A MULTISCALE HYBRID MIXED METHOD FOR TIME-HARMONIC MAXWELL'S EQUATIONS IN TWO DIMENSIONS

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1. Notations

In the following, $\Omega \subset \mathbb{R}^2$ is a polygonal domain. We consider a family of meshes $(\mathcal{T}_H)_{H>0}$ of Ω such that, for all $H > 0$, the elements $K \in \mathcal{T}_H$ are triangles satisfying

$$\operatorname{diam}(K) = \sup_{x,y \in K} |x - y| \leq H,$$

In addition, we assume that there exists a constant $\beta > 0$, such that

$$\frac{\operatorname{diam}(K)}{\rho(K)} \geq \beta,$$

for all $K \in \mathcal{T}_H$ and $H > 0$, where

$$\rho(K) = \sup \left\{ r > 0 \mid \exists \mathbf{x} \in K;\ B(\mathbf{x}, r) \subset K \right\}.$$

Finally, for all $H > 0$ and for all pairs of distinct elements $K_+, K_- \in \mathcal{T}_H$, we assume that $\partial K_+ \cap \partial K_-$ is either empty, or a vertex or or a full edge of K_+ and K_-.

We denote by $\mathcal{E}_H^{int}$ and $\mathcal{E}_H^{ext}$ the set of internal and external edges of $\mathcal{T}_H$. We also employ the notation $\mathcal{E}_H = \mathcal{E}_H^{int} = \cup \mathcal{E}_H^{ext}$.

If $\boldsymbol{\phi}$ and ϕ are vectorial and scalar functions, we define

$$\operatorname{curl} \boldsymbol{\phi} = \frac{\partial \boldsymbol{\phi}_2}{\partial \mathbf{x}_1} - \frac{\partial \boldsymbol{\phi}_1}{\partial \mathbf{x}_2}, \quad \mathbf{curl} \phi = \left(\frac{\partial \phi}{\partial \mathbf{x}_2}, -\frac{\partial \phi}{\partial \mathbf{x}_1} \right),$$

where the derivatives are taken in the sense of distributions.

If $\mathcal{O} \subset \mathbb{R}^2$ is a connected open set with Lipschitz boundary $\partial \mathcal{O}$, $H^1(\mathcal{O})$ is the space of scalar functions $\phi \in L^2(\mathcal{O})$ such that $\boldsymbol{\nabla} \phi \in \mathbf{L}^2(\mathcal{O})$ and $\mathbf{H}(\operatorname{curl}, \mathcal{O})$ is the space of vectorial functions $\boldsymbol{\phi} \in \mathbf{L}^2(\mathcal{O})$ such that $\operatorname{curl} \boldsymbol{\phi} \in L^2(\mathcal{O})$. These spaces are equipped with the norms

$$\|\phi\|_{1,\mathcal{O}}^2 = \|\phi\|_{0,\mathcal{O}}^2 + \|\boldsymbol{\nabla} \phi\|_{0,\mathcal{O}}^2, \quad \|\boldsymbol{\phi}\|_{\operatorname{curl},\mathcal{O}}^2 = \|\boldsymbol{\phi}\|_{0,\mathcal{O}}^2 + \|\operatorname{curl} \boldsymbol{\phi}\|_{0,\mathcal{O}}^2.$$

If $\phi \in H^1(\mathcal{O})$, its trace on $\partial \mathcal{O}$ is defined as an element of $H^{1/2}(\partial \mathcal{O})$. The space $H^{1/2}(\partial \mathcal{O})$ is equipped with the quotient norm

$$\|\psi\|_{1/2,\partial \mathcal{O}} = \inf_{\phi \in H^1(\mathcal{O}} \left\{ \|\phi\|_{1,\mathcal{O}} \mid \phi|_{\partial \mathcal{O}} = \psi \right\}.$$

