Stability Analysis and Output-Feedback Control Design for Time-Delay Systems ⋆

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Abstract: The aim of this paper is to present new results on $\mathcal{H}_\infty$ control synthesis, via output-feedback, for time-delay linear systems. We extend the use of the finite order LTI system, called *comparison system*, to design a controller which depends not only on the output at the present time and maximum delay, but also on an arbitrary number of values between those. This approach allows us to increase the maximum stable delay without requiring any additional information. All methods presented here consider time-delay systems control design with classical numeric routines based on $\mathcal{H}_\infty$ theory. An illustrative example is presented.

Keywords: Time-delay Systems, Output-feedback, $\mathcal{H}_\infty$-norm, Comparison System.

1. INTRODUCTION

Time delays are intrinsically present in the structure of feedback control loops. Even though these delays are sometimes neglected, they can be responsible for poor performance and they may even lead the system to instability. For that reason, several studies considering the so called time-delay systems have being made through the last decades. Important theoretical results on time-delay systems control can be seen in the books (Gu et al., 2003) and (Niculescu and Gu, 2004) and in the survey paper (Richard, 2003).

Filtering and output-feedback for time-delay systems can be seen in (Fridman and Shaked, 2002b) and the design of observers in (Sename and Briat, 2006). State and output feedback stability were dealt in (Mahmoud and Zribi, 1999). A modified Riccati equation is used in (Lee et al., 1994) for the design of a memoryless $\mathcal{H}_\infty$ controller. The $\mathcal{H}_\infty$ control problem for multiple input-output delays was also discussed in (Meinsma and Mirkin, 2005). Other important works on the field are, for example, (Choi and Chungs, 1995), (Ge et al., 1996), (Fridman and Shaked, 2002a) and (de Oliveira and Geromel, 2004). In this work, our goal is to increase the maximum delay allowed in time-delay linear systems for a given $\mathcal{H}_\infty$ level γ through output-feedback control design. Furthermore, if the delay is given, our goal is to minimize γ. In (Korogui et al., 2012), which is the main work that we rely on, the Rekasius substitution (Rekasius, 1980) for $k = 1$ was successfully applied to obtain a finite order LTI system, called *comparison system* which was used to calculate an inferior bound for the $\mathcal{H}_\infty$ norm of the time-delay system. Here, we extend this approach finding the linear dependence on the matrices of the system with its *comparison system* for a substitution of order N. Therefore, our contribution is in designing a controller which depends not only on the output at the present time and maximum delay, but also on an arbitrary number of intermediate values in between, for both minimising $\mathcal{H}_\infty$ norm or maximising the allowed delay. Hence, we are able to increase the maximum stable delay using information that is already in the buffer. The new parametrisation for the comparison system proposed here altogether with the design procedure, when compared with results in the literature, is simpler to be implemented and provides more accurate results. One illustrative example is presented.

Notation. Matrices are denoted by capital letters, whilst small letters represent scalars and vectors. For real matrices or vectors the symbol $(')$ indicates transpose. Additionally, the maximum singular value of a matrix is denoted by $\sigma(\cdot)$. The sets of real, nonnegative real and natural numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$. The kronecker product is represented by $\otimes$, the null space of a matrix L by $\ker(L) = \{v \in V | L(v) = 0\}$ and vectorisation of a matrix A by $\mathrm{vec}(A) = [a_{1,1}, \ldots, a_{m,1}, a_{1,2}, \ldots, a_{m,n}]'$. Moreover, $\mathrm{vec}^{-1}(\mathrm{vec}(A)) = A$. In particular, rational transfer functions of LTI systems are denoted as

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C\left(sI - A\right)^{-1} B + D, \tag{1}$$

in which all matrices are real and of compatible dimensions. The binomial coefficient is denoted by

$$\binom{N}{k} = \frac{N!}{k!(N-k)!}, \tag{2}$$

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which is zero whenever the integers k and N satisfy $k < 0$ or $k > N$.

