Constrained L2-approximation by polynomials on subsets of the circle

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To the memory of André Boivin

1 Abstract

We study best approximation to a given function, in the least square sense on a subset of the unit circle, by polynomials of given degree which are pointwise bounded on the complementary subset. We show that the solution to this problem, as the degree goes large, converges to the solution of a bounded extremal problem for analytic functions which is instrumental in system identification. We provide a numerical example on real data from a hyperfrequency filter.

2 Introduction

This paper deals with best approximation to a square summable function, on a finite union I of arcs of the unit circle $\mathbb{T}$, by a polynomial of fixed degree which is bounded by 1 in modulus on the complementary system of arcs $J = \mathbb{T} \setminus I$. This we call, for short, the polynomial problem. We are also concerned with the natural limiting version when the degree goes large, namely best approximation in $L^2(I)$ by a Hardy function of class H^2 which is bounded by 1 on J. To distinguish this issue from the polynomial problem, we term it the analytic problem. The latter is a variant, involving mixed norms, of constrained extremal problems for analytic functions considered in [12, 3, 2, 13, 18]. As we shall see, solutions to the polynomial problem converge to those of the analytic problem as the degree tends to infinity, in a sense to be made precise below. This is why solving for high degree the polynomial problem (which is finite-dimensional) is an interesting way to regularize and approximately solve the analytic problem (which is infinite-dimensional). This is the gist of the present work.

Constrained extremal problems for analytic functions, in particular the analytic problem defined above, can be set up more generally in the context of weighted approximation, *i.e.* seeking best approximation in $L^2(I, w)$ where w is a weight on I. In fact, that kind of generalization is useful for applications as we shall see. As soon as w is invertible in $L^\infty(I)$, though, such a weighted problem turns out to be equivalent to another one with unit weight, hence the present formulation warrants most practical situations. This property allows one to carry the analytic problem over to more general curves than the circle. In particular, in view of the isomorphism between Hardy spaces of the disk and the half-plane arising by composition with a Möbius transform [10, ch. 10], best approximation in $L^2(I)$ from H^2 of the disk can be converted to weighted best approximation in $L^2(\mathfrak{I}, w)$ from the Hardy space $\mathfrak{h}^2$ of a half-plane with $\mathfrak{I}$ a finite union of bounded intervals on the line and w a weight arising from the derivative of the Möbius transform. Since this weight is boundedly invertible on $\mathfrak{I}$, it follows that the analytic problem on the circle and its analog on the line are equivalent. One may also define another Hardy space $\mathcal{H}^2$, say of the right half-plane as the space of analytic functions whose L^2-means over vertical lines are uniformly bounded. Then, best approximation in $L^2(I)$ from H^2 is equivalent to best approximation from $\mathcal{H}^2$ in $L^2(\mathfrak{I})$, *i.e.* weight is no longer needed. Of course, such considerations hold for many other domains and boundary curves than the half-plane and the line, but the latter are of special significance to us as we now explain.

Indeed, on the line, constrained extremal problems for analytic functions naturally arise in Engineering when studying deconvolution issues, in particular those pertaining to system identification and design. This motivation is stressed in [12, 4, 5, 19, 2], whose results are effectively used today to identify microwave

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devices [1, 14]. More precisely, recall that a linear time-invariant dynamical system is just a convolution operator, hence the Fourier-Laplace transform of its output is that of its input times the Fourier-Laplace transform of its kernel. The latter is called the transfer-function. Now, by feeding periodic inputs to a stable system, one can essentially recover the transfer function pointwise on the line, but typically in a restricted range of frequencies only, corresponding to the passband of the system, say $\mathfrak{I}$ [9]. Here, the type of stability under consideration impinges on the smoothness of the transfer function as well as on the precise kind of recovery that can be achieved, and we refer the reader to [6, Appendix 2] for a more thorough analysis. For the present discussion, it suffices to assume that the system is stable in the L^2 sense, *i.e.* that it maps square summable inputs to square summable outputs. Then, its transfer function lies in H^∞ of the half-plane [15], and to identify it we are led to approximate the measurements on $\mathfrak{I}$ by a Hardy function with a bound on its modulus. Still, on $\mathfrak{I}$, a natural criterion from the stochastic viewpoint is $L^2(\mathfrak{I}, w)$, where the weight w is the reciprocal of the pointwise covariance of the noise assumed to be additive [16]. Since this covariance is boundedly invertible on I, we face an analytic problem on the line upon normalizing the bound on the transfer function to be 1. This stresses how the analytic problem on the line, which can be mapped back to the circle, connects to system identification. Now, this analytic problem is convex but infinite-dimensional. Moreover, as Hardy functions have no discontinuity of the first kind on the boundary [11, ch. II, ex. 7] and since the solution to an analytic problem generically has exact modulus 1 on J, as we prove later on, it will typically oscillate at the endpoints of I, J which is unsuited. One way around these difficulties is to solve the polynomial problem for sufficiently high degree, as a means to regularize and approximately solve the analytic one. This was an initial motivation by the authors to write the present paper, and we provide the reader in Section 6 with a numerical example on real data from a hyperfrequency filter. It must be said that the polynomial problem itself has numerical issues: though it is convex in finitely many variables, bounding the modulus on J involves infinitely many convex constraints which makes it of so-called semi-infinite programming type. A popular technique to handle such problems is through linear matrix inequalities, but we found it easier to approximate from below the polynomial problem by a finite-dimensional one with finitely many constraints, in a demonstrably convergent manner as the number of these constraints gets large.

The organization of the paper is as follows. In section 3 we set some notation and we recall standard properties of Hardy spaces. We state the polynomial and analytic problems in Section 4, where we also show they are well-posed. Section 5 deals with the critical point equations characterizing the solutions, and with convergence of the polynomial problem to the analytic one. Finally, we report on some numerical experiment in Section 6.

3 Notations and preliminaries

Throughout we let $\mathbb{T}$ be the unit circle and $I \subset \mathbb{T}$ a finite union of nonempty open arcs whose complement $J = \mathbb{T} \setminus I$ has nonempty interior. If h_1 (resp. h_2) is a function defined on a set containing I (resp. J), we put $h_1 \vee h_2$ for the concatenated function, defined on the whole of $\mathbb{T}$, which is h_1 on I and h_2 on J.

For $E \subset \mathbb{T}$, we let ∂E and $\overset{\circ}{E}$ denote respectively the boundary and the interior of E when viewed as a subset of $\mathbb{T}$; we also let χ_E for the characteristic function of E and $h_{|_E}$ for the restriction of h to E. Lebesgue measure on $\mathbb{T}$ is just the image of Lebesgue measure on $[0, 2\pi)$ under the parametrization $\theta \mapsto e^{i\theta}$. We denote by $|E|$ the measure of a measurable subset $E \subset \mathbb{T}$, and if $1 \le p \le \infty$ we write $L^p(E)$ for the familiar Lebesgue space of (equivalence classes of a.e. coinciding) complex-valued measurable functions on E with norm

$$\|f\|_{L^p(E)} = \left(\frac{1}{2\pi}\int_E |f(e^{i\theta})|^p \, d\theta\right)^{1/p} < \infty \quad \text{if } 1 \le p < \infty, \qquad \|f\|_{L^\infty(E)} = \operatorname{ess.sup}_{\theta \in E} |f(e^{i\theta})| < \infty.$$

We sometimes indicate by $L^p_{\mathbb{R}}(E)$ the real subspace of real-valued functions. We also set

$$\langle f, g\rangle_E = \frac{1}{2\pi}\int_E f(e^{i\theta})\overline{g(e^{i\theta})}\, d\theta \tag{1}$$

whenever $f \in L^p(E)$ and $g \in L^q(E)$ with $1/p + 1/q = 1$. If f and g are defined on a set containing E, we write for simplicity $\langle f, g\rangle_E$ to mean $< f_{|_E}, g_{|_E} >$ and $\|f\|_{L^p(E)}$ to mean $\|f_{|_E}\|_{L^p(E)}$. Hereafter $C(E)$ stands for the space of bounded complex-valued continuous functions on E endowed with the sup norm, while $C_{\mathbb{R}}(E)$ indicates real-valued continuous functions.

