and Ω_1 and we obtain that their dimensions are 1 and 2, respectively.

Third Step We compute ${}^1\phi_1$ and ${}^2\phi_1$ through algorithm 1. They vanish as in the previous case. As a result, all the subsequent steps of algorithm 1 provide null vectors. Therefore, the assumptions of lemma 1 are trivially met. We set $m' = 0$ and we skip to the sixth step.

Sixth Step By using algorithm 2 we obtain that Ω_2 has dimension equal to 3. Additionally, by a direct computation, it is possible to check that $\Omega_3 = \Omega_2$ meaning that $m^* = 2$ and $\Omega^* = \Omega_2$.

γ	Output	State observability
known	$y = r$	yes
known	$y = \beta$	yes
known	$y = \phi$	yes
unknown	$y = r$	no
unknown	$y = \beta$	no
unknown	$y = \phi$	yes

Table 1: Weak local observability of the state in all the considered scenarios

Seventh Step The dimension of the observable codistribution is $3 < 4$. We conclude that the state is not weakly locally observable. In particular, since the differential of every state component does not belong to Ω_2, we conclude that no state component is observable.

$y = \tan \phi$

First Step We obviously obtain the same expression for L_g^1 as in the case $y = \tan \phi$ of section 4.1.

Second Step We compute Ω_0 and Ω_1 and we obtain that their dimensions are 1 and 2, respectively.

Third Step We compute ${}^1\phi_1$ and ${}^2\phi_1$ through algorithm 1. They vanish as in the previous case. As a result, all the subsequent steps of algorithm 1 provide null vectors. Therefore, the assumptions of lemma 1 are trivially met. We set $m' = 0$ and we skip to the sixth step.

Sixth Step By using algorithm 2 we obtain that Ω_2 has dimension equal to 4, which is the dimension of the state. Hence, algorithm 2 has converged.

Seventh Step We conclude that the state is weakly locally observable.

Table 1 summarizes the results of the observability analysis carried out in this section. We conclude this section by remarking that these results agree with our expectation. By using the observability rank condition in [6], we easily obtain that, in absence of the disturbance, the dimension of the observable codistribution is 2 for the first two observations ($y = r$ and $y = \beta$) and 3 for the last one ($y = \phi$). In particular, for the first two observations, all the initial states rotated around the vertical axis are indistinguishable. In other words, in these two cases, the system exhibits a continuous symmetry (the reader is addressed to [14, 15] for the meaning of continuous symmetry). In presence of the disturbance, when γ is known, the aforementioned system invariance is broken and the entire state becomes observable. When γ is unknown, the symmetry still remains (and obviously also concerns the new state component γ). Note also a very important aspect. The presence of a disturbance improves the observability properties of a system (this regards the case when γ is known). In particular, if $w = 0$ (absence of disturbance), the state becomes unobservable even if it is known that it is zero, while, when $w \neq 0$, the state is observable even if it is unknown. Note that the analytic derivation of the analytic criterion holds when the disturbance does not vanish [18].

5 Simulations

The scope of this section is to show that, by using a very simple estimator based on an Extended Kalman Filter, we obtain results which agree with our observability analysis. Note that the goal of this paper is not to derive a new observer for a system characterized by an unknown input. On the other hand, the results obtained by using the aforementioned estimator, are useful to further validate the analytic criterion in section 3.

5.1 Simulated trajectories and robot sensors

The trajectories are simulated as follows. The equations in (2.1) are discretized with a time step of $5 * 10^{-4}$ s and they last for 10 s. The linear speed, i.e. v, is constant and equal to 0.1 m s^{-1}. The angular speed, i.e. ω_v, is generated randomly. Specifically, its value is settled at each step and follows a Gaussian distribution with mean value $\frac{2\pi}{10}$ rad s^{-1} and variance $\left(\frac{6\pi}{10}\right)^2$ rad^2 s^{-2}. A typical example of trajectory, obtained with this setting, is displayed in fig. 2, left side. The disturbance is generated as follows. The parameter γ is set equal to 0.7 π. The unknown input w is generated randomly. Specifically, its value is settled at each step and follows a Gaussian distribution with mean value 0.05 m s^{-1} and variance $(0.005)^2$ m^2 s^{-2}. A typical example of trajectory is displayed in fig. 2, right side.

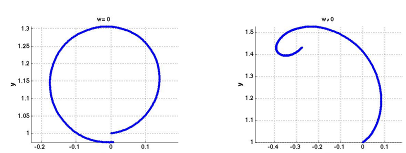

Figure 2: Typical simulated trajectories. Left side without disturbance, right side with disturbance.

The robot is equipped with proprioceptive sensors able to measure at each time step the linear and the angular speed. These measurements are affected by errors. Specifically, each measurement is generated at every time step of $5 * 10^{-4}$ s by adding to the true value a random error that follows a Gaussian distribution.

