Delaunay triangulation of a random sample of a good sample has linear size*

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Abstract

The *randomized incremental construction* (RIC) for building geometric data structures has been analyzed extensively, from the point of view of worst-case distributions. In many practical situations however, we have to face nicer distributions. A natural question that arises is: do the usual RIC algorithms automatically adapt when the point samples are nicely distributed. We answer positively to this question for the case of the Delaunay triangulation of ε-nets.

ε-nets are a class of nice distributions in which the point set is such that any ball of radius ε contains at least one point of the net and two points of the net are distance at least ε apart. The Delaunay triangulations of ε-nets are proved to have linear size; unfortunately this is not enough to ensure a good time complexity of the randomized incremental construction of the Delaunay triangulation. In this paper, we prove that a uniform random sample of a given size that is taken from an ε-net has a linear sized Delaunay triangulation in any dimension. This result allows us to prove that the randomized incremental construction needs an expected linear size and an expected $O(n \log n)$ time.

Further, we also prove similar results in the case of non-Euclidean metrics, when the point distribution satisfies a certain *bounded expansion* property; such metrics can occur, for example, when the points are distributed on a low-dimensional manifold in a high-dimensional ambient space.

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1 Introduction

An ε-net of some compact domain D is a point set of size n in D such that any ball centered in D of radius ε contains at least a point and two points of the net are distance at least ε apart. When we enforce such a hypothesis of "nice" distribution of the points in space, a volume counting argument ensures that the local complexity of the Delaunay triangulation around a vertex is bounded by a constant (dependent on the dimension d but not on the number of points). Unfortunately, to be able to control the complexity of the usual randomized

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incremental algorithms [9, 2, 5, 1], it is not enough to control the final complexity of the Delaunay triangulation, but also the complexity of the triangulation of a random subset.

One would expect that a random subsample of size k of an ε-net is also an ε'-net for $\varepsilon' = \varepsilon\sqrt[d]{\frac{n}{k}}$ with high probability. Actually this is not quite true, it may happen with reasonable probability that a ball of radius $O(\varepsilon')$ contains $\Omega(\log k/\log\log k)$ points or that a ball of radius $\Omega(\varepsilon'\sqrt[d]{\log k})$ does not contain any point. Thus this approach can transfer the complexity of an ε-net to the one of a random subsample of an ε-net but losing log factors.

In this paper, we study the Delaunay triangulation of a random sample of an ε-net and deduce results about the complexity of randomized incremental constructions. To avoid technicalities due to boundary, we assume without real loss of generality that the ε-net is taken from the flat torus of dimension d which is a compact manifold without boundary. If we equip the flat torus with the Euclidean metric, we will show that the expected number of d-simplices of the star of any point in the sample can be bounded by a constant that does not depend on the number of points in the sample. It will follow that the complexity of the randomized incremental construction of the Delaunay triangulation of an ε-net in general position takes time $O(n\log n)$ in any dimension.

We will extend those results to non-Euclidean metrics that satisfy a certain *bounded expansion* property; such metrics can occur, for example, when the points are well distributed on a low-dimensional manifold in a high-dimensional ambient space.

The rest of the paper is as follows: In Section 2 we introduce the basic concepts of Delaunay complex, net, growth-restricted measure, random sample, and state our results. In Section 3, we bound the size of the d-skeleton of the Delaunay complex of a uniform random sample of a given size extracted from an ε-net. In Section 4, we extend this result to growth-restricted metrics. Finally, in Section 5, we use those size bounds to compute the space and time complexity of the randomized incremental construction for constructing Delaunay complexes of ε-nets.

2 Definitions, Notations, and Results

In this paper, we consider a finite set of points $\mathcal{X}$ in the flat torus $\mathbb{T}^d$ of dimension d, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Since $\mathbb{T}^d$ is a compact manifold without boundary, it will be possible to define finite nets without having to care about boundary effects. We can associate to the flat torus its infinite sheeted covering by $\mathbb{R}^d$ obtained by periodically copying the points in $[0,1]^d$ by translations with integer coordinates.

We denote by $\Sigma(p,r)$ and $B(p,r)$ the sphere and the ball of center p and radius r respectively and $\mathrm{int}(B)$ the interior of a set B.

The volume of the unit Euclidean ball of dimension d is denoted V_d and the area of the boundary of such a ball is denoted S_{d-1}. It is known that $V_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}$ and $S_d = 2\pi V_{d-1}$, where $\Gamma(t) := \int_0^\infty e^{-x}x^{t-1}dx$, $(t>0)$ denotes the *Gamma function*. For $d \in \mathbb{Z}^+$, $\Gamma(d+1) = d!$. We note that $2^d d^{-d/2} \le V_d \le d^{-d/2}$ (see e.g. [13]).

Given a discrete set A, $\sharp(A)$ denotes its cardinality and, for $k \in \mathbb{Z}^+$, $\binom{A}{k}$ denotes the set of k-sized subsets of A.

2.1 Delaunay complexes

► **Definition 1** (Delaunay complex). Given a set $\mathcal{X}$ in some ambient metric space, the Delaunay complex of $\mathcal{X}$ is the (abstract) simplicial complex with vertex set $\mathcal{X}$ which is the nerve of the Voronoi diagram of $\mathcal{X}$: a simplex σ belongs to $Del(\mathcal{X})$ iff the Voronoi cells of its vertices have a non empty common intersection. Equivalently, σ can be circumscribed by an empty ball, i.e. a ball whose bounding sphere contains the vertices of σ and whose interior contains no points of $\mathcal{X}$.

Our combinatorial results (Theorems 14, 15, 16) do not assume that the points of $\mathcal{X}$ are in general position and we will consider Euclidean as well as more general metrics. Accordingly, we won't assume that the Delaunay complex embeds in the ambient space and we will provide bounds on the size of the d-skeleton of the Delaunay complex, i.e. the subcomplex consisting of the faces of the Delaunay complex of dimension at most d. Differently, our algorithmic result (Theorem 18) assumes that the Delaunay complex is a triangulation of $\mathcal{X}$, i.e. a geometric simplicial complex embedded in the ambient space containing $\mathcal{X}$.