Recall that the Hardy space H^p is the closed subspace of $L^p(\mathbb{T})$ consisting of functions whose Fourier coefficients of strictly negative index do vanish. We refer the reader to [11] for standard facts on Hardy spaces, in particular those recorded hereafter. Hardy functions are the nontangential limits a.e. on $\mathbb{T}$ of functions holomorphic in the unit disk $\mathbb{D}$ having uniformly bounded L^p means over all circles centered at 0 of radius less than 1:

$$\|f\|_{H^p} = \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)^{1/p} \quad \text{if } 1\le p<\infty, \qquad \|f\|_{H^\infty} = \sup_{z\in\mathbb{D}}|f(z)|. \tag{2}$$

The correspondence between such a holomorphic function f and its non tangential limit $f^\sharp$ is one-to-one and even isometric, namely the supremum in (2) is equal to $\|f^\sharp\|_p$, thereby allowing us to identify f and $f^\sharp$ and to drop the superscript $\sharp$. Under this identification, we regard members of H^p both as functions in $L^p(\mathbb{T})$ and as holomorphic functions in the variable $z\in\mathbb{D}$, but the argument (which belongs to $\mathbb{T}$ in the former case and to $\mathbb{D}$ in the latter) helps preventing confusion. It holds in fact that $f_r(e^{i\theta}) = f(re^{i\theta})$ converges as $r\to 1^-$ to $f(e^{i\theta})$ in $L^p(\mathbb{T})$ when $f\in H^p$ and $1\le p<\infty$. It follows immediately from (2) and Hölder's inequality that, whenever $g_1\in H^{p_1}$ and $g_2\in H^{p_2}$, we have $g_1g_2\in H^{p_3}$ if $1/p_1+1/p_2=1/p_3$.

Given $f\in H^p$, its values on $\mathbb{D}$ are obtained from its values on $\mathbb{T}$ through a Cauchy as well as a Poisson integral [17, ch. 17, thm 11], namely:

$$f(z) = \frac{1}{2\,i\,\pi}\int_{\mathbb{T}}\frac{f(\xi)}{\xi-z}\,d\xi\,, \qquad \text{and also} \qquad f(z) = \frac{1}{2\,\pi}\int_{\mathbb{T}}\mathrm{Re}\left\{\frac{e^{i\theta}+z}{e^{i\theta}-z}\right\}f(e^{i\theta})\,d\theta\,, \qquad z\in\mathbb{D}, \tag{3}$$

where the right hand side of the first equality in (3) is a line integral. The latter immediately implies that the Fourier coefficients of a Hardy function on the circle are the Taylor coefficients of its power series expansion at 0 when viewed as a holomorphic function on $\mathbb{D}$. In this connection, the space H^2 is especially simple to describe: it consists of those holomorphic functions g in $\mathbb{D}$ whose Taylor coefficients at 0 are square summable, namely

$$g(z) = \sum_{k=0}^{\infty}a_kz^k: \quad \|g\|_{H^2}^2 := \sum_{k=0}^{\infty}|a_k|^2 < +\infty, \qquad g(e^{i\theta}) = \sum_{k=0}^{\infty}a_ke^{ik\theta}, \tag{4}$$

where the convergence of the last Fourier series holds in $L^2(\mathbb{T})$ by Parseval's theorem (and also pointwise a.e. by Carleson's theorem but we do not need this deep result). Incidentally, let us mention that for no other value of p is it known how to characterize H^p in terms of the size of its Fourier coefficients.

By the Poisson representation (*i.e.* the second integral in (3)), a Hardy function g is also uniquely represented, up to a purely imaginary constant, by its real part h on $\mathbb{T}$ according to:

$$g(z) = i\mathrm{Im}g(0) + \frac{1}{2\,\pi}\int_{\mathbb{T}}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,h(e^{i\theta})\,d\theta\,, \qquad z\in\mathbb{D}. \tag{5}$$

The integral in (5) is called the *Riesz-Herglotz transform* of h and, whenever $h\in L^1_{\mathbb{R}}(\mathbb{T})$, it defines a holomorphic function in $\mathbb{D}$ which is real at 0 and whose nontangential limit exists a.e. on $\mathbb{T}$ with real part equal to h. Hence the Riesz-Herglotz transform (5) assumes the form $h(e^{i\theta})+i\widetilde{h}(e^{i\theta})$ a.e. on $\mathbb{T}$, where the real-valued function $\widetilde{h}$ is said to be *conjugate* to h. It is a theorem of M. Riesz [11, chap. III, thm 2.3] that if $1<p<\infty$, then $\widetilde{h}\in L^p_{\mathbb{R}}(\mathbb{T})$ when $h\in L^p_{\mathbb{R}}(\mathbb{T})$. This neither holds for $p=1$ nor for $p=\infty$.

A nonzero $f\in H^p$ can be uniquely factored as $f=jw$ where

$$w(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|f(e^{i\theta})|\,d\theta\right\} \tag{6}$$

belongs to H^p and is called the *outer factor* of f, while $j\in H^\infty$ has modulus 1 a.e. on $\mathbb{T}$ and is called the *inner factor* of f. That $w(z)$ in (6) is well-defined rests on the fact that $\log|f|\in L^1$ if $f\in H^1\setminus\{0\}$; it entails that a H^p function cannot vanish on a subset of strictly positive Lebesgue measure on $\mathbb{T}$ unless it is identically zero. For simplicity, we often say that a function is outer (resp. inner) if it is equal, up to a unimodular multiplicative constant, to its outer (resp. inner) factor.

Closely connected to Hardy spaces is the Nevanlinna class N^+, consisting of holomorphic functions in $\mathbb{D}$ that can be factored as jE, where j is an inner function and E an outer function of the form

$$E(z) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\rho(e^{i\theta})\,d\theta\right\}, \tag{7}$$

with ρ a positive function such that $\log\rho\in L^1(\mathbb{T})$ (though ρ itself may not be summable). Such a function has nontangential limits of modulus ρ a.e. on $\mathbb{T}$. The Nevanlinna class is instrumental in that

$N^+ \cap L^p(\mathbb{T}) = H^p$, see [10, thm 2.11] or [11, 5.8, ch.II]. Thus, formula (7) defines a H^p-function if and only if $\rho \in L^p(\mathbb{T})$.

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. The Hardy space $\bar{H}^p$ of $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ can be given a treatment parallel to H^p upon changing z into $1/z$. Specifically, $\bar{H}^p$ consists of functions in $L^p(\mathbb{T})$ whose Fourier coefficients of strictly positive index do vanish; these are, a.e. on $\mathbb{T}$, the complex conjugates of H^p-functions, and they can also be viewed as nontangential limits of functions analytic in $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ having uniformly bounded L^p means over all circles centered at 0 of radius bigger than 1. We further single out the subspace $\bar{H}^p_0$ of $\bar{H}^p$, consisting of functions vanishing at infinity or, equivalently, having vanishing mean on $\mathbb{T}$. Thus, a function belongs to $\bar{H}^p_0$ if, and only if it is of the form $e^{-i\theta}\overline{g(e^{i\theta})}$ for some $g \in H^p$. For $G \in \bar{H}^p_0$, the Cauchy formula assumes the form :

$$G(z) = \frac{1}{2\,i\,\pi} \int_{\mathbb{T}} \frac{G(\xi)}{z - \xi}\, d\xi\,, \qquad z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}. \tag{8}$$