If $\boldsymbol{\phi} \in \mathbf{H}(\operatorname{curl}, \mathcal{O})$ its tangential trace is defined in $H^{-1/2}(\partial \mathcal{O})$ as

$$\int_{\partial \mathcal{O}} \boldsymbol{\phi} \times \mathbf{n} \overline{\psi} = \int_{\mathcal{O}} \operatorname{curl} \boldsymbol{\phi} \overline{\psi} - \boldsymbol{\phi} \cdot \mathbf{curl} \overline{\psi},$$

for all $\psi \in H^1(\Omega)$, and we have

$$\left| \int_{\partial\mathcal{O}} \boldsymbol{\phi} \times \mathbf{n}\overline{\psi} \right| \leq \|\boldsymbol{\phi}\|_{\mathrm{curl},\mathcal{O}} \|\psi\|_{1,\mathcal{O}}.$$

For every $H > 0$, the "broken spaces" $H^1(\mathcal{T}_H)$ (resp. $\mathbf{H}(\mathrm{curl}, \mathcal{T}_H)$) are defined as space of $\phi \in L^2(\Omega)$ (resp. $\boldsymbol{\phi} \in \mathbf{L}^2(\Omega)$) such that $\phi|_K \in H^1(K)$ (resp. $\boldsymbol{\phi}|_K \in \mathbf{H}(\mathrm{curl}, K)$) for all $K \in \mathcal{T}_H$. They are equipped with the norms

$$\|\phi\|^2_{1,\mathcal{T}_H} = \sum_{K\in\mathcal{T}_H} \|\phi|_K\|^2_{1,K}, \quad \|\boldsymbol{\phi}\|^2_{\mathrm{curl},\mathcal{T}_H} = \sum_{K\in\mathcal{T}_H} \|\boldsymbol{\phi}|_K\|^2_{\mathrm{curl},K}.$$

In addition, we introduce the space

$$H^{1/2}(\mathcal{E}_H) = \left\{ \eta \in \prod_{K\in\mathcal{T}_H} H^{1/2}(\partial K) \mid \exists \phi \in H^1(\Omega);\ \phi|_{\partial K} = \eta|_{\partial K} \quad \forall K \in \mathcal{T}_H \right\},$$

equipped with the norm

$$\|\eta\|^2_{1/2,\mathcal{E}_H} = \sum_{K\in\mathcal{T}_H} \|\eta|_K\|^2_{1/2,\partial K}.$$

We easily check that

$$\|\eta\|_{1/2,\mathcal{E}_H} = \inf_{\phi\in H^1(\Omega)} \left\{ \|\phi\|_{1,\Omega} \mid \phi|_{\partial K} = \eta|_{\partial K}; \quad \forall K \in \mathcal{T}_H \right\}.$$

If $\phi \in H^1(\Omega)$ we will use the notation $\phi|_{\mathcal{E}_H}$ to denote the element $\eta \in H^{1/2}(\mathcal{E}_H)$ such that $\eta|_{\partial K} = \phi|_{\partial K}$ for all $K \in \mathcal{T}_H$.

2. Maxwell's equations

We consider time-harmonic Maxwell's equations in the 2D domain Ω at frequency $\omega > 0$. For the sake of simplicity, we assume that Ω is an isotropic linear medium, so that it can be characterized by three measurable scalar functions $\epsilon, \mu, \sigma : \Omega \to \mathbb{R}$ that respectively represent the permittivity, the permeability, and the conductivity. We assume that there exist some real constants such that

$$0 < \epsilon_\star \leq \epsilon(\mathbf{x}) \leq \epsilon^\star < \infty, \quad 0 < \mu_\star \leq \mu(\mathbf{x}) \leq \mu^\star < \infty, \quad 0 < \sigma_\star \leq \sigma(\mathbf{x}) \leq \sigma^\star < \infty$$

for a.e. $\mathbf{x} \in \Omega$.