Delaunay [7] proved that, if the ambient space is $\mathbb{R}^d$ equipped with the Euclidean metric and if $\mathcal{X}$ is in general position[1], the Delaunay complex of $\mathcal{X}$ is a triangulation called the Delaunay triangulation of $\mathcal{X}$. The case of the flat torus equipped with the Euclidean metric is slightly more complicated. The Delaunay complex then embeds in the infinite sheeted covering as a geometric triangulation when the points are in general position. However, the Delaunay complex does not embed in the flat torus in general since a point can have a copy of itself as one of its neighbours. Nevertheless, as shown by Caroli and Teillaud [4], the complex always embeds in the 3^d-sheeted covering of $\mathbb{T}^d$ (consisting of 3^d copies of each point in $(\mathbb{R}/3\mathbb{Z})^d$) and embed in the one sheeted covering provided that the largest Delaunay sphere has diameter less than half the systole of the space, which is 1 for $\mathbb{T}^d$. More generally, if the faces of the Voronoi diagram of $\mathcal{X}$ satisfy the so-called *closed ball property* (i.e. the faces are topological balls of the right dimension), then the Delaunay complex is a triangulation under the general position assumption [11]. A popular way to ensure the closed ball property is to assume that $\mathcal{X}$ is a sufficiently dense net (see [10] for instance) which is consistent with the general approach taken in this paper.

2.2 ε-nets over the Euclidean metric

▶ **Definition 2** (ε-net)**.** A set $\mathcal{X}$ of n points in $\mathbb{T}^d$ is an *ε-net* if any ball of radius ε contains at least one point, and any two points are at least distance ε apart.

This definition applies for any metric. In the case of the Euclidean metric, we have the following properties.

▶ **Lemma 3** (Maximum packing size)**.** *Let $\rho \le 1$, then any packing of the ball of radius $r \ge \rho$ in dimension d by disjoint balls of radius $\rho/2$ has a number of balls smaller than $\left(\frac{3r}{\rho}\right)^d$.*

Proof. Consider a maximal set of disjoint balls of radius $\frac{\rho}{2}$ with center inside the ball $B(r)$ of radius r. Then the balls with the same centers and radius ρ cover the ball $B(r)$ (otherwise it contradicts the maximality). By a volume argument we get that the number of balls is bounded from above by $\frac{V_d \times \left(r+\frac{\rho}{2}\right)^d}{V_d \times \left(\frac{\rho}{2}\right)^d} \le \left(\frac{3r}{\rho}\right)^d$ for $\rho < 1$. ◀

▶ **Lemma 4** (Minimum cover size)**.** *Any covering of a ball of radius r in dimension d by balls of radius ρ has a number of balls greater than $\left(\frac{r}{\rho}\right)^d$.*

Proof. The volume argument gives a lower bound of $\frac{V_d \times r^d}{V_d \times \rho^d} = \left(\frac{r}{\rho}\right)^d$. ◀

[1] i.e. no subset of $d+2$ points of $\mathcal{X}$ lie on a same hypersphere.

A key property of ε-nets is that the d-skeletons of their Delaunay complexes have linear size as stated in the next lemma.

▶ **Lemma 5.** *Let $\varepsilon \in (0,1]$ be given, and let $\mathcal{X}$ be an ε-net over $\mathbb{T}^d$, where $d \in \mathbb{Z}^+$ is any positive integer. Then the d-skeleton of the Delaunay complex of $\mathcal{X}$, $Del(\mathcal{X})$ has $4^{d^2}\varepsilon^{-d}$ simplices.*

Proof. Observe that, by the minimum distance property of the points in $\mathcal{X}$, the balls of radius $\varepsilon/2$ centered around each point in $\mathcal{X}$, are disjoint, and by a volume argument there can be at most $\frac{1}{V_d \times (\varepsilon/2)^d} \leq 2^{-d} d^{d/2} (\varepsilon/2)^{-d} = d^{d/2}\varepsilon^{-d}$ such balls in $\mathbb{T}^d$. The balls of radius ε centered around each point in $\mathcal{X}$ cover the space thus their number is at least $\frac{1}{V_d \times \varepsilon^d} \geq d^{d/2} 2^{-d} \cdot \varepsilon^{-d}$. Thus

$$\sharp(\mathcal{X}) \in [d^{d/2} 2^{-d} \cdot \varepsilon^{-d}, d^{d/2}\varepsilon^{-d}]. \tag{1}$$

Next, observe that the circumradius of any simplex in $Del(\mathcal{X})$ cannot be greater than ε, since this would imply the existence of a ball in $\mathbb{T}^d$ of radius at least ε, containing no points from $\mathcal{X}$. Therefore given a point $p \in \mathcal{X}$, any point which lies in a Delaunay simplex incident to p, must be at most distance 2ε from p. Again by a volume argument, the number of such points is at most $\frac{V_d \times (2\varepsilon + \varepsilon/2)^d}{V_d \times ((\varepsilon/2)^d)} = 5^d$. Thus, the number of Delaunay simplices of dimension at most d that contain p, is at most the complexity of the d-skeleton of the Delaunay complex in $\mathbb{T}^d$ on 5^d vertices. This is at most $(5^d)^{\lceil d/2 \rceil}$.Thus we can conclude that the number of simplices in $Del(\mathcal{X})$ is at most the cardinality of $\mathcal{X}$, times the maximum number of simplices incident to any given point $p \in \mathcal{X}$, or $(5^d)^{\lceil d/2 \rceil} \cdot \sharp(\mathcal{X}) \leq (5^d)^{\frac{d+1}{2}} 2^{-d} d^{d/2} \varepsilon^{-d} \leq 4^{d^2}\varepsilon^{-d}$. ◀

2.3 Growth-restricted measures

We shall denote by $(\mathcal{M}, \mathcal{X})$ a metric measure space $\mathcal{M} = (U, d(.,.), \mu)$, where $(U, d(.,.))$ is a metric space, and μ is a measure over the Borel algebra of U, together with an ε-net $\mathcal{X}$ over $\mathcal{M}$. We shall use for μ the counting measure with respect to $\mathcal{X}$, that is, for a Borel set $S \subset \mathcal{M}$, $\mu(S) = \sharp(\mathcal{X} \cap S)$. Abusing notation slightly, we shall use $\mu(c, r)$ to mean $\mu(B(c, r))$.

▶ **Definition 6** (see e.g. [6], [12]). A measure space $\mathcal{M} = (U, d(.,.), \mu)$ is said to be (ρ, dim)-*growth-restricted*[2], for $dim, \rho > 0$, if and only if for all $x \in \mathcal{M}$ and all $r \geq \rho$,

$$\mu(x, 2r) \leq 2^{dim} \cdot \mu(x, r).$$

2^{dim} will be referred to as the *expansion constant* of $\mathcal{M}$ and dim as the expansion dimension. A (ρ, dim)-growth-restricted measure space $\mathcal{M}$ is *strongly growth-restricted* if there also exist constants $\eta > 0$ and $r_0 > 0$, such that for all $x \in \mathcal{M}$ and $r < r_0$

$$\mu(x, 2r) \quad \geq \quad (1+\eta)\mu(x, r). \tag{2}$$

We shall refer to a (ρ, dim)-growth-restricted measure space simply as a growth-restricted space, when (ρ, dim) are not explicitly required or are clear from the context.