It follows at once from the Cauchy formula that the duality product $\langle\ ,\ \rangle_{\mathbb{T}}$ makes H^p and $\bar{H}^q_0$ orthogonal to each other, and it reduces to the familiar scalar product when $p = q = 2$. In particular, we have the orthogonal decomposition :

$$L^2(\mathbb{T}) = H^2 \oplus \bar{H}^2_0. \tag{9}$$

For $f \in C(\mathbb{T})$ and $\nu \in \mathcal{M}$, the space of complex Borel measures on $\mathbb{T}$, we set

$$\nu.f = \int_{\mathbb{T}} f(e^{i\theta})\, d\nu(\theta) \tag{10}$$

and this pairing induces an isometric isomorphism between $\mathcal{M}$ (endowed with the norm of the total variation) and the dual of $C(\mathbb{T})$ [17, thm 6.19]. If we let $\mathcal{A} \subset H^\infty$ designate the disk algebra of functions analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, and if $\mathcal{A}_0$ indicates those functions in $\mathcal{A}$ vanishing at zero, it is easy to see that $\mathcal{A}_0$ is the orthogonal space under (10) to those measures whose Fourier coefficients of strictly negative index do vanish. Now, it is a fundamental theorem of F. and M. Riesz that such measures are absolutely continuous, that is have the form $d\nu(\theta) = g(e^{i\theta})\, d\theta$ with $g \in H^1$. The Hahn-Banach theorem implies that H^1 is dual *via* (10) to the quotient space $C(\mathbb{T})/\mathcal{A}_0$ [11, chap. IV, sec. 1]. Equivalently, $\bar{H}^1_0$ is dual to $C(\mathbb{T})/\overline{\mathcal{A}}$ under the pairing arising from the line integral :

$$(\dot{f}, F) = \frac{1}{2i\pi} \int_{\mathbb{T}} f(\xi) F(\xi)\, d\xi\,, \tag{11}$$

where F belongs to $\bar{H}^1_0$ and $\dot{f}$ indicates the equivalence class of $f \in C(\mathbb{T})$ modulo $\overline{\mathcal{A}}$. Therefore, contrary to $L^1(\mathbb{T})$, the spaces H^1 and $\bar{H}^1_0$ enjoy a weak-* compactness property of their unit ball.

We define the analytic and anti-analytic projections $\mathbf{P}_+$ and $\mathbf{P}_-$ on Fourier series by :

$$\mathbf{P}_+\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{n=0}^{\infty} a_n e^{in\theta}, \qquad \mathbf{P}_-\left(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}\right) = \sum_{n=-\infty}^{-1} a_n e^{in\theta}\,.$$

It is a theorem of M. Riesz theorem [11, ch. III, sec, 1] that $\mathbf{P}_+ : L^p \to H^p$ and $\mathbf{P}_- : L^p \to \bar{H}^p_0$ are bounded for $1 < p < \infty$, in which case they coincide with the Cauchy projections:

$$\mathbf{P}_+(h)(z) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{h(\xi)}{\xi - z}\, d\xi, \quad z \in \mathbb{D}, \qquad \mathbf{P}_-(h)(s) = \frac{1}{2i\pi} \int_{\mathbb{T}} \frac{h(\xi)}{s - \xi}\, d\xi, \quad s \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}. \tag{12}$$

When restricted to $L^2(\mathbb{T})$, the projections $\mathbf{P}_+$ and $\mathbf{P}_-$ are just the orthogonal projections onto H^2 and $\bar{H}^2_0$ respectively. Although $\mathbf{P}_\pm(h)$ needs not be the Fourier series of a function when h is merely in $L^1(\mathbb{T})$, it is Abel summable almost everywhere to a function lying in $L^s(\mathbb{T})$ for $0 < s < 1$ and it can be interpreted as a function in the Hardy space of exponent s that we did not introduce [10, cor. to thm 3.2]. To us it will be sufficient, when $h \in L^1$, to regard $\mathbf{P}_\pm(f)$ as the Fourier series of a distribution.

Finally, we let P_n denote throughout the space of complex algebraic polynomials of degree at most n. Clearly, $P_n \subset H^p$ for all p.

4 Two extremal problems

We first state the polynomial problem discussed in Section 2. We call it $PBEP(n)$ for "Polynomial Bounded Extremal Problem":

PBEP(n)

For $f \in L^2(I)$, find $k_n \in P_n$ such that $|k_n(e^{i\theta})| \leq 1$ for a.e. $e^{i\theta} \in J$ and

$$\|f - k_n\|_{L^2(I)} = \inf_{\substack{g \in P_n \\ |g| \leq 1 \text{ a.e. on } J}} \|f - g\|_{L^2(I)}. \tag{13}$$

Next, we state the analytic problem from Section 2 that we call $ABEP$ for "Analytic Bounded Extremal Problem":

ABEP

Given $f \in L^2(I)$, find $g_0 \in H^2$ such that $|g_0(e^{i\theta})| \leq 1$ a.e. on J and

$$\|f - g_0\|_{L^2(I)} = \inf_{\substack{g \in H^2 \\ |g| \leq 1 \text{ a.e. on } J}} \|f - g\|_{L^2(I)}. \tag{14}$$

Note that, in $ABEP$, the constraint $|g| \leq 1$ on J could be replaced by $|g| \leq \rho$ where ρ is a positive function in $L^2(J)$. For if $\log \rho \in L^1(J)$ then, denoting by $w_{1 \vee (1/\rho)}$ the outer factor having modulus 1 on I and $1/\rho$ on J, we find that $g \in H^2$ satisfies $|g| \leq \rho$ on J if and only if $h = g w_{1 \vee (1/\rho)}$ lies in H^2 and satisfies $|h| \leq 1$ on J. It is so because, for g as indicated, h lies in the Nevanlinna class by construction and $|h|_{|_I} = |g|_{|_I}$ while $|h|_{|_J} = |g|_{|_J}/\rho$. If, however, $\log \rho \notin L^1(J)$, then we must have $\int_J \log \rho = -\infty$ because $\rho \in L^2(J)$, consequently the set of candidate approximants reduces to $\{0\}$ anyway because a nonzero Hardy function has summable log-modulus. Altogether, it is thus equivalent to consider $ABEP$ for the product f times $(w_{1 \vee \rho^{-1}})_{|_I}$. A similar argument shows that we could replace the error criterion $\|.\|_{L^2(I)}$ by a weighted norm $\|.\|_{L^2(I,w)}$ for some weight w which is non-negative and invertible in $L^\infty(I)$. Then, the problem reduces to $ABEP$ for $f(w_{\rho^{1/2} \vee 0})_I$.

Such equivalences do not hold for $PBEP(n)$ because the polynomial character of k_n is not preserved under multiplication by outer factors. Still, the results to come continue to hold if we replace in $PBEP(n)$ the constraint $|k_n| \leq 1$ by $|k_n| \leq \rho$ on J and the criterion $\|.\|_{L^2(I)}$ by $\|.\|_{L^2(I,w)}$, provided that $\rho \in C(J)$ and that w is invertible in $L^\infty(I)$. Indeed, we leave it to the reader to check that proofs go through with obvious modifications.

After these preliminaries, we are ready to state a basic existence and uniqueness result.

Theorem 1 . *Problems $PBEP(n)$ and $ABEP$ have a unique solution. Moreover, the solution g_0 to $ABEP$ satisfies $|g_0| = 1$ almost everywhere on J, unless $f = g_{|_I}$ for some $g \in H^2$ such that $\|g\|_{L^\infty(J)} \leq 1$.*

Proof. Consider the sets

$$E_n = \{g_{|_I} : \ g \in P_n, \ \|g\|_{L^\infty(J)} \leq 1\},$$

$$F = \{g_{|_I} : \ g \in H^2, \ \|g\|_{L^\infty(J)} \leq 1\}.$$

Clearly $E_n \subset F$ are convex and nonempty subsets of $L^2(I)$, as they contain 0. To prove existence and uniqueness, it is therefore enough to show they are closed, for we can appeal then to well-known properties of the projection on a closed convex set in a Hilbert space. Since $E_n = P_n \cap F$, it is enough in fact to show that F is closed. For this, let g_m be a sequence in H^2 with $|g_m|_{|_J} \leq 1$ and such that $(g_m)_{|_I}$ converges in $L^2(I)$. Obviously g_m is a bounded sequence in $L^2(\mathbb{T})$, some subsequence of which converges weakly to $h \in H^2$. We continue to denote this subsequence with g_m. The restrictions $(g_m)_{|_I}$ a fortiori converge weakly to $h_{|_I}$ in $L^2(I)$, and since the strong and the weak limit must coincide when both exist we find that $(g_m)_{|_I}$ converges to $h_{|_I}$ in $L^2(I)$. Besides, $(g_m)_{|_J}$ is contained in the unit ball of $L^\infty(J)$ which is dual to $L^1(J)$, hence some subsequence (again denoted by $(g_m)_{|_J}$) converges weak-* to some $h_1 \in L^\infty(J)$ with $\|h_1\|_{L^\infty(J)} \leq 1$. But since $(g_m)_{|_J}$ also converges weakly to $h_{|_J}$ in $L^2(J)$, we have that

$$\langle h_1, \varphi \rangle_J = \lim_{m \to \infty} \langle g_m, \varphi \rangle_J = \langle h_{|_J}, \varphi \rangle_J$$

for all $\varphi \in L^2(J)$ which is dense in $L^1(J)$. Consequently $h_1 = h_{|_J}$, thereby showing that $\|h\|_{L^\infty(J)} \leq 1$, which proves that F is closed.

