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Abstract

Helton and Nie conjectured that every convex semialgebraic set over the field of real numbers can be written as the projection of a spectrahedron. Recently, Scheiderer disproved this conjecture. We show, however, that the following result, which may be thought of as a tropical analogue of this conjecture, is true: over a real closed nonarchimedean field of Puiseux series, the convex semialgebraic sets and the projections of spectrahedra have precisely the same images by the nonarchimedean valuation. The proof relies on game theory methods.

Keywords: Convex algebraic geometry; spectrahedra; nonarchimedean fields; tropical geometry; semidefinite programming

1. Introduction

Convex semialgebraic sets appear in various guises in computational optimization (Blekherman et al., 2013). They include spectrahedra, i.e., feasible sets of semidefinite programs (SDPs). A long-standing problem is to characterize the convex semialgebraic sets that are SDP representable, meaning that they can be represented as the image of a spectrahedron by a (linear) projector. The notion of SDP representability originates from the monograph of Nesterov and Nemirovskii (1994). Nemirovski (2007) asked whether every convex semialgebraic set is SDP representable. Helton and Nie conjectured that the answer is positive.

Conjecture 1 (Helton and Nie, 2009)**.** *Every convex semialgebraic set in $\mathbb{R}^n$ is a projection of a spectrahedron.*

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Several classes of convex semialgebraic sets for which the answer is positive have been identified (Helton and Nie, 2009; Helton and Vinnikov, 2007; Helton and Nie, 2010; Lasserre, 2009; Gouveia et al., 2010; Gouveia and Netzer, 2011; Nie et al., 2008). In particular, it is known that the conjecture is true in dimension 2 (Scheiderer, 2018a). The conjecture has been recently disproved by Scheiderer (2018b), who showed that the cone of positive semidefinite forms cannot be expressed as a projection of spectrahedra, except in some particular cases. A comprehensive list of references can be found in this work.

Theorem 2 (Scheiderer, 2018b)**.** *The cone of positive semidefinite forms of degree $2d$ in n variables can be expressed as a projection of a spectrahedron only when $2d = 2$ or $n \leqslant 2$ or $(n, 2d) = (3, 4)$.*

The notion of convex and semialgebraic sets make sense over any real closed field, in particular over the nonarchimedean field $\mathbb{K}$ of real Puiseux series, equipped with the total order induced by its nonnegative cone $\mathbb{K}_{\geqslant 0}$, consisting of series with a nonnegative leading coefficient. Our main result shows that the next statement, which may be thought of as a "Helton–Nie conjecture for valuations," is valid.

Theorem 3. *The image by the valuation of every convex semialgebraic subset of $\mathbb{K}^n$ coincides with the image by the valuation of a projected spectrahedron over $\mathbb{K}$.*

Our approach relies on tropical methods. Tropical semialgebraic sets can be defined as the images by the nonarchimedean valuation of semialgebraic sets over $\mathbb{K}$. The quantifier elimination techniques in real valued fields developed by Pas (1989), building on work of Denef (1986), imply that tropical semialgebraic sets are semilinear. Moreover, the image by the nonarchimedean valuation of a convex set over $\mathbb{K}$ is a tropical convex set, i.e., a set stable by taking tropical convex combinations. In a previous work (Allamigeon et al., 2016) we studied tropical spectrahedra, defined as the images by the nonarchimedean valuation of spectrahedra over $\mathbb{K}$, and gave a combinatorial characterization of generic tropical spectrahedra.

The proof relies on the recently developed relations between tropical convex programming and zero-sum games (Akian et al., 2012; Allamigeon et al., 2018b). In particular, in the latter reference, we demonstrated a class of generic tropical spectrahedra that corresponds precisely to the sets of subharmonic vectors (subfixed points) of a class of nonlinear Markov operators (Shapley operators of stochastic mean payoff games). In that way, one obtains an explicit construction for these tropical spectrahedra.

The tropical perspective proved to be useful to find counterexamples to classical conjectures in real algebraic geometry. For instance, Itenberg and Viro (1996) disproved the Ragsdale conjecture as an application of the tropical patchworking method. More recently, Allamigeon et al. (2018a) contradicted, by a tropical method, the continuous analogue of the Hirsch conjecture proposed by Deza et al. (2008). The validity of the tropical analogue of the Helton–Nie conjecture raises the question whether a counterexample could be found by a tropical approach, for instance, by studying images of convex semialgebraic sets and spectrahedra through a map carrying more information than the valuation.

We finally note that semilinear sets that are tropically convex have been studied recently by Bodirsky and Mamino (2016) from a different perspective, motivated by a class of satisfiability problems. They showed in particular that feasibility and infeasibility certificates for these problems can be obtained from stochastic games. The tropical convex sets they consider differ from ours in two respects: the $-\infty$ element is not allowed in their approach, whereas it appears as the image of the zero element by the nonarchimedean valuation; moreover, the tropicalizations of convex semialgebraic sets are always closed, and so, definable by weak inequalities, whereas systems including both strict and weak inequalities are considered in (Bodirsky and Mamino, 2016).

2. Preliminaries

2.1. Puiseux series and tropical algebra

A *(generalized formal real) Puiseux series* is a formal series of the form

$$\boldsymbol{x} = \sum_{i=1}^{\infty} c_{\lambda_i} t^{\lambda_i} , \tag{1}$$

where t is a formal parameter, $(\lambda_i)_{i \geqslant 1}$ is a strictly decreasing sequence of real numbers that is either finite or unbounded, and $c_{\lambda_i} \in \mathbb{R} \setminus \{0\}$ for all λ_i. There is also a special, empty series, which is denoted by 0. The Puiseux series can be added and multiplied in the natural way. They form a real closed field (Markwig, 2010), which we denote here by $\mathbb{K}$. If $\boldsymbol{x}$ is a Puiseux series as in (1), then by $\mathsf{lc}(\boldsymbol{x})$ we denote its *leading coefficient*, $\mathsf{lc}(\boldsymbol{x}) = c_1$ (with the convention that $\mathsf{lc}(0) = 0$). The (unique) total order on $\mathbb{K}$ is given by the relation $\boldsymbol{x} \geqslant \boldsymbol{y} \iff \mathsf{lc}(\boldsymbol{x} - \boldsymbol{y}) \geqslant 0$.

The field of Puiseux series is equipped with a *(nonarchimedean) valuation* $\mathsf{val}\colon \mathbb{K} \to \mathbb{R} \cup \{-\infty\}$. If $\boldsymbol{x} \in \mathbb{K}$ is as in (1), then we define $\mathsf{val}(\boldsymbol{x})$ as the leading exponent of $\boldsymbol{x}$, $\mathsf{val}(\boldsymbol{x}) = \lambda_1$ (with the convention that $\mathsf{val}(0) = -\infty$). It follows from the definition that we have the relations

$$\mathsf{val}(\boldsymbol{x} + \boldsymbol{y}) \leqslant \max(\mathsf{val}(\boldsymbol{x}), \mathsf{val}(\boldsymbol{y})) \tag{2}$$

$$\mathsf{val}(\boldsymbol{x}\boldsymbol{y}) = \mathsf{val}(\boldsymbol{x}) + \mathsf{val}(\boldsymbol{y}) \tag{3}$$

$$0 \leqslant \boldsymbol{x} \leqslant \boldsymbol{y} \implies \mathsf{val}(\boldsymbol{x}) \leqslant \mathsf{val}(\boldsymbol{y}) \tag{4}$$

Furthermore, the inequality in (2) becomes an equality when the leading terms of $\boldsymbol{x}$ and $\boldsymbol{y}$ do not cancel, which is the case if $\mathsf{val}(\boldsymbol{x}) \neq \mathsf{val}(\boldsymbol{y})$ or if $\boldsymbol{x}, \boldsymbol{y} \geqslant 0$. We denote by $\mathbb{K}_{\geqslant 0}$ the set of nonnegative Puiseux series (the series which fulfill the inequality $\boldsymbol{x} \geqslant 0$).

Remark 4. We chose the specific field $\mathbb{K}$ for simplicity of exposition. Actually, a quantifier elimination argument allows one to deduce that our main results stated over $\mathbb{K}$ are also valid over other real closed nonarchimedean fields, see Remark 43.

2.2. Tropical semifield

The *tropical semifield* $\mathbb{T}$ describes the algebraic structure of $\mathbb{K}$ under the valuation map. The underlying set of $\mathbb{T}$ is defined as $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$, the tropical addition is defined as $x \oplus y = \max(x, y)$, and the tropical multiplication is defined as $x \odot y = x + y$. The structure $\mathbb{T}$ constitutes only a semifield, for the tropical addition does not have an inverse operation. We use the notation $\bigoplus_{i=1}^{n} a_i = a_1 \oplus \cdots \oplus a_n$ and $a^{\odot n} = a \odot \cdots \odot a$ (n times). We also endow $\mathbb{T}$ with the standard order $\geqslant$. The properties (2)–(4) imply that val is an order-preserving morphism of semifields from $\mathbb{K}_{\geqslant 0}$ to $\mathbb{T}$.

When dealing with semialgebraic sets, it is convenient to keep track not only of the valuations of the elements of $\mathbb{K}$, but also of their signs. To this end, we introduce the sign function $\mathsf{sign}\colon \mathbb{K} \to \{-1, 0, +1\}$ defined as $\mathsf{sign}(\boldsymbol{x}) = 1$ if $\boldsymbol{x} > 0$, $\mathsf{sign}(\boldsymbol{x}) = -1$ if $\boldsymbol{x} < 0$, and $\mathsf{sign}(0) = 0$. The set of *signed tropical numbers* is then defined as $\mathbb{T}_{\pm} := (\{+1, -1\} \times \mathbb{R}) \cup \{(0, -\infty)\}$, and the *signed valuation* is defined as $\mathsf{sval}\colon \mathbb{K} \to \mathbb{T}_\pm$, $\mathsf{sval}(\boldsymbol{x}) = (\mathsf{sign}(\boldsymbol{x}), \mathsf{val}(\boldsymbol{x}))$. The *modulus* function $|\cdot|\colon \mathbb{T}_\pm \to \mathbb{T}$ is defined as the projection which forgets the first coordinate. The *sign* function $\mathsf{sign}\colon \mathbb{T}_\pm \to \{-1, 0, 1\}$ is defined as the projection which forgets the second coordinate. The elements of the form $(1, a)$ of $\mathbb{T}_\pm$ are called *positive tropical numbers* and are denoted by $\mathbb{T}_+$. Similarly, the elements of the form $(-1, a)$ of $\mathbb{T}_\pm$ are called *negative tropical numbers* and are denoted by $\mathbb{T}_-$. By convention, we denote the positive tropical number $(1, a)$ by a, the negative tropical number $(-1, a)$ by $\ominus a$, and the element $(0, -\infty)$ by $-\infty$. Here, $\ominus$ is a formal symbol. Note that we can extend the definition of tropical multiplication to $\mathbb{T}_\pm$ using the usual rules for signs (e.g., we have $(\ominus 3) \odot 7 = \ominus 10$ and $(\ominus 3) \odot (\ominus 7) = 10$). However, we only partially extend the tropical addition to the elements of $\mathbb{T}_\pm$ which have the same sign (e.g., we have $3 \oplus 7 = 7$ and $(\ominus 3) \oplus (\ominus 7) = \ominus 7$, but $(\ominus 3) \oplus 7$ is not defined). One can extend the set $\mathbb{T}_\pm$ even further to get a semiring with a well-defined tropical addition (Akian et al., 2009), or work with hyperfields (Viro, 2010; Connes and Consani, 2011) instead of semifields, but we do not need that here. Furthermore, note that the tropical semiring $\mathbb{T}$ is isomorphic to $\mathbb{T}_+ \cup \{-\infty\}$.

A *tropical (signed) polynomial* over the variables $X_1, \dots, X_n$ is a formal expression of the form

$$P(X) = \bigoplus_{\alpha \in \Lambda} a_\alpha \odot X_1^{\odot \alpha_1} \odot \cdots \odot X_n^{\odot \alpha_n} , \tag{5}$$

where $\Lambda \subset \{0, 1, 2, \dots\}^n$, and $a_\alpha \in \mathbb{T}_\pm \setminus \{-\infty\}$ for all $\alpha \in \Lambda$. If P is given as in (5), we define P^+ (resp. P^-) as the tropical polynomial generated by the terms $|a_\alpha| \odot X_1^{\odot \alpha_1} \odot \dots \odot X_n^{\odot \alpha_n}$ where $a_\alpha \in \mathbb{T}_+$ (resp. $\mathbb{T}_-$). Note that the quantities $P^+(x)$ and $P^-(x)$ are well defined for all $x \in \mathbb{T}^n$, because the tropical polynomials P^+ and P^- have only tropically positive coefficients.

We extend the functions val and sval to vectors and matrices by applying them coordinate-wise.

2.3. Tropical convexity

In this section, we recall some basic facts about convexity in the usual and tropical sense. A set $\boldsymbol{X} \subset \mathbb{K}^n$ is called *convex* if for every $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and every $\boldsymbol{\lambda} \in \mathbb{K}$ such that $0 \leqslant \boldsymbol{\lambda} \leqslant 1$ we have $\boldsymbol{\lambda} + (1 - \boldsymbol{\lambda})\boldsymbol{y} \in \boldsymbol{X}$. Since the intersection of any number of convex sets in convex, for every set $\boldsymbol{X} \subset \mathbb{K}^n$ we can define its *convex hull* (denoted $\operatorname{conv}(\boldsymbol{X})$) as the smallest (inclusionwise) convex set that contains $\boldsymbol{X}$. This set is characterized by Carathéodory's theorem.

Theorem 5. *If $\boldsymbol{X} \subset \mathbb{K}^n$, then we have the equality*

$$\operatorname{conv}(\boldsymbol{X}) = \Big\{ \sum_{k=1}^{n+1} \boldsymbol{\lambda}_k \boldsymbol{x}_k \in \mathbb{K}^n \colon \forall k,\, \boldsymbol{x}_k \in \boldsymbol{X} \,\wedge\, \forall k,\, \boldsymbol{\lambda} \geqslant 0 \,\wedge\, \sum_{k=1}^{n+1} \boldsymbol{\lambda}_k = 1 \Big\}.$$

We refer to (Schrijver, 1987, Corollary 7.1j) for a proof of Theorem 5 that is valid over every ordered field.

Let us now move to tropical convexity, referring the reader to (Cohen et al., 2004; Develin and Sturmfels, 2004) for background. We say that a set $X \subset \mathbb{T}^n$ is *tropically convex* if for every $x, y \in X$ and every $\lambda, \mu \in \mathbb{T}$ such that $\lambda \oplus \mu = 0$ the point $(\lambda \odot x) \oplus (\mu \odot y)$ belongs to X. The latter quantity corresponds to the tropical analogue of a convex combination of x and y. Indeed, the scalars λ and μ are implicitly "nonnegative" in the tropical sense, as they are greater than or equal to the tropical zero element $-\infty$. Besides, their tropical sum equals the tropical unit 0. The intersection of any number of tropically convex sets is tropically convex. Hence, for any set

$X \subset \mathbb{T}^n$ we can define its *tropical convex hull* (denoted $\mathrm{tconv}(X)$) as the smallest (inclusionwise) tropically convex set that contains X. Alternatively, one may work with a *tropical (convex) cone* X, defined by requiring $(\lambda \odot x) \oplus (\mu \odot y) \in X$ for all $\lambda, \mu \in \mathbb{T}$. Tropical convex sets can be identified to cross sections of tropical convex cones (Cohen et al., 2004). Carathéodory's theorem is still true in the tropical setting:

Theorem 6 (Helbig (1988), Briec and Horvath (2004), Develin and Sturmfels (2004))**.** *If $X \subset \mathbb{T}^n$, then we have the equality*

$$\mathrm{tconv}(X) = \Big\{ \bigoplus_{k=1}^{n+1} (\lambda_k \odot x_k) \colon \forall k,\ x_k \in X \ \wedge \ \bigoplus_{k=1}^{n+1} \lambda_k = 0 \Big\} .$$

A relation between the convexity in $\mathbb{K}$ and the tropical convexity is shown in the next two lemmas.

Lemma 7. *If $\boldsymbol{X} \subset \mathbb{K}^n$ is a convex set, then $\mathsf{val}(\boldsymbol{X})$ is tropically convex.*

Proof. Let $x, y \in \mathsf{val}(\boldsymbol{X})$ and take any $\lambda, \mu \in \mathbb{T}$ such that $\lambda \oplus \mu = 0$. Without loss of generality, suppose that $\lambda = 0$. Take any points $\boldsymbol{x} \in \boldsymbol{X} \cap \mathsf{val}^{-1}(x)$ and $\boldsymbol{y} \in \boldsymbol{X} \cap \mathsf{val}^{-1}(y)$. Let us look at two cases. If $\mu < 0$, then for any real positive constant c, we have $1 - ct^{\mu} > 0$, and so the point $\boldsymbol{z} = (1 - ct^{\mu})\boldsymbol{x} + ct^{\mu}\boldsymbol{y}$ belongs to $\boldsymbol{X}$. We already noted that the equality holds in the inequality (2) if the leading terms do not cancel. Hence, choosing c such that $c \neq -\mathsf{lc}(\boldsymbol{x}_k)/\,\mathsf{lc}(\boldsymbol{y}_k)$ for all $k \in [n]$ satisfying $\boldsymbol{y}_k \neq 0$, we deduce that $\mathsf{val}(\boldsymbol{z}) = (\lambda \odot x) \oplus (\mu \odot y)$. If $\mu = 0$, then we take now a real constant $c \in (0, 1)$ such that for all $k \in [n]$ satisfying $\boldsymbol{y}_k \neq 0$ we have $c/(1 - c) \neq -\mathsf{lc}(\boldsymbol{x}_k)/\,\mathsf{lc}(\boldsymbol{y}_k)$. Then, the point $\boldsymbol{z} = (1 - c)\boldsymbol{x} + c\boldsymbol{y}$ belongs to $\boldsymbol{X}$ and we deduce as above that $\mathsf{val}(\boldsymbol{z}) = (\lambda \odot x) \oplus (\mu \odot y)$. ☐

The next lemma shows that a tighter relation holds for sets included in the nonnegative orthant of $\mathbb{K}$.

Lemma 8. *If $\boldsymbol{X} \subset \mathbb{K}^n_{\geqslant 0}$ is any set, then we have* $\mathsf{val}(\mathrm{conv}(\boldsymbol{X})) = \mathrm{tconv}(\mathsf{val}(\boldsymbol{X}))$.

Proof. We start by proving the inclusion $\subset$. Take a point $\boldsymbol{y} \in \mathrm{conv}(\boldsymbol{X})$. By Theorem 5, there exist $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{n+1} \geqslant 0$ and $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n+1} \in \boldsymbol{X}$ such that $\boldsymbol{y} = \boldsymbol{\lambda}_1\boldsymbol{x}_1 + \dots + \boldsymbol{\lambda}_{n+1}\boldsymbol{x}_{n+1}$. Hence, by (2) and (3) (and using the fact that $\boldsymbol{X} \subset \mathbb{K}^n_{\geqslant 0}$) we have

$$\mathsf{val}(\boldsymbol{y}) = (\mathsf{val}(\boldsymbol{\lambda}_1) \odot \mathsf{val}(\boldsymbol{x}_1)) \oplus \dots \oplus (\mathsf{val}(\boldsymbol{\lambda}_{n+1}) \odot \mathsf{val}(\boldsymbol{x}_{n+1})) .$$

Furthermore, we have $\sum_{k=1}^{n+1} \boldsymbol{\lambda}_k = 1$ and hence $\bigoplus_{k=1}^{n+1} \mathsf{val}(\boldsymbol{\lambda}_k) = 0$. Therefore, $\mathsf{val}(\boldsymbol{y}) \in \mathrm{tconv}(X)$ by Theorem 6. Conversely, take any point $y \in \mathrm{tconv}(X)$. By Theorem 6, there exist $\lambda_1, \dots, \lambda_{n+1} \in \mathbb{T}$, $\bigoplus_{k=1}^{n+1} \lambda_k = 0$ and $x_1, \dots, x_{n+1} \in X$ such that $y = (\lambda_1 \odot x_1) \oplus \dots (\lambda_{n+1} \odot x_{n+1})$. We define $\boldsymbol{\lambda}_k := t^{\lambda_k}/(\sum_{l=1}^{n+1} t^{\lambda_l})$. Observe that for all k, $\mathsf{val}(\boldsymbol{\lambda}_k) = \lambda_k$ because the term $\sum_{l=1}^{n+1} t^{\lambda_l}$ has valuation $\bigoplus_{l=1}^{n+1} \lambda_l = 0$. Moreover, we have $\boldsymbol{\lambda}_k \geqslant 0$ and $\sum_{k=1}^{n+1} \boldsymbol{\lambda}_k = 1$. Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n+1}$ be any points of $\boldsymbol{X}$ such that $\mathsf{val}(\boldsymbol{x}_i) = x_i$. Then, the point $\boldsymbol{y} = \boldsymbol{\lambda}_1\boldsymbol{x}_1 + \dots + \boldsymbol{\lambda}_{n+1}\boldsymbol{x}_{n+1}$ belongs to $\mathrm{conv}(\boldsymbol{X})$ and verifies $\mathsf{val}(\boldsymbol{y}) = y$. ☐

We also need the following lemma.

Lemma 9. *Suppose that sets $X, Y \subset \mathbb{T}^n$ are tropically convex. Then we have the equality*

$$\mathrm{tconv}(X \cup Y) = \{ (\lambda \odot x) \oplus (\mu \odot y) \in \mathbb{T}^n \colon x \in X, y \in Y, \lambda \oplus \mu = 0 \} .$$

Proof. The inclusion $\supset$ follows immediately from the definition of tropical convex hull. The other inclusion holds because the set on the right-hand side contains X and Y and is tropically convex. $\square$

2.4. Tropicalization of convex semialgebraic sets

A set $\boldsymbol{S} \subset \mathbb{K}^n$ is called *basic semialgebraic* if it is of the form

$$\{\boldsymbol{x} \in \mathbb{K}^n : \forall i = 1, \dots, p,\ \boldsymbol{P}_i(\boldsymbol{x}) > 0 \wedge \forall i = p+1, \dots, q,\ \boldsymbol{P}_i(\boldsymbol{x}) = 0\}, \tag{6}$$

where $\boldsymbol{P}_i \in \mathbb{K}[X_1, \dots, X_n]$ are polynomials. A set $\boldsymbol{S} \subset \mathbb{K}^n$ is called *semialgebraic* if it is a finite union of basic semialgebraic sets. In this section, we characterize the sets that arise as images by valuation of convex semialgebraic sets.

Lemma 10. *If $\boldsymbol{S} \subset \mathbb{K}^n$ is a semialgebraic set, then* $\mathrm{conv}(\boldsymbol{S})$ *is also semialgebraic.*

Proof. This is an immediate consequence of Theorem 5 and of the quantifier elimination in real closed fields (Marker, 2002, Theorem 3.3.15). $\square$

Let us make the following definition.

Definition 11. We say that a set $\mathcal{S} \subset \mathbb{T}^n$ is a *tropicalization of a convex semialgebraic* set if there exists a convex semialgebraic set $\boldsymbol{S} \subset \mathbb{K}^n$ such that $\mathsf{val}(\boldsymbol{S}) = \mathcal{S}$.

Given $d \geqslant 1$, the *support* of a point $y \in \mathbb{T}^d$ is defined as the set of indices $k \in [d]$ such that $y_k \neq -\infty$. Given a nonempty subset $K \subset [d]$, and a set $Y \subset \mathbb{T}^d$, we define the *stratum of Y associated with K* as the subset of $\mathbb{R}^K$ formed by the projection $(y_k)_{k \in K}$ of the points $y \in Y$ with support K. The stratum associated with the set $[d]$ is referred to as the *main stratum*.

We say that a set $\mathcal{S} \subset \mathbb{R}^d$ is a *basic semilinear* set if it is a relatively open polyhedron of the form

$$\{x \in \mathbb{R}^d : \forall i = 1, \dots, p,\ \langle A_i, x\rangle > b_i \wedge \forall i = p+1, \dots, q,\ \langle A_i, x\rangle = b_i\},$$

where the matrix $A \in \mathbb{Q}^{q \times d}$ is rational, the vector $b \in \mathbb{R}^q$ is real, and $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^d$. We say that a set is *semilinear* if it is a finite union of basic semilinear sets. Note that $\mathcal{S} \subset \mathbb{R}^d$ is a closed semilinear set if and only if it is a finite union of polyhedra of the form $Ax \geqslant b$, where the matrix $A \in \mathbb{Q}^{q \times d}$ is rational and the vector $b \in \mathbb{R}^q$ is real. The following proposition characterizes the tropicalizations of convex semialgebraic sets. This result is based on the Denef–Pas quantifier elimination in the theory on real closed valued fields.

Proposition 12. *A set $\mathcal{S} \subset \mathbb{T}^n$ is a tropicalization of a convex semialgebraic set if and only if $\mathcal{S}$ is tropically convex and every stratum of $\mathcal{S}$ is a closed semilinear set.*

Proof. The "only if" part follows from (Allamigeon et al., 2016, Theorems 4 and 10) and Lemma 7. To prove the opposite implication, suppose that $\mathcal{S}$ is tropically convex and has closed semilinear strata. Therefore, it is a finite union of sets of the form $\mathcal{W} = \{x \in \mathbb{T}^n : Ax_K \geqslant b,\ x_{[n]\setminus K} = -\infty\}$, where, for every $L \subset [n]$, x_L denotes the vector formed by the coordinates of x taken from L, the matrix $A \in \mathbb{Q}^{m \times |K|}$ is rational and the vector $b \in \mathbb{R}^m$ is real. Take any such set $\mathcal{W}$ and consider the set

$$\boldsymbol{\mathcal{W}} := \{\boldsymbol{x} \in \mathbb{K}^n_{\geqslant 0} : \forall i \in [m], \prod_{k \in K} \boldsymbol{x}_k^{A_{ik}} \geqslant t^{b_i} \wedge \boldsymbol{x}_{[n]\setminus K} = 0\}.$$

Note that $\boldsymbol{\mathcal{W}}$ is semialgebraic. Moreover, by (2)–(4) we have $\mathsf{val}(\boldsymbol{\mathcal{W}}) \subset \mathcal{W}$. Furthermore, if $x \in \mathcal{W}$, and we take $\boldsymbol{x}_k := t^{x_k}$ for all $k \in [n]$ (with the convention that $t^{-\infty} = 0$), then we have $\boldsymbol{x} \in \boldsymbol{\mathcal{W}}$. Therefore $\mathsf{val}(\boldsymbol{\mathcal{W}}) = \mathcal{W}$. Let $\boldsymbol{U}$ denote the union of all sets $\boldsymbol{\mathcal{W}}$ that arise in this way. We have $\mathsf{val}(\boldsymbol{U}) = \mathcal{S}$ and $\boldsymbol{U}$ is semialgebraic. Thus, if we take $\boldsymbol{\mathcal{S}} := \mathrm{conv}(\boldsymbol{U})$, then $\boldsymbol{\mathcal{S}}$ is convex and semialgebraic by Lemma 10. Moreover, Lemma 8 shows that $\mathsf{val}(\boldsymbol{\mathcal{S}}) = \mathcal{S}$. □

2.5. Tropical Metzler spectrahedra

Let us recall that a real symmetric matrix is positive semidefinite if it admits a Cholesky decomposition. This is equivalent to the nonnegativity of its principal minors, its smallest eigenvalue, and the associated quadratic form. All of these properties are still equivalent for symmetric matrices defined over arbitrary real closed fields, such as Puiseux series (this is a consequence of the completeness of the theory of such fields, see Marker, 2002, Corollary 3.3.16). This implies that the definition of a spectrahedron is valid over $\mathbb{K}$.

Definition 13. Suppose that $\boldsymbol{Q}^{(0)}, \dots, \boldsymbol{Q}^{(n)} \in \mathbb{K}^{m\times m}$ are symmetric matrices. Then, the *spectrahedron* associated with these matrices is defined as

$$\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} \in \mathbb{K}^n \colon \boldsymbol{Q}^{(0)} + \boldsymbol{x}_1\boldsymbol{Q}^{(1)} + \dots + \boldsymbol{x}_n\boldsymbol{Q}^{(n)} \succcurlyeq 0\}\,,$$

where the symbol $\succcurlyeq$ denotes the Loewner order on symmetric matrices. (By definition, $X \succcurlyeq Y$ if $X - Y$ is positive semidefinite.)

In our previous works (Allamigeon et al., 2016, 2018b) we introduced the notion of tropical spectrahedra and a special subclass of these objects called tropical Metzler spectrahedra. The latter have a simpler combinatorial description; moreover, any generic tropical spectrahedron can be represented by a boolean combination of tropical Metzler spectrahedra (Allamigeon et al., 2016, Sections 5.3–5.4). Tropical spectrahedra are defined as follows.

Definition 14. We say that a set $\mathcal{S} \subset \mathbb{T}^n$ is a *tropical spectrahedron* if there exists a spectrahedron $\boldsymbol{\mathcal{S}} \subset \mathbb{K}^n_{\geqslant 0}$ such that $\mathcal{S} = \mathsf{val}(\boldsymbol{\mathcal{S}})$.

A square matrix $\boldsymbol{M}$ is called a *(negated) Metzler matrix* if its off-diagonal entries are nonpositive. Similarly, a matrix $M \in \mathbb{T}_{\pm}^{m\times m}$ is called a *tropical Metzler matrix* if its off-diagonal entries belong to $\mathbb{T}_{-} \cup \{-\infty\}$. Fix a sequence of symmetric tropical Metzler matrices $Q^{(0)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m\times m}$. For every pair $(i, j) \in [m]^2$ we consider the tropical polynomial $Q_{ij}(X)$ defined as

$$Q_{ij}(X) := Q_{ij}^{(0)} \oplus (Q_{ij}^{(1)} \odot X_1) \oplus \ldots \oplus (Q_{ij}^{(n)} \odot X_n)\,.$$

Definition 15. The *tropical Metzler spectrahedron* described by $Q^{(0)}, \dots, Q^{(n)} \in \mathbb{T}_{\pm}^{m\times m}$, denoted $\mathcal{S}(Q^{(0)}|Q^{(1)}, \dots, Q^{(n)})$, is the set of all points $x \in \mathbb{T}^n$ which satisfy the following two conditions:

- $Q_{ii}^{+}(x) \geqslant Q_{ii}^{-}(x)$ for every $i \in [m]$
- $Q_{ii}^{+}(x) \odot Q_{jj}^{+}(x) \geqslant (Q_{ij}(x))^{\odot 2}$ for every $i, j \in [m]^2, i \neq j$.

Note that the function $Q_{ij}(x)$ is well-defined for all $x \in \mathbb{T}^n$ because every $Q^{(k)}$ is a tropical Metzler matrix. If the matrix $Q^{(0)}$ is equal to $-\infty$, then we say that $\mathcal{S}(-\infty|Q^{(1)}, \dots, Q^{(n)})$ is a *tropical Metzler spectrahedral cone*. We say that a tropical Metzler spectrahedron $\mathcal{S}(Q^{(0)}|Q^{(1)}, \dots, Q^{(n)})$ is *real* if it is included in $\mathbb{R}^n$.

Remark 16. The definition above differs slightly from the one of (Allamigeon et al., 2016, 2018b). Indeed, in these references it was enough to work with tropical Metzler spectrahedral cones, while the use of affine tropical spectrahedra is indispensable in the context of the Helton–Nie conjecture. The connection between the two notions is given in (Allamigeon et al., 2016, Lemma 20).

The name "tropical Metzler spectrahedron" is justified by the fact that these sets are indeed tropical spectrahedra, as shown in (Allamigeon et al., 2016, Proposition 23 and Lemma 20).

Proposition 17. *Every tropical Metzler spectrahedron is a tropical spectrahedron.*

3. The tropical analogue of the Helton–Nie conjecture

As stated in the introduction, Scheiderer (2018b) has shown that the cone of positive semidefinite forms over $\mathbb{R}$ is a counterexample to the Helton–Nie conjecture. We first note that this yields a counterexample to the analogue of this conjecture over Puiseux series.

Corollary 18 (of Scheiderer, 2018b, Corollary 4.25)**.** *The cone of positive semidefinite forms of degree $2d$ in n variables over $\mathbb{K}$ can be expressed as a projection of a spectrahedron over $\mathbb{K}$ only when $2d = 2$ or $n \leqslant 2$ or $(n, 2d) = (3, 4)$.*

Proof. Consider a real closed field $\mathscr{K}$, and integers d, m, n, and p. The statement "the cone of positive semidefinite forms of degree $2d$ in n variables over $\mathscr{K}$ is the projection of a spectrahedron in $\mathscr{K}^p$ associated with matrices of size $m \times m$" is a sentence in the language of ordered rings. Since the theory of real closed fields is complete (Marker, 2002, Corollary 3.3.16), this sentence is true over $\mathbb{R}$ if and only if it is true over $\mathscr{K}$. □

We next state the main result of this paper. We shall prove a special case of this result in Section 4, and derive the general case in Section 5.

Theorem 19. *Fix a set $\mathcal{S} \subset \mathbb{T}^n$. Then, the following conditions are equivalent:*

(a) $\mathcal{S}$ is a tropicalization of a convex semialgebraic set

(b) $\mathcal{S}$ is tropically convex and has closed semilinear strata

(c) $\mathcal{S}$ is tropically convex and every stratum of $\mathcal{S}$ is a projection of a real tropical Metzler spectrahedron

(d) $\mathcal{S}$ is a projection of a tropical Metzler spectrahedron.

We point out that Theorem 3 is a corollary of Theorem 19.

Proof of Theorem 3. Let $\boldsymbol{S} \subset \mathbb{K}^n$ be any convex semialgebraic set and let $\mathcal{S} := \mathsf{val}(\boldsymbol{S}) \subset \mathbb{T}^n$. By Theorem 19, the set $\mathcal{S} \subset \mathbb{T}^n$ is a projection of a tropical Metzler spectrahedron. In other words, there exists $n' \geqslant 0$ and a tropical Metzler spectrahedron $S \subset \mathbb{T}^{n+n'}$ such that $\pi(S) = \mathcal{S}$, where $\pi\colon \mathbb{T}^{n+n'} \to \mathbb{T}^n$ denotes the projection on the first n coordinates. By Proposition 17, there is a spectrahedron $\boldsymbol{S}' \subset \mathbb{K}_{\geqslant 0}^{n+n'}$ such that $\mathsf{val}(\boldsymbol{S}') = S$. Let $\boldsymbol{\pi}\colon \mathbb{K}^{n+n'} \to \mathbb{K}^n$ denote the projection on the first n coordinates. Then $\mathsf{val}(\boldsymbol{\pi}(\boldsymbol{S}')) = \pi(\mathsf{val}(\boldsymbol{S}')) = \pi(S) = \mathcal{S} = \mathsf{val}(\boldsymbol{S})$. □

4. Tropical Helton–Nie conjecture for real tropical cones

In this section, we show that the tropical analogue of Helton–Nie conjecture is true for real tropical cones. We say that a set $X \subset \mathbb{R}^n$ is a *real tropical cone* if for every $x, y \in X$ and every $\lambda, \mu \in \mathbb{R}$ we have $(\lambda \odot x) \oplus (\mu \odot y) \in X$. A real tropical cone is nothing but the main stratum of a tropical cone as defined in Section 2.3. Indeed, if Y is a tropical cone, then $Y \cap \mathbb{R}^n$ is a real tropical cone, whereas if X is a real tropical cone, then $X \cup \{-\infty\}$ is a tropical cone.

4.1. Preliminaries on semilinear monotone homogeneous operators

A function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is said to be *piecewise affine* if there exist full-dimensional polyhedra $\mathcal{W}^{(1)}, \dots, \mathcal{W}^{(p)} \subset \mathbb{R}^n$ satisfying $\cup_{s=1}^{p} \mathcal{W}^{(s)} = \mathbb{R}^n$ and such that the restriction of F to $\mathcal{W}^{(s)}$ is affine, i.e., $F_{|\mathcal{W}^{(s)}}(x) = A^{(s)}x + b^{(s)}$ for some matrix $A^{(s)} \in \mathbb{R}^{m\times n}$ and vector $b^{(s)} \in \mathbb{R}^m$. In particular, piecewise affine functions are continuous (since the polyhedra $\mathcal{W}^{(1)}, \dots, \mathcal{W}^{(p)}$ are closed). We shall say that the family $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_s$ is a *piecewise description* of the function F.

We recall the following minimax representation result proved by Ovchinnikov (2002), in which we denote $F(x) = (F_1(x), \dots, F_m(x))$.

Theorem 20 (Ovchinnikov, 2002)**.** *Suppose that the function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is piecewise affine, and let $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ be a piecewise description of F. Then, for every $k \in [n]$ there exists a number $M_k \geqslant 1$ and a family $\{S_{ki}\}_{i\in[M_k]}$ of subsets of $[p]$ such that for all $x \in \mathbb{R}^n$ we have*

$$F_k(x) = \min_{i\in[M_k]} \max_{s\in S_{ki}} (A_k^{(s)}x + b_k^{(s)}) \,.$$

We say that a function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is *semilinear* if its graph $\{(x, y) \in \mathbb{R}^{n\times m} \colon y = F(x)\}$ is a semilinear set. The next lemma shows that continuous semilinear functions are piecewise affine.

Lemma 21. *Suppose that the continuous function $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is semilinear. Then, it is piecewise affine. Moreover, it admits a piecewise description $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ such that the polyhedra $\mathcal{W}^{(s)}$ are semilinear, and the matrices $A^{(s)}$ are rational.*

Proof. Since F is continuous and semilinear, the graph of F is a closed semilinear set. Therefore, it is a finite union of semilinear polyhedra. Let $\{(x, y)\colon Bx + Cy \geqslant d\}$, where $B \in \mathbb{Q}^{p\times n}$, $C \in \mathbb{Q}^{p\times m}$, $d \in \mathbb{R}^p$ be one of these polyhedra. If we fix $\bar{x}$, then, by the definition of a graph, the polyhedron consisting of all y such that $Cy \geqslant d - B\bar{x}$ reduces to a point $\bar{y}$. Thus, there exists an invertible submatrix $C_I \in \mathbb{Q}^{m\times m}$ of C such that $\bar{y} = C_I^{-1}(d_I - B_I\bar{x}) = C_I^{-1}d_I - C_I^{-1}B_I\bar{x}$. In other words, the graph of F is a finite union of polyhedra of the form

$$\mathcal{W} = \{(x, y)\colon Bx + Cy \geqslant d, y = C_I^{-1}d_I - C_I^{-1}B_I x\} \,,$$

where C_I is an invertible submatrix of C. As a result, if $\pi\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ denotes the projection on the first n coordinates, and $x \in \pi(\mathcal{W})$ is any point, then we have $F(x) = C_I^{-1}d_I - C_I^{-1}B_I x$. By eliminating the polyhedra $\pi(\mathcal{W})$ that are not full dimensional, we obtain a piecewise description of F satisfying the expected requirements. □

We say that a selfmap $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is *monotone* if $F(x) \leqslant F(y)$ as soon as $x \leqslant y$, where $\leqslant$ denotes the coordinatewise partial order over $\mathbb{R}^n$. Such a function is said to be *(additively) homogeneous* if $F(\lambda + x) = \lambda + F(x)$ for all $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Here, if $z \in \mathbb{R}^n$, then $\lambda + z$ stands for the vector with entries $\lambda + z_k$.

The following observation is well known (Crandall and Tartar, 1980).

Lemma 22. *Every monotone homogeneous operator is nonexpansive in the supremum norm.*

Proof. Observe that $x \leqslant \|x-y\|+y$. Therefore, we get $F(x) \leqslant F(\|x-y\|+y) = \|x-y\|+F(y)$. $\square$

Kolokoltsov (1992) showed that every monotone homogeneous operator F has a minimax representation as a dynamic programming operator of a zero-sum game. When F is semilinear, the following result shows that we have a finite representation of the same nature.

Lemma 23. *If $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is semilinear, monotone, and homogeneous, then it can be written in the form*

$$\forall k,\ F_k(x) = \min_{i\in[M_k]} \max_{s\in S_{ki}} (A_k^{(s)}x + b_k^{(s)})\,, \tag{7}$$

where $A^{(1)},\dots,A^{(p)} \in \mathbb{Q}^{n\times n}$ is a sequence of stochastic matrices, $b^{(s)} \in \mathbb{R}^n$ for all $s \in [n]$, $M_k \geqslant 1$ for all $k \in [n]$, and S_{ki} is a subset of $[p]$ for every $k \in [n]$ and $i \in [M_k]$.

Proof. Lemma 22 shows that F is continuous. Let $(\mathcal{W}^{(s)}, A^{(s)}, b^{(s)})_{s\in[p]}$ a piecewise description of F as provided by Lemma 21. In particular, every matrix $A^{(s)}$ is rational. We want to show that it is stochastic. To this end, take any $x \in \operatorname{int}(\mathcal{W}_s)$. Let y be the sum of the columns of $A^{(s)}$. Since F is homogeneous, for any $\rho > 0$ small enough we have $F(\rho+x) = A^{(s)}x+b^{(s)}+\rho y = \rho+F(x)$. In other words, the sum of every line of $A^{(s)}$ is equal to 1. Let ϵ_k denote the kth vector of standard basis in $\mathbb{R}^n$. Since F is monotone, for $\rho > 0$ small enough we have $F(x+\rho\epsilon_k) = A^{(s)}x + b^{(s)} + \rho A^{(s)}\epsilon_k \geqslant F(x)$. In other words, the matrix $A^{(s)}$ has nonnegative entries in its kth column. Since k was arbitrary, $A^{(s)}$ is stochastic. Therefore, the claim follows from Theorem 20. $\square$

We now characterize the class of closed semilinear real tropical cones. To this end, we use the model-theoretic definition of semilinear sets. Let $\mathcal{L}_{\mathrm{og}} := (0, +, \leqslant)$ denote the language of ordered groups. Then, the elimination of quantifiers in divisible ordered abelian groups (Marker, 2002, Theorem 3.1.17), shows that a set $\mathcal{S} \subset \mathbb{R}^n$ is semilinear if and only if there exists a number $m \geqslant 0$, an $\mathcal{L}_{\mathrm{og}}$-formula $\psi(x_1,\dots,x_{n+m})$, and a vector $\bar{b} \in \mathbb{R}^m$ such that

$$\mathcal{S} = \{x \in \mathbb{R}^n \colon \psi(x_1,\dots,x_n,\bar{b}) \text{ is true in } \mathbb{R}\}\,.$$

Proposition 24. *A set $\mathcal{S} \subset \mathbb{R}^n$ is a closed semilinear real tropical cone if and only if there exists a semilinear monotone homogeneous operator $F\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{S} = \{x \in \mathbb{R}^n \colon x \leqslant F(x)\}$.*

Proof. To prove the first implication, we consider two cases. If $\mathcal{S}$ is empty, then we take $F(x) = x - (1,\dots,1)$. Otherwise, we define F by $F_k(x) := \sup\{y_k \colon y \in \mathcal{S}\,,\ y \leqslant x\}$ for all $k \in [n]$. We claim that every supremum is attained. Indeed, the set $\{y \in \mathcal{S} \colon y \leqslant x\}$ is nonempty (take an arbitrary $z \in \mathcal{S}$, and consider $\lambda + z$ for $\lambda \in \mathbb{R}$ small enough), closed, and bounded by x. Let $y^{(k)} \in \mathcal{S}$ attaining the maximum in $F_k(x)$. Then the point $y^{(1)} \oplus \dots \oplus y^{(n)}$ is an element of $\mathcal{S}$ smaller than or equal to x. We deduce that it coincides with $F(x)$. Subsequently, $F(x)$ belongs to $\mathcal{S}$.

The operator F is semilinear because the supremum is definable in the language $\mathcal{L}_{\mathrm{og}}$, and $\mathcal{S}$ is semilinear. Besides, F is obviously monotone. It is also homogeneous because if $y \in \mathcal{S}$, then $\lambda + y \in \mathcal{S}$ for all $\lambda \in \mathbb{R}$. Finally, the inclusion $\mathcal{S} \subset \{x \in \mathbb{R}^n \colon x \leqslant F(x)\}$ is straightforward, while the inverse inclusion follows from the fact that if $x \leqslant F(x)$, then $x = F(x)$.

Conversely, fix a semilinear monotone homogeneous operator F and take the set $\mathcal{S} = \{x \in \mathbb{R}^n \colon x \leqslant F(x)\}$. This set is semilinear. Moreover, $\mathcal{S}$ is closed because F is continuous. To prove

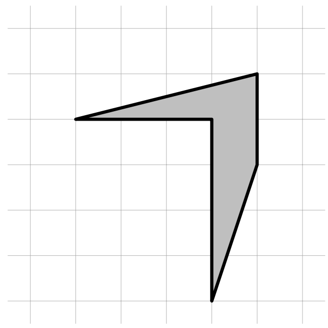

Figure 1: A real tropical cone from Example 26 (for $x_3 = 0$).

that this is an real tropical cone, fix a pair $\lambda, \mu \in \mathbb{R}$ and $x, y \in \mathcal{S}$. Since F is monotone and homogeneous, we have $F(\max\{\lambda + x, \mu + y\}) \geqslant F(\lambda + x) = \lambda + F(x) \geqslant \lambda + x$ and similarly $F(\max\{\lambda + x, \mu + y\}) \geqslant \mu + y$. Hence $\max\{\lambda + x, \mu + y\} \in \mathcal{S}$. □

Remark 25. One could ask if there is a more direct way to obtain a piecewise description of the operator F given a real tropical cone $\mathcal{S}$ (without the use of model theory). This can be done in the following way. We first decompose $\mathcal{S} = \bigcup_{s=1}^{p} \{x \in \mathbb{R}^n : A^{(s)}x \leqslant b^{(s)}\}$ where the matrix $A^{(s)}$ has rational entries, and $b^{(s)}$ is a real vector. Then, given $x \in \mathbb{R}^n$ we denote by $P(x) \subset [p]$ the set of all $s \in [p]$ such that the polyhedron $\{y : A^{(s)}y \leqslant b^{(s)}, y \leqslant x\}$ is nonempty. By the strong duality of linear programming (and the fact that $F(x)$ is well defined for all $x \in \mathbb{R}^n$) we have

$$\begin{aligned} F_k(x) &= \max_{s \in P(x)} \max\{y_k : A^{(s)}y \leqslant b^{(s)}, y \leqslant x\} \\ &= \max_{s \in P(x)} \min\{z^\intercal b^{(s)} + w^\intercal x : (A^{(s)})^\intercal z + w = \epsilon_k, z \geqslant 0, w \geqslant 0\}. \end{aligned}$$

For every $s \in [p]$, let $V_k^{(s)}$ denote the set of vertices of the rational polyhedron $\{(z, w) : (A^{(s)})^\intercal z + w = \epsilon_k, z \geqslant 0, w \geqslant 0\}$. Hence

$$F_k(x) = \max_{s \in P(x)} \min_{(z,w) \in V_k^{(s)}} \{z^\intercal b^{(s)} + w^\intercal x\}. \tag{8}$$

Moreover, by Farkas' Lemma, the polyhedron $\{y : A^{(s)}y \leqslant b^{(s)}, y \leqslant x\}$ is nonempty if and only if for all (z, w) such that $(A^{(s)})^\intercal z + w = 0$, $z \geqslant 0$, $w \geqslant 0$, we have $z^\intercal b^{(s)} + w^\intercal x \geqslant 0$. If $U^{(s)}$ consists of precisely one representative of every extreme ray of the rational cone $\{(z, w) : (A^{(s)})^\intercal z + w = 0, z \geqslant 0, w \geqslant 0\}$, this amounts to the finite system of linear inequalities $z^\intercal b^{(s)} + w^\intercal x \geqslant 0$ for all $(z, w) \in U^{(s)}$. The inequalities of this form yield an arrangement of hyperplanes, and the value of $P(x)$ is constant when x varies in the relative interior of any cell of this arrangement. By fixing the terms achieving the maximum and minimum in (8) we refine the latter arrangement in such a way that F is affine on every cell of the refinement. Since F is continuous (by Lemma 22), we can then restrict ourselves to those of these cells that are full dimensional, and this gives the piecewise description of F.

Example 26. We illustrate our results on the following example. Take $n = 3$, $p = 2$, $M_k = 1$,

$S_{k,1} = \{1, 2\}$ for all $k \in \{1, 2, 3\}$,

$$A^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 1/4 & 0 & 3/4 \\ 1 & 0 & 0 \end{bmatrix} \quad b^{(1)} = \begin{bmatrix} 1 \\ 3/4 \\ 0 \end{bmatrix},$$

$$A^{(2)} = \begin{bmatrix} 0 & 1/3 & 2/3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad b^{(2)} = \begin{bmatrix} 4/3 \\ 2\pi \\ 0 \end{bmatrix}.$$

Then, the operator $F \colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$F_1(x) = \max\Big\{x_3 + 1, \frac{1}{3}x_2 + \frac{2}{3}x_3 + \frac{4}{3}\Big\},$$
$$F_2(x) = \max\Big\{\frac{1}{4}x_1 + \frac{3}{4}x_3 + \frac{3}{4}, x_3 + 2\pi\Big\},$$
$$F_3(x) = \max\{x_1, x_2\}.$$

The real tropical cone $\{x \in \mathbb{R}^3 \colon x \leqslant F(x)\}$ is depicted in Fig. 1.

4.2. Description of real tropical cones by directed graphs

We now describe how semilinear monotone homogeneous operators can be encoded by directed graphs. To this end we take a directed graph $\vec{\mathcal{G}} := (V, E)$, where the set of vertices is divided into Max vertices, Min vertices, and Random vertices, i.e., $V := V_{\mathrm{Min}} \uplus V_{\mathrm{Rand}} \uplus V_{\mathrm{Max}}$, where the symbol $\uplus$ denotes the disjoint union of sets. We suppose that the sets of Max vertices and Min vertices are nonempty. If $v \in V$ is a vertex of $\vec{\mathcal{G}}$, then by $\mathrm{In}(v) := \{(w, v) \colon (w, v) \in E\}$ we denote the set of its incoming edges, and by $\mathrm{Out}(v) := \{(v, w) \colon (v, w) \in E\}$ we denote the set of its outgoing edges. We suppose that the every vertex has at least one outgoing edge. If v is a Min vertex or a Max vertex and $e \in \mathrm{Out}(v)$ is its outgoing edge, then we equip this edge with a real number r_e. Furthermore, if v is a Random vertex, then we equip its set of outgoing edges with a rational probability distribution. More precisely, every edge $e \in \mathrm{Out}(v)$ is equipped with a strictly positive rational number $q_e \in \mathbb{Q}$, $q_e > 0$, and we suppose that $\sum_{e \in \mathrm{Out}(v)} q_e = 1$. We also make the following assumptions:

Assumption 27. (i) Every path between any two Min vertices contains at least one Max vertex;

(ii) Every path between any two Max vertices contains at least one Min vertex;

(iii) From every Random vertex, there is a path to a Min or a Max vertex.

We now construct a semilinear monotone homogeneous operator from such a graph. We define a Markov chain with state space V, and transition probabilities $p_{vv} := 1$ for all $v \in V_{\mathrm{Max}} \uplus V_{\mathrm{Min}}$, $p_{vw} := q_{(v,w)}$ if $v \in V_{\mathrm{Rand}}$ and $(v, w) \in \mathrm{Out}(v)$, and $p_{vw} := 0$ otherwise. Therefore, every state of $V_{\mathrm{Max}} \uplus V_{\mathrm{Min}}$ is absorbing, and a trajectory of the Markov chain visits the states of V_{Rand} by picking at random, for each vertex $v \in V_{\mathrm{Rand}}$, one edge in $\mathrm{Out}(v)$ according to the probability law given by $q_{(v,\cdot)}$, until it reaches a state of $V_{\mathrm{Max}} \uplus V_{\mathrm{Min}}$. In this way, after leaving a Min vertex, the trajectory reaches a Max vertex, and vice versa. If e is an edge and v is a Max or Min vertex, then we denote by p_v^e the conditional probability to reach the absorbing state v from the head of e. Note that every p_v^e is rational since we have assumed that the q_e are in $\mathbb{Q}$ (Kemeny and Snell, 1976, Theorem 3.3.7). For the sake of simplicity, we assume that $V_{\mathrm{Min}} = [n]$ and $V_{\mathrm{Max}} = [m]$.

Definition 28. The *operator encoded by* $\vec{\mathcal{G}}$ is the function $F\colon \mathbb{R}^n \to \mathbb{R}^n$ defined as

$$\forall v \in [n],\ (F(x))_v := \min_{e \in \mathrm{Out}(v)} \Big(r_e + \sum_{w \in [m]} p_w^e \max_{e' \in \mathrm{Out}(w)} \big(r_{e'} + \sum_{u \in [n]} p_u^{e'} x_u \big) \Big) . \tag{9}$$

Lemma 29. *The operator encoded by* $\vec{\mathcal{G}}$ *is semilinear monotone homogeneous.*

Proof. Let F be the operator encoded by $\vec{\mathcal{G}}$. It is obviously semilinear (as a definable function in $\mathcal{L}_{\mathrm{og}}$) and monotone. We already observed that the Max and Min vertices are absorbing states in the Markov chain constructed from $\vec{\mathcal{G}}$. Besides, Assumption 27 (iii) still holds in the subgraph obtained by removing the edges going out of the Max and Min vertices. As a consequence, for every Random vertex v, the probability to reach a Min or Max vertex starting from v is positive. We deduce that the Max and Min vertices are the only final classes in the Markov chain. Let v be a Min vertex, and $e \in \mathrm{Out}(v)$. We claim that if $u \in [n]$ is a Min state, then $p_u^e = 0$. Indeed, by Assumption 27 (i), any path from the head of e to u in $\vec{\mathcal{G}}$ contains a Max vertex. As a consequence, there is no path from the head of e to u in the subgraph in which we have removed the edges going out of the Max and Min vertices. We deduce that for every Min vertex v and edge $e \in \mathrm{Out}(v)$, we have $\sum_{w \in [m]} p_w^e = 1$. Analogously, we can show that for all Max vertices w and edge $e' \in \mathrm{Out}(w)$, $\sum_{u \in [n]} p_u^{e'} = 1$. We deduce that the operator F is homogeneous. □

In the following lemma, we show that any semilinear monotone homogeneous operator is encoded by some digraph:

Lemma 30. *Let* $F\colon \mathbb{R}^n \to \mathbb{R}^n$ *be a semilinear monotone homogeneous operator. Then, there exists a directed graph* $\vec{\mathcal{G}}$ *satisfying Assumption 27 such that* F *is encoded by* $\vec{\mathcal{G}}$.

Proof. The idea is to identify the representation (7) to a special case of (9), in which the probabilities p_w^e with $e \in \mathrm{Out}(v)$ and $v \in V_{\mathrm{Min}}$ take only the values 0 and 1. Formally, let $A^{(1)}, \dots, A^{(p)} \in \mathbb{Q}^{n \times n}$ and $b^{(1)}, \dots, b^{(p)} \in \mathbb{R}^n$ such that Lemma 23 holds. We build $\vec{\mathcal{G}}$ as the graph in which the set of Min vertices is $[n]$, the set of Max vertices is $\uplus_k [M_k]$, and the set of Random vertices is $\uplus_{k \in [n],\, i \in [M_k]} S_{ki}$. Let k be a Min vertex. We add an edge (k, i) for every $i \in [M_k]$, with $r_{(k,i)} := 0$. Moreover, for every $i \in [M_k]$, we add an edge (i, s) for each $s \in S_{ki}$, with $r_{(i,s)} := b_k^{(s)}$. Finally, if $i \in [M_k]$ and $s \in S_{ki}$, we add an edge (s, l) with $q_{(s,l)} := A_{kl}^{(s)}$ for every $l \in [n]$ such that $A_{kl}^{(s)} > 0$. The requirements of Assumption 27 are straightforwardly satisfied. □

Example 31. The graph presented in Fig. 2 encodes the operator from Example 26.

4.3. Construction of tropical Metzler spectrahedra

The following proposition characterizes the semilinear monotone homogeneous operators associated with tropical Metzler spectrahedral cones. We discussed this family of operators in our previous work (Allamigeon et al., 2018b), where we interpreted them as the dynamic programming operators of a zero-sum game.

Proposition 32. *Suppose that the graph* $\vec{\mathcal{G}}$ *fulfills Assumption 27 and has the following properties:*

- *every Random vertex has exactly two outgoing edges and the probability distribution associated with these edges is equal to* $(1/2, 1/2)$

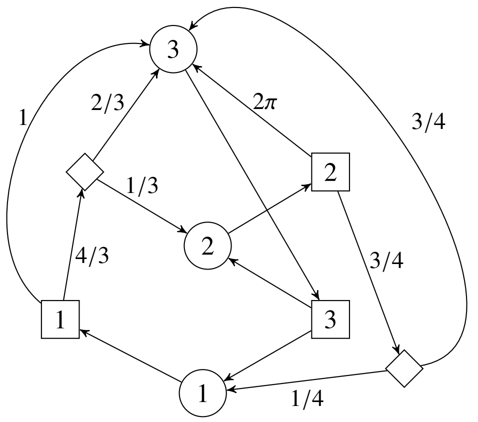

Figure 2: Graph that encodes the operator from Example 26. Min vertices are depicted by circles, Max vertices are depicted by squares, Random vertices are depicted by diamonds. We put $r_e = 0$ for every edge $e \in E$ that has no label.

- *every edge outgoing from a Random vertex has a Max vertex as its head.*

Let F denote the semilinear monotone homogeneous operator encoded by $\vec{\mathcal{G}}$. Then, the set $\{x \in \mathbb{T}^n : x \leqslant F(x)\}$ is a tropical Metzler spectrahedral cone.

Proof. Consider the Markov chain introduced before Definition 28, take a Min vertex $v \in [n]$ and an outgoing edge $e \in \mathrm{Out}(v)$. Under the assumptions over the graph $\vec{\mathcal{G}}$, the absorbing states reachable from the head of e form a set $\{w_e, w'_e\} \subset [m]$ of cardinality at most 2 (we use the convention $w_e = w'_e$ if there is only one such absorbing state). Moreover, if $w_e \neq w'_e$, then $p^e_{w_e} = p^e_{w'_e} = 1/2$. Furthermore, observe that if $w \in [m]$ is a Max vertex and $e' \in \mathrm{Out}(w)$ is an outgoing edge, then our assumptions imply that the head of e' is a Min vertex. We denote it by $u_{e'}$. With this notation, we have:

$$(F(x))_v = \min_{e \in \mathrm{Out}(v)} \Big(r_e + \frac{1}{2} \big(\max_{e' \in \mathrm{Out}(w_e)} (r_{e'} + x_{u_{e'}}) + \max_{e' \in \mathrm{Out}(w'_e)} (r_{e'} + x_{u_{e'}}) \big) \Big) . \tag{10}$$

The operators of the form given in (10) are studied in (Allamigeon et al., 2018b, Sections 4.2 and 5.1). In particular, the claim follows from (Allamigeon et al., 2018b, Lemma 52). ☐

We want to show that every real tropical cone associated with a graph $\vec{\mathcal{G}}$ is a projection of a tropical Metzler spectrahedron. The idea of the proof is to take an arbitrary graph $\vec{\mathcal{G}}$ and transform it (by adding auxiliary states) into a graph $\vec{\mathcal{G}}'$ that fulfills the conditions of Proposition 32. Furthermore, our construction needs to preserve the projection. A key ingredient is the following construction, which was used by Zwick and Paterson (1996) to show the reduction from discounted games to simple stochastic games.

Lemma 33 (Zwick and Paterson, 1996)**.** *One can transform an arbitrary graph $\vec{\mathcal{G}}$ into a graph $\vec{\mathcal{G}}'$ such that*

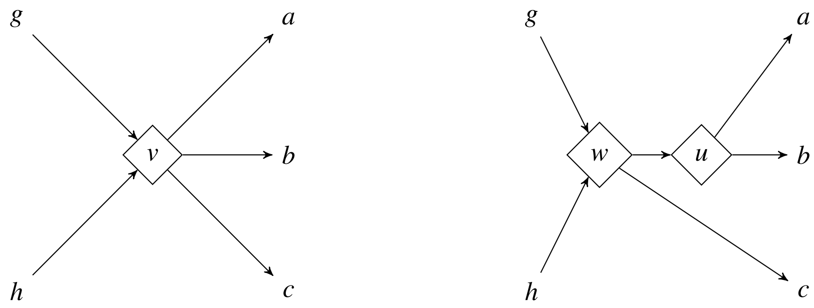

Figure 3: Lowering the degree. Random vertices are depicted by diamonds.

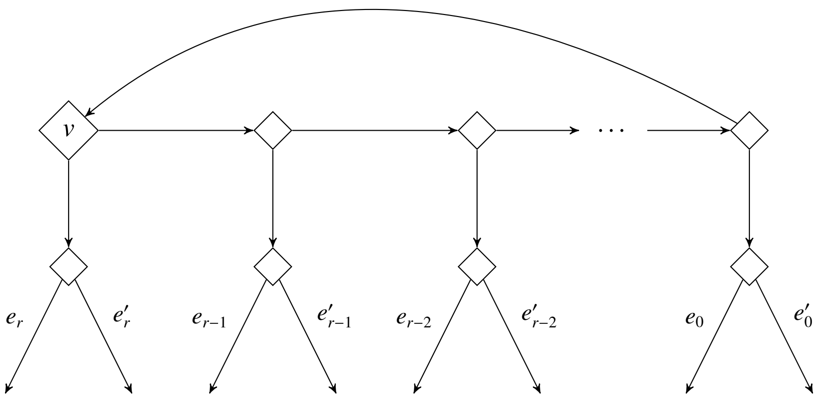

Figure 4: The construct of Zwick and Paterson.

- *every Random vertex of $\vec{\mathcal{G}}'$ has exactly two outgoing edges and the probability distribution associated with these edges is equal to* $(1/2, 1/2)$
- *$\vec{\mathcal{G}}$ and $\vec{\mathcal{G}}'$ encode the same operator.*

Let us present the construction of Zwick and Paterson for the sake of completeness.

Proof. Fix a Random vertex v belonging to $\vec{\mathcal{G}}$. If this vertex has only one outgoing edge e, then we can delete v by joining all incoming edges $\mathrm{In}(v)$ with the head of e.

If v has at least three outgoing edges, then we enumerate the outgoing edges $\mathrm{Out}(v)$ by $\{e_1, \dots, e_d\}$, $d \geqslant 3$. Let us recall that the vertex v is equipped with a probability distribution $(q_{e_s})_{s=1}^{d}$. We now perform the transformation presented on Fig. 3. We replace the vertex v by a pair of vertices (w, u) such that all incoming edges of v are connected to w and w has two outgoing edges: one going to the head of e_1 with probability q_{e_1} and the other going to u with probability $1 - q_{e_1}$. Finally, u has $d - 1$ outgoing edges, the head of the sth outgoing edge is the head of e_s, and the associated probability is equal to $q_{e_s}/(1 - q_{e_1})$. We repeat this transformation until we reach a graph in which all Random vertices have exactly two outgoing edges.

If v has exactly two outgoing edges, then we denote the heads of these edges by w and u, and the associated probability distribution by $(q, 1 - q)$, where $q = a/b$, $a, b \in \mathbb{N}^*$ and $a < b$. If $q \neq 1/2$, then we take $r \geqslant 1$ such that $2^r \leqslant b < 2^{r+1}$. We write a and $b-a$ in binary, $a = \sum_{s=0}^{r} c_s 2^s$ and $b - a = \sum_{s=0}^{r} d_s 2^s$ for $c_s, d_s \in \{0, 1\}$. We now replace the outgoing edges of vertex v by the construct presented on Fig. 4. In this construction, every Random node has exactly two outgoing edges and the associated probability distribution is equal to $(1/2, 1/2)$. Furthermore, for any s, if $c_s = 1$, then head of e_s is w and if $c_s = 0$, then the head of e_s is v. Similarly, if $d_s = 1$, then head of e'_s is u and if $d_s = 0$, then the head of e'_s is v. Suppose that the Markov chain reaches v. Then, with probability $a/2^{r+2}$ the Markov chain goes to w without coming back to v. Similarly, with probability $(b - a)/2^{r+2}$ the Markov chain moves to u without coming back to v. Therefore, the probability that the Markov chain finally reaches w is equal to a/b and the probability that it finally reaches u is equal to $(b - a)/b$. We repeat this procedure for every Random vertex of our graph.

To finish the proof, observe that the operations described above do not affect the associated semilinear monotone homogeneous operator. □

We now describe how to transform a graph given in Lemma 33 into a graph that verifies the conditions of Proposition 32. More precisely, we transform the graph $\vec{\mathcal{G}}$ (which has n Min vertices) into a graph $\vec{\mathcal{G}}'$ (which has n' Min vertices, where $n' \geqslant n$) in such a way that the real tropical cone $\{x \in \mathbb{R}^n : x \leqslant F(x)\}$ associated with $\vec{\mathcal{G}}$ is a projection of the real tropical cone $\{x \in \mathbb{R}^{n'} : x \leqslant F'(x)\}$ associated with $\vec{\mathcal{G}}'$. The main difficulty here is that the operators arising from tropical (Metzler) spectrahedra have a special structure, of "Player I – Chance – Player II" type, to adopt a game theoretical terminology, meaning that arcs in the graph connect Min vertices to Random vertices, Random vertices to Max vertices, and Max vertices to Min vertices, as is apparent from (10). By comparison, the Zwick–Paterson construction (Lemma 33) leads to a graph with consecutive sequences of Random nodes. We shall see, however, that the latter situation can be reduced from the former one by applying, as a basic ingredient, two transformations, the validity of which is expressed in Lemmas 34 and 35.

The first transformation that we execute is presented on Fig. 5. It is given as follows. Suppose that we are given a graph $\vec{\mathcal{G}}$. Denote $V_{\text{Min}} = [n]$ and $V_{\text{Max}} = [m]$. Furthermore, let $E_{\text{Max}} \subset E$ denote the set of all edges that have a Max vertex as their tail. Let F denote the operator associated with $\vec{\mathcal{G}}$. For every Max vertex $v \in [m]$ and outgoing edge $e \in \text{Out}(v)$, we insert a Min vertex between v and the head of e, as illustrated in Fig. 5. In a similar way, for every Min vertex v and incoming edge $e \in \text{In}(v)$, we insert a Max vertex between the tail of e and v. We denote the transformed graph by $\vec{\mathcal{G}}'$. Observe that this graph fulfills Assumption 27. We refer to the Min vertices in $\vec{\mathcal{G}}'$ as follows: the vertices that were present in $\vec{\mathcal{G}}$ are denoted by $[n]$, whereas the added Min vertices are denoted by $e \in E_{\text{Max}}$.

Lemma 34. *Suppose that the operator F' is obtained from F by the first transformation above. Then, the real tropical cone $\{x \in \mathbb{R}^n : x \leqslant F(x)\}$ is the projection of the real tropical cone $\{(x, x') \in \mathbb{R}^n \times \mathbb{R}^{|E_{\text{Max}}|} : (x, x') \leqslant F'(x, x')\}$.*

Proof. Denote the operator F as

$$\forall v \in [n],\ (F(x))_v = \min_{e \in \text{Out}(v)} \Big(r_e + \sum_{w \in [m]} p_w^e \max_{e' \in \text{Out}(w)} \big(r_{e'} + \sum_{u \in [n]} p_u^{e'} x_u \big) \Big).$$

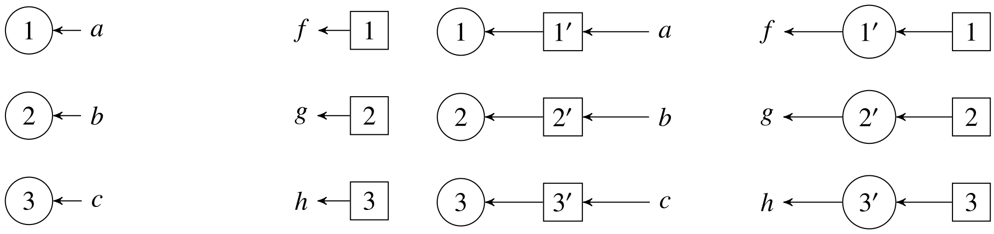

Figure 5: First transformation of a graph. Min vertices are presented by circles, Max vertices are presented by squares.

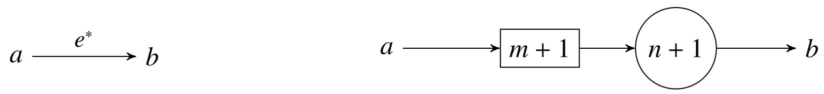

Figure 6: Second transformation of a graph.

Observe that for every $v \in [n]$ we have

$$(F'(x, x'))_v = \min_{e \in \mathrm{Out}(v)} \Big(r_e + \sum_{w \in [m]} p^e_w \max_{e' \in \mathrm{Out}(w)} (r_{e'} + x_{e'}) \Big) .$$

Furthermore, for every $e \in E_{\mathrm{Max}}$ we have

$$(F'(x, x'))_e = \sum_{v \in [n]} p^e_v x_v .$$

Therefore, if $x \leqslant F(x)$ and for every $e \in E_{\mathrm{Max}}$ we set $x_e = \sum_{v \in [n]} p^e_v x_v$, then for every $v \in [n]$ we have $x_v \leqslant (F(x))_v = (F'(x, x'))_v$ and for every $e \in E_{\mathrm{Max}}$ we have $x_e = (F'(x, x'))_e$. Conversely, if $(x, x') \leqslant F'(x, x')$, then we have $x_v \leqslant (F'(x, x'))_v \leqslant (F(x))_v$ for every $v \in [n]$. ☐

The second transformation is given as follows. As previously, suppose that we are given a graph $\vec{\mathcal{G}}$. Denote $V_{\mathrm{Min}} = [n]$ and $V_{\mathrm{Max}} = [m]$. Furthermore, let $E_{\mathrm{Max}} \subset E$ denote the set of all edges that have a Max vertex as their tail. Let F denote the operator associated with $\vec{\mathcal{G}}$. Moreover, suppose that $\vec{\mathcal{G}}$ is such that every edge $e \in E_{\mathrm{Max}}$ has a Min vertex as its head. Suppose that $e^* \in E$ is a fixed edge in $\vec{\mathcal{G}}$ that connects two Random vertices. We add a Max vertex $m + 1$ and a Min vertex $n + 1$ onto e^* as presented on Fig. 6. We denote the transformed graph by $\vec{\mathcal{G}}'$. Since every edge $e \in E_{\mathrm{Max}}$ has a Min vertex as its head, every path that joins a Max vertex with a Min vertex has length 1. In particular, e^* does not belong to any such path. Hence, the transformed graph $\vec{\mathcal{G}}'$ fulfills Assumption 27.

Lemma 35. *Suppose that the operator F' is obtained from F by the second transformation above. Then, the real tropical cone $\{x \in \mathbb{R}^n : x \leqslant F(x)\}$ is a projection of the real tropical cone $\{(x, x_{n+1}) \in \mathbb{R}^{n+1} : (x, x_{n+1}) \leqslant F'(x, x_{n+1})\}$.*

Proof. Denote the operator F as

$$\forall v \in [n],\ (F(x))_v = \min_{e \in \mathrm{Out}(v)} \Big(r_e + \sum_{w \in [m]} p^e_w \max_{e' \in \mathrm{Out}(w)} (r_{e'} + \sum_{u \in [n]} p^{e'}_u x_u) \Big) .$$

Let us introduce the following notation. For every $e \in E$, we denote by p^e_{e*} the conditional probability that the Markov chain reaches the head of e^* from the head of e. Moreover, for every Max vertex w and every edge $e \in E$, we denote by p^e_{w2} the conditional probability that the Markov chain reaches w from the head of e without passing by the head of e^*. Thus, for every Max vertex w and every $e \in E$ we have $p^e_w = p^e_{e*} p^{e^*}_w + p^e_{w2}$. Therefore, for any $x_{n+1} \in \mathbb{R}$ we have

$$(F'(x, x_{n+1}))_{n+1} = \sum_{w \in [m]} p^{e^*}_w \max_{e' \in \mathrm{Out}(w)} (r_{e'} + \sum_{u \in [n]} p^{e'}_u x_u) = (F'(x, 0))_{n+1}$$

and

$$\begin{aligned}(F(x))_v &= \min_{e \in \mathrm{Out}(v)} \Big(r_e + \sum_{w \in [m]} (p^e_{e^*} p^{e^*}_w + p^e_{w2}) \max_{e' \in \mathrm{Out}(w)} (r_{e'} + \sum_{u \in [n]} p^{e'}_u x_u) \Big) \\ &= \min_{e \in \mathrm{Out}(v)} \Big(r_e + p^e_{e^*} (F'(x, 0))_{n+1} + \sum_{w \in [m]} p^e_{w2} \max_{e' \in \mathrm{Out}(w)} (r_{e'} + \sum_{u \in [n]} p^{e'}_u x_u) \Big) .\end{aligned}$$

Furthermore, for every $v \in [n]$ we have

$$(F'(x, x_{n+1}))_v = \min_{e \in \mathrm{Out}(v)} \Big(r_e + p^e_{e^*} x_{n+1} + \sum_{w \in [m]} p^e_{w2} \max_{e' \in \mathrm{Out}(w)} (r_{e'} + \sum_{u \in [n]} p^{e'}_u x_u) \Big) .$$

Therefore, if $x \leqslant F(x)$ and we set $x_{n+1} = (F'(x, 0))_{n+1}$, then $(x, x_{n+1}) \leqslant F'(x, x_{n+1})$. Conversely, if $(x, x_{n+1}) \leqslant F'(x, x_{n+1})$, then $x_{n+1} \leqslant (F'(x, 0))_{n+1}$ and hence $x_v \leqslant (F(x))_v$ for all $v \in [n]$. □

Proposition 36. *Every closed semilinear real tropical cone is a projection of a real tropical Metzler spectrahedron.*

Proof. Take any closed semilinear real tropical cone $\mathcal{S} = \{x \in \mathbb{R}^n : x \leqslant F(x)\}$. Let $\vec{\mathcal{G}}$ denote the graph associated with F. By Lemma 33 we may suppose that the probabilities associated with Random vertices in $\vec{\mathcal{G}}$ are equal to $1/2$. We perform the first transformation on the graph $\vec{\mathcal{G}}$. Denote the transformed graph by $\vec{\mathcal{G}}_1$. We perform the second transformation on every edge in $\vec{\mathcal{G}}_1$ that joins two Random vertices. Denote the transformed graph by $\vec{\mathcal{G}}'$ and the associated operator as F'. By Lemmas 34 and 35, the real tropical cone $\{x \in \mathbb{R}^n : x \leqslant F(x)\}$ is the projection of the real tropical cone $\{(x, x') \in \mathbb{R}^n \times \mathbb{R}^{n'} : (x, x') \leqslant F'(x, x')\}$. Furthermore, $\vec{\mathcal{G}}'$ fulfills the conditions of Proposition 32. Therefore, the set $\mathcal{S}' = \{(x, x') \in \mathbb{T}^n \times \mathbb{T}^{n'} : (x, x') \leqslant F'(x, x')\}$ is a tropical Metzler spectrahedral cone. Finally, we take the set

$$\begin{aligned}\mathcal{S}'' &= \{(x, x', y) \in \mathbb{T}^n \times \mathbb{T}^{n'} \times \mathbb{T}^{n+n'} : (x, x') \leqslant F'(x, x') \land (x, x') + y \geqslant 0\} \\ &= \{(x, x', y) \in \mathbb{R}^n \times \mathbb{R}^{n'} \times \mathbb{R}^{n+n'} : (x, x') \leqslant F'(x, x') \land (x, x') + y \geqslant 0\} .\end{aligned}$$

The set $\mathcal{S}''$ is a real tropical Metzler spectrahedron. Moreover, $\mathcal{S}$ is a projection of $\mathcal{S}''$. □

Example 37. Take the graph from Fig. 2 and consider the Random vertex that has Min vertices 2 and 3 as its neighbors. Figure 7 presents the outcome of the procedure described in the lemmas above when applied to this vertex.

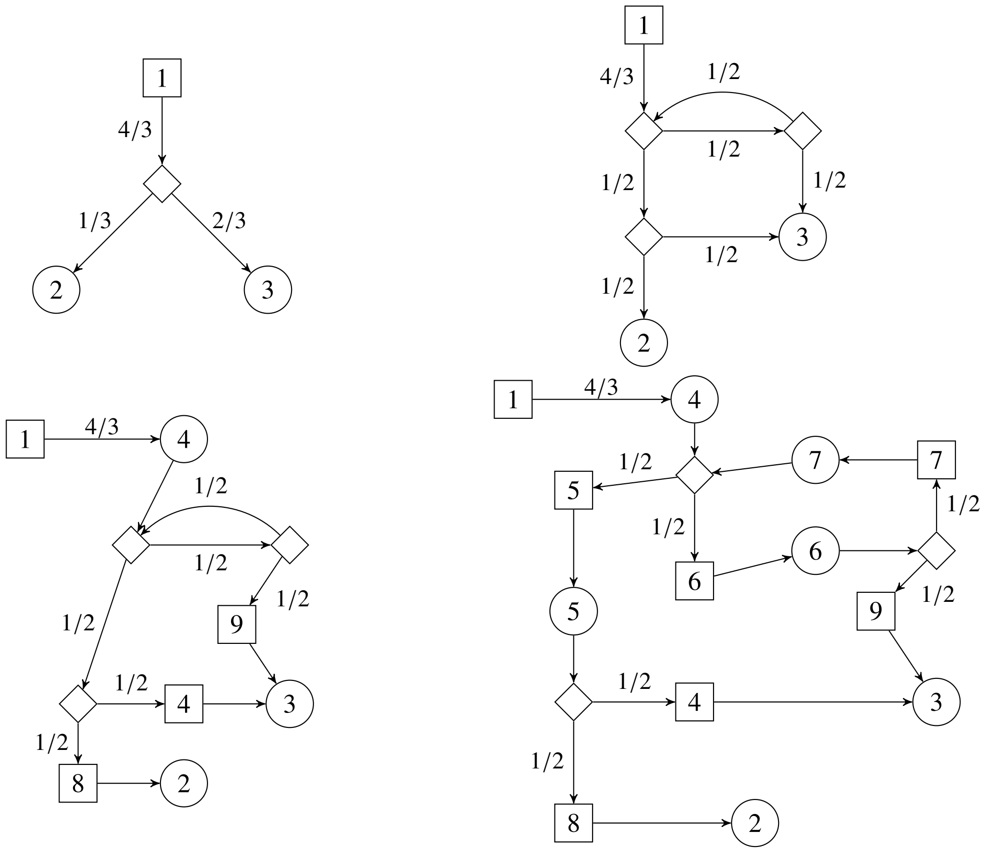

Figure 7: The transformation of Lemmas 33 to 35 applied to one Random vertex from the graph presented in Fig. 2. Top left: the initial graph. Top right: the graph after the application of Lemma 33. Bottom left: the graph after the application of Lemma 34. Bottom right: the graph after the application of Lemma 35.

5. General case of the tropical Helton–Nie conjecture

We now generalize Proposition 36 to tropically convex sets in $\mathbb{T}^n$. In order to study this case, we use the notion of homogenization of a convex set. There are many possible homogenizations of a given set. We need to use three different notions.

Definition 38. If $\mathcal{S}$ is a tropically convex set with only finite points (i.e., $\mathcal{S} \subset \mathbb{R}^n$), then we define its *real homogenization* as

$$\mathcal{S}^{rh} = \{(x_0, x_0 + x) \in \mathbb{R}^{n+1} \colon x \in \mathcal{S}\}.$$

The set $\mathcal{S}^{rh}$ is a real tropical cone. If $\mathcal{S} \subset \mathbb{T}^n$ is a tropically convex set, then we define its *homogenization* as

$$\mathcal{S}^{h} = \{(x_0, x_0 + x) \in \mathbb{T}^{n+1} \colon x \in \mathcal{S}\}.$$

The set $\mathcal{S}^h$ is a tropical cone. If $\mathcal{S}(Q^{(0)}|Q^{(1)}, \dots, Q^{(n)}) \subset \mathbb{T}^n$ is a tropical Metzler spectrahedron, then we define its *formal homogenization* as the tropical Metzler spectrahedron $\mathcal{S}^{fh} \subset \mathbb{T}^{n+1}$, $\mathcal{S}^{fh} := \mathcal{S}(-\infty|Q^{(0)}, Q^{(1)}, \dots, Q^{(n)})$. The set $\mathcal{S}^{fh}$ is a tropical Metzler spectrahedral cone.

Lemma 39. *Every closed semilinear tropically convex set in* $\mathbb{R}^n$ *is a projection of a tropical Metzler spectrahedron.*

Proof. Take any closed semilinear tropically convex set $\mathcal{S} \subset \mathbb{R}^n$ and consider its real homogenization $\mathcal{S}^{rh}$. This is a closed semilinear real tropical cone in $\mathbb{R}^{n+1}$. By Proposition 36, $\mathcal{S}^{rh}$ is a projection of a tropical Metzler spectrahedron $\mathcal{S}_1 \subset \mathbb{T} \times \mathbb{T}^n \times \mathbb{T}^{n'}$. Consider the set

$$\mathcal{S}_2 = \{(x_0, x, y) \in \mathbb{T} \times \mathbb{T}^n \times \mathbb{T}^{n'} : (x_0, x, y) \in \mathcal{S}_1 \land x_0 = 0\}.$$

The set $\mathcal{S}_2$ is a tropical Metzler spectrahedron. Furthermore, $\mathcal{S}$ is its projection. □

We now want to extend this result to tropically convex sets in $\mathbb{T}^n$. In order to do this, we proceed stratum-by-stratum. This requires us to show that a tropical convex hull of finitely many projected Metzler spectrahedra is a projected Metzler spectrahedron. In the classical case of real spectrahedra, it is known that a convex hull of finitely many projected spectrahedra is a projected spectrahedron. This fact has a very short proof presented in (Netzer and Sinn, 2009). The proof in the tropical case is exactly the same (we only change the classical notation to the tropical one). Let us present this proof for the sake of completeness.

Lemma 40. *A tropically convex set* $\mathcal{S} \subset \mathbb{T}^n$ *is a projected tropical Metzler spectrahedron if and only if its homogenization is a projected tropical Metzler spectrahedron.*

Proof. First, suppose that $\mathcal{S}^h$ is a projection of a tropical Metzler spectrahedron $\mathcal{S}_1 \subset \mathbb{T} \times \mathbb{T}^n \times \mathbb{T}^{n'}$. Consider the set

$$\mathcal{S}_2 = \{(x_0, x, y) \in \mathbb{T} \times \mathbb{T}^n \times \mathbb{T}^{n'} : (x_0, x, y) \in \mathcal{S}_1 \land x_0 = 0\}.$$

The set $\mathcal{S}_2$ is a tropical Metzler spectrahedron. Furthermore, $\mathcal{S}$ is its projection. Conversely, suppose that $\mathcal{S}$ is a projection of a tropical Metzler spectrahedron $\mathcal{S}_1 \subset \mathbb{T}^n \times \mathbb{T}^{n'}$. Consider its formal homogenization $\mathcal{S}_1^{fh} \subset \mathbb{T}^{1+n+n'}$ and take the set

$$\mathcal{S}_2 = \{(x_0, x, y, z) \in \mathbb{T} \times \mathbb{T}^n \times \mathbb{T}^{n'} \times \mathbb{T}^n : (x_0, x, y) \in \mathcal{S}_1^{fh} \ \land\ \forall k \in [n], x_0 + z_k \geqslant 2x_k\}.$$

The set $\mathcal{S}_2$ is a tropical Metzler spectrahedron. We will show that $\mathcal{S}^h$ is a projection of $\mathcal{S}_2$. Take any point $x \in \mathcal{S}$. Then, there exists y such that $(0, x, y) \in \mathcal{S}_1^{fh}$. Therefore, $(x_0, x + x_0 e, y + x_0 e) \in \mathcal{S}_1^{fh}$ for any $x_0 \in \mathbb{R}$. If we take z_k large enough, then $(x_0, x + x_0 e, y + x_0 e, z) \in \mathcal{S}_2$. Moreover, we have $-\infty \in \mathcal{S}_2$. This shows that $\mathcal{S}^h$ is included in the projection of $\mathcal{S}_2$. Conversely, suppose that $(x_0, x, y, z) \in \mathcal{S}_2$. If $x_0 = -\infty$, then $x = -\infty$ and hence $(x_0, x) \in \mathcal{S}^h$. If $x_0 \neq -\infty$, then we have $(0, x - x_0 e, y - x_0 e, z - x_0 e) \in \mathcal{S}_2$. Hence $(0, x - x_0 e, y - x_0 e) \in \mathcal{S}_1^{fh}$, $(x - x_0 e, y - x_0 e) \in \mathcal{S}_1$, and $x - x_0 e \in \mathcal{S}$. Therefore $(x_0, x) \in \mathcal{S}^h$. □

Lemma 41. *Suppose that* $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{T}^n$ *are projected tropical Metzler spectrahedra. Then* $\operatorname{tconv}(\mathcal{S}_1 \cup \mathcal{S}_2)$ *is a projected tropical Metzler spectrahedron.*

Proof. Let $\mathcal{S} = \operatorname{tconv}(\mathcal{S}_1 \cup \mathcal{S}_2)$ and consider

$$\mathcal{S}_1^h \oplus \mathcal{S}_2^h = \{x \in \mathbb{T}^{n+1} : \exists (u, w) \in \mathcal{S}_1^h \times \mathcal{S}_2^h,\ x = u \oplus w\}.$$

Observe that we have the identity $\mathcal{S}^h = \mathcal{S}_1^h \oplus \mathcal{S}_2^h$. Indeed, since $\mathcal{S}_1 \subset \mathcal{S}$, we have $\mathcal{S}_1^h \subset \mathcal{S}^h$. Similarly, $\mathcal{S}_2^h \subset \mathcal{S}^h$. Therefore, we have $\mathcal{S}_1^h \oplus \mathcal{S}_2^h \subset \mathcal{S}^h$. Conversely, take a point $z \in \mathcal{S}^h$. By Lemma 9, we can write z as

$$z = \Big(z_0, z_0 \odot \big((\lambda \odot x) \oplus (\mu \odot y)\big)\Big) \in \mathcal{S}^h ,$$

where $\lambda \oplus \mu = 0$, $x \in \mathcal{S}_1$, and $y \in \mathcal{S}_2$. Then $z = \tilde{x} \oplus \tilde{y}$, where

$$\tilde{x} = (\lambda \odot z_0, (\lambda \odot z_0) \odot x) \in \mathcal{S}_1^h \,,$$
$$\tilde{y} = (\mu \odot z_0, (\mu \odot z_0) \odot y) \in \mathcal{S}_2^h \,.$$

Hence $\mathcal{S}^h = \mathcal{S}_1^h \oplus \mathcal{S}_2^h$ and the claim follows from Lemma 40. □

We are now ready to present the proof of Theorem 19.

Proof of Theorem 19. The equivalence between Theorem 19 (a) and Theorem 19 (b) is given in Proposition 12. The implication from Theorem 19 (b) to Theorem 19 (c) follows from Lemma 39. We now prove the implication from Theorem 19 (c) to Theorem 19 (d). Let $\mathcal{S} \subset \mathbb{T}^n$ be as in Theorem 19 (c). If $\mathcal{S}$ is empty, then it is a tropical Metzler spectrahedron defined by a single inequality $-\infty \geqslant 0$. Otherwise let $K \subset [n]$ be any nonempty set such that the stratum $\mathcal{S}_K \subset \mathbb{R}^{|K|}$ is nonempty. The set $\mathcal{S}_K$ is a projection of a tropical Metzler spectrahedron $S_K \subset \mathbb{T}^{|K|} \times \mathbb{T}^{n'}$. For any $x \in \mathbb{T}^n$ we denote by $x_K \in \mathbb{T}^{|K|}$ the subvector formed by the coordinates of x with indices in K. Furthermore, let $X_K \subset \mathbb{T}^n$ denote the set

$$X_K = \{x \in \mathbb{T}^n \colon x_k \neq -\infty \iff k \in K\} \,.$$

The set $\mathcal{S} \cap X_K$ is a projection of a tropical Metzler spectrahedron defined as

$$\tilde{S}_K = \{(x, y) \in \mathbb{T}^n \times \mathbb{T}^{n'} \colon (x_K, y) \in S_K \wedge \forall k \notin K, -\infty \geqslant x_k\} \,.$$

Moreover, for $K = \emptyset$, let us denote $X_\emptyset = -\infty$. Note that the intersection $\mathcal{S} \cap X_\emptyset$ is either empty or is equal to $-\infty$, and that $-\infty$ is a tropical Metzler spectrahedron (defined by the inequalities $-\infty \geqslant x_k$ for all $k \in [n]$). Hence, we have $\mathcal{S} = \cup_{K \subset [n]} \mathcal{S} \cap X_K = \operatorname{tconv}(\cup_{K \subset [n]} \mathcal{S} \cap X_K)$. Therefore, the claim follows from Lemma 41. Finally, to prove the implication Theorem 19 (d) to Theorem 19 (a), let $\mathcal{S} \subset \mathbb{T}^n$ be a projection of a tropical Metzler spectrahedron $S \subset \mathbb{T}^n \times \mathbb{T}^{n'}$. By Proposition 17, there is a spectrahedron $\boldsymbol{S} \in \mathbb{K}_{\geqslant 0}^{n+n'}$ such that $\mathsf{val}(\boldsymbol{S}) = S$. Let $\boldsymbol{\pi} \colon \mathbb{K}^{n+n'} \to \mathbb{K}^n$ denote the projection on the first n coordinates. Then $\mathcal{S} = \mathsf{val}(\boldsymbol{\pi}(\boldsymbol{S}))$. □

Remark 42. Consider a convex semialgebraic subset $\boldsymbol{S}$ over $\mathbb{K}^n$. Theorem 3 shows that there exist integers p and m such that $\boldsymbol{S}$ has the same image by the valuation as a projection of some spectrahedron over $\mathbb{K}^p$ associated with matrices of size $m \times m$. The integers p and m appearing in the proof of this theorem have the following remarkable uniformity property: if $\boldsymbol{S}$ is given as a union of finitely many basic semialgebraic sets of the form (6), then p, m are bounded from above by a number N that depends only on the degrees and the number of polynomials involved in the description of $\boldsymbol{S}$ (i.e., that N is independent of the coefficients of these polynomials). The proof, however, is quite involved. First, one should observe that, given only the degrees and the number of polynomials describing $\boldsymbol{S}$, the Denef–Pas quantifier elimination creates a finite set of $\mathcal{L}_{\text{og}}$-formulas such that every stratum of $\mathsf{val}(\boldsymbol{S})$ is described by a formula from this set. Second, Theorem 19 gives a tropical Metzler spectrahedron S such that $\mathsf{val}(\boldsymbol{S})$ is its projection. A careful examination of the proof presented here shows that the dimension and the size of the matrices defining S can be bounded by a quantity that depends only on the aforementioned $\mathcal{L}_{\text{og}}$-formulas (and not on the particular choice of their parameters). This gives the desired bound N.

Remark 43. Given the bound of Remark 42, the Denef–Pas quantifier elimination implies that our main result (Theorem 3) is valid not only over the field of Puiseux series considered here, but over every real closed valued field equipped with a nontrivial and convex valuation.

6. Concluding remarks

We showed that the convex semialgebraic sets and the projections of spectrahedra over the nonarchimedean field of real Puiseux series have the same images by the nonarchimedean valuation. We gave an explicit representation for these images, as the subfixed point sets of semilinear monotone homogeneous maps (dynamic programming operators of zero-sum stochastic games with perfect information). One may ask whether more insight on the projections of spectrahedra over nonarchimedean fields or over the field of real numbers can be gotten by tropical methods. In this respect, we note that we considered the simplest possible tropicalization, looking at the image of Puiseux series by their ordinary valuations. We also leave it as a further work to see whether more sophisticated tropicalizations, capturing also the sign, or higher order approximations of Puiseux series (spaces of jets) may be exploited.

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