▶ **Lemma 7.** *If $(\mathcal{M}, \mathcal{X})$ has expansion dimension at most dim, then for all $c \in U$, $r > 0$,*

$$\mu(c, r) \leq \left(\frac{4r}{\varepsilon}\right)^{dim}.$$

[2] Also known as *doubling measure, Federer measure*, or *diametrically regular measure* in the literature.

Proof. Let $c \in U$ be any point, and $r > 0$. Consider the ball $B = B(c, \varepsilon/2)$. Since $\mathcal{X}$ is an ε-net, by the triangle inequality, there can be at most a single point of $\mathcal{X}$ inside B, that is, $\mu(c, \varepsilon/2) \leq 1$. Therefore, by the bound on dim, we have that $\mu(c, \varepsilon) \leq 2^{dim}$. Applying this argument repeatedly $k = \lceil \log(r/\varepsilon) \rceil$ times, we have that $\mu(c, r) \leq \mu(c, 2^k \varepsilon) \leq 2^{k \cdot dim} \leq 2^{(\log(r/\varepsilon)+1)dim} \leq (2r/\varepsilon)^{dim}$. ◀

▶ **Lemma 8.** *Given $(\mathcal{M}, \mathcal{X})$, $p, c \in U$, and $r > r_1 > 0$, then if $B(c, r_1) \subseteq B(p, r)$, then $\mu(c, r_1) \geq 2^{-\lceil \log(2r/r_1) \rceil \cdot dim} \cdot \mu(p, r)$.*

Proof. Any point $q \in B(p, r)$, satisfies $d(c, q) \leq 2r$. Therefore, $B(p, r) \subseteq B(c, 2r) = B(c, 2^k r_1)$, where $k = \lceil \log(2r/r_1) \rceil$. Applying the restricted growth condition k times, we get $\mu(c, r_1) \geq 2^{-k \cdot dim} \mu(c, 2r) \geq 2^{-kdim} \mu(p, r)$. ◀

We now give a lemma that shows the connection between growth-restricted and strongly growth-restricted measure spaces, when the underlying domain is compact.

▶ **Lemma 9.** *Given a (ρ, dim) growth-restricted metric measure space $(\mathcal{M}, \mathcal{X})$, $\mathcal{M} = (U, d(.,.), \mu)$ where U is a compact domain, then $(\mathcal{M}, \mathcal{X})$ is also strongly growth-restricted, i.e. for $r \leq r_{max}/2$, where r_{max} is at most half the systole of U, any $p \in U$ and $r > \rho$ satisfy*

$$\mu(p, 2r) \geq (1 + \eta) \cdot \mu(p, r), \qquad \text{where } \eta \geq 2^{-4dim}.$$

Proof. Consider the ball $B_1 = B(p, 2r)$, where $r \geq 4\rho$. Since $r \leq r_{max}/2$, where r_{max} is at most half the systole of U, $\Sigma(p, 2r)$ is not self-intersecting. Therefore, no point in $B(p, 2r)$ overlaps itself. Now, since U is a continuous domain, there exists a point $c \in B(p, 2r)$ such that $d(p, c) = 3r/2$. Then the ball $B(c, r/4) \subseteq B(p, 2r)$, since any point $s \in B(c, r/4)$ satisfies $d(p, s) \leq d(p, c) + d(c, s) \leq 7r/4$. Further, $B(p, r) \cap B(c, r/4) = \emptyset$, i.e. the balls $B(c, r/2)$ and $B(p, r)$ must be disjoint, since if there existed any point $q \in B(c, r/2) \cap B(p, r)$, then $d(p, q) \leq r$ and $d(q, c) \leq r/4$, and therefore we would have that $d(p, c) \leq 5r/4 < 3r/2$, which would contradict our assumption $d(p, c) = 3r/2$. Applying Lemma 8(ii) to $B(c, r/4) \subset B(p, 2r)$, we have that

$$\mu(c, r/4) \geq 2^{-\lceil \log\left(\frac{2(2r)}{(r/4)}\right) \rceil dim} \cdot \mu(p, 2r) = 2^{-4dim} \mu(p, 2r) \geq 2^{-4dim} \mu(p, 2r) \geq 2^{-4dim} \mu(p, r).$$

Since $B(p, r) \cap B(c, r/4) = \emptyset$ and $B(p, r) \cup B(c, r/4) \subset B(p, 2r)$, we have that

$$\mu(p, 2r) \geq \mu(c, r/4) + \mu(p, r) \geq (1 + 2^{-4dim}) \mu(p, r),$$

which completes the proof of the lemma. ◀

2.4 Random samples

In this paper, we consider two types of random subsets of ε-nets.

▶ **Definition 10** (Bernoulli random sample)**.** A subset $\mathcal{Y}$ of a set $\mathcal{X}$ is a Bernoulli sample of $\mathcal{X}$ of parameter α if each point of $\mathcal{X}$ belongs to $\mathcal{Y}$ with probability α independently.

▶ **Definition 11** (Uniform random sample)**.** A subset $\mathcal{Y}$ of set $\mathcal{X}$ is a uniform random sample of $\mathcal{X}$ of size s if $\mathcal{Y}$ is any possible subset of $\mathcal{X}$ of size s with equal probability.

In order to work with uniform random samples, we shall require some lemmas on the uniformly random sampling distribution stated below.

▶ **Lemma 12** (j-moment)**.** *Given $a, b, c \in \mathbb{Z}^+$, and a set C of size c, a fixed subset $B \subseteq C$ of size $b \leq c$ and a uniformly random sample A of size a, then for $j \leq \min\{c/2, \sqrt{c}\}$, the j-th moment of the random variable $X := \sharp(A \cap B)$ is given by $\mathbb{E}\left[X^j\right] \leq 3\,\mathbb{E}\left[X\right]^j$.*

The proof is provided in the Appendix.