Assume now that f is not the trace on I of an H^2-function which is less than 1 in modulus on I. To prove that $|g_0| = 1$ a.e. on J, we argue by contradiction. If not, there is a compact set K of positive measure, lying interior to J, such that $\|g_0\|_{L^\infty(K)} \leq 1 - \delta$ for some $0 < \delta < 1$; it is so because, by hypothesis, J must consist of finitely many closed arcs, of which one at least has nonempty interior. For K' an arbitrary subset of K, consider the Riesz-Herglotz transform of its characteristic function:

$$h_{K'}(z) = \frac{1}{2\pi} \int_{K'} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\theta\,, \qquad z \in \mathbb{D}, \tag{15}$$

and put $w_t = \exp(t h_{K'})$ for $t \in \mathbb{R}$, which is the outer function with modulus $\exp t$ on K' and 1 elsewhere. By construction, $g_0 w_t$ is a candidate approximant in $ABEP$ for all $t < -\log(1-\delta)$. Thus, the map $t \mapsto \|f - g_0 w_t\|^2_{L^2(I)}$ attains a minimum at $t = 0$. Because K is at strictly positive distance from I, we may differentiate this expression with respect to t under the integral sign and equate the derivative at $t = 0$ to zero which gives us $2\mathrm{Re}\langle f - g_0, h_{K'} g_0\rangle_I = 0$. Replacing $g_0 w_t$ by $i g_0 w_t$, which is a candidate approximant as well, we get a similar equation for the imaginary part so that

$$0 = \langle f - g_0\,,\, h_{K'} g_0\rangle_I = \langle (f-g_0)\bar{g}_0\,,\, h_{K'}\rangle_I. \tag{16}$$

Let e^{it_0} be a density point of K and I_l the arc centered at e^{it_0} of length l, so that $|I_l \cap K|/l \to 1$ as $l \to 0$. Since

$$\left| \frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}} - \frac{e^{it_0}+e^{i\theta}}{e^{it_0}-e^{i\theta}} \right| \leq \frac{2l}{\mathrm{dist}^2(K,I)} \qquad \text{for } e^{it} \in I_l \cap K, \;\; e^{i\theta} \in I, \tag{17}$$

it follows by dominated convergence that

$$\lim_{l\to 0} \frac{1}{|I_l \cap K|} \int_{I_l \cap K} \left| \frac{e^{it}+e^{i\theta}}{e^{it}-e^{i\theta}} - \frac{e^{it_0}+e^{i\theta}}{e^{it_0}-e^{i\theta}} \right| dt = 0, \quad \text{uniformly w.r. to } e^{i\theta} \in I,$$

and therefore that

$$\lim_{l\to 0} \frac{h_{I_l\cap K}(e^{i\theta})}{|I_l \cap K|} = \frac{e^{it_0}+e^{i\theta}}{e^{it_0}-e^{i\theta}} \quad \text{uniformly w.r. to } e^{i\theta} \in I.$$

Applying now (16) with $K' = I_l \cap K$ and taking into account that $(e^{it_0}+e^{i\theta})/(e^{it_0}-e^{i\theta})$ is pure imaginary on I, we find in the limit, as $l \to 0$ that

$$\frac{1}{2\pi} \int_I \frac{e^{it_0}+e^{i\theta}}{e^{it_0}-e^{i\theta}} \Big((f-g_0)\bar{g}_0 \Big)(e^{i\theta})\, d\theta = 0. \tag{18}$$

Next, let us consider the function

$$F(z) = \frac{1}{2\pi} \int_I \frac{e^{i\theta}+z}{e^{i\theta}-z} \Big((f-g_0)\bar{g}_0 \Big)(e^{i\theta})\, d\theta = -\frac{1}{2\pi} \int_I \Big((f-g_0)\bar{g}_0 \Big)(e^{i\theta}) d\theta + \frac{1}{i\pi} \int_I \frac{\Big((f-g_0)\bar{g}_0 \Big)(\xi) d\xi}{\xi - z}$$

which is the sum of a constant and of twice the Cauchy integral of $(f - (g_0)_{|_I})(\bar{g}_0)_{|_I} \in L^1(I)$, hence is analytic in $\hat{\mathbb{C}} \setminus I$. Equation (18) means that F vanishes at every density point of K, and since a.e. point in K is a density point F must vanish identically because its zeros accumulate in the interior of J. Denoting by F^+ and F^- the nontangential limits of F from sequences of points in $\mathbb{D}$ or $C \setminus \overline{\mathbb{D}}$ respectively, we now get from the Plemelj-Sokhotski formulas [11, ch. III] that

$$0 = F^+(\xi) - F^-(\xi) = (f-g_0)(\xi)\overline{g_0(\xi)}, \qquad a.e.\ \xi \in I.$$

Thus, either g_0 is nonzero a.e. on I, in which case $f = (g_0)_{|_I}$ and we reach the desired contradiction, or else $g_0 \equiv 0$. In the latter case, if we put id for the identity map on $\mathbb{T}$, we find that $t \mapsto \|f - t\,\mathrm{id}^k\|^2_{L^2(I)}$ has a minimum at $t = 0$ for each integer $k \geq 0$, since $e^{i\theta} \mapsto t e^{ik\theta}$ is a candidate approximant for $t \in [-1, 1]$. Differentiating with respect to t and expressing that the derivative at $t = 0$ is zero, we deduce that all Fourier coefficients of non-negative index of $(f - (g_0)_{|_I}) \vee 0$ do vanish. This means this last function lies in $\bar{H}^2$, but as it vanishes on J it is identically zero, therefore $f = (g_0)_{|_I}$ in all cases. ∎

Remark: the theorem shows that the constraint $|g_0| \leq 1$ on J is saturated in a very strong sense for problem $ABEP$, namely $|g_0| = 1$ a.e. on J unless f is already the trace of the solution on I. In contrast, it is not true that $\|k_n\|_{L^\infty(J)} = 1$ unless $f = g_{|_I}$ for some $g \in P_n$ such that $\|g\|_{L^\infty(J)} < 1$. To see this, observe that the set E_n is not only closed but compact. Indeed, if we pick distinct points $\xi_1, \cdots, \xi_{n+1}$ in J and form the Lagrange interpolation polynomials $L_j \in P_n$ such that $L_j(\xi_j) = 1$ and $L_j(\xi_\ell) = 0$ if $\ell \neq j$, we get a basis of P_n in which the coordinates of every $g \in P_n$ meeting $\|g\|_{L^\infty(J)} \leq 1$ are bounded by 1 in modulus. Hence E_n is bounded in P_n, and since it is closed by the proof of Theorem 1 it is compact. Thus, each $f \in L^2(I)$ has a best approximant from E_n, and if $(p_n)_I$ is a best approximant to f with $p_n \in P_n$, then for $\lambda > \|p_n\|_{L^\infty(J)}$ we find that p_n/λ is a best approximant to f/λ in $L^2(I)$ which is strictly less than 1 on J. This justifies the remark.