The mean value of this error is zero and the standard deviation is 0.01 times the true value for both the linear and the angular speed.
Regarding the exteroceptive measurements, they are generated at a lower frequency. Specifically, the measurements are generated each $2.5 * 10^{-2}$ s. Also these measurements are affected by errors. Specifically, the measurement is generated by adding to the true value a random error that follows a Gaussian distribution. The mean value of this error is zero and the standard deviation is 0.01 m for the range measurements and 1 deg for the two angular measurements (bearing of the origin in the robot frame and bearing of the robot in the global frame).

5.2 Estimation results We adopt an Extended Kalman Filter that estimates an extended state that includes the unknown input together with its first order time derivative. In other words, in the case when γ is known, the estimated state is: $[x_v,\ y_v,\ \theta_v,\ w,\ w^{(1)}]$. Its dynamics is obtained from (2.1) and are:

$$(5.6) \qquad \begin{cases} \dot{x}_v &= v\cos\theta_v + w\cos\gamma \\ \dot{y}_v &= v\sin\theta_v + w\sin\gamma \\ \dot{\theta}_v &= \omega_v \\ \dot{w} &= w^{(1)} \\ \dot{w}^{(1)} &= w^{(2)} \end{cases}$$

When γ is unknown, the estimated state also includes γ and the dynamics are given by (5.6) and the additional equation $\dot{\gamma} = 0$.

In order to implement the prediction phase of our Extended Kalman Filter we have to provide the value of $w^{(2)}$, which is unknown. We set this quantity to zero. Note that the simulated trajectory does not satisfy this hypothesis since the disturbance is randomly generated. However, the estimator is able to provide good performance as it is shown in figures 3-5.

Figure 3 displays the estimated trajectory in the case when the output is provided by the range sensor ($h = r$). Dots blue are the ground truth, red circles the estimated trajectory by only using the knowledge of the proprioceptive measurements (i.e., the measurements of v and ω_v), black stars the trajectory estimated by our estimator. Left side is the case when γ is known and right side when it is unknown. In accordance with our observability analysis, the estimator follows the true trajectory only in the case when γ is known. Figure 4 displays the estimated trajectory in the case when the output is provided by the on-board bearing sensor ($h = \beta$). Also in this case, the estimator follows the true trajectory only in the case when γ is known, in accordance with our observability analysis. Finally, figure 5 displays the estimated trajectory in the case when the output is provided by the bearing sensor at the origin ($h = \phi$). In this case, the estimator follows the true trajectory both in the case when γ is known and when it is unknown. Also this result agrees with our observability analysis.

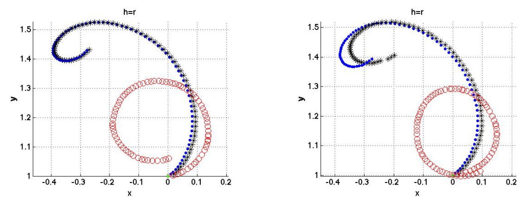

Figure 3: Estimated trajectory in the case when the output is $h = r$. Dots blue are the ground truth, red circles the estimated trajectory by only using the knowledge of the proprioceptive measurements, black stars the trajectory estimated by our estimator. On the left side is the case when γ is known and on the right side when it is unknown.

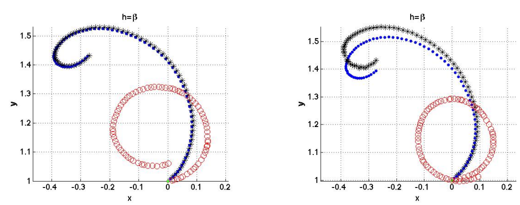

Figure 4: The same as in figure 3 in the case when the output is $h = \beta$.

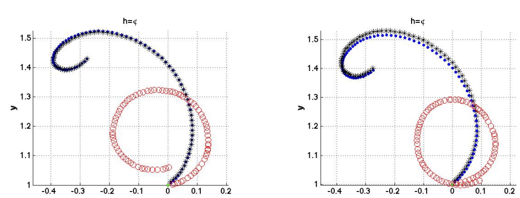

Figure 5: The same as in figure 3 in the case when the output is $h = \phi$.

We conclude this section by remarking that the results of our simulations fully agree with the observability analysis provided in section 4. Note that we showed the result of a single trial. However, we found always similar results by running many trials.

6 Conclusion

In this paper we investigated the observability properties of the unicycle in presence of a disturbance. This is a nonlinear system whose dynamics are driven by two known inputs (the linear and the angular speed that would result in absence of the disturbance) and an external unknown input. The unknown input consists of an additional unknown and time dependent robot speed, along a fixed direction. We considered both the case when this fixed direction is known and unknown. The goal of the paper was to obtain the observability properties of the state that characterizes the robot configuration. The novelty of this observability analysis was that it accounted for the presence of an unknown and time dependent disturbance. In order to account for the time dependent disturbance, the paper used the analytic criterion recently introduced in [18]. Note that this analytic criterion is the solution of a fundamental open problem in control theory. The analytic criterion was previously introduced in [17] for the easier driftless case and characterized by a single unknown input. We showed that the application of this criterion is very simple and can be done automatically. Additionally, we simulated the aforementioned systems. In particular, in the simulations, the disturbance was generated time dependent. We showed that, a simple estimator based on an Extended Kalman Filter, provides results that fully agree with what we could expect from the observability analysis.

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