▶ **Lemma 13.** *Given $a, b, c \in \mathbb{Z}^+$, with $b \leq \min(\frac{a}{2}, \sqrt{c})$, the probability that the random sample A having cardinality a of a set C of cardinality c, contains B having cardinality b, and is disjoint from another fixed set T, which is disjoint from B and has cardinality t, is at most $\left(1 + \frac{2b^2}{c}\right)\left(\frac{a}{c}\right)^b e^{-t\frac{a}{2c}}$.*

Proof. The total number of ways of choosing the random sample A is $\binom{c}{a}$. The number of ways of choosing A such that $B \subset A$ and $T \cap A = \emptyset$, is $\binom{c-b-t}{a-b}$. Therefore the required probability is

$$
\begin{aligned}
\mathbb{P}\left[B \subset A,\ \ T \cap A = \emptyset\right] &= \frac{\binom{c-b-t}{a-b}}{\binom{c}{a}} \\
&= \frac{\prod_{i=0}^{b-1}(a-i)\prod_{i=b}^{a-1}(a-i)}{\prod_{i=0}^{b-1}(c-i)\prod_{i=b}^{a-1}(c-i)} \cdot \frac{\prod_{i=0}^{a-b-1}(c-b-t-i)}{\prod_{i=0}^{a-b-1}(a-b-i)} \\
&= \frac{\prod_{i=0}^{b-1}(a-i)}{\prod_{i=0}^{b-1}(c-i)}\,\frac{\prod_{i=0}^{a-b-1}(c-b-t-i)}{\prod_{i=0}^{a-b-1}(c-b-i)} \\
&= (a/c)^b \frac{\prod_{i=0}^{b-1}(1-i/a)}{\prod_{i=0}^{b-1}(1-i/c)} \prod_{i=0}^{a-b-1}\left(1 - \frac{t}{c-b-i}\right) \\
&\leq (a/c)^b \frac{\prod_{i=0}^{b-1}(1-i/a)}{\prod_{i=0}^{b-1}(1-i/c)} \left(1 - \frac{t}{c-b}\right)^{a-b} \\
&= (a/c)^b \frac{\prod_{i=0}^{b-1}(1-i/a)}{\exp(\sum_{i=0}^{b-1}\ln(1-i/c))} \left(1 - \frac{t}{c-b}\right)^{a-b} \\
&\leq (a/c)^b \frac{1}{\exp(\sum_{i=0}^{b-1}\ln(1-i/c))} \left(\exp\left(-\frac{t}{c-b}\right)\right)^{a-b} \qquad (3) \\
&= (a/c)^b \exp\left(-\sum_{i=0}^{b-1}\ln(1-i/c)\right) \left(\exp\left(-\frac{t}{c-b}\right)\right)^{a-b} \\
&\leq (a/c)^b \exp\left(\sum_{i=0}^{b-1}\frac{2i}{c}\right) \left(\exp\left(-\frac{t}{c-b}\right)\right)^{a-b} \qquad (4) \\
&= (a/c)^b \exp\left(\frac{b(b-1)}{c}\right) \exp\left(-\frac{t(a-b)}{c-b}\right) \\
&\leq (a/c)^b \left(1 + \frac{2b^2}{c}\right) e^{\left(-\frac{ta}{2c}\right)}, \qquad (5)
\end{aligned}
$$

where (3) uses $1 - x < e^{-x}$ for $x > 0$, (4) comes from the fact that $-\ln(1-x) \leq 2x$ for $x \in (0, 1/2]$, and that $b < a/2 < c/2$, and (5) comes from $e^x < 1 + 2x$ for $x \in [0, 1]$ and $b^2 < c$ for the second factor and for the third factor the fact that when $0 \leq b \leq a/2$ we have $\frac{a-b}{c-b} \geq \frac{a-a/2}{c-0} = \frac{a}{2c}$. ◀

2.5 Our Results

We first consider the Euclidean case and provide a bound on the number of Delaunay simplices containing a given point in a random subsample of an ε-net (Theorem 14). Although the bound for the Euclidean case given will be generalized in Theorem 15 and Theorem 16, we state it and prove it separately since its proof is simpler and the constants are better.

▶ **Theorem 14** (Euclidean metric)**.** *Given an ε-net $\mathcal{X}$ in $(\mathbb{T}^d, \|.\|)$, where $\varepsilon \in (0, \frac{1}{4}]$, the expected number of simplices incident to a point $p \in \mathcal{X}$, in the d-skeleton of the Delaunay complex of a uniform random sample $S \subset \mathcal{X}$ of size $s \geq (2\sqrt{d})^d d^3 + 1$ containing p, is less than $2^{d(4d+1)+2}$.*

Theorem 14 will be proved in Section 3. Next, we generalize Theorem 14 to hold for growth-restricted metrics.

▶ **Theorem 15.** *Given $\varepsilon \in (0, 1]$, a metric distance function $d(.,.)$ over $\mathbb{T}^d$, and an ε-net $\mathcal{X}$, such that $(\mathbb{T}^d, d(.,.), \mu)$ is a growth-restricted measure space, having expansion constant at most 2^{dim}, it holds that the expected number of d-simplices incident to a point $p \in \mathcal{X}$, in the Delaunay complex of a uniform random sample $S \subset \mathcal{X}$ of size $s \geq 4d$, is at most*

$$\mathbb{E}\left[\sharp\left(star(p)\right)\right] \leq 2^{2d \cdot dim + 3d} + \left(\frac{2^{d+2+3(d \cdot dim)}}{d!}\right) \cdot \left[\sum_{k=1}^{\infty} g(k)^d \cdot e^{-g(k)}\right],$$

where $g(k) = q \cdot \frac{\mu(p, 2^k \delta)}{2^{2dim+1}}$, with $q := \frac{s-1}{n-1}$, and δ as the least y such that $q \cdot \mu(p, 2y) \geq 2^{2dim+1} d$.

The following theorem is a corollary of Theorem 15.

▶ **Theorem 16.** *For a strongly growth-restricted metric space $(\mathcal{M}, \mathcal{X})$, if $\mathcal{M}$ has a compact domain U then the expected number of simplices in the star of $p \in \mathcal{X}$ is at most*

$$\mathbb{E}\left[\sharp\left(star(p)\right)\right] \leq 2^{2ddim+3d} + 2^{d+3+3(d \cdot dim)}.$$

Our most general result, on growth-restricted metrics over $\mathbb{T}^d$, follows as a simple consequence of Theorem 16 and Lemma 9.

▶ **Corollary 17.** *Given $\varepsilon \in (0, 1]$, a metric distance function $d(.,.)$ over $\mathbb{T}^d$, and an ε-net $\mathcal{X}$, such that $(\mathbb{T}^d, d(.,.), \mu)$ is a growth-restricted measure space, having expansion constant at most 2^{dim}, then*

$$\mathbb{E}\left[\sharp\left(star(p)\right)\right] \leq 2^{2ddim+3d} + 2^{d+3+3(d \cdot dim)}.$$

We next use the above bounds to get the space and time complexity of the randomized incremental construction of the d-skeleton of the Delaunay complex of an ε-net:

▶ **Theorem 18** (Randomized incremental construction)**.** *Let $\mathcal{X}$ be an ε-net over a strongly growth-restricted metric space $(\mathbb{T}^d, d(.,.), \mu)$, where $\varepsilon \in (0, 1]$. If the faces of the Voronoi diagram of $\mathcal{X}$ satisfy the closed-ball property, then the randomized incremental construction of the d-skeleton of the Delaunay complex needs $O(n \log n)$ expected time and $O(n)$ expected space, where $n = \sharp(\mathcal{X})$ and d is considered as a constant in the big O.*

Theorem 18 will be proved in Section 5.