2. COMPARISON SYSTEM

Consider the time-delay linear system with M commensurate delays, whose minimal realisation is given by

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{j=1}^{M} A_j x(t-\tau_j) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{j=1}^{M} C_{zj} x(t-\tau_j), \end{aligned} \tag{3}$$

in which, for all $t \in \mathbb{R}_+$, $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^m$ is the exogenous input, $z(t) \in \mathbb{R}^p$ is the output of interest and $\tau_j = \tau(M-j+1)/M, j = 1 \ldots M$, for a given constant time delay $\tau \geq 0$. In this work, we address the problem of the commensurate delayed system (3) by applying the following substitution to the time delay operator concerning the largest delay:

$$e^{-\tau s} = \left(\frac{\lambda - s}{\lambda + s}\right)^N, \tag{4}$$

which is an exact relation for $s = j\omega$, with $\tau, \lambda, \omega \in \mathbb{R}_+$ and $N \in \mathbb{N}$ such that

$$\omega\tau = 2N \arctan\left(\frac{\omega}{\lambda}\right). \tag{5}$$

When $N = 1$ this is known as Rekasius substitution (Rekasius, 1980). For the following developments it will be necessary that the number of delays be the same as the order of the approximation (4). Note however, that whenever $N = hM$ for some $h \in \{1, 2, \ldots\}$, system (3) can be equivalently restated as

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + \sum_{k=1}^{N} A_k x(t-\tau_k) + E_0 w(t), \\ z(t) &= C_{z0} x(t) + \sum_{k=1}^{N} C_{zk} x(t-\tau_k), \end{aligned} \tag{6}$$

with $\tau_k = (N-k+1)/N$, by a suitable change of indices and considering null matrices whenever the respective delay is not present in the original formulation (3). Thus, without loss of generality, hereafter we are going to work with the rearranged system (6) which satisfies $N = hM$ for some $h \in \{1, 2, \ldots\}$.

One of our goals is to determine the maximal time-delay $\tau^\star > 0$ that ensures the system is globally asymptotically stable for any $\tau \in [0, \tau^\star)$. To achieve this, one must analyse the non-rational transfer function of (6), which is given by

$$\begin{aligned} T(s,\tau) = &\left(C_{z0} + \sum_{k=1}^{N} C_{zk} e^{-\tau_k s}\right) \times \\ &\times \left(sI - A_0 - \sum_{k=1}^{N} A_k e^{-\tau_k s}\right)^{-1} E_0, \end{aligned} \tag{7}$$

Applying the substitution (4) to the transfer function $T(s,\tau)$ in (7), we can define the *comparison system* with transfer function $H(s,\lambda)$ such that $H(j\omega,\lambda) = T(j\omega,\tau)$, whenever (5) holds. In this case, the comparison system's transfer function is given by the following two Lemmas

Lemma 1. (Cardeliquio et al. (2016)) For any finite $s \in \mathbb{C}$ and matrices $C_k \in \mathbb{R}^{p\times n}$, $A_k \in \mathbb{R}^{n\times n}$ and $E_0 \in \mathbb{R}^{n\times m}$

$$\begin{aligned} H(s) &= \left(\sum_{k=0}^{N} C_k s^k\right)\left(s^{N+1} I - \sum_{k=0}^{N} A_k s^k\right)^{-1} E_0 \\ &= \begin{bmatrix} C_0' \\ C_1' \\ \vdots \\ C_N' \end{bmatrix}' \left(sI - \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I \\ A_0 & A_1 & A_2 & \cdots & A_{N-1} & A_N \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_0 \end{bmatrix}. \end{aligned} \tag{8}$$

Lemma 2. The transfer function (7), through the substitutions (4) and (5) can be written in an equivalent form as

$$\begin{aligned} H(s,\lambda) &= \left[\begin{array}{c|c} A_\lambda & E \\ \hline C_z & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ \sum_{k=0}^{N} \alpha_k(0) A_k & \sum_{k=0}^{N} A_k \Gamma_k - \lambda \Gamma_\lambda & E_0 \\ \hline \sum_{k=0}^{N} \alpha_k(0) C_{zk} & \sum_{k=0}^{N} C_{zk} \Gamma_k & 0 \end{array}\right], \end{aligned} \tag{9}$$

in which $\Gamma_k, \Gamma_\lambda \in \mathbb{R}^{n\times Nn}$ are given by

$$\Gamma_k = [\alpha_k(1)\ \alpha_k(2)\ \alpha_k(3)\ \cdots\ \alpha_k(N-1)\ \alpha_k(N)] \otimes I, \tag{10}$$

$$\Gamma_\lambda = [\alpha_0(0)\ \alpha_0(1)\ \alpha_0(2)\ \cdots\ \alpha_0(N-1)] \otimes I, \tag{11}$$

and $\alpha_0(i)$, $\alpha_k(i)$, for $k = 0$ and $k \geq 1$, respectively, are given by

$$\alpha_0(i) = \binom{N}{i}, \tag{12}$$

$$\alpha_k(i) = \sum_{\ell=0}^{k-1} \binom{k-1}{\ell}\binom{N-k+1}{i-\ell}(-1)^{i-\ell}. \tag{13}$$

Proof. Substituting the Rekasius expression (4) in (7), we get

$$\begin{aligned} H(s,\lambda) = &\left(C_{z0} + \sum_{k=1}^{N} C_{zk}\left(\frac{\lambda - s}{\lambda + s}\right)^{N-k+1}\right) \times \\ &\times \left(sI - A_0 - \sum_{k=1}^{N} A_k \left(\frac{\lambda - s}{\lambda + s}\right)^{N-k+1}\right)^{-1} E_0. \end{aligned} \tag{14}$$

Then, we can multiply $H(s,\lambda)$ by $\frac{(\lambda+s)^N}{(\lambda+s)^N}$ to obtain

$$\begin{aligned} H(s,\lambda) = &\left(C_{z0}(\lambda+s)^N + \textstyle\sum_{k=1}^{N} C_{zk}(\lambda-s)^{N-k+1}(\lambda+s)^{k-1}\right) \times \\ &\times \left((sI - A_0)(\lambda+s)^N - \textstyle\sum_{k=1}^{N} A_k(\lambda-s)^{N-k+1}(\lambda+s)^{k-1}\right)^{-1} E_0. \end{aligned} \tag{15}$$

Expanding the binomials, the previous expression becomes

$$H(s,\lambda) = C_z(s,\lambda)(A(s,\lambda))^{-1} E_0, \tag{16}$$

where

$$
\begin{aligned}
C_z(s,\lambda) = {} & C_{z0}\sum_{i=0}^{N}\binom{N}{i}\lambda^{N-i}s^i + C_{z1}\sum_{i=0}^{N}\binom{N}{i}\lambda^{N-i}(-s)^i + \\
& + C_{z2}\sum_{i=0}^{N-1}\binom{N-1}{i}\lambda^{N-1-i}(-s)^i\sum_{\ell=0}^{1}\binom{1}{\ell}\lambda^{1-\ell}s^\ell + \cdots \\
& + C_{zN}\sum_{i=0}^{1}\binom{1}{i}\lambda^{1-i}(-s)^i\sum_{\ell=0}^{N-1}\binom{N-1}{\ell}\lambda^{N-1-\ell}s^\ell \\
& = C_{z0}\sum_{i=0}^{N}\binom{N}{i}\lambda^{N-i}s^i + \sum_{k=1}^{N}C_{zk}\sum_{i=0}^{N-k+1}\sum_{\ell=0}^{k-1}\times \\
& \times\binom{N+1-k}{i}\binom{k-1}{\ell}\lambda^{N-i-\ell}s^{i+\ell}(-1)^i,
\end{aligned}
\tag{17}
$$

and

$$
\begin{aligned}
A(s,\lambda) = {} & (sI-A_0)\sum_{i=0}^{N}\binom{N}{i}\lambda^{N-i}s^i - \sum_{k=1}^{N}A_k\sum_{i=0}^{N-k+1}\sum_{\ell=0}^{k-1}\times \\
& \times\binom{N+1-k}{i}\binom{k-1}{\ell}\lambda^{N-i-\ell}s^{i+\ell}(-1)^i.
\end{aligned}
\tag{18}
$$

One can immediately see that the powers of s are in the interval $[0 \quad N]$ and that the power of s and the power of λ always adds up to N. Hence, it is possible to group the terms that multiply the same power of s together

$$
\begin{aligned}
H(s,\lambda) = {} & \left(\sum_{i=0}^{N}\tilde{C}_{zi}\lambda^{N-i}s^i\right)\times \\
& \times\left(s^{N+1}I - \sum_{i=0}^{N}\tilde{A}_i\lambda^{N-i}s^i\right)^{-1}E_0,
\end{aligned}
\tag{19}
$$

where

$$
\tilde{C}_{zi} = \sum_{k=0}^{N}C_{zk}\alpha_k(i), \tag{20}
$$

$$
\tilde{A}_i = \sum_{k=0}^{N}A_k\alpha_k(i) - \lambda\alpha_0(i-1), \tag{21}
$$

where $\alpha_k(i)$ is given by (12) when $k=0$ and by (13) when $k \geq 1$. Now, to apply Lemma 1, all we need to do is a similarity transformation on (8) using

$$
L = \mathbf{diag}(\lambda^N I, \lambda^{N-1}I, \cdots, \lambda I, I). \tag{22}
$$

As Lemma 1 is applied to the transformed matrices, we observe that the terms depending on λ in (19) are cancelled. Finally, using (10)-(13), (20) and (21) we achieve (9) which concludes the proof.

2.1 $\mathcal{H}_\infty$ Norm Calculation

We will now show how to approximate

$$
\|T(s,\tau)\|_\infty = \sup_{\omega\in\mathbb{R}}\sigma(T(j\omega,\tau)) \tag{23}
$$

for a given $\tau \in [0,\tau^\star)$. The purpose is to show that the rational transfer function $H(s,\lambda)$ can be successfully used for $\mathcal{H}_\infty$ norm calculation.

In the light of the results presented in (Korogui et al., 2011), we extract an important property relating the $\mathcal{H}_\infty$ norm for both the comparison system and the original time-delay one. To this end, we need to define the scalar $\lambda_o = \inf\{\lambda \mid A_\lambda \text{ is Hurwitz}\}$, which allows us to state the following theorem, extending Theorem 1 of (Korogui et al., 2011).

Theorem 1. Assume that $\sum_{i=0}^{N} A_i$ is Hurwitz and, for each $\lambda \in (\lambda_0, \infty)$, let

$$
\alpha = \arg\sup_{\omega\in\mathbb{R}}\sigma(H(j\omega,\lambda)), \tag{24}
$$

and determine the time delay $\tau(\lambda,\alpha)$ that satisfies $\alpha/\lambda = \tan(\alpha\tau/2N)$. If $\tau(\lambda,\alpha) \in [0,\tau^\star)$ then,

$$
\|H(s,\lambda)\|_\infty \leq \|T(s,\tau(\lambda,\alpha))\|_\infty. \tag{25}
$$

See (Korogui et al., 2012) for the case $N=1$. Similar arguments hold for $N \geq 2$ using the Rekasius substitution (4).

3. OUTPUT FEEDBACK DESIGN

In this section we address the main problem of this work, the output-feedback design. Lets consider the following time-delay system with minimal realisation:

$$
\begin{aligned}
\dot{x}(t) &= A_0x(t) + \textstyle\sum_{j=1}^{M}A_jx(t-\tau_j) + B_0u(t) + E_0w(t), \\
z(t) &= C_{z0}x(t) + \textstyle\sum_{j=1}^{M}C_{zj}x(t-\tau_j) + D_{zu}u(t), \\
y(t) &= C_{y0}x(t) + \textstyle\sum_{j=1}^{M}C_{yj}x(t-\tau_j) + D_{yw}w(t),
\end{aligned}
\tag{26}
$$

where, in addition to the assumptions and the variables defined in previous sections, $y(t) \in \mathbb{R}^p$ is the measured signal. The aim at this point is to design a full order dynamic output feedback controller with the following structure

$$
\begin{aligned}
\dot{\hat{x}}(t) &= \hat{A}_0\hat{x}(t) + \sum_{k=1}^{N}\hat{A}_k\hat{x}(t-\tau_k) + \hat{B}_0y(t), \\
u(t) &= \hat{C}_0\hat{x}(t) + \sum_{k=1}^{N}\hat{C}_k\hat{x}(t-\tau_k),
\end{aligned}
\tag{27}
$$

where $\hat{x}(t) \in \mathbb{R}^n$ for all $t \in \mathbb{R}_+$ and $N = hM$ for some $h \in \{1,2,\ldots\}$. We, once again, through a suitable change of indices and considering null the matrices where the respective delay is not present, rewrite (26) as

$$
\begin{aligned}
\dot{x}(t) &= A_0x(t) + \textstyle\sum_{k=1}^{N}A_kx(t-\tau_k) + B_0u(t) + E_0w(t), \\
z(t) &= C_{z0}x(t) + \textstyle\sum_{k=1}^{N}C_{zk}x(t-\tau_k) + D_{zu}u(t), \\
y(t) &= C_{y0}x(t) + \textstyle\sum_{k=1}^{N}C_{yk}x(t-\tau_k) + D_{yw}w(t).
\end{aligned}
\tag{28}
$$

After connecting (27) to (28), we obtain

$$
\begin{aligned}
\dot{\xi}(t) &= F_0\xi(t) + \sum_{k=1}^{N}F_k\xi(t-\tau_k) + G_0w(t), \\
z(t) &= J_0\xi(t) + \sum_{k=1}^{N}J_k\xi(t-\tau_k),
\end{aligned}
\tag{29}
$$

where $\xi(t) = [x(t)' \quad \hat{x}(t)']' \in \mathbb{R}^{2n}$ is the state and the indicated matrices stand for

$$
\begin{aligned}
F_k &= \begin{bmatrix} A_k & B_0\hat{C}_k \\ \hat{B}_0C_{yk} & \hat{A}_k \end{bmatrix}, \quad J_k = \begin{bmatrix} C_{zk} & D_{zu}\hat{C}_k \end{bmatrix}, \\
G_0 &= \begin{bmatrix} E_0' & D_{yw}'\hat{B}_0' \end{bmatrix}'.
\end{aligned}
\tag{30}
$$

The transfer function $T_C(s,\tau)$ from the external input $w(t)$ to the controlled output $z(t)$ becomes exactly (4) if with consider $J_k \leftarrow C_{zk}$, $F_k \leftarrow A_k$ and $J_0 \leftarrow E_0$ where the subindex C indicates its dependence on a given controller of the form (27). Hence, the goal is to design a controller such that $\|T_C(s,\tau)\|_\infty < \gamma$ for a given $\gamma > 0$, which is accomplished by the definition of the following rational comparison system

$$
\begin{aligned}
H_C(s,\lambda) &= \left[\begin{array}{c|c} F_\lambda & G \\ \hline J & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} 0 & \lambda I & 0 \\ \sum_{k=0}^{N} \alpha_k(0) F_k & \sum_{k=0}^{N} F_k \Gamma_k - \lambda \Gamma_\lambda & G_0 \\ \hline \sum_{k=0}^{N} \alpha_k(0) J_k & \sum_{k=0}^{N} J_k \Gamma_k & 0 \end{array}\right]. \qquad (31)
\end{aligned}
$$

With this system, we can solve the corresponding $\mathcal{H}_\infty$ output feedback design problem for each $\lambda > 0$ and extract the corresponding time-delay $\tau(\lambda)$. Note that, even though the matrices of the state space realisation of $H_C(s,\lambda)$ depend on an intricate manner of the control state space realisation matrices by applying the following similarity transformation