The electric field $\mathbf{u} : \Omega \to \mathbb{C}^2$ is solution to

$$\left\{ \begin{array}{rcll} (i\omega\sigma - \epsilon\omega^2)\,\mathbf{u} + \mathbf{curl}\,(\mu^{-1}\,\mathrm{curl}\,\mathbf{u}) & = & \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

where the action of an electric source $\mathbf{f} \in \mathbf{L}^2(\Omega)$

Classically, we recast (1) into a variational problem: find $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega), \tag{2}$$

where the sesquilinear form a is given by

$$a(\mathbf{u}, \mathbf{v}) = \left(\left(i\omega\sigma - \epsilon\omega^2\right)\mathbf{u}, \mathbf{v} \right) + \left(\mu^{-1}\,\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v} \right).$$

We can easily show that a is coercive, i.e. that there exists a $\theta \in [0, 2\pi)$ such that

$$|a(\mathbf{v}, e^{i\theta}\mathbf{v})| \gtrsim \|\mathbf{v}\|^2_{\mathrm{curl},\Omega}, \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega). \tag{3}$$

The existence and uniqueness of a solution $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ to (2) is thus a direct consequence of the Lax-Migram lemma.

3. An hybrid formulation

We introduce an hybrid formulation of (1), in which the tangential continuity of $\mathbf{u}$ is relaxed, and weakly imposed using a Lagrange multiplier λ. The relaxed space for $\mathbf{u}$ is $\mathbf{V} = \mathbf{H}(\mathrm{curl}, \mathcal{T}_H)$. As shown in Lemma 1, the correct space for the Lagrange multiplier is $\Lambda = H^{1/2}(\mathcal{E}_H)$.

Lemma 1. *Let $\mathbf{v} \in \mathbf{V}$. Then $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ if and only if*

$$b(\eta, \mathbf{v}) = 0, \quad \forall \eta \in \Lambda, \tag{4}$$

where

$$b(\eta, \mathbf{v}) = \sum_{K \in \mathcal{T}_H} \int_{\partial K} \overline{\mathbf{v} \times \mathbf{n}} \eta.$$

Proof. The key point of the proof is a convenient way of redefining $b(\eta, \mathbf{v})$ for $\eta \in \Lambda$ and $\mathbf{v} \in \mathbf{V}$. If $E \in \mathcal{E}_H^{int}$ is the edge shared by two elements $K_+, K_- \in \mathcal{T}_H$, since η have the same value on both side of the edge, it makes sense of defining the restriction $\eta|_E$ of η to E. On the other hand, since $\mathbf{v}$ has two independent definitions on K_- and K_+ we introduce the notation $\mathbf{v}_\pm \times \mathbf{n}_\pm = (\mathbf{v}|_{K_\pm}) \times \mathbf{n}_\pm|_E$, where $\mathbf{n}_\pm$ is the unit vector normal to E pointing outside $K_\pm$. Then, we can write

$$b(\eta, \mathbf{v}) = \sum_{E \in \mathcal{E}_H^{ext}} \int_E \overline{\mathbf{v} \times \mathbf{n}} \eta + \sum_{E \in \mathcal{E}_H^{int}} \int_E \overline{\mathbf{v}_+ \times \mathbf{n}_+ + \mathbf{v}_- \times \mathbf{n}_-} \eta. \tag{5}$$

Assume that $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$. Because of the essential boundary condition, we have $\mathbf{v}|_E \times \mathbf{n} = 0$ on all exterior edges $E \in \mathcal{E}_H^{ext}$. On the other hand, because the tangential component of $\mathbf{v}$ is continuous, if $E \in \mathcal{E}_H^{int}$ is the edge shared by two elements $K_+, K_- \in \mathcal{T}_H$, we have

$$\mathbf{v}_+ \times \mathbf{n}_+ + \mathbf{v}_- \times \mathbf{n}_- = 0. \tag{6}$$

As a result, (4) directly follows from (5).

On the other hand, consider $\mathbf{v} \in \mathbf{V}$ such that (4) holds. We first show that $\mathbf{v} \in \mathbf{H}(\mathrm{curl}, \Omega)$. To this end, we introduce an arbitrary test function $\phi \in H^1_0(\Omega)$. We have

$$\begin{aligned}\int_\Omega \overline{\mathbf{v}} \cdot \mathbf{curl}\phi &= \sum_{K\in\mathcal{T}_H} \int_K \overline{\mathbf{v}} \cdot \mathbf{curl}\phi \\ &= \sum_{K\in\mathcal{T}_H} \int_K \overline{\mathrm{curl}\,\mathbf{v}}\phi + \sum_{\partial K} \overline{\mathbf{v}\times\mathbf{n}}\phi \\ &= \sum_{K\in\mathcal{T}_H} \int_K \overline{\mathrm{curl}\,\mathbf{v}}\phi + b(\eta, \mathbf{v}),\end{aligned}$$