▶ Remark. 1. Theorem 14 and Theorem 16 also work for the case when the random sample is a Bernoulli sample of parameter q. Observe that in this case $\mathbb{P}\left[E^{(\tau)}\right]$ is just the probability that the d points of τ are chosen in $\mathcal{Y}$, and the points inside $B(c_\sigma, r_\sigma)$ are not chosen in $\mathcal{Y}$. Again, the number of points in $B(c_\sigma, r_\sigma) \cap \mathcal{X}$ is at least $(r_\sigma/\varepsilon)^d$. Therefore, $\mathbb{P}\left[E^{(\tau)}\right]$ is simply $q^d (1-q)^{(r_\sigma/\varepsilon)^d} < q^d e^{-q(r_\sigma/\varepsilon)^d}$, i.e. less than the bound in inequality (6). The rest of the proof follows as before.

2. Our results can be extended to other types of good samples, e.g. the weaker notion of (ε, κ)-*samples* for which any ball of radius ε contains at least one point and at most κ points. If we fix $\kappa = \kappa_0 = 2^{O(d)}$, we get exactly the same result. The bounds can be straightforwardly adapted to accomodate other values of κ.

3 The Euclidean case (Proof of Theorem 14)

In this section, we prove that a subsample of a given size s, drawn randomly from an ε-net $\mathcal{X} \subset \mathbb{T}^d$, has a Delaunay complex with a d-skeleton of linear complexity, with a constant of proportionality bounded by 2^{cd^2}, where c is a constant independent of ε and d.

We shall focus on computing the expected number of d-dimensional simplices. (The expected number of i-dimensional simplices can be computed analogously for each i). Let n denote $\sharp(\mathcal{X})$; from the volume argument in the proof of Lemma 5 we have upper and lower bounds on n at Equation (1). Let the uniform random sample of size s denoted by $\mathcal{Y}$. Also, let us fix a point $p \in \mathcal{Y}$; we shall upper bound the number of d-simplices incident in the Delaunay complex $Del(\mathcal{Y})$ incident to p, that is $star^{(d)}_{Del(\mathcal{Y})}(p)$, or $star(p)$ in short. Consider a d-tuple of points in $\mathcal{X}$: $\tau \in \mathcal{X}^d$, such that the d-simplex formed by the points in $\sigma := \tau \cup \{p\}$, whose circumcenter and circumradius are denoted c_σ and r_σ. Then, given that $p \in \mathcal{Y}$, the event $E^{(\tau)} := \sigma \in Del(\mathcal{Y})$ could occur only if the following events occur

(i) $E_1^{(\sigma)} := \forall p' \in \tau,\ \ p' \in \mathcal{Y}$, and

(ii) $E_2^{(\sigma)} := \mathrm{int}(B(c_\sigma, r_\sigma)) \cap \mathcal{Y} = \emptyset$.

Observe that once the points of σ are fixed, the points in $\mathrm{int}(B(c_\sigma, r_\sigma)) \cap \mathcal{X}$ are uniquely determined. Given that $p \in \mathcal{Y}$, the distribution of $\mathcal{Y} \setminus \{p\}$ is now that of a uniformly random sample of size $s-1$ from $\mathcal{X} \setminus \{p\}$. The event $E^{(\tau)}$ now fits the setting of Lemma 13, with the universe $C = \mathcal{X} \setminus \{p\}$, the random sample $A = \mathcal{Y} \setminus \{p\}$, the set of points required to be contained in the sample A as $B = \tau$, and the disjoint set of points required to be not contained in the sample, $T = B(c_\sigma, r_\sigma) \cap \mathcal{X}$. Since $\mathcal{X}$ is an ε-net, we have an ε-covering of $B(c_\sigma, r_\sigma)$. Therefore from Lemma 4, we have that $t = \sharp(T) = \sharp(B(c_\sigma, r_\sigma) \cap \mathcal{X}) \geq \left(\frac{r_\sigma}{\varepsilon}\right)^d$. We can also assume that $c = n-1$, $a = s-1$, and $b = d$ satisfy the conditions of Lemma 13, that (i) $b \leq \min(\frac{a}{2}, \sqrt{c})$ since otherwise $s \leq \max(2d+1, d^2+1)$, and so the worst-case complexity is $s^{d+1} \leq 2d^{2(d+1)}$, which is a constant; and (ii) $b \leq a$, since $s \geq (2\sqrt{d})d^3 + 1$. Let $q := \frac{s-1}{n-1}$ and $\delta := \varepsilon \cdot \left(\frac{2d}{q}\right)^{1/d}$. Therefore, applying Lemma 13 with $t \geq (r_\sigma/\varepsilon)^d$, the probability that $\sigma \in Del(\mathcal{Y})$ given $p \in \mathcal{Y}$ can be upper-bounded by:

$$\begin{aligned}\mathbb{P}\left[E^{(\tau)} | p \in \mathcal{Y}\right] = \mathbb{P}\left[E_1^{(\sigma)} \wedge E_2^{(\sigma)}\right] \quad &\leq \quad (1 + \tfrac{2d^2}{n-1}) \times q^d \times \exp(-\frac{q}{2}(r_\sigma/\varepsilon)^d) \\ &\leq \quad 3 \times q^d \times \exp(-\frac{q}{2}(r_\sigma/\varepsilon)^d).\end{aligned}$$

The first inequality follows by applying Lemma 13 with $c = n-1,\ a = s-1,\ b = d$, and $t \geq (r_\sigma/\varepsilon)^d$. Then we use the fact that we are working in the range $b \leq \sqrt{c}$, i.e. $\frac{2b^2}{c} = \frac{2d^2}{n-1} \leq 2$.

Let $I_0 := [0, \delta)$, $I_k := [2^{k-1}\delta, 2^k\delta)$ for $k \in \mathbb{N}$. By the triangle inequality, if $\sigma \in Del(\mathcal{Y})$ has a circumradius r_σ, then all the points in σ must lie in the ball $B(p, 2r_\sigma)$. This ball is not self intersecting in $\mathbb{T}^d$ if $r_\sigma \leq \frac{1}{4}$, which allows to relate the number of points inside to its volume. Therefore by Lemma 3, the number of potential d-tuples which can contribute to $star(p)$ is at most $\binom{(3\cdot 2r_\sigma/\varepsilon)^d}{d} \leq \frac{(6r_\sigma/\varepsilon)^{d^2}}{d!}$.