5 Critical point equations and convergence of approximants

At this point, it is worth recalling informally some basic principles from convex optimization, for which the reader may consult [7]. The solution to a strictly convex minimization problem is characterized by a variational inequality expressing that the *criterium* increases under admissible increments of the variable. If the problem is smooth enough, such increments admit a tangent space at the point under consideration (*i.e.* the solution) in the variable space. We term it the tangent space to the constraints, and its orthogonal in the dual space to the variable space is called the orthogonal space to the constraints (at the point under consideration). The variation of the objective function must vanish on the tangent space to the constraints to the first order, thereby giving rise to the so-called *critical point equation.* It says that the gradient of the objective function, viewed as an element of the dual space to the variable space, lies in the orthogonal space to the constraints. If a basis of the latter is chosen, the coordinates of the gradient in this basis are known as the *Lagrange parameters.* More generally, one can form the Lagrangian which is a function of the variable and of the Lagrange parameters, not necessarily optimal ones. It is obtained by adding the gradient of the criterion, at the considered value of the variable, with the member of the orthogonal space to the constraints defined by the chosen Lagrange parameters. By what precedes, the Lagrangian must vanish at the solution for appropriate values of the Lagrange parameters. One can further define a function of the Lagrange parameters only, by minimizing the Lagrangian with respect to the variable. This results in a concave function which gets maximized at the optimal value of the Lagrange parameters for the original problem. This way, one reduces the original constrained convex minimization problem to an unconstrained concave maximization problem, called the dual problem. In an infinite-dimensional context, the arguments needed to put this program to work may be quite subtle.

Below we derive the critical point equation for $PBEP(n)$ described in (13). For $g \in P_n$ define

$$E(g) = \{x \in J,\ |g(x)| = ||g||_{L^\infty(J)}\},$$

which is the set of extremal points of g on J.

Theorem 2 *A polynomial $g \in P_n$ is the solution to $PBEP(n)$ iff the following two conditions hold:*

- $||g||_{L^\infty(J)} \le 1,$
- *there exists a set of r distinct points $x_1, \cdots, x_r \in E(g)$ and non-negative real numbers $\lambda_1, \cdots, \lambda_r$, with $0 \le r \le 2n+2$, such that*

$$\langle g - f, h\rangle_I + \sum_{j=1}^{r} \lambda_j g(x_j)\overline{h(x_j)} = 0, \qquad \forall h \in P_n. \tag{19}$$

Moreover the λ_j's meet the following bound

$$\sum_{j=1}^{r} \lambda_j \le 2||f||^2_{L^2(I)}. \tag{20}$$

We emphasize that the set of extremal points $\{x_j,\ j = 1, \dots, r\}$ is possibly empty (i.e $r = 0$).

Proof. Suppose g verifies the two conditions and differs from the solution k_n. Set $h = k_n - g \in P_n$ and observe that

$$\mathrm{Re}\left(g(x_i)\overline{h(x_i)}\right) = \mathrm{Re}\left(g(x_i)\overline{k_n(x_i)} - 1\right) \le 0,\ \ i = 1 \dots r. \tag{21}$$

From the uniqueness and optimality of k_n we deduce that

$$\begin{aligned} ||k_n - f||^2_{L^2(I)} &= ||g - f + h||^2_{L^2(I)} \\ &= ||g - f||^2_{L^2(I)} + ||h||^2_{L^2(I)} + 2\mathrm{Re}\langle g - f, h\rangle_I \\ &< ||g - f||^2_{L^2(I)}. \end{aligned}$$

Consequently $\mathrm{Re}\langle g - f, h\rangle_{L^2(I)} < 0$ which, combined with (21), contradicts (19).

Conversely, suppose that g is the solution to $PBEP(n)$ and let ϕ_0 be the $\mathbb{R}$-linear forms on P_n given by

$$\phi_0(h) = \mathrm{Re}\langle g - f, h\rangle_I, \qquad h \in P_n.$$

For each extremal point $x \in E(g)$, define further a $\mathbb{R}$-linear form ϕ_x by

$$\phi_x(h) = \mathrm{Re}\left(g(x)\overline{h(x)}\right), \quad h \in P_n.$$

Put K for the union of these forms:

$$K = \{\phi_0\} \cup \{\phi_x,\ x \in E(g)\}.$$

If we let $P_n^{\mathbb{R}}$ indicate P_n viewed as a real vector space, K is a subset of the dual $(P_n^{\mathbb{R}})^*$. As J is closed by definition, simple inspection shows that K is closed and bounded in $(P_n^{\mathbb{R}})^*$ (it is in fact finite unless g is a constant), hence it is compact and so is its convex hull $\hat{K}$ as $(P_n^{\mathbb{R}})^*$ is finite-dimensional. Suppose for a contradiction that $0 \notin \hat{K}$. Then, since $(P_n^{\mathbb{R}})^{**} = P_n^{\mathbb{R}}$ because $P_n^{\mathbb{R}}$ is finite-dimensional, there exists by the Hahn-Banach theorem an $h_0 \in P_n$ such that,

$$\phi(h_0) \geq \tau > 0, \quad \forall \phi \in \hat{K}.$$

The latter and the continuity of g and h_0 ensure the existence of a neighborhood V of $E(g)$ on $\mathbb{T}$ such that for x in $U = J \cap V$ we have $\mathrm{Re}\left(g(x)\overline{h_0(x)}\right) \geq \frac{\tau}{2} > 0$, whereas for x in $J \backslash U$ it holds that $|g(x)| \leq 1 - \delta$ for some $\delta > 0$. Clearly, for $\epsilon > 0$ with $\epsilon ||h_0||_{L^\infty(J)} < \delta$, we get that

$$\sup_{J \backslash U} |g(x) - \epsilon h_0(x)| \leq 1. \tag{22}$$

Moreover, assuming without loss of generality that $\epsilon < 1$, it holds for $x \in U$ that

$$\begin{aligned} |g(x) - \epsilon h_0(x)|^2 &= |g(x)|^2 - 2\mathrm{Re}\left(g(x)\overline{h_0(x)}\right) + \epsilon^2 |h_0(x)|^2 \\ &\leq |g(x)|^2 - 2\mathrm{Re}\left(\epsilon g(x)\overline{h_0(x)}\right) + \epsilon^2 |h_0(x)|^2 \\ &\leq 1 - \epsilon\tau + \epsilon^2 ||h_0||^2_{L^\infty(J)}. \end{aligned}$$

The latter combined with (22) shows that, for ϵ sufficiently small, we have

$$||g - \epsilon h_0||_{L^\infty(J)} \leq 1. \tag{23}$$

However, since

$$\begin{aligned} ||f - g - \epsilon h_0||^2_{L^2(J)} &= ||f - g||^2_{L^2(J)} - 2\epsilon\phi_0(h_0) + \epsilon^2 ||h_0||^2_{L^2(J)} \\ &\leq ||f - g||^2_{L^2(J)} - 2\epsilon\tau + \epsilon^2 ||h_0||^2_{L^2(J)}, \end{aligned} \tag{24}$$

we deduce in view of (23) that for ϵ small enough the polynomial $g - \epsilon h_0$ performs better than g in $FBEP$, thereby contradicting optimality. Hence $0 \in \hat{K}$, therefore by Carathedory's theorem [8, ch. 1, sec. 5] there are r' elements γ_j of K, with $1 \leq r' \leq 2(n+1) + 1$ (the real dimension of $P_n^{\mathbb{R}}$ plus one), such that

$$\sum_{j=1}^{r'} \alpha_j \gamma_j = 0 \tag{25}$$

for some positive α_j satisfying $\sum \alpha_j = 1$. Of necessity ϕ_0 is a γ_j, otherwise evaluating (25) at g yields the absurd conclusion that

$$0 = \sum_{j=1}^{r'} \alpha_j \gamma_j(g) = \sum_{j=1}^{r'} \alpha_j |g(x_j)|^2 = 1.$$