where η is the element of Λ defined by $\eta|_{\partial K} = \phi|_{\partial K}$ for all $K \in \mathcal{T}_H$. Since we assume that (4) holds, it follows that

$$\int_\Omega \overline{\mathbf{v}} \cdot \mathbf{curl}\phi = \int_\Omega \left(\sum_{K\in\mathcal{T}_H} \overline{\mathrm{curl}\,\mathbf{v}|_K} \right) \phi,$$

which means that

$$\mathrm{curl}\,\mathbf{v} = \sum_{K\in\mathcal{T}_H} \mathrm{curl}\,\mathbf{v}|_K \in L^2(\Omega),$$

in the sense of distribution. As a result, $\mathbf{v} \in \mathbf{H}(\mathrm{curl}, \Omega)$.

Since we established that $\mathbf{v} \in \mathbf{H}(\mathrm{curl}, \Omega)$, (6) holds, and $\mathbf{v} \times \mathbf{n} \in H^{-1/2}(\partial\Omega)$. Thus, using (5), for all $\phi \in H^1(\Omega)$, if $\eta = \phi|_{\mathcal{E}_H}$, we have

$$0 = b(\eta, \mathbf{v}) = \int_{\partial\Omega} \overline{\mathbf{v}\times\mathbf{n}}\eta.$$

Since this last equality is true for every element of $H^{1/2}(\partial\Omega)$, we conclude that $\mathbf{v} \times \mathbf{n} = 0$ on $\partial\Omega$, and $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$. □

The hybrid formulation is simply obtained by relaxing the tangential continuity and introducing the Lagrange multiplier: find $(\mathbf{u}, \lambda) \in \mathbf{V} \times \Lambda$ such that

$$\left\{ \begin{array}{rcll} a(\mathbf{u}, \mathbf{v}) + b(\lambda, \mathbf{v}) &=& (\mathbf{f}, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}, \\ b(\eta, \mathbf{u}) &=& 0, & \forall \eta \in \Lambda, \end{array} \right. \tag{7}$$

where we have extended the definition of a over $\mathbf{V}$ by

$$a(\mathbf{u}, \mathbf{v}) = \sum_{K\in\mathcal{T}_H} \left\{ \left(\left(i\omega\sigma - \epsilon\omega^2 \right) \mathbf{u}|_K, \mathbf{v}|_K \right) + (\mathrm{curl}\,\mathbf{u}|_K, \mathrm{curl}\,\mathbf{v}|_K) \right\}.$$

Theorem 1. *$\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ is solution to (2) if and only if $(\mathbf{u}, (\mu^{-1}\,\mathrm{curl}\,\mathbf{u})|_{\mathcal{E}_H}) \in \mathbf{V} \times \Lambda$ is solution to (7).*

Proof. Assume that $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ is solution to (2). Then, by Lemma 1, $b(\eta, \mathbf{u}) = 0$ for all $\eta \in \Lambda$, and the second equation of (7) is satisfied.

Since $\mathbf{u}$ satisfies the first equation of (1) in the sense of distributions, we have

$$\mathbf{curl}\left(\mu^{-1}\,\mathrm{curl}\,\mathbf{u}\right) = \mathbf{f} + \left(\epsilon\omega^2 - i\omega\sigma\right)\mathbf{u} \in \mathbf{L}^2(\Omega).$$

It follows that $\mu^{-1}\operatorname{curl}\mathbf{u} \in H^1(\Omega)$, and it makes sense to consider the restriction $(\mu^{-1}\operatorname{curl}\mathbf{u})|_{\mathcal{E}_H}$ as an element of Λ.

Consider $\mathbf{v} \in \mathbf{V}$ multiplying the first equation of (1) and by $\mathbf{v}$ and performing an integration by parts over an arbitrary element $K \in \mathcal{T}_H$, we have

$$\left(\left(i\omega\sigma - \epsilon\omega^2\right)\mathbf{u}|_K, \mathbf{v}|_K\right) + (\mu^{-1}\operatorname{curl}\mathbf{u}|_K, \operatorname{curl}\mathbf{v}|_K) + \int_{\partial K} \overline{\mathbf{v}|_K \times \mathbf{n}_K}\mu^{-1}\operatorname{curl}\mathbf{u}|_K = (\mathbf{f}|_K, \mathbf{v}|_K).$$

Then, by summation over $K \in \mathcal{T}_H$, we obtain that

$$a(\mathbf{u}, \mathbf{v}) + b((\mu^{-1}\operatorname{curl}\mathbf{u})|_{\mathcal{E}_H}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V},$$

which is the first equation of (7). We have thus established that $(\mathbf{u}, (\mu^{-1}\operatorname{curl}\mathbf{u})|_{\mathcal{E}_H} \in \mathbf{V} \times \Lambda$ is solution to (7).

Now, assume that $(\mathbf{u}, \lambda) \in \mathbf{V} \times \Lambda$ is solution to (7). Then, by Lemma 1, we have that $\mathbf{u} \in \mathbf{H}_0(\operatorname{curl}, \Omega)$. Using again Lemma 1, we see that for all $\mathbf{v} \in \mathbf{H}_0(\operatorname{curl}, \Omega)$

$$a(\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) + b(\lambda, \mathbf{v}) = (\mathbf{f}, \mathbf{v}),$$

so that $\mathbf{u}$ is solution to (2).

It remains to identify λ. If $\mathbf{v} \in \mathbf{H}(\operatorname{curl}, K)$, we have

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_K \left(i\omega\sigma - \epsilon\omega^2\right)\mathbf{u} \cdot \overline{\mathbf{v}} + \int_K \mu^{-1}\operatorname{curl}\mathbf{u}\overline{\operatorname{curl}\mathbf{v}} \\ &= \int_K \left(\left(i\omega\sigma - \epsilon\omega^2\right)\mathbf{u} + \mathbf{curl}\left(\mu^{-1}\operatorname{curl}\mathbf{u}\right)\right) \cdot \overline{\mathbf{v}} + \int_{\partial K} \overline{\mathbf{v} \times \mathbf{n}}\mu^{-1}\operatorname{curl}\mathbf{u}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) - b(\lambda, \mathbf{v}) \\ &= \int_K \mathbf{f} \cdot \overline{\mathbf{v}} + \int_{\partial K} \overline{\mathbf{v} \times \mathbf{n}}\lambda, \end{aligned}$$

and we identify that

$$\mu^{-1}\operatorname{curl}\mathbf{u} = \lambda \text{ on } \partial K.$$

Because the above identification is valid for every element $K \in \mathcal{T}_H$, we obtain that $\lambda = (\mu^{-1}\operatorname{curl}\mathbf{u})|_{\mathcal{E}_H}$, which is the desired result. □

As a direct consequence of Theorem 1, we have:

Corollary 1. *There exists a unique pair* $(\mathbf{u}, \lambda) \in \mathbf{V} \times \Lambda$ *solution to* (7).

4. The Multiscale Hybrid Mixed formulation

The Multiscale Hybrid Mixed (MHM) formulation is formally obtained from hybrid formulation (7) by substituting $\mathbf{u}$ by λ in the first equation. Assuming that $\lambda \in \Lambda$ is known, we can write

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) - b(\lambda, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}.$$

Since a is coercive over $\mathbf{V}$, we see that $\mathbf{u}$ is uniquely determined given λ. In addition, by linearity, we can formally write that

$$\mathbf{u} = \mathbf{T}\lambda + \hat{\mathbf{T}}\mathbf{f}, \tag{8}$$

for two linear operators $\mathbf{T}$ and $\hat{\mathbf{T}}$. Then, plugging (8) into the second equation of (7), we obtain a variational problem that is solely expressed in terms of λ:

$$b(\eta, \mathbf{T}\lambda) = -b(\eta, \hat{\mathbf{T}}\mathbf{f}), \quad \forall \eta \in \Lambda.$$

We start by properly defining the substitution operators $\mathbf{T}$ and $\hat{\mathbf{T}}$ and discuss their basic properties. Then, we establish that the MHM formulation is well-posed, and that its solution coincides with the solution of the original problem.

4.1. **Local operators.** For all $\eta \in \Lambda$ and $\mathbf{h} \in \mathbf{L}^2(\Omega)$ the image of η and $\mathbf{h}$ through $\mathbf{T}$ and $\hat{\mathbf{T}}$ are defined as the solutions to

$$a(\mathbf{T}\eta, \mathbf{v}) = -b(\eta, \mathbf{v}), \quad a(\hat{\mathbf{T}}\mathbf{h}, \mathbf{v}) = (\mathbf{h}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}. \tag{9}$$

Since a is coercive over $\mathbf{V}$, it is clear that (9) uniquely defines $\mathbf{T}\eta$ and $\hat{\mathbf{T}}\mathbf{h}$. As a result, $\mathbf{T} : \Lambda \to \mathbf{V}$ and $\hat{\mathbf{T}} : \mathbf{L}^2(\Omega) \to \mathbf{V}$ are well defined linear operators.

Lemma 2. *For all $\eta \in \Lambda$ and $\mathbf{h} \in \mathbf{L}^2(\Omega)$, we have*

$$\|\mathbf{T}\eta\|_{\mathbf{V}} \lesssim \|\eta\|_{\Lambda}, \quad \|\hat{\mathbf{T}}\mathbf{h}\|_{\mathbf{V}} \lesssim \|\mathbf{h}\|_{0,\Omega}.$$

Proof. Let $\eta \in \Lambda$. We have

$$\|\mathbf{T}\eta\|_{\mathbf{V}}^2 \lesssim |a(\mathbf{T}\eta, e^{i\theta}\mathbf{T}\eta)| = |b(\eta, \mathbf{T}\eta)|.$$

For all $\phi \in H^1(\Omega)$, since

$$\left| \int_{\partial K} \overline{\mathbf{T}\eta \times \mathbf{n}} \phi \right| \leq \|\phi\|_{1,K} \|\mathbf{T}\eta\|_{\mathrm{curl},K},$$

we have

$$|b(\phi, \mathbf{T}\eta)| \leq \|\phi\|_{1,\Omega} \|\mathbf{T}\eta\|_{\mathbf{V}},$$

for all $\phi \in H^1(\Omega)$. Hence, if $\phi \in H^1(\Omega)$ satisfies $\phi|_{\partial K} = \eta|_{\partial} K$ for all $K \in \mathcal{T}_H$, we have

$$\|\mathbf{T}\eta\|_{\mathbf{V}} \lesssim \|\phi\|_{1,\Omega},$$

but then, by taking the infinimum, we obtain

$$\|\mathbf{T}\eta\|_{\mathbf{V}} \lesssim \inf_{\phi \in H^1(\Omega)} \{\|\phi\|_{1,\Omega} \mid \phi|_{\partial K} = \eta|_{\partial K} \; \forall K \in \mathcal{T}_H\} = \|\eta\|_{\Lambda}.$$

Now, if $\mathbf{h} \in \mathbf{L}^2(\Omega)$, we have

$$\begin{aligned} \|\hat{\mathbf{T}}\mathbf{h}\|_{\mathbf{V}}^2 &\lesssim |a(\hat{\mathbf{T}}\mathbf{h}, e^{i\theta}\hat{\mathbf{T}}\mathbf{h})| \\ &= |(\mathbf{h}, \hat{\mathbf{T}}\mathbf{h})| \\ &\leq \|\mathbf{h}\|_{0,\Omega} \|\hat{\mathbf{T}}\mathbf{h}\|_{0,\Omega} \\ &\leq \|\mathbf{h}\|_{0,\Omega} \|\hat{\mathbf{T}}\mathbf{h}\|_{\mathbf{V}}, \end{aligned}$$

and the result follows. □

Proposition 1. *For all $\eta \in \Lambda$, we have*

$$\|\eta\|_\Lambda \lesssim \|\mathbf{T}\eta\|_{\mathbf{V}}.$$

Proof. Let $\eta \in \Lambda$. For each element $K \in \mathcal{T}_H$, we have

$$(i\omega\sigma - \epsