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Antoine Lejay* and Paolo Pigato†

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In financial markets, low prices are generally associated with high volatilities and vice-versa, this well known stylized fact usually being referred to as leverage effect. We propose a local volatility model, given by a stochastic differential equation with piecewise constant coefficients, which accounts of leverage and mean-reversion effects in the dynamics of the prices. This model exhibits a regime switch in the dynamics accordingly to a certain threshold. It can be seen as a continuous time version of the Self-Exciting Threshold Autoregressive (SETAR) model. We propose an estimation procedure for the volatility and drift coefficients as well as for the threshold level. Tests are performed on the daily prices of 21 assets. They show empirical evidence for leverage and mean-reversion effects, consistent with the results in the literature.

Keywords. Oscillating Brownian motion; leverage effect; realized volatility; mean-reversion; Self-Exciting Threshold Autoregressive model, Regime-Switch.

*Université de Lorraine, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54600, France
CNRS, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54600, France
Inria, Villers-lès-Nancy, F-54600, France
E-mail: `Antoine.Lejay@univ-lorraine.fr`
†Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, Berlin, 10117, Germany
E-mail: `paolo.pigato@wias-berlin.de`

1 Introduction

Despite the predominance of the Black & Scholes model for the dynamics of asset prices, its deficiencies to reflect all the phenomena observed in the markets are well documented and subject to many studies. Some *stylized facts* not consistent with Black & Scholes model are non normality of log-returns, asymmetry, heavy tails, varying conditional volatilities, volatility clustering, ... [9]. Regime switching is also consistently observed [2, 41]. Besides, some assets and indices exhibit mean-reverting effects (see *e.g.* [31–33, 37, 44, 46]).

By considering only the asset's price at discrete, fixed times $\{k\Delta t\}_{k=0,1,2,\dots}$, the log-returns $r_t = \log(S_{t+1}/S_t)$ of the Black & Scholes model $\{S_t\}_{t\geq 0}$ are nothing more than the simple time series

$$r_{t+1} = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\epsilon_t \text{ with } \epsilon_t \sim \mathcal{N}(0,1). \tag{1}$$

Several models alternative to (1) have been proposed to take some of the stylized facts into account. The most popular are the ARCH and GARCH models and their numerous variants that reproduce volatility clustering effects [12].

In this article, we consider *Leverage effects*, a term which refers, in the financial literature, to a correlation between the prices and the volatility. As observed for a long time, the lower the price, the higher the volatility. First explanations were given in [4, 8]. Processes such as the *constant elasticity volatility* (CEV) were proposed to account of these phenomena [8]. The origin of leverage effects is still subject to discussion (see *e.g.* [16]). Similarly, also *psychological barriers* lead to threshold models [19]. Other types of barriers on stock prices are given in [22].

In the early '80s, H. Tong have proposed a broad class of time series — the *threshold autoregressive models* (TAR) — with non-linear effects reproducing cyclical data [47–49]. This class contains *Hidden Markov chains* (HMM) and *self-exciting threshold autoregressive* models (SETAR) producing a wide range of behaviors. The former rely on a temporal segmentation (HMM models are good for crisis detection), while the latter rely on a spatial segmentation with a regime change when the price goes below or above a threshold.

Time series of SETAR type fit leverage and mean-reverting effects by defining a threshold which separates two regimes (high/low volatility, positive/negative trend). Unlike models such as HMM, no external nor latent randomness is used.

In finance, various aspects of SETAR like models have been considered [7, 32, 39, 42, 51]. An alternative form to SETAR model is provided by *threshold stochastic volatility* models [43, 50], where the volatility depends non-linearly on the price through a threshold model. In the continuous-time setting, self-exciting variants of Vasiček and Cox-Ingersoll-Ross models have also been proposed for interest rate models [11, 35].

Continuous times models could be seen as the limit of time series as the time step goes to 0. They have some advantages over time series, for allowing irregularly sampling, the use of stochastic calculus tools and possibly analytic or semi-analytic formulas for fast evaluation of option prices and risk estimation. Continuous time threshold models (or *threshold diffusion*) have been studied in [42, 45] for option valuation, in [32] for portfolio optimization, etc. In [5], a continuous time equivalent of an integrated SETAR model is constructed and applied to financial data. In [13, 34], M. Esquível and P. Mota have proposed two continuous time models which aim at mimicking the SETAR time series. In [34], one of these models, referred to as the *Delay Threshold Regime Switching model* (DTRS), is tested on the daily prices of 21 companies over almost 5 years. For almost all the stocks, they found a regime-change for the volatility.

The present paper has then three goals. First, we present a local volatility model called the *Geometric Oscillating Brownian motion* (GOBM), with piecewise constant volatility and drift, according to a threshold, as in [15,30]. The GOBM is an instance of the *tiled volatility model* of [30]. The market is complete under such a model, which reproduces leverage effects and psychological barriers. The GOBM is simpler to manipulate than the DTRS of [34], although having similar features.

The GOBM is also easily simulated by a standard Euler scheme [6, 52]. Option valuation could be performed as well using semi-analytic approaches [15,30]. Its ex-ante volatility could also be estimated from the call prices by solving Sturm-Liouville problems [30].

Second, we show how to estimate the ex-post volatilities, drifts and thresholds from discrete observations of the stocks prices under historical measure. The estimation procedures used in [5, 13, 34] are all derived from the ones designed for SETAR time series. In our case, stochastic calculus is used. The estimator for the volatility is the subject of the article [28], while the drift is treated in [27]. We discuss several issues regarding their quality. In addition, we provide a hypothesis test to decide whether or not the volatility is constant.

Third, we test our model against the same dataset as [34], finding similar results. This confirms that both leverage effects and mean-reversion effects hold for most of the stocks. This validates the use of the GOBM as a simple model which captures such effects, while the Black & Scholes model does not.

Outline. The GOBM is presented in Section 2. In Section 3 we consider the estimation procedures for the volatilities (Section 3.1), the drift (Section 3.2) and the threshold (Section 3.3). In Section 4, we present the results of [34] on the DTRS model (in Section 4.1) as well as our empirical findings (in Section 4.2). In Section 4.3, we present a hypothesis test to decide whether some leverage effect is present. The article ends with a global conclusion in Section 5.

2 The (geometric) Oscillating Brownian motion

The model. The Geometric Oscillating Brownian motion (GOBM) is the solution of an SDE of type

$$S_t = x + \int_0^t \sigma(S_s) S_s \, \mathrm{d}B_s + \int_0^t \mu(S_s) S_s \, \mathrm{d}s, \tag{2}$$

where, for a threshold $m \in \mathbb{R}$,

$$\sigma(x) = \begin{cases} \sigma_+ & \text{if } x \geq m, \\ \sigma_- & \text{if } x < m \end{cases} \text{ and } \mu(x) = \begin{cases} \mu_+ & \text{if } x \geq m, \\ \mu_- & \text{if } x < m. \end{cases} \tag{3}$$

Remark that this is a local volatility model, meaning that the coefficients (and in particular the volatility coefficient σ) only depend on the stock price.

We use a solution S of (2) as a model for the evolution of the price of an asset. The *log-price* $X = \log(S)$ satisfies the SDE

$$X_t = x + \int_0^t \sigma(X_s) \, \mathrm{d}B_s + \int_0^t b(X_s) \, \mathrm{d}s, \tag{4}$$

with

$$\sigma(x) = \begin{cases} \sigma_+ & \text{if } x \geq r, \\ \sigma_- & \text{if } x < r \end{cases} \text{ and } b(x) = \begin{cases} b_+ = \mu_+ - \sigma_+^2/2 & \text{if } x \geq r, \\ b_- = \mu_- - \sigma_-^2/2 & \text{if } x < r \end{cases} \tag{5}$$

for a threshold $r = \log(m)$. Notice the slight abuse of notation in (3) and (5), due to the change of the value for the threshold when taking the logarithm.

When the drift $b = 0$ and $m = 0$, X is called an *Oscillating Brownian motion* (OBM, [21]), a name we keep even in presence of a two-valued drift and a threshold. When $\sigma_+ = \sigma_-$ and $b_+ = b_-$, then the price follows the Black & Scholes model. By extension, we still call the solution to (4) a *GOBM*.

The effect of the drift is discussed in Section 3.2. When $b_+ < 0$ and $b_- > 0$, the process is ergodic and mean-reverting. The convergence toward equilibrium differs from the ones in the Vašičeck and Heston models in which the drift is linear.

Existence and uniqueness. The solution to (4) is an instance of a more general class of processes with discontinuous coefficients which was studied in [24].

Proposition 1 ([24]). *There exists a unique strong solution to* (4)*, hence to* (2)*.*

The (geometric)-OBM can be easily manipulated with the standard tool of stochastic analysis, sometimes relying on the Itô-Tanaka formula instead of the sole Itô formula (See *e.g.* [14]).

Properties of the market. Unlike in some regime switching models, there is no hidden randomness leading to incomplete markets, while offering some regime change properties.

Proposition 2. *Assuming the GOBM model for the returns process with a constant risk-free rate, the market is viable and complete.*

Proof. Using the results of [24], the Girsanov Theorem can be applied to the equation for the log-price. Hence, as for the Black & Scholes model, it is possible to reduce the discounted log-price to a martingale by removing the drift. Hence, there exists an equivalent martingale measure, meaning that the market is viable [20, Theorem 2.1.5.4, p. 89].

As any absolutely continuous measure could only be reached through a Girsanov transform [24], the risk neutral measure is unique, meaning that the market is complete. □

Remark 1. Through a simple transform, the OBM is strongly related to the Skew Brownian motion (SBM). As shown in [40], the situation is radically different for the SBM where arbitrage could exists. The GOBM may be generalized by considering a log-price solution to $\mathrm{d}X_t = x + \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t + \eta\,\mathrm{d}L_t^r(X)$, where $L^r(X)$ is the local time of X at the threshold r. The effect of the coefficient $\eta \in (-1, 1)$ would be to "push upward" (if $\eta > 0$) or downward (if $\eta < 1$) the price, which corresponds to some directional predictability effect. However, considering $\eta \neq 0$ radically changes the structure of the market.

Application to options pricing. The simple form of the coefficients in the GOBM allows one to perform explicit computations. For example, the resolvent has a rather simple closed-form expression. However, for a non-vanishing drift, the analytic form of the density could become cumbersome [25]. Notwithstanding, the explicit expressions of the density or the generator could be used to perform option pricing [10, 15, 19] or to estimate implied volatility [30].

Monte Carlo simulation. From [6] or [52], the continuous time Euler scheme

$$\overline{X}_t = \overline{X}_{\frac{kT}{n}} + \int_{kT/n}^{t} \sigma\left(\overline{X}_{\frac{kT}{n}}\right) \mathrm{d}B_s + b\left(\overline{X}_{\frac{kT}{n}}\right)\left(t - \frac{kT}{n}\right) \text{ for } \frac{kT}{n} \leq t < \frac{(k+1)T}{n}$$

provides us with an approximation of X. Thus, the GOBM at times kT/n, $k = 0, 1, 2, \dots, n$ is very simple to simulate by $\overline{S}_{kT/n} = \exp(\overline{X}_{kT/n})$ through the recursive equation

$$\overline{X}_{\frac{(k+1)T}{n}} = \overline{X}_{\frac{kT}{n}} + \sigma\left(\overline{X}_{\frac{kT}{n}}\right)\xi_k + b\left(\overline{X}_{\frac{kT}{n}}\right)\frac{T}{n},\ k = 0, \dots, n-1$$

for a sequence $\xi_0, \dots, \xi_n$ of independent random variables with distribution $\mathcal{N}(0, T/n)$.

3 Estimation of the parameters from the observations of the stock prices

The GOBM X is defined on five parameters (volatility, drift and threshold, see Table 1) which we are willing to estimate. In Sections 3.1 and 3.2 we consider the estimation of $(\sigma_\pm, b_\pm)$ for fixed threshold r, by considering first the estimation of the volatility and then of the drift. Afterwards in Section 3.3, the threshold is chosen through a model selection principle.

S	price of the stock
$X = \log(S)$	log-price
$\xi = X - r$	shifted log-price for a threshold r
r	threshold of X
$m = \exp(r)$	threshold of S
σ_-	volatility of X below r
σ_+	volatility of X above r
b_-	drift of X below r
b_+	drift of X above r
$\mu_- = b_- + \frac{\sigma_-^2}{2}$	appreciation rate of S below m
$\mu_+ = b_+ + \frac{\sigma_+^2}{2}$	appreciation rate of S above m
d	delay (DTRS only)

Table 1: Notations for the GOBM and DTRS models.

3.1 Estimation of the volatility

In this section we consider the estimation of the volatility for prices given by the model in (2), when the threshold $r = \log(m)$ is known. We recall the estimators and the theoretical convergence results presented in [28], and discuss their application in the framework of volatility modeling. Remark that the process $\xi := X - r = \log(S) - r$ is a drifted OBM.

The data. Our observations are $n+1$ daily data $\{\xi_k\}_{k=0,\dots,n}$ with $\xi_k = \log(S_k) - r$ for an *a priori* known threshold r.

Our aim is to estimate (σ_+, σ_-) from such observations.

Discrete brackets. For two processes Z, Z', we define the discrete brackets by

$$[Z, Z']_n := \sum_{i=1}^{n}(Z_i - Z_{i-1})(Z'_i - Z'_{i-1}) \text{ and } [Z]_n := [Z, Z]_n.$$

Occupation times. The occupation times below and above the threshold play a central role in our study.

Using the shifted log-price $\xi = \log(S) - r$, the positive and negative *occupation times* up to time T are $Q_T^{\pm} = \int_0^T \mathbf{1}_{\pm\xi_s \geq 0}\,\mathrm{d}s$.

We then define a Riemann type approximation of $Q_n^{\pm}$ by A Riemann approximation is then

$$Q_{\pm}(n) := \sum_{i=1}^{n} \mathbf{1}_{\pm\xi_i \geq 0}. \tag{6}$$

The estimators. For a process ξ, we write $\xi^+ = \max\{\xi, 0\}$ and $\xi^- = -\min\{\xi, 0\}$, its positive and negative part. Our estimators $\sigma_{\pm}(n)^2$ for $\sigma_{\pm}^2$ are

$$\sigma_{\pm}(n)^2 = \frac{[\xi^{\pm}, \xi]_n}{Q_{\pm}(n)}.$$

These estimators are natural generalizations of the *realized volatility estimators* [3].

Proposition 3 ([28]). *When $b_+ = b_- = 0$, then $(\sigma_-(n)^2, \sigma_+(n)^2)$ is a consistent estimator of (σ_-^2, σ_+^2). Besides, there exists a pair of unit Gaussian random variables (G_-, G_+) independent from the underlying Brownian motion B (hence of ξ) such that*

$$\begin{bmatrix} \sqrt{n}\sqrt{\dfrac{Q_-(n)}{n}}\left(\sigma_-(n)^2 - \sigma_-^2\right) \\ \sqrt{n}\sqrt{\dfrac{Q_+(n)}{n}}\left(\sigma_+(n)^2 - \sigma_+^2\right) \end{bmatrix} = \begin{bmatrix} \dfrac{[\xi^-, \xi]_n - Q_-(n)\sigma_-^2}{\sqrt{Q_-(n)}} \\ \dfrac{[\xi^+, \xi]_n - Q_+(n)\sigma_+^2}{\sqrt{Q_+(n)}} \end{bmatrix} \xrightarrow[n\to\infty]{law} \begin{bmatrix} \sqrt{2}\sigma_-^2 G_- \\ \sqrt{2}\sigma_+^2 G_+ \end{bmatrix}. \tag{7}$$

Dealing with a drift. Proposition 3 is actually proved on high-frequency data $\xi_{k,n} := \xi_{k/n}$, $k = 0, \dots, n$ on the time interval $[0, 1]$.

Using a scaling argument, for any constant $c > 0$, $\{c^{-1/2}\xi_{ct}\}_{t\geq 0}$ is equal in distribution to $\zeta^{(c)}$ solution to the SDE

$$\mathrm{d}\zeta_t^{(c)} = \sigma(\zeta_t^{(c)})\,\mathrm{d}W_t + \sqrt{c}b(\zeta_t^{(c)})\,\mathrm{d}t,\ \zeta_0^{(c)} = c^{-1/2}\xi_0$$

for a Brownian motion W. With $c = n$, the problem of estimating the coefficients of $\{\xi_k\}_{k=0,\dots,n}$ is the same as the *high frequency* estimation of the coefficients of $\{\sqrt{n}\zeta_{k,n}^{(n)}\}_{k=0,\dots,n}$ on the time range $[0, 1]$.

Without drift, observing $\{\xi_{k,n}\}_{k=0,\dots,n}$ or $\{\xi_k\}_{k=0,\dots,n}$ leads to the same estimation. Using the Girsanov theorem, Proposition 3 stated for the high-frequency regime, that is on the observations $\{\zeta^{(1)}_{k,n}\}_{k=0,\dots,n}$ (since all the $\zeta^{(n)}$ are equal in distribution), is also valid in presence of a bounded drift.

With our data, the drift is very small compared to the volatility and the number n of observations is finite so that we still apply Proposition 3.

3.2 Estimation of the drift coefficients

To estimate the values $b_\pm$ of the drift, we consider that $\sigma_\pm$ has already been estimated and that the threshold $r = \log m$ is known (this issue is treated in Sect. 3.3). For the sake of simplicity, we still consider the shifted log-price process $\xi = X - r = \log S - r$.

Maximum likelihood estimation of the drift. A way to estimate the drift is to consider the drift among the possible ones which maximizes the Girsanov density $G(\xi)$ with respect to the solution to the driftless SDE $\mathrm{d}\xi_t = \sigma(\xi_t)\,\mathrm{d}W_t$ for a Brownian motion W. We then construct a maximum likelihood estimator [18]. The Girsanov density is

$$G(\xi) = \exp\left(\int_0^t \frac{b(\xi_s)}{\sigma(\xi_s)}\,\mathrm{d}W_s - \frac{1}{2}\int_0^t \frac{b^2(\xi_s)}{\sigma^2(\xi_s)}\,\mathrm{d}s\right).$$

As $b_\pm$ and $\sigma_\pm$ are piecewise constant, we can transform the stochastic integral $\int b(\xi_s)\sigma^{-1}(\xi_s)\,\mathrm{d}W_s$ using the Itô-Tanaka formula. It is then straightforward to establish that for our choice of model, the maximum of $G(\xi)$ with respect to a piecewise constant b is realized for

$$\beta_\pm(T) = \pm\frac{\xi_T^\pm - \xi_0^\pm - L_T/2}{Q_T^\pm(\xi)}, \tag{8}$$

where L_T is the symmetric local time of ξ at 0 and $Q_T^\pm$ are the occupation times of $\mathbb{R}^\pm$.

When the coefficients are constant, as for the log-price in the Black & Scholes model, where $\mathrm{d}X_t = \sigma\,\mathrm{d}B_t + b\,\mathrm{d}t$, the coefficient b may be estimated through $b(T) = (X_T - X_0)/T$. Our estimator (8) generalizes this formula; the local time term appears because of the discontinuity in the coefficients.

As (8) is applied to ξ, solution to $\mathrm{d}\xi_t = \sigma(\xi_t)\,\mathrm{d}B_t + b(\xi_t)\,\mathrm{d}t$, a direct application of the Itô-Tanaka formula to $x \mapsto x^\pm$ in (8) implies that for the martingales $M_t^\pm = \int_0^t \sigma_\pm(\xi_s)\mathbf{1}_{\pm\xi_s\geq 0}\,\mathrm{d}B_s$,

$$\beta_\pm(T) = b_\pm + \frac{M_T^\pm}{Q_T^\pm} \text{ with } \langle M^\pm\rangle_T = Q_T^\pm \text{ and } \langle M^+, M^-\rangle_T = 0. \tag{9}$$

Estimators. Although neither $Q_T^{\pm}$ nor L_T are observed, they can be approximated from the observations. The occupation time $Q_T^{\pm}$ is approximated by $Q_{\pm}(n)$ given by (6). The local time L_T could be approximated by [28]

$$L(n)_T = \frac{3\sqrt{\pi}}{2\sqrt{2}} \frac{\sigma_+ + \sigma-}{\sigma_+\sigma_-} [\xi^+, \xi^-]_T.$$

We then approximate $\beta_{\pm}$ by

$$\beta_{\pm} \approx b_{\pm}(n) := \pm \frac{\xi_T^{\pm} - \xi_0^{\pm} - L(n)_T/2}{Q_{\pm}(n)_T}.$$

The $b_{\pm}(n)$ are discrete times approximations of the continuous time estimators, which are easily constructed from the observations.

Asymptotic properties. The drift estimator shall be studied for long time horizon. The asymptotic properties of $\beta_{\pm}$ as $T \to \infty$, hence of $b_{\pm}(n)$, depend on the asymptotic behaviors of $Q_T^{\pm}$ from (9). We summarize in Table 2 the different cases that depend solely on the respective signs of b_+ and b_-.

	$b_+ < 0$	$b_+ = 0$	$b_+ > 0$
$b_- > 0$	ergodic (**E**)	null recurrent (**N1**)	transient (**T0**)
$b_- = 0$	null recurrent (**N1**)	null recurrent (**N0**)	transient (**T0**)
$b_- < 0$	transient (**T0**)	transient (**T0**)	transient (**T1**)

Table 2: Regime of X according to the sign of $b_{\pm}$.

The ergodic case, which corresponds to a mean-reverting process, is of course the most favorable one. In the transient case, the estimators may not converge. We present quickly some of the results in [27].

E The ergodic case is equivalent to the mean-reverting case. Thus $Q_T^{\pm}/T$ converges almost surely as $T \to \infty$. Therefore (β_-, β_+) converges almost surely to (b_-, b_+). For two independent unit Gaussian random variables (N^-, N^+),

$$\sqrt{T} \begin{bmatrix} \beta_- - b_- \\ \beta_+ - b_+ \end{bmatrix} \xrightarrow[T\to\infty]{\text{law}} \sqrt{|b_-| + |b_+|} \begin{bmatrix} \frac{\sigma_-}{\sqrt{b_-}} N^- \\ \frac{\sigma_+}{\sqrt{b_+}} N^+ \end{bmatrix}.$$

T0 If $b_+ > 0$, $b_- \geq 0$, $\lim_{T\to\infty} Q_T^- < +\infty$. Therefore, β_+ converges to b_+ and $\sqrt{T}(\beta_+ - b_+)$ converges in distribution to $\sigma_+ N^+$ for $N^+ \sim \mathcal{N}(0, 1)$. The estimator β_- of b_- does not converge to b_- and is then meaningless. The case $b_- < 0$, $b_+ \leq 0$ is treated by symmetry.

T1 If $b_+ > 0$ and $b_- < 0$, then with probability $p = \sigma_- b_+/(\sigma_+ b_- + \sigma_- b_+)$ it holds that $\lim_{T\to\infty} Q_T^+/T = 1$ a.s. and $\lim_{T\to\infty} Q_T^- < +\infty$, while with probability $1-p$, $\lim_{T\to\infty} Q_T^-/T = 1$ a.s. and $\lim_{T\to\infty} Q_T^+ < +\infty$. This asymptotic behavior is due to the fact that after a given random time, the process does not cross the threshold anymore. Given $\lim_{T\to\infty} Q_T^+/T = 1$, β_+ converges almost surely to b_+ and $\sqrt{T}(\beta_+ - b_+)$ converges in distribution to $\sigma_+ N^+$ for $N^+ \sim \mathcal{N}(0,1)$. The alternative situation, happening with probability $1-p$, is treated by symmetry.

N0 Whatever $T > 0$, Q_T^+/T follows a variant of the ArcSine distribution [21, 28]. Therefore, the distribution of $\sqrt{T}(\beta_-, \beta_+)$ does not depend on T. Then $\beta_\pm$ are consistent estimators of $b_\pm = 0$.

N1 If $b_+ = 0$, $b_- > 0$, then $\lim_{T\to\infty} Q_T^+/T = 1$ almost surely. In addition $Q_T^-/\sqrt{T}$ converges in distribution to $\sigma_+|N|/b_-$ for $N \sim \mathcal{N}(0,1)$. Therefore, (β_-, β_+) converges almost surely to (b_-, b_+). Besides, there exist independent unit Gaussian random variables N^- and N^+, also independent from N, such that

$$\begin{bmatrix} T^{1/4}(\beta_- - b_-) \\ T^{1/2}\beta_+ \end{bmatrix} \xrightarrow[T\to\infty]{\text{law}} \begin{bmatrix} \sqrt{b_- \frac{\sigma_-}{\sigma_+}} \frac{N^-}{\sqrt{|N|}} \\ \sigma_+ N^+ \end{bmatrix}.$$

The case $b_+ < 0$, $b_- = 0$ is treated by symmetry.

3.3 Estimation of the threshold

The above estimators for σ and b assume that the value m of the threshold is known. Following [48] (see also [38, p. 79]), we estimate m using a principle of *model selection* relying on the ideas of the Akaike Information Principle (AIC) [1]. Since the AIC involves the likelihood function, for which we do not necessarily have closed form expressions, we will need to work with approximations.

Approximation of the density. Given a threshold m as well as volatility and drift functions $x \mapsto \sigma(x)$ and $x \mapsto b(x)$, we first consider the density $y \mapsto p(\Delta t, x, y; m, \sigma, b)$ of $X_{t+\Delta t}$ given $X_t = x$ (the process is time-homogeneous so that p only depends on Δt, not on t). For a vanishing drift, a close form expression for p is know [21]. In presence of drift, the expression may become cumbersome if not intractable [25]. However, p can be approximated in a short time via the related Green function, easier to compute (See [29, Chapter 2]). Alternatively, we assume that the drift is constant over the time interval $[t, t+\Delta t]$ and replace p by the density of $Y_{t+\Delta t} + b(x)\Delta t$ given $Y_t = x$, where Y has the same volatility of X yet with a vanishing drift. In the implementation, we use the latter approximation of p which we denote by $\widetilde{p}(t, x, \cdot; m, \sigma, b)$.

Selection of the threshold. The procedure to select the "best" threshold is then

1/ We fix $m^{(1)}, \dots, m^{(k)}$ possible thresholds in the range of the observed values $\{X_{t_i}\}_{i=0,\dots,T}$ of the log-price X.

2/ For each threshold $m^{(j)}$, we estimate the drift and volatilities $\widehat{\sigma}^{(j)}$ and $\widehat{b}^{(j)}$.

3/ We compute the approximate log-likelihood

$$\Lambda^{(j)} = \sum_{i=0}^{T-1} \log \widetilde{p}(t, X_{t_i}, X_{t_{i+1}}; m^{(j)}, \widehat{\sigma}^{(j)}, \widehat{b}^{(j)}). \tag{10}$$

4/ We select as threshold $\widehat{m}$ the value $m^{(\tilde{j})}$ where $\tilde{j}$ is the indice for which $\{\Lambda^{(j)}\}_{j=1,\dots,k}$ is minimal.

Comparison with other models. In the model selection based on the AIC, the best model is the one for which the log-likelihood corrected by a value depending on the number of parameters is minimized. Here, the number of parameters is fixed to 4 so that it is sufficient to use only approximations of the log-likelihoods. A similar procedure is used in [32], yet with a density estimated through Monte Carlo, which is time-consuming. On the contrary, our procedure avoids any simulation step. With respect to the estimation for the SETAR model [38, 48], as well as the one of the DTRS model presented below, based on least squares [34], there is no delay so that the dimension of the model is reduced by 1.

4 Empirical evidences

We apply our estimators to financial data. We benchmark our model against the *Delay and Threshold Regime Switching model* (DTRS) of [34] by using the same data. Before this, we shortly present this model.

4.1 The Delay and Threshold Regime Switching model

In [34], M. Esquível and P. Mota introduce a *regime switching model with delay and threshold* (DTRS). First, they consider two sets of (functional) parameters (μ_1, σ_1) and (μ_2, σ_2), as well as a diffusion solution to the stochastic differential equation

$$\mathrm{d}S_t = \mu_{J_t}(t, S_t)\,\mathrm{d}t + \sigma_{J_t}(t, S_t)\,\mathrm{d}B_t$$

for a Brownian motion B, where J is a non-anticipative process with values in the set of indices $\{1, 2\}$.

The rule for J to switch is based on a threshold m, a delay d as well as a small parameter $\epsilon > 0$. Assume $S_0 \leq m$ and $J_0 = 1$ (resp. $S_0 \geq m$ and $J_0 = 2$). Let τ be

GOOG	Google	HP	Hewlett-Packard	AAPL	Apple
ADBE	Adobe	CA	CA	C	CitiGroup
KO	Coca-cola	CSCO	Cisco	IBM	IBM
JPM	JP Morgan	MCD	McDonalds	SBUX	Starbucks
PM	Philip Morris	PG	P & G	PFE	Pfizer
PCG	PG&E	NYT	New-York Times	MSFT	Microsoft
MSI	Motorola	MON	Monsanto	AMZN	Amazon

Table 3: Abbreviations of the names of the stocks (in Yahoo Finance).

the first time τ the process reaches $m+\epsilon$ (resp. $m-\epsilon$). Then $J_t = 1$ (resp. $J_t = 2$) on the time interval $[0, \tau+d)$ and then switches to $J_{\tau+d} = 2$ (resp. $J_{\tau+d} = 1$) before starting with a refreshed dynamics.

The parameter ϵ prevents an accumulation of "immediate" switches so that S could be constructed on rigorous basis [13]. With respect to simulation or estimation, ϵ is of no real importance as S is only known or simulated at discrete times.

More specifically, the DTRS model considered in [34] assumes that the μ_i and σ_i ($i = 1, 2$) are

$$\begin{cases} \sigma_1(t,x) = \sigma_- \cdot x & \text{if } x < m, \\ \sigma_2(t,x) = \sigma_+ \cdot x & \text{if } x \geq m \end{cases} \text{ and } \begin{cases} \mu_1(t,x) = \mu_- \cdot x & \text{if } x < m, \\ \mu_2(t,x) = \mu_+ \cdot x & \text{if } x \geq m. \end{cases}$$

for some constants $\sigma_\pm > 0$ and $\mu_\pm$. Hence, on each regime, the price S follows a dynamic of Black & Scholes type. We also define $b_\pm = \mu_\pm - \sigma_\pm^2/2$ so that $b_\pm$ are the possible values of the drift for the log-price.

Adapting the estimation approach for the SETAR [48], M. Esquível and P. Mota proposed a consistent estimation procedure of the parameters which is based on least squares [34].

Results for the DTRS. This estimator is then applied to the daily log-prices of 21 stocks (presented in Table 3) prices from January 2005 to November 2009. In Table 4, we report the estimated values of $\sigma_\pm$, m, $\mu_\pm$ (or $b_\pm$) and d found in [34]. These values have to be compared with the ones of Table 5.

For most of the data, a leverage effect is observed: the volatility below the threshold is higher than above it. In [34], option prices on European calls are also computed using a Monte Carlo procedure. The resulting prices are in good agreement with the ones of the market.

Comparison between the DTRS model and the GOBM. In spirit, the GOBM is similar to DTRS of [34] or to the one in [13]. Yet it avoids all the difficulties

Delay Threshold Regime Switching (DTRS) [34]									
Index	d	m [\$]	σ_- [%]	σ_+ [%]	μ_- [‰]	μ_+ [‰]	b_- [‰]	b_+ [‰]	signs
GOOG	13	642.0	2.33	2.53	1.61	−5.90	1.34	−6.22	+−
HP	1	46.9	2.47	1.71	1.68	−3.13	1.38	−3.27	+−
AAPL	8	173.5	3.41	2.87	2.28	−4.46	1.70	−4.87	+−
ADBE	1	41.5	2.77	1.58	1.26	−2.95	0.87	−3.07	+−
CA	1	22.1	3.34	1.55	1.76	−0.92	1.20	−1.04	+−
C	2	43.1	7.60	1.06	1.58	−0.14	−1.30	−0.20	−−
KO	1	10.0	4.95	1.83	15.80	−0.21	14.58	−0.38	+−
CSCO	1	16.3	3.53	1.97	12.44	0.00	11.81	−0.20	+−
IBM	1	124.3	1.58	1.28	0.51	−3.71	0.38	−3.79	+−
JPM	2	25.0	8.27	2.99	28.39	−0.01	24.97	−0.46	+−
MCD	1	54.6	1.41	1.62	1.54	−1.17	1.44	−1.30	+−
SBUX	15	33.6	2.86	1.64	0.46	−1.58	0.05	−1.71	+−
PM	1	42.0	2.80	2.01	4.81	−1.60	4.42	−1.80	+−
PG	1	61.9	1.28	1.27	0.76	−1.11	0.68	−1.19	+−
PFE	2	16.7	2.53	1.51	2.66	−0.58	2.34	−0.69	+−
PCG	6	35.3	3.11	1.36	6.75	−0.16	6.27	−0.26	+−
NYT	4	32.5	3.12	1.09	−0.39	−3.55	−0.88	−3.61	−−
MSFT	1	22.9	3.40	1.58	3.22	0.24	2.64	0.12	++
MSI	14	21.9	3.12	1.79	0.31	−1.90	−0.17	−2.06	−−
MON	1	112.0	2.96	3.11	2.28	−5.77	1.85	−6.25	+−
AMZN	1	77.7	3.24	2.86	2.15	−18.34	1.63	−18.75	+−

Table 4: Estimated parameters found in [34] for the DTRS model on the data from January 2005 to November 2009 with the notations given in Table 1. The last column *signs* contains the respective signs of b_-, b_+ (a $+-$ indicates a mean-reversion effect).

related to the "gluing" and regime change that involves a very thin layer which serves to avoid infinitely many immediate switches. The article [17] discusses the asymptotic behavior of the process as the width of the layer decreases to 0.

The GOBM has 5 parameters while the DTRS has 6 parameters because it also involves a delay. For most of the data, the estimated delay in the DTRS is $d = 1$, which means that the switching occurs without delay. Otherwise, the delay means a slow decreasing auto-correlation, or a long memory effect. Yet, for long delay, how to discriminate a leverage effect from sudden changes due to external parameters such as crisis? The presence of a delay increases the possibility of miss-specifications in the estimation procedure.

4.2 Estimation of the parameters of the GOBM

In Table 5, we estimate the parameters for the GOBM over the same stocks as for the DTRS. The complete numerical results may be found in the side report [26].

Although we use the same source (Yahoo Finance) as [34], it seems that KO is a different time series than in this article.

The volatilities (σ_-, σ_+) are in good agreement for both models. The respective signs of b_- and b_+ are consistent with the ones of [34] and suggest a mean-reversion effect ($b_- > 0$, $b_+ < 0$) for most of the stocks' prices. The magnitudes of b_- and b_+ are also consistent with the ones of [34]. As the number of data is rather small ($n = 1217$) and the considered period is only 5 years, it is not reasonable to aim for a more accurate description of the drift.

The threshold estimations are also in good agreement for 11 stocks out of 21.

For both the DTRS and GOBM models, $\mu_- > 0$ excepted for C, NYT and MSI for the GOBM model, and NYT for the DTRS model. Besides, $\sigma_- > \sigma_+$ unless for MCD for both models and GOOG for the DTRS model. In the later situation, σ_- is close to σ_+. This indicates that below the threshold, the volatility is higher and the drift is upward oriented.

4.3 Is there some leverage effect?

Testing if $\sigma_- = \sigma_+$. Our aim is to test whether or not $\sigma_+ = \sigma_-$ when $b_- = b_- = 0$. Our Hypothesis test is then

(H_0) (null hypothesis) $\sigma_- = \sigma_+$;

(H_1) (alternative hypothesis) $\sigma_- \neq \sigma_+$.

Construction of a confidence region. For the sake of simplicity, let us set $S_\pm := \sigma_\pm^2$ and $S(n)_\pm := (\sigma_\pm(n))^2$.

For two elements $f_\pm$, such as $S_\pm$, we also define the two dimensional vector $\mathbf{f} := (f_-, f_+)'$.

Oscillating Brownian motion (OBM)								
Index	m [\$]	σ_- [%]	σ_+ [%]	μ_- [‰]	μ_+ [‰]	b_- [‰]	b_+ [‰]	signs
GOOG	378.1	2.81	2.07	3.41	0.19	3.02	−0.02	+−
HP	57.9	4.18	2.53	1.78	−3.58	0.91	−3.90	+−
AAPL	117.0	3.78	2.56	2.19	0.11	1.48	−0.22	+−
ADBE	25.9	4.37	3.00	5.03	−0.48	4.07	−0.93	+−
CA	21.6	3.20	1.61	2.00	−0.56	1.49	−0.69	+−
C	40.4	7.47	1.09	−1.24	−0.48	−4.03	−0.54	−−
KO	47.6	1.49	1.13	0.54	0.14	0.43	0.08	++
CSCO	17.1	3.65	1.92	10.01	−0.44	9.35	−0.63	+−
IBM	115.4	1.64	1.27	0.71	−0.87	0.57	−0.95	+−
JPM	32.2	8.33	2.63	12.66	−0.34	9.19	−0.68	+−
MCD	51.6	1.28	1.77	1.33	−0.06	1.25	−0.22	+−
SBUX	13.3	4.52	2.92	2.00	−0.55	0.98	−0.98	+−
PM	45.3	2.66	1.76	0.76	−0.31	0.41	−0.47	+−
PG	52.2	1.81	1.27	2.48	0.03	2.31	−0.05	+−
PFE	18.9	2.51	1.30	0.63	−0.36	0.32	−0.44	+−
PCG	33.9	7.09	1.45	24.20	0.08	21.69	−0.02	+−
NYT	15.6	4.98	1.64	0.08	−1.16	−1.16	−1.29	−−
MSFT	23.0	3.28	1.64	6.17	−0.83	5.64	−0.96	+−
MSI	14.3	4.18	1.64	−0.35	−0.02	−1.22	−0.15	−−
MON	119.2	3.41	2.73	1.32	−5.72	0.74	−6.09	+−
AMZN	39.4	2.42	3.44	3.28	0.63	2.98	0.04	++

Table 5: Estimated parameters for the GOBM model on the data from January 2005 to November 2009 with the notations given in Table 1. The last column *signs* contains the respective signs of b_-, b_+ (a $+-$ indicates a mean-reversion effect).

Given the occupation time $Q_T^\pm$ below and above the threshold, we define $O_\pm = Q_T^\pm/T$ as the renormalized occupation time. The asymptotic result (7) of Proposition 3 is rewritten as

$$\mathbf{S}^n \approx \mathbf{S} + \frac{\sqrt{2}}{\sqrt{n}} \begin{bmatrix} S_- \frac{G_-}{\sqrt{O_-}} \\ S_+ \frac{G_+}{\sqrt{O_+}} \end{bmatrix} = \mathbf{S} + \frac{1}{\sqrt{n}} M\mathbf{G} \text{ with } M = \sqrt{2} \begin{bmatrix} S_-/\sqrt{O_-} & 0 \\ 0 & S_+/\sqrt{O_+} \end{bmatrix}, \tag{11}$$

where $\mathbf{G} = \mathcal{N}(0, \mathrm{Id})$ is a Gaussian vector independent from the process X. The stable convergence means that the limit term in (11) involves a double randomness, and M is a measurable function of X. Replacing $\mathbf{S}$ by its approximation $\mathbf{S}(n)$ as well as $\mathbf{O}$ by its Riemann sum approximation $\mathbf{O}(n)$ constructed from the observations, we set

$$M^n = \sqrt{2}\begin{bmatrix} S_-(n)/\sqrt{O_-(n)} & 0 \\ 0 & S_+(n)/\sqrt{O_+(n)} \end{bmatrix}.$$

Thanks to the isotropy of the Gaussian vector $\mathbf{G}$, we define for a level of confidence α the quantity q_α by $\mathbb{P}[|\mathbf{G}| \leq q_\alpha] = 1 - \alpha$. This quantity is easily computed since $|\mathbf{G}|^2$ follows a χ^2 distribution with two degrees of freedom. Our *confidence region* of level α is the ellipsis

$$\mathcal{R}_\alpha = \left\{ \mathbf{S}(n) + \frac{q_\alpha}{\sqrt{n}} M^n \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \,\middle|\, \theta \in [0, 2\pi) \right\}.$$

The rule of decision. Our rule of decision is then: reject the Null Hypothesis (H_0) if the diagonal line $s : [0, +\infty) \mapsto (s, s)$ does not cross $\mathcal{R}_\alpha$.

Empirical result. As the drift is small, it should not affect this test. Therefore we assume through all this section that $b_- = b_+ = 0$.

In Figure 1, we apply this rule to our data. The null hypothesis (H_0) is rejected for all the stocks except for PCG, meaning that a leverage effect should be considered for 20 out of 21 stocks. The normalized occupation time O_+ for PCG is close to 99 %. This may explain the elongated shape of the associated confidence region.

In Figure 2, we plot the approximated log-likelihood $\Lambda^{(i)}$ in function of $r^{(i)} = \log m^{(i)}$ for 3 stocks. We see that $\Lambda^{(i)}$ may have one main peak (for CSCO), two main peaks (for GOOG) or be “flat” as for PCG. A steep peak means that σ_- is likely to differ from σ_+ and that a leverage effect occurs.

Comparison with a non-parametric estimator. Non-parametric estimation of the coefficients assume nothing on the underlying volatility and drift coefficients [18, 23]. The Nadaraya-Watson estimator provides us with such an estimator [18]. We then compared graphically our estimations with the non-parametric estimation of the coefficients of the log-price. For this, we used the `R` package `sde` [18]. In Figure 3, we presents the results for the 3 stocks already used in Figure 2 (more figures may be found in [26]). Most of the stocks seem to exhibit a behavior similar to the one presented here, with a sharp variation of both the volatility and the drift. Again, this reinforces the idea that regime switching holds for most of the stocks.

5 Conclusion

Leverage effects in finance have been the subject of a large literature with many empirical evidences. The Black & Scholes model does not account of this stylized

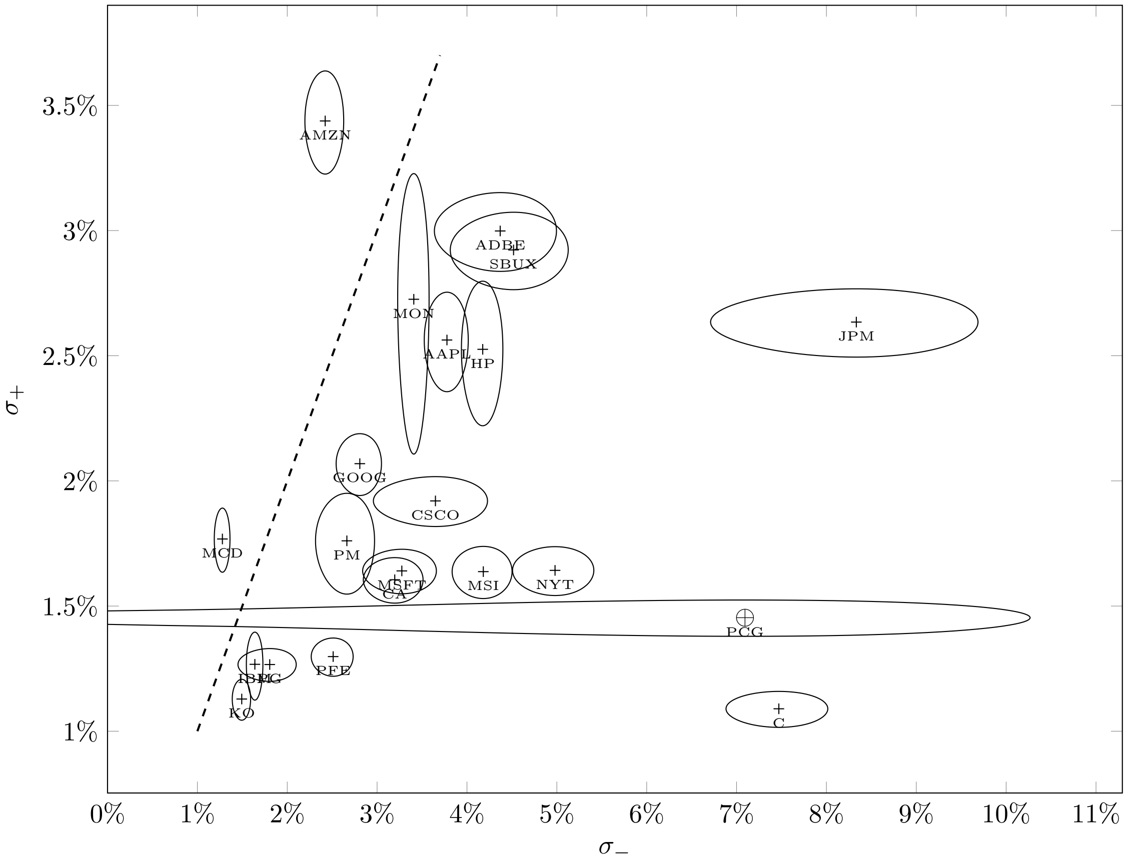

Figure 1: Confidence regions for (σ_-, σ_+): Each point is the value of the stock in the (σ_-, σ_+)-plane. Confidence regions at 95 % are the ellipsis in the (σ_-, σ_+)-plane around the points. Points marked by $\oplus$ are the ones for which the Hypothesis $\sigma_- = \sigma_+$ is not rejected. Points marked by $+$ are the ones for which this Hypothesis is rejected.

fact.

The *Geometric Oscillating Brownian motion* (GOBM) studied in this article mimics such leverage effect. This model can be thought as a continuous time version of the self-exciting threshold autoregressive model (SETAR).

We showed its validity on real data and exhibited evidence in favor of leverage effects. Our estimations are consistent with the ones of M. Esquível and P. Mota based on least squares.

Our model is simple and does not aim at capturing other stylized facts. It could serve as a basic building brick for more complex models. Our rationale is that the GOBM is really tractable while offering more flexibility than the Black & Scholes

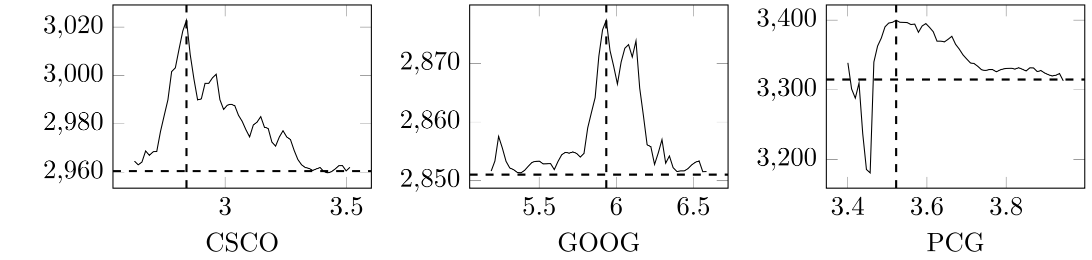

Figure 2: The (approximated) log-likelihood $\Lambda^{(i)}$ given by (10) in function of possible threshold $r^{(i)} = \log m^{(i)}$ for the stocks CSCO, GOOG and PCG. The vertical dashed line represents the threshold $r^{(i)}$ which maximizes $\Lambda^{(i)}$, hence the estimated r. The horizontal dashed line represents the value of the log-likelihood of the drifted Brownian motion (that is $\sigma_- = \sigma_+$ and $b_- = b_+$).

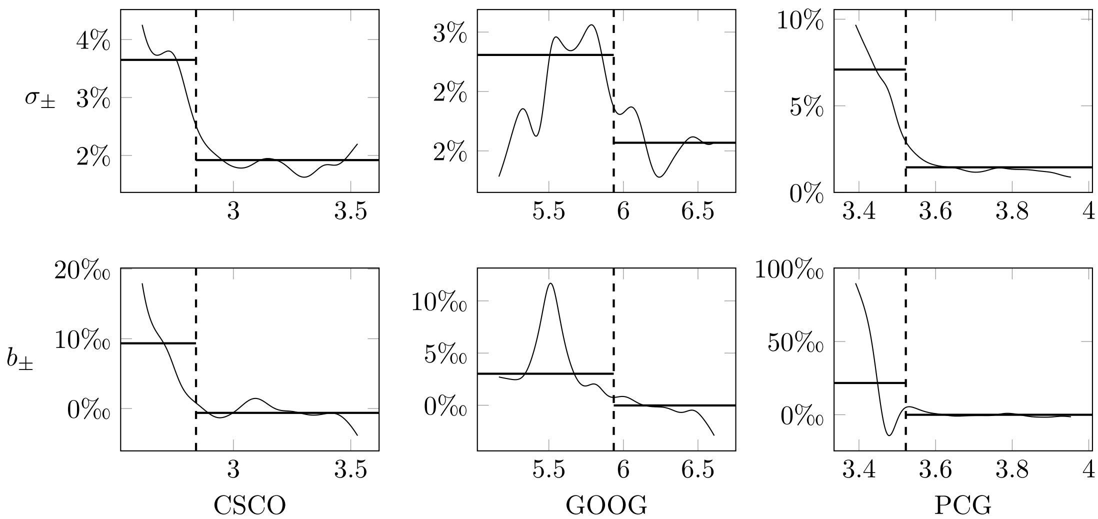

Figure 3: Non-parametric estimation of σ and b for the log-price of the stocks CSCO, GOOG and PCG with a Nadaraya-Watson estimator. The vertical dashed line represents the choice of the threshold. The horizontal lines represent the estimated values of $(\sigma_-(n), \sigma_+(n))$ (top) and $(b_-(n), b_+(n))$ (bottom).

model:

- The estimation procedure is simple to set up.
- Simulations are easily performed.

- The market is complete.
- Option pricing could be performed through analytic or semi-analytic approach without relying on Monte Carlo simulations.

In addition, our model and estimation procedure could serve other purposes. In this model the leverage effect is a consequence of a spatial segmentation in which the dynamics of the price changes according to a threshold. The same estimation procedure could also be applied in short time windows in order to detect sharp changes, hence reflecting temporal changes, as for regime switching models involving Hidden Markov models.

Another possible application of the GOBM, and more generally of local volatilities with discontinuities, would be to introduce such features in more complex models. The properties we showed in the present paper and their capability of reproducing extreme skews in implied volatility (cf. [36]) suggest that such discontinuities could be a tractable way to introduce asymmetries and regime changes in other models (cf. [11]).

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