Equation (25) can therefore be rewritten as

$$\alpha_1 \mathrm{Re}\langle f - g, h\rangle_I + \sum_{j=2}^{r'} \alpha_j \mathrm{Re}(g(x_j)\overline{h(x_j)}) = 0 \quad \forall h \in P_n, \quad \alpha_1 \neq 0.$$

Dividing by α_1 and noting that the last equation is also true with ih instead of h yields (19) with $r = r' - 1$. Finally, replacing h by g in (19) we obtain

$$\sum_{j=1}^{r}|\lambda_j| = \sum_{j=1}^{r}\lambda_j| = \langle f-g,g\rangle_I \leq \ \langle f-g,f-g\rangle_I + |\langle f-g,f\rangle_I|$$
$$\leq ||f-g||^2_{L^2(I)} + ||f-g||_{L^2(I)}||f||_{L^2(I)}$$
$$\leq 2||f||^2_{L^2(I)}$$

where the next to last majorization uses the Schwarz inequality and the last that 0 is a candidate approximant for $PBEP(n)$ whereas g is the optimum. ■

The next result describes the behavior of k_n when n goes to infinity, in connection with the solution g_0 to $ABEP$.

Theorem 3 *Let k_n be the solution to $PBEP(n)$ defined in* (13)*, and g_0 the solution to $ABEP$ described in* (14)*. When $n\to\infty$, the sequence $(k_n)_{|_I}$ converges to $(g_0)_{|_I}$ in $L^2(I)$, and the sequence $(k_n)_{|_J}$ converges to $(g_0)_{|_J}$ in the weak-* topology of $L^\infty(J)$, as well as in $L^p(J)$-norm for $1\leq p<\infty$ if f is not the race on I of a H^2-function which is at most 1 in modulus on J. Altogether this amounts to:*

$$\lim_{n\to\infty}||g_0-k_n||_{L^p(\mathbb{T})}=0, \quad 1\leq p\leq 2, \tag{26}$$

$$\lim_{n\to\infty}\langle k_n,h\rangle_J=\langle g_0,h\rangle_J \quad \forall h\in L^1(J), \tag{27}$$

$$\text{if } f\neq g_0 \text{ on } I,\ \lim_{n\to\infty}||g_0-k_n||_{L^p(J)}=0, \quad 1\leq p<\infty. \tag{28}$$

Proof. Our first objective is to show that g_0 can be approximated arbitrary close in $L^2(I)$ by polynomials that remain bounded by 1 in modulus on J. By hypothesis I is the finite union of $N\geq 1$ open disjoint sub-arcs of $\mathbb{T}$. Without loss of generality, it can thus be written as

$$I=\bigcup_{i=1}^{N}(e^{ia_i},e^{ib_i}), \qquad 0=a_1\leq b_1\leq a_2\cdots\leq b_N\leq 2\pi.$$

Let (ϵ_n) be a sequence of positive real numbers decreasing to 0. We define a sequence (v_n) in H^2 by

$$v_n(z)=g_0(z)\exp\left(-\frac{1}{2\pi}\left(\sum_{i=1}^{N}\int_{a_i}^{a_i+\epsilon_n}\frac{e^{it}+z}{e^{it}-z}\log|g_0|dt+\int_{b_i-\epsilon_n}^{b_i}\frac{e^{it}+z}{e^{it}-z}\log|g_0|dt\right)\right)$$

Note that indeed $v_n\in H^2$ for n large enough because then it has the same modulus as g_0 except over the arcs $(a_i,a_i+\epsilon_n)$ and $(b_i-\epsilon_n,b_i)$ where it has modulus 1. *We claim* that $(v_n)_{|_I}$ converges to g_0 in $L^2(I)$ as $n\to\infty$. To see this, observe that v_n converges a.e. on I to g_0, for each $z\in I$ remains at some distance from the sub-arcs $(a_i,a_i+\epsilon_n)$ and $(b_i,b_i+\epsilon_n)$ for all n sufficiently large, hence the argument of the exponential in (29) converges to zero as $n\to\infty$ by absolute continuity of $\log|g_0|dt$. Now, we remark that by construction $|v_n|\leq|g_0|+1$, hence by dominated convergence, we get that

$$\lim_{n\to\infty}||g_0-v_n||_{L^2(I)}=0.$$

This proves the claim. Now, let $\epsilon>0$ and $0<\alpha<1$ such that $||g_0-\alpha g_0||_{L^2(I)}\leq\frac{\epsilon}{4}$. Let also n_0 be so large that $||v_{n_0}-g_0||_{L^2(I)}\leq\frac{\epsilon}{4}$. For $0<r<1$ define $u_r\in\mathcal{A}$ (the disk algebra) by $u_r(z)=v_{n_0}(rz)$ so that, by Poisson representation,

$$u_r(e^{i\theta})=\int_{\mathbb{T}}P_r(\theta-t)v_{n_0}(re^{it})dt,$$

where P_r is the Poisson kernel. Whenever $e^{i\phi}\in J$, we note by construction that $|v_n|=1$ a.e on the sub-arc $(e^{i(\phi-\epsilon_{n_0})},e^{i(\phi+\epsilon_{n_0})})$. This is to the effect that

$$|u_r(e^{i\phi})|\leq\int_{\mathbb{T}}P_r(\phi-t)|v_{n_0}(re^{it})|dt$$
$$\leq P_r(\epsilon_{n_0})\int_{\mathbb{T}}|v_{n_0}(re^{it})|dt+\int_{-\epsilon_{n_0}}^{+\epsilon_{n_0}}P_r(t)dt$$
$$\leq P_r(\epsilon_{n_0})||v_{n_0}||_{L^1(\mathbb{T})}+1\leq P_r(\epsilon_{n_0})||v_{n_0}||_{L^2(\mathbb{T})}+1$$

by Hölder's inequality. Hence, for r sufficiently close to 1, we certainly have that $|u_r| \leq 1/\alpha^2$ on J and otherwise that $||u_r - v_{n_0}||^2_{L^2(I)} \leq \frac{\epsilon}{4}$ since $u_r \to v_{n_0}$ in H^2. Finally, call q the truncated Taylor expansion of u_r (which converges uniformly to the latter on $\mathbb{T}$), where the order of truncation has been chosen large enough to ensure that $|q| \leq 1/\alpha$ on J and that $||q - u_r||^2_{L^2(I)} \leq \frac{\epsilon}{4}$. Then, we have that

$$
\begin{aligned}
||\alpha q - g_0||_{L^2(I)} &\leq \alpha \left(||q - u_r||_{L^2(I)} + ||u_r - v_{n_0}||_{L^2(I)} + ||v_{n_0} - g_0||_{L^2(I)} \right) + ||g_0 - \alpha g_0||_{L^2(I)} \\
&\leq \epsilon.
\end{aligned}
$$

Thus, we have found a polynomial (namely αq) which is bounded by 1 in modulus on J and close by ϵ to g_0 in $L^2(I)$. By comparison, this immediately implies that

$$
\lim_{n\to\infty} ||f - k_n||_{L^2(I)} = ||f - g_0||_{L^2(I)}, \tag{29}
$$

from which (26) follows by Hölder's inequality. Moreover, being bounded in H^2, the sequence (k_n) has a weakly convergent sub-sequence. The traces on J of this subsequence are in fact bounded by 1 in $L^\infty(J)$-norm, hence up to another subsequence we obtain (k_{n_m}) converging also in the weak-* sense on J. Let g be the weak limit (H^2 sense) of k_{n_m}, and observe that $g_{|J}$ is necessarily the weak-* limit of $(k_{n_m})_{|J}$ in $L^\infty(J)$, as follows by integrating against functions from $L^2(J)$ which is dense in $L^1(J)$. Since balls are weak-* closed in $L^\infty(J)$, we have that $||g||_{L^\infty(J)} \leq 1$, and it follows from (29) that $||f - g||_{L^2(I)} = ||f - g_0||_{L^2(I)}$. Thus, $g = g_0$ by the uniqueness part of Theorem 1. Finally, if $f \neq g_0$ on J, then we know from Theorem 1 that $|g_0| = 1$ a.e. on J. In this case, (29) implies that $\limsup ||k_{n_m}||_{L^2(\mathbb{T})} \leq ||g_0||_{L^2(\mathbb{T})}$, and since the norm of the weak limit is no less than the limit of the norms it follows that $(k_{n_m})_{|J}$ converges strongly to $(g_0)_{|J}$ in the strictly convex space $L^2(J)$. The same reasoning applies in $L^p(J)$ for $1 < p < \infty$. Finally we remark that the preceding arguments hold true when k_n is replaced by any subsequence of itself; hence k_n contains no subsequence not converging to g_0 in the sense stated before, which achieves the proof. ■

We come now to an analog of theorem 2 in the infinite dimensional case. We define $H^{2,\infty}_J$ and $H^{2,1}_I$ to be the following vector spaces:

$$
H^{2,\infty}_J = \{h \in H^2,\ ||h||_{L^\infty(J)} < \infty\},
$$

$$
H^{2,1}_I = \{h \in H^1,\ ||h||_{L^2(I)} < \infty\},
$$

endowed with the natural norms. We begin with an elementary lemma.

Lemma 1 *Let $v \in L^1(J)$ such that $\mathbf{P}_+(0 \vee v) \in H^{2,1}_I$. Then:*

$$
\forall h \in H^{2,\infty}_J,\ \langle \mathbf{P}_+(0 \vee v), h\rangle_{\mathbb{T}} = \langle v, h\rangle_J.
$$

Proof. Let u be the function defined on $\mathbb{T}$ by

$$
u = (0 \wedge v) - \mathbf{P}_+(0 \vee v).
$$

By assumption $u \in L^1(\mathbb{T})$, and by its very definition all Fourier coefficients of u of non-negative index vanish. Hence $u \in \bar{H}^1_0$, and since it is L^2 integrable on I where it coincides with $-\mathbf{P}_+(0 \vee v)$, .we conclude that $\overline{u} \in H^{2,1}_I$ and that $\overline{u}(0) = 0$ Now, for $h \in H^{2,\infty}_J$ we have that

$$
\begin{aligned}
\langle v\chi_J, h\rangle_{\mathbb{T}} &= \langle u, h\rangle_{\mathbb{T}} + \langle \mathbf{P}_+(0 \vee v), h\rangle_{\mathbb{T}} \\
&= \overline{u}(0)h(0) + \langle \mathbf{P}_+(0 \vee v), h\rangle_{\mathbb{T}} \\
&= \langle \mathbf{P}_+(0 \vee v), h\rangle_{\mathbb{T}}
\end{aligned} \tag{30}
$$

where the second equality follows from the Cauchy formula because $(\overline{u}h) \in H^1$. ■

Theorem 4 *Suppose that $f \in L^2(I)$ is not the trace on I of a H^2-function of modulus less or equal to 1 a.e on J. Then, $g \in H^2$ is the solution to ABEP iff the following two conditions hold.*

- $|g(e^{i\theta})| = 1$ *for a.e.* $e^{i\theta} \in J$,

- *there exists a nonnegative real function $\lambda \in L^1_{\mathbb{R}}(J)$ such that,*

$$\forall h \in H^{2,\infty}_J,\ \langle g - f, h\rangle_I + \langle \lambda g, h\rangle_J = 0. \tag{31}$$

Proof. Suppose g verifies the two conditions and differs from g_0. Set $h = (g_0 - g) \in H^{2,\infty}_J$ and observe that

$$\mathrm{Re}\langle \lambda g, h\rangle_J = \frac{1}{2\pi}\int_J \lambda(\mathrm{Re}(\overline{g}g_0) - 1) \leq 0. \tag{32}$$

In another connection, since $-h$ is an admissible increment from g_0, the variational inequality characterizing the projection onto a closed convex set gives us (*cf.* Theorem 1) $\mathrm{Re}\langle g_0 - f, h\rangle_I \leq 0$, whence

$$\mathrm{Re}\langle g - f, h\rangle_I = \mathrm{Re}\langle g_0 - f, h\rangle_I - \langle h, h\rangle_I < 0$$

which, combined with (32), contradicts (31).

Suppose now that g is the solution of *ABEP*. The property that $|g| = 1$ on J has been proven in Theorem 1. In order to let n tend to infinity, we rewrite (19) with self-explaining notations as

$$\langle k_n - f, e^{im\theta}\rangle_I + \sum_{j=1}^{r(n)} \lambda_j^n k_n(e^{i\theta_j^n})\overline{e^{im\theta_j^n}} = 0, \quad \forall m \in \{0 \dots n\},\ . \tag{33}$$

We define (Λ_n), $n \in \mathbb{N}$, to be a family of linear forms on $C(J)$ defined as

$$\Lambda_n(u) = \sum_{j=1}^{r(n)} \lambda_j^n k_n(e^{i\theta_j^n})u(e^{\theta_j^n}), \quad \forall u \in C(J).$$

Equation (20) shows that (Λ_n) is a bounded sequence in the dual $C(J)^*$ which by the Banach-Alaoglu theorem admits a weak-* converging subsequence whose limit we call Λ. Moreover, the Riesz representation theorem ensures the existence of a complex measure μ to represent Λ so that, appealing to Theorem 3 and taking the limit in (33), we obtain

$$\langle g_0 - f, e^{im\theta}\rangle_I + \int_J \overline{e^{im\theta}} d\mu = 0, \quad \forall m \in \mathbb{N}.. \tag{34}$$

Now, the F. and M. Riesz theorem asserts that the measure which is μ on J and $(g_0 - f)d\theta$ on I is absolutely continuous with respect to Lebesgue measure, because its Fourier coefficients of nonnegative index do vanish, by (34). Therefore there is $v \in L^1(J)$ such that,

$$\langle g_0 - f, e^{im\theta}\rangle_I + \langle v, e^{im\theta}\rangle_J = 0, \quad \forall m \in \mathbb{N}\,,$$

which is equivalent to

$$\langle g_0 - f, e^{im\theta}\rangle_I + \langle \lambda g_0, e^{im\theta}\rangle_J = 0, \quad \forall m \in \mathbb{N}, \tag{35}$$

where we have set $\lambda(z) = v(z)\overline{g_0(z)}\ \forall z \in J$. Equation (35) means that

$$\mathbf{P}_+((g_0 - f)\chi_I) = -\mathbf{P}_+(0 \vee \lambda g_0),$$

which indicates that $\mathbf{P}_+(0 \vee \lambda g_0)$ lies in H^2. Thus, thanks to Lemma 1, we get that

$$\langle g_0 - f, u\rangle_I + \langle \lambda g_0, u\rangle_J = 0, \quad \forall u \in H^{2,\infty}_J. \tag{36}$$

In order to prove the realness as well as the nonnegativity of λ, we pick $h \in C^\infty_{c,\mathbb{R}}(I)$, the space of smooth real-valued functions with compact support on I, and we consider its Riesz-Herglotz transform

$$b(z) = \frac{1}{2\pi}\int_I \frac{e^{it} + z}{e^{it} - z} h(e^{it})\, dt = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{e^{it} + z}{e^{it} - z} \chi_I(e^{it})\, h(e^{it})\, dt\,. \tag{37}$$

It is standard that b is continuous on $\overline{\mathbb{D}}$ [11, ch. III, thm. 1.3]. For $t \in \mathbb{R}$, define $\omega_t = \exp(tb)$ which is the outer function whose modulus is equal to $\exp th$ on I and 1 on J. The function $g_0\,\omega_\lambda$ is a candidate approximant in problem *ABEP*, hence $t \mapsto \|f - g_0\,\omega_t\|^2_{L^2(I)}$ reaches a minimum at $t = 0$. By the boundedness of b, we may differentiate this function with respect to t under the integral sign, and equating the derivative to 0 at $t = 0$ yields

$$0 = \mathrm{Re}\langle (f - g_0)\overline{g_0}, b\rangle_I = \mathrm{Re}\langle (f - g_0), bg_0\rangle_I.$$

In view of (36), it implies that

$$0 = \mathrm{Re}\langle \lambda g_0, bg_0\rangle_J = \mathrm{Re}\langle \lambda, b\rangle_J,$$

where we used that $|g_0| \equiv 1$ on J. Remarking that b is pure imaginary on J, this means

$$\langle Im(\lambda), b\rangle_{L^2(J)} = 0, \quad \forall h \in C^\infty_{c,\mathbb{R}}(I).$$

Letting $h = h_m$ range over a sequence of smooth positive functions which are approximate identies, namely of unit $L^1(I)$-norm and supported on the arc $[\theta - 1/m, \theta + 1/m]$ with $e^{i\theta} \in I$, we get in the limit, as $m \to \infty$, that

$$\langle Im(\lambda), (e^{i\theta} + .)/(e^{i\theta} - .)\rangle_J = 0, \quad e^{i\theta} \in I.$$

Then, appealing to he Plemelj-Sokhotski formulas as in the proof of Theorem 1, this time on J, we obtain that $Im(\lambda) = 0$ which proves that λ is real-valued. Note that the argument based on the Plemelj-Sokhotski formulas and the Hahn-Banach theorem together imply that the space generated by $\xi \mapsto (e^{i\theta}+\xi)/(e^{i\theta}-\xi)$, as $e^{i\theta}$ ranges over an infinite compact subset lying interior to J, is dense in $L^p(I)$ for $1 < p < \infty$. In fact using the F. and M. Riesz theorem and the Plemelj-Sokhoski formulas, it is easy to see that such functions are also uniformly dense in $C(\overline{I})$. Then, using that $ABEP$ is a convex problem, we obtain upon differentiating once more that

$$\mathrm{Re}\langle (g_0 - f)\bar{g}_0, b^2\rangle_I \ \geq 0,$$

which leads us by (36) to

$$\mathrm{Re}\langle \lambda, ((e^{i\theta} + .)/(e^{i\theta} - .))^2\rangle_J = \mathrm{Re}\langle \lambda g_0, g_0((e^{i\theta} + .)/(e^{i\theta} - .))^2\rangle_J \ \leq 0, \qquad e^{i\theta} \in I.$$

By the density property just mentioned this implies that $((e^{i\theta} + .)/(e^{i\theta} - .))^2_{|_{\overline{I}}}$ is dense in the set of nonpositive continuous functions on $\overline{I}$, therefore $\lambda \geq 0$. Note also that (35) implies $(f - g_0) \vee \lambda g_0 \in \bar{H}^1$, hence it cannot vanish on a subset of $\mathbb{T}$ of positive measure unless it is the zero function. But this would imply $f = g$ a.e on I which contradicts the hypothesis. This yields $\lambda > 0$ a.e on J. ■

6 A numerical example

For practical applications the continuous constraint of $PBEP$ on the arc J is discretized in $m+1$ points. Suppose that $J = \{e^{it}, t \in [-\theta, \theta]\}$, for some $\theta \in [0, \pi]$. Call J_m the discrete version of the arc J defined by

$$J_m = \{e^{it}, t \in \{-\theta + \frac{2k\theta}{m}, k \in \{0 \dots m\}\}$$

we define following auxiliary extremal problem:

DBEP(n,m)
For $f \in L^2(I)$, find $k_{n,m} \in P_n$ such that $\forall t \in J_m$ $|k_{n,m}(t)| \leq 1$ and

$$\|f - k_{n,m}\|_{L^2(I)} = \min_{\substack{g \in P_n \\ |g| \leq 1 \text{ a.e. on } J_m}} \|f - g\|_{L^2(I)}\,. \qquad (38)$$

For the discretized problem **DBEP(n,m)**, the following holds.

Theorem 5 *For $\lambda = (\lambda_0, \dots, \lambda_m) \in \mathbb{R}^{m+1}$ and $g \in P_n$ define the Lagrangian*

$$L(\lambda, g) = \|f - g\|_{L^2(I)} + \sum_{k=0}^{m} \lambda_k(|g(e^{i(-\theta + \frac{2k\theta}{m})}|^2 - 1)$$

, then

- *Problem* **DBEP(n,m)** *has a unique solution $k_{n,m}$,*

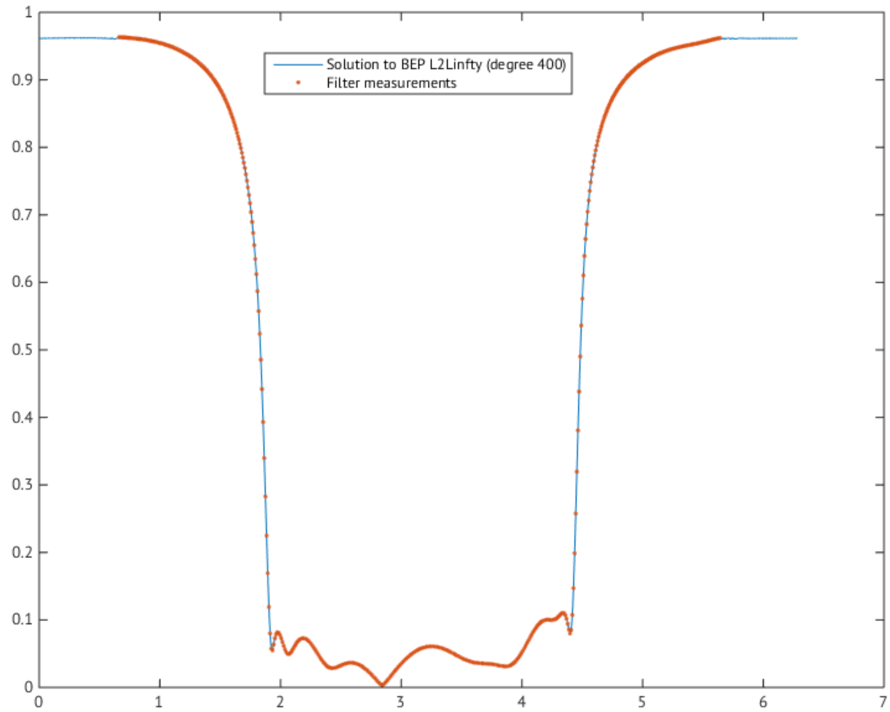

Figure 1: Solution of DBEP at hand of partial scattering measurements of a microwave filter

- *$k_{n,m}$ is also the unique solution of the concave maximisation problem:*

$$\text{to find } g_{opt} \text{ and } \lambda_{opt} \text{ solving for} \quad \max_{\lambda \geq 0} \min_{g \in P_n} L(\lambda, g), \tag{39}$$

where $\lambda \geq 0$ means that each component of λ is non negative.

- *For a fixed n, $\lim_{m \to \infty} k_{n,m} = k_n$ in P_n.*

The proof of Theorem 5 follows from standard convex optimization theory, using in addition that the *sup*-norm of the derivative of a polynomial of degree n on $\mathbb{T}$ is controlled by the values it assumes at a set of $n+1$ points. This depends on Bernstein's inequality and on the argument using Lagrange interpolation polynomials used in the Remark after Theorem 1.

In the minmax problem (39) , the minimization is a quadratic convex problem. It can be tackled efficiently by solving the critical point equation which is a linear system of equations similar to (19). Eventually, an explicit expression of the gradient and of the hessian of the concave maximization problem (39) allows us for a fast converging computational procedure to estimate $k_{n,m}$.

Figure (1) represents a solution to problem **DBEP(n,m)**, where f is obtained from partial measurement of the scattering reflexion parameter of a wave-guide microwave filter by the CNES (French Space Agency). The problem is solved for $n = 400$ and $m = 800$, while the constraint on J has been renormalized to 0.96 (instead of 1). The modulus of $k_{400,800}$ is plotted as a blue continuous line while the measurements $|f|$ appear as red dots. As the reader can see, the fit is extremely good.

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