Let

$$Z_p(k) := \sharp\left(\{\sigma \in Del(\mathcal{Y}) \ \ : \ \ p \in \sigma,\ r_\sigma \in I_k\}\right), p \in \mathcal{Y}.$$

denote the number of Delaunay simplices incident to $p \in \mathcal{Y}$ and which have circumradius $r_\sigma \in I_k$.

Bound on $Z_p(0)$

Firstly, consider the range $r_\sigma \in I_0$. If $q \in \sigma \in star(p)$ such that $r_\sigma \in I_0$, then by the triangle inequality, q lies in the ball $B(p, 2\delta) \cap \mathcal{X}$. By Lemma 3, the expected number of points in this ball is at most

$$\begin{aligned}\mathbb{E}\left[\sharp\left(\mathrm{int}(B(p,2\delta)) \cap \mathcal{Y}\right)\right] &\leq (6\delta/\varepsilon)^d q \\ &= 6^d \left(\frac{(2d/q)^{1/d}\cdot\varepsilon}{\varepsilon}\right)^d q \\ &= 2d \cdot 6^d \leq 2^{3d+1}\cdot d.\end{aligned}$$

By Lemma 12, we have that

$$\mathbb{E}\left[\sharp\left(\mathrm{int}(B(p,2\delta)) \cap \mathcal{Y}\right)^d\right] \leq 3\,\mathbb{E}\left[\sharp\left(\mathrm{int}(B(p,2\delta)) \cap \mathcal{Y}\right)\right]^d \leq 2^{3d^2+d}\cdot d^d.$$

For $k = 0$, we have $\mathbb{E}\left[Z_p(0)\right] \leq \frac{\mathbb{E}[\sharp(\mathrm{int}(B(p,2\delta))\cap\mathcal{Y})]^d}{d!} \leq 3 \cdot 2^{3d^2+d}\cdot e^{d-1} < 2^{3d^2+3d}$, where in the penultimate inequality we used Stirling's approximation, $d! \geq e(d/e)^d$.

Bound on $Z_p(k), k \geq 1$

For $k \geq 1$, to apply the above bound on the number of potential simplices yielding a sphere of radius in I_k, we need to limit $r_\sigma \leq \frac{1}{4}$. In this case we have

$$\begin{aligned}\mathbb{E}\left[Z_p(k)\right] \leq \sum_{\tau \in \binom{\mathcal{X}\cap \mathrm{int}(B(p,2^k\delta))}{d}} \mathbb{P}\left[E^{(\tau)}\right] &\leq \frac{(6\cdot 2^{k+1}\delta/\varepsilon)^{d^2}}{d!}\cdot 3\cdot q^d e^{-\frac{q}{2}(2^{k-1}\delta/\varepsilon)^d} \\ &\leq 3\cdot 2^{3d^2}\frac{(2^k\delta/\varepsilon)^{d^2}}{d!}\cdot q^d\cdot e^{-\frac{q}{2}(2^{k-1}\delta/\varepsilon)^d}.\end{aligned}$$

Bound on $k_{\max}$

Let $I_{k_{\max}} = [2^{k_{\max}}\delta, \infty) \supset [1/4, \infty)$, so that $k_{\max} := \lfloor \log(1/4\delta)\rfloor$. From Equation (1), we have that $n = \sharp(\mathcal{X}) \leq 2^{-d}d^{d/2}\varepsilon^{-d}$. Therefore, by Lemma 4, any ball B of radius at least $2^{k_{\max}}\delta \geq 1/8$, has at least $(2^{k_{\max}}\delta/\varepsilon)^d = (1/4\varepsilon)^d \geq \left(\frac{n}{2^d d^{d/2}}\right)$ points in its interior, i.e.

$$\sharp\left(\mathrm{int}B \cap \mathcal{X}\right) \geq \left(\frac{n}{(2\sqrt{d})^d}\right).$$

The maximum number of d-tuples which can possibly form a Delaunay d-simplex with p, is at most $\binom{n-1}{d} \leq (n-1)^d/d!$. Each of these simplices yields less than 3^d possible Delaunay sphere in the $\mathbb{T}^d$. Therefore, the expected number of simplices having radius at least $2^{k_{\max}}\delta$, is at most

$$\begin{aligned}\mathbb{E}\left[Z_p(k_{\max})\right] &= 3^d\frac{(n-1)^d}{d!}\cdot\mathbb{P}\left[E^{(\tau)}|p\in\mathcal{Y}\right] \\ &= 3^d\frac{(n-1)^d}{d!}\cdot q^d\cdot\exp\left(-\frac{q}{2}\cdot\frac{n}{(2\sqrt{d})^d}\right) \\ &\leq 3^d\frac{(s-1)^d}{d!}\cdot\exp\left(-\frac{s-1}{2(2\sqrt{d})^d}\right). \qquad (6)\end{aligned}$$

For $s > s_0 = 2(2\sqrt{d})^d\cdot d^3 + 1$ this function is decreasing in term of s and it is easy to check that the value in s_0 is smaller than 4. Thus we have $\mathbb{E}\left[Z_p(k_{\max})\right] \leq 4$.

Therefore for $s \geq s_0$, we only need to sum k upto $k_{\max}$.

Summing $\mathbb{E}[Z_p(k)]$

$$
\begin{aligned}
\sum_{k=1}^{k_{\max}-1} \mathbb{E}[Z_p(k)] &\leq \sum_{k=1}^{\infty} \frac{2^{3d^2}}{d!} e^{-\frac{q}{2}(2^{k-1}\delta/\varepsilon)^d} \cdot 3 \cdot ((2^k\delta/\varepsilon)^d q)^d \\
&\leq \sum_{k=1}^{\infty} \frac{2^{3d^2}}{d!} e^{-g(k)} \cdot 3 \cdot (g(k))^d \cdot 2^{d^2+d} \\
&\leq 3 \cdot \frac{2^{4d^2+d}}{d!} \sum_{k=1}^{\infty} e^{-g(k)} \cdot (g(k))^d,
\end{aligned}
$$

where $g(k) := (2^{k-1}\delta/\varepsilon)^d q/2$. Observe that by the definition of δ, we have that $g(1) = (\delta/\varepsilon)^d q/2 = d$. Further, for all $k \in \mathbb{Z}^+$, $g(k+1) = 2^d \times g(k)$, i.e. $g(k)$ is a strictly increasing function of k. Therefore, for all $k > 1$, $g(k) > d$. Since the function $f(x) = x^d e^{-x}$ is maximised at $x = d$, and is monotone decreasing for $x > d$, the summation $\sum_{k=1}^{\infty}(g(k))^d e^{-g(k)}$ can be upper bounded by the integral $\int_{x=0}^{\infty} x^d e^{-x} dx$, by substituting $x = g(k)$. Define $Z_p := \sum_{k=1}^{\infty} Z_p(k)$ to be the number of Delaunay simplices in $star(p)$. Thus we get

$$
\begin{aligned}
\mathbb{E}[Z_p] &\leq Z_p(0) + Z_p(k_{\max}) + 3 \cdot \frac{2^{4d^2+d}}{d!} \int_{x=0}^{\infty} e^{-x} \cdot x^d dx \\
&\leq 2^{3d^2+3d+2} + 4 + 3 \cdot \frac{2^{4d^2+d}}{d!} \Gamma(d+1) \qquad (7) \\
&= 2^{3d^2+3d+2} + 4 + 3 \cdot 2^{4d^2+d} \leq 2^{4d^2+d+2},
\end{aligned}
$$

where in inequality (7) we used the identity $\Gamma(d+1) = d!$. This completes the proof of Theorem 14. Therefore, the expected size of $Del(\mathcal{Y})$ is given by

$$
\begin{aligned}
\mathbb{E}[\sharp(Del(\mathcal{Y}))] &\leq \sum_{p\in\mathcal{X}} \mathbb{P}[p \in \mathcal{Y}]\, \mathbb{E}[Z_p | p \in \mathcal{Y}] \\
&\leq \sum_{p\in\mathcal{X}} \left(\frac{s}{n}\right) \cdot 2^{(4d+1)d+2} \leq s \cdot 2^{4d^2+d+2}. \qquad (8)
\end{aligned}
$$

4 Growth-Restricted Measures

In this section we generalise Theorem 14 to growth-restricted metrics, and prove Theorem 15.

Proof of Theorem 16. By the definition of δ in the statement of Theorem 15, $g(1) = d$. Therefore, $g(k) \geq d$ for all $k \in [1, k_{max}]$. Also, by the condition (2), for $k \in [1, k_{max}]$, we have

$$
g(k+1) = \frac{q\mu(p, 2^{k+1}\delta)}{2^{2dim+1}} \geq (1+\eta) \cdot \left(\frac{q\mu(p, 2^k\delta)}{2^{2dim+1}}\right) = (1+\eta) \cdot g(k).
$$

Since $g(k) \geq d$ inductively, we get that $g(k)$ is a strictly increasing function of k. Substituting $x = g(k)$, we get $x \in [d, \infty)$ for $k \geq 1$. Since $x^d e^{-x}$ is decreasing for all $x \in [d, \infty)$, therefore the sum $\sum_{k=1}^{\infty}(g(k))^d e^{-g(k)}$ can be upper bounded by the integral $\int_{x=0}^{\infty} x^d e^{-x} dx = \Gamma(d+1) = d!$. Using Stirling's approximation now gives the theorem. ◀

Proof of Theorem 15. The proof proceeds in similar fashion to that of Theorem 14. Consider a simplex $\sigma \in \binom{\mathcal{X}}{d+1}$, such that $p \in \sigma$, $\sigma \setminus \{p\} = \tau$, having circumcentre c_σ and circumradius r_σ.

As in the proof of Theorem 14, we can again assume that $c = n-1$, $a = s-1$, and $b = d$ satisfy the conditions of Lemma 13, that (i) $b \leq \min(\frac{a}{2}, \sqrt{c})$ and (ii) $b \leq a$, since $s \geq 4d+1$. Again applying Lemma 13, we have

$$\mathbb{P}\left[\sigma \in Del_d(S)\right] \leq \left(1 + \frac{2d^2}{n-1}\right) \cdot q^d e^{-q\cdot\mu(c_\sigma, r_\sigma)/2} \leq 3 \cdot q^d e^{-q\cdot\mu(c_\sigma,r_\sigma)/2}. \quad (9)$$

Let $Z_p(0)$ denote the number of Delaunay simplices incident to p in the uniformly random sample $S \setminus \{p\}$, with circumradius $r_\sigma \in [0, \delta)$. For $k \in \mathbb{Z}^+$, let $Z_p(k)$ denote the number of Delaunay simplices incident to p in $S \setminus \{p\}$ with circumradius $r_\sigma \in [2^{k-1}\delta, 2^k\delta)$, i.e.

$$Z_p(k) = \sharp\left(\{\sigma \in star(p) : r_\sigma \in [2^{k-1}\delta, 2^k\delta)\}\right).$$

Let $p' \in \sigma \in star(p)$, then $p' \in B(c_\sigma, r_\sigma)$. Applying Lemma 8, we get that $B(c_\sigma, r_\sigma) \subset B(p, 2r_\sigma) \subset B(c_\sigma, 4r_\sigma)$. Therefore we get

$$\begin{aligned}
\mathbb{E}\left[Z_p(0)\right] &\leq \frac{\mathbb{E}\left[\sharp\left(B(p,2\delta)\cap\mathcal{Y}\right)^d\right]}{d!} \\
&\leq 3\frac{\mathbb{E}\left[\sharp\left(B(p,2\delta)\cap\mathcal{Y}\right)\right]^d}{d!} && (10)\\
&\leq 3\frac{(q\mu(p,2\delta))^d}{d!} \leq 3\frac{(2^{2dim+1}d)^d}{d!} && (11)\\
&\leq 3\cdot(2^{2d\cdot dim+d})\frac{d^d}{e(d/e)^d} \leq 2^{2d\cdot dim+d+2d} \leq 2^{2d\cdot dim+3d}.
\end{aligned}$$

where the inequality (10) was by applying Lemma 12, inequality (11) followed from the definition of δ, and the last line followed from the defintion of δ and Stirling's approximation. Next, to bound $\mathbb{E}\left[Z_p(k)\right]$ for non-zero values of k, by the definition of expansion dimension, we get

$$\mu(c_\sigma, r_\sigma) \geq 2^{-2dim}\cdot\mu(c_\sigma, 4r_\sigma) \geq 2^{-2dim}\cdot\mu(p, 2r_\sigma), \quad (12)$$

where the second inequality is from containment. Thus the expected value of $Z_p(k)$, $k \geq 1$ is bounded by

$$\begin{aligned}
\mathbb{E}\left[Z_p(k)\right] &\leq \sum_{\sigma\in\binom{\text{int}B(p,2^{k+1}\delta)\cap\mathcal{X}}{d+1};\ p\in\sigma} \mathbb{P}\left[\sigma\in Del_d(S)\right] \\
&\leq \sum_{\sigma\in\binom{\text{int}B(p,2^{k+1}\delta)\cap\mathcal{X}}{d+1};\ p\in\sigma} 3\cdot q^d e^{-q\cdot\mu(c_\sigma,r_\sigma)/2} && (13)\\
&\leq \sum_{\sigma\in\binom{\text{int}B(p,2^{k+1}\delta)\cap\mathcal{X}}{d+1};\ p\in\sigma} 3\cdot q^d e^{-q\cdot\mu(p,2^k\delta)/(2\cdot 2^{2dim})} && (14)\\
&\leq 3\cdot\left(\frac{\mu(p,2^{k+1}\delta)^d}{d!}\right)\cdot q^d e^{-q\cdot\mu(p,2^k\delta)/(2\cdot 2^{2dim})},
\end{aligned}$$

where line (13) follows from (9), and line (14) follows from (12).

Bounding $\mu(p, 2^{k+1}\delta)$ from above by $2^{dim}\cdot\mu(p,2^k\delta)$, and using the definition of $g(k) = \frac{q\mu(p,2^k\delta)}{2^{2dim+1}}$, we get for $k \geq 1$

$$\mathbb{E}\left[Z_p(k)\right] \leq 3\cdot\frac{2^{d+3d\cdot dim}}{d!}(g(k))^d e^{-g(k)} \leq \left(\frac{2^{2+d+3d\cdot dim}}{d!}\right)(g(k))^d e^{-g(k)} \quad (15)$$

Therefore, the number of simplices in $star(p)$ is given by

$$Z(p) := \sharp\left(star(p)\right) = \sum_{k=0}^{\infty} Z_p(k).$$

Taking expectations, we get

$$\begin{aligned}\mathbb{E}\left[Z(p)\right] &\leq \sum_{k=0}^{\infty} \mathbb{E}\left[Z_p(k)\right], \\ &\leq 2^{2d\cdot dim+3d} + \left(\frac{2^{3d\cdot dim+d+2}}{d!}\right)\sum_{k=1}^{\infty}(g(k))^d \cdot e^{-g(k)},\end{aligned}$$

which completes the proof of Theorem 15.

◀

5 Randomized Incremental Construction (Proof of Theorem 18)

In this section we show how the results in Section 3 imply bounds on the algorithmic complexity of d-skeleton of the Delaunay complex of ε-nets. We state a general version of a theorem for the complexity, in terms of time and space requirements, of the randomized incremental construction of the d-skeleton of the Delaunay complex of a given point set in $\mathbb{R}^d$.

▶ **Theorem 19** (Boissonnat-Yvinec [3], Devillers [8]). *Let $F(s)$ denote the expected number of simplices that appear in the d-skeleton of the Delaunay complex of a uniform random sample of size s, from a given point set P. Then*

(i) [3], Theorem 5.2.3(1):
The expected number of simplices that appear in the d-skeleton of the Delaunay complex of a point set $P \in \mathbb{R}^d$ during the randomized incremental construction is

$$O\left(\sum_{s=1}^{n} \frac{F(s)}{s}\right),$$

(ii) [8], Theorem 5(1):
If $F(s) = O(s)$, the expected space complexity is $O(n)$.

(iii) [8], Theorem 5(2):
If $F(s) = O(s)$, then the expected time complexity is $\sum_{s=1}^{n} \frac{n-s}{s} = O(n \log n)$.

We can now prove Theorem 18.

Proof of Theorem 18. We first observe that if the faces of the Voronoi diagram of $\mathcal{X}$ satisfy the closed-ball property, then the usual randomized incremental algorithm is correct. We analyze now its complexity. If $\epsilon > \frac{1}{3\sqrt{d}}$ then, by Equation (1), $s = O(1)$ and the number of Delaunay simplices is also bounded by a constant (for a fixed d).

Otherwise, the expected space complexity follows from Theorem 14 and Theorem 16, by applying Theorem 19(i): the expected number of simplices that appear at any time in the duration of the algorithm is $\sum_{s\geq 1} \frac{F(s)}{s} \leq \sum_{s\geq 1}^{(2\sqrt{d})^d d^3} \frac{1}{s}(d^3(2\sqrt{d})^d)^d + \sum_{s\geq 4d}^{n} 2^{d(4d+1)+2} \leq n2^{d(4d+1)+2} = O(n)$. From Theorem 19 (ii), the expected space complexity is also bounded by $O(n)$. The time complexity also follows directly by application of Theorem 19 (iii). We thus get Theorem 18. ◀

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Appendix

Proof of Lemma 12. The j-th moment $\mathbb{E}\left[X^j\right]$ is the sum over all ordered j-tuples $l = (l_1, l_2, \ldots, l_j) \in B^j$ with $l_i \in B$, $i = 1, \ldots, j$, of the probability that a j-tuple gets chosen in the random sample A. This probability is

$$\begin{aligned}
\mathbb{P}\left[l \in A^j\right] &= \frac{\binom{c-j}{a-j}}{\binom{c}{a}} = \frac{\prod_{i=0}^{j-1}(a-i)}{\prod_{i=0}^{j-1}(c-i)} = \frac{a^j}{c^j} \times \frac{\prod_{i=0}^{j-1}(1-i/a)}{\prod_{i=0}^{j-1}(1-i/c)} \\
&\leq \frac{a^j}{c^j} \times \frac{1}{\prod_{i=0}^{j-1}\exp(\ln(1-i/c))} = \frac{a^j}{c^j} \times \exp\left(-\sum_{i=0}^{j-1}\ln(1-i/c)\right) \\
&\leq \frac{a^j}{c^j} \times \exp\left(\sum_{i=0}^{j-1}\frac{2i}{c}\right) \leq \frac{a^j}{c^j} \times e^{\frac{j^2}{c}} \qquad (16) \\
&\leq \frac{a^j}{c^j} \times (1 + \frac{2j^2}{c}) \leq \frac{3a^j}{c^j}. \qquad (17)
\end{aligned}$$

where in step (16) we used that $-\ln(1-x) \leq 2x$, for $x \in [0, 1/2]$, and in step (17) we used $e^x \leq 1 + 2x$, for $x \in [0, 1]$. Now since the number of such tuples A^j is no more than b^j, the expected number of chosen tuples is given by $\mathbb{E}\left[X^j\right] \leq 3 \cdot \left(\frac{ab}{c}\right)^j = 3(\mathbb{E}[X])^j$. ◀

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