$$
S = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \qquad (32)
$$

one can rewrite (31) in the equivalent form

$$
\begin{aligned}
H_C(s,\lambda) &= \left[\begin{array}{c|c} S F_\lambda S^{-1} & SG \\ \hline J S^{-1} & 0 \end{array}\right] \\
&= \left[\begin{array}{cc|c} A_\lambda & B\hat{C} & E \\ \hat{B} C_y & \hat{A}_\lambda & \hat{B} D_{yw} \\ \hline C_z & D_{zu}\hat{C} & 0 \end{array}\right], \qquad (33)
\end{aligned}
$$

where the system matrices (A_λ, E, C_z) have been defined in (9),

$$
B' = [0 \quad B_0'], \quad C_y = \begin{bmatrix} \sum_{k=0}^{N} \alpha_k(0) C_{yk} & \sum_{k=0}^{N} C_{yk} \Gamma_k \end{bmatrix}, \qquad (34)
$$

and the controller matrices are given by

$$
\begin{aligned}
\hat{A}_\lambda &= \begin{bmatrix} 0 & \lambda I \\ \sum_{k=0}^{N} \alpha_k(0) \hat{A}_k & \sum_{k=0}^{N} \hat{A}_k \Gamma_k - \lambda \Gamma_\lambda \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ \hat{B}_0 \end{bmatrix}, \\
\hat{C} &= \begin{bmatrix} \sum_{k=0}^{N} \alpha_k(0) \hat{C}_k & \sum_{k=0}^{N} \hat{C}_k \Gamma_k \end{bmatrix}, \qquad (35)
\end{aligned}
$$

indicating that they are in the comparison form. Hence, the controller (27) whenever connected to the time-delay system (28) produces an LTI comparison system associated to the regulated output (29) whose transfer function can be alternatively determined from the connection of the LTI comparison system of the system (28) and the LTI comparison system of the controller (27).

Given the particular structure of (35), we propose a strategy similar to the one presented in (Korogui et al., 2012) such that the controller matrices $(\hat{A}_\lambda, \hat{B}, \hat{C})$ will be replaced by general matrix variables (A_C, B_C, C_C). These two realisations are coupled by a nonsingular matrix $V \in \mathbb{R}^{(k+1)n \times (k+1)n}$ which defines the similarity transformation

$$
\hat{A}_\lambda = V A_C V^{-1}, \qquad (36)
$$

$$
\hat{B} = V B_C, \qquad (37)
$$

$$
\hat{C} = C_C V^{-1}. \qquad (38)
$$

These equalities hold under the conditions stated on the following Theorem.

Theorem 2. Let V be a nonsigular matrix such that

$$
\mathrm{vec}(V) \in \ker \begin{bmatrix} B_C' \otimes \begin{bmatrix} I & 0 \end{bmatrix} \\ A_C' \otimes \begin{bmatrix} I & 0 \end{bmatrix} - I \otimes \begin{bmatrix} 0 & \lambda I \end{bmatrix} \end{bmatrix} \qquad (39)
$$

and

$$
2n + n/k > q + 1. \qquad (40)
$$

Then, V satisfies

$$
(\hat{A}_\lambda, \hat{B}, \hat{C}) = (V A_C V^{-1}, V B_C, C_C V^{-1}). \qquad (41)
$$

Proof. From (39) we have that

$$
\begin{bmatrix} B_C' \otimes \begin{bmatrix} I & 0 \end{bmatrix} \\ A_C' \otimes \begin{bmatrix} I & 0 \end{bmatrix} - I \otimes \begin{bmatrix} 0 & \lambda I \end{bmatrix} \end{bmatrix} \mathrm{vec}(V) = 0 \qquad (42)
$$

Using de fact that $\mathrm{vec}(ABC) = (C' \otimes A)\mathrm{vec}(B)$, we can rewrite the matrix equality (42) into

$$
\begin{bmatrix} I & 0 \end{bmatrix} V B_C = 0 \qquad (43)
$$

and

$$
\begin{bmatrix} I & 0 \end{bmatrix} V A_C = \begin{bmatrix} 0 & \lambda I \end{bmatrix} V \qquad (44)
$$

The first equation ensures that the first kn rows of VB_C are zero. Hence, we have knq equations. The second equation says that the first kn rows of VA_CV^{-1} are in the form $\begin{bmatrix} 0 & \lambda I \end{bmatrix}$. Hence, we have more $kn(k+1)n$ equalities. Together, those equations guarantee the desired form of (35) and imply that

$$
((k+1)n)^2 > knq + kn(k+1)n, \qquad (45)
$$

which can be easily simplified to (40). Therefore, V can be obtained from $\mathrm{vec}^{-1}(\mathrm{vec}(V))$.

To obtain the matrices A_C, B_C e C_C we just solve the traditional LTI $\mathcal{H}_\infty$ problem for output-feedback. This can be achieved through Riccati equations (Doyle et al., 1989) under the usual assumptions $D_{zu}' C_z = 0$, $E D_{yw}' = 0$, $D_{yw} D_{yw}' = I$ and $D_{zu}' D_{zu} = I$, or via LMIs such as in (Gahinet, 1996).

Now, to finally recover the controller matrices we define

$$
\hat{A} = \begin{bmatrix} \sum_{k=0}^{N} \alpha_k(0) \hat{A}_k & \sum_{k=0}^{N} \hat{A}_k \Gamma_k \end{bmatrix}, \qquad (46)
$$

where $\hat{A}$ is obtained from the n last rows of $\hat{A}_\lambda$ added to $[0 \quad \Gamma_\lambda]$, and

$$
\tilde{\Gamma} = \begin{bmatrix} \alpha_0(0) & \alpha_0(1) & \cdots & \alpha_0(N) \\ \alpha_1(0) & \alpha_1(1) & \cdots & \alpha_1(N) \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_N(0) & \alpha_N(1) & \cdots & \alpha_N(N) \end{bmatrix}, \qquad (47)
$$

which is non singular, see (Cardeliquio et al., 2016). Thus, we may recover the desired matrices as follows

$$\begin{bmatrix} \hat{A}_0 & \hat{A}_1 & \cdots & \hat{A}_N \end{bmatrix} = \hat{A}\left(\tilde{\Gamma}\otimes I\right)^{-1}, \quad (48)$$

$$\begin{bmatrix} \hat{C}_0 & \hat{C}_1 & \cdots & \hat{C}_N \end{bmatrix} = \hat{C}\left(\tilde{\Gamma}\otimes I\right)^{-1}, \quad (49)$$

and $\hat{B}_0$ is immediately obtained from $\hat{B}$.

Once we have the controller matrices at hand, it is a simple matter of computation to determine whether $\|T_C(s,\tau(\lambda))\|_\infty < \gamma$ holds, see (Fioravanti et al., 2012) and (Avanessoff et al., 2014).

4. EXAMPLE

To illustrate the results for output feedback design we consider a second order example borrowed from (Fridman and Shaked, 2002a) and (Korogui et al., 2012) where the matrices corresponding to the state space realisation (28) are as follows

$$\left[\, A_0 \,\middle|\, A_1 \,\middle|\, E_0 \,\right] = \left[\begin{array}{cc|cc|c|cc} 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -0.9 & 1 & 1 & 0 \end{array}\right],$$

$$\left[\, B_0 \,\middle|\, C_{z0} \,\middle|\, C_{z1} \,\middle|\, D_{zu} \,\right] = \left[\begin{array}{c|cc|cc|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0.1 \end{array}\right],$$

$$\left[\, C_{y0} \,\middle|\, C_{y1} \,\middle|\, D_{yw} \,\right] = \left[\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0.1 \end{array}\right].$$

We can now solve two distinct problems, the maximum delay problem and the minimum norm problem. In the first one, for a fixed pre-specified $\mathcal{H}_\infty$ level γ, we find the output-feedback controller that maximise the delay. In the second one, for a fixed pre-specified maximum delay τ, we find the output-feedback controller matrices that minimise the $\mathcal{H}_\infty$ level γ. In both cases, $T(s,\tau)$ is stable and $\|T(s,\tau)\|_\infty \le \gamma$.

To illustrate the first problem, lets set $\gamma = 1$. Using $N = 1$ in the expansion for the comparison system, we achieve for the maximum delay $\tau = 1.2324$. This result and the behaviour of both norms, $\|T_c(s,\tau(\lambda))\|$ and $\|H_c(s,\lambda)\|$, as a function of τ is exactly the same as in (Korogui et al., 2012). However, increasing the expansion by using $N = 2$ and $N = 3$ we get for the maximum delay $\tau = 1.4268$ and $\tau_\gamma = 1.5117$ respectively. Besides the fact that we increased the maximum delay allowed by a factor of 22%, we also have, that for every $0 < \tau < \tau_\gamma$, the norm of the comparison system is the same as the norm of the system with delays as depicted in Figure 1. This results can be better visualised in the table below.

N	τ_{max}
1	1.2324
2	1.4268
3	1.5117

For the maximum delay

$$\|T_c(s,\tau(\lambda))\| = \|H_c(s,\lambda)\| = 0.2675. \quad (50)$$

We have obtained a sequence of stabilising controllers for each pair $(\lambda_N,\tau(\lambda_N))$ such that $\lambda_N \in (\lambda_\gamma,\infty)$ and $\tau(\lambda_N) \in [0,\tau_\gamma)$.

Now, setting $\tau = 1$. We get $\|T_c(s,\tau(\lambda))\| = 0.2010$ which is 26% smaller than the $\mathcal{H}_\infty$ norm obtained by (Korogui et al., 2012) and 76% smaller than the $\mathcal{H}_\infty$ norm obtained by (Fridman and Shaked, 2002b). We also have exactly the same norm for the comparison system, $\|H_c(s,\lambda)\| = 0.2010$. Finally, the controller matrices for this case are

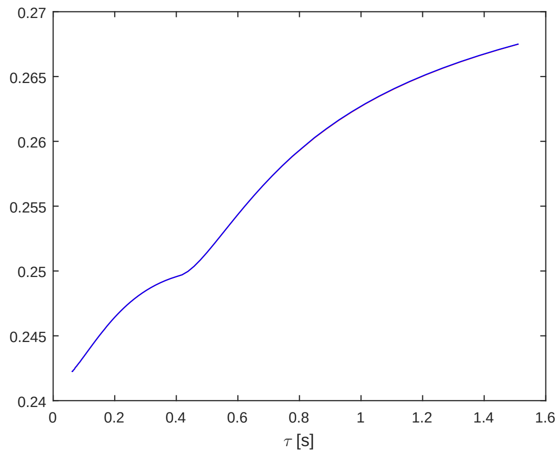

Fig. 1. $\mathcal{H}_\infty$ performance versus time delay for $\gamma = 1$

$$\left[\hat{A}_0 \middle| \hat{A}_1\right] = \left[\begin{array}{cc|cc} -15.1593 & 9.5743 & 1.2467 & -1.7115 \\ 18.5286 & -11.8869 & -0.6430 & 0.8165 \end{array}\right],$$

$$\left[\hat{A}_2 \middle| \hat{A}_3\right] = \left[\begin{array}{cc|cc} 0.7977 & -2.8727 & -1.2294 & -1.1746 \\ -1.4215 & 2.7861 & 1.2287 & 1.6388 \end{array}\right],$$

$$\left[\hat{C}'_0 \middle| \hat{C}'_1\right] = \left[\begin{array}{c|c} 39.1349 & -5.6235 \\ -24.8696 & 4.4252 \end{array}\right],$$

$$\left[\hat{C}'_2 \middle| \hat{C}'_3\right] = \left[\begin{array}{c|c} -1.1655 & 3.0810 \\ 7.2716 & 2.7257 \end{array}\right],$$

$$\hat{B}_0 = \left[-2.0019 \;\; 6.2982\right]'.$$

5. CONCLUSION

This work is an extension of (Cardeliquio et al., 2016). This time, we extended the procedure for time-delay control design based on a comparison system, obtained by Rekasius substitution, to implement output-feedback. To the best of the authors knowledge, this is the first procedure able to better use the buffer necessary for implementing delayed output-feedback, and obtaining simultaneously more stability margin and lower $\mathcal{H}_\infty$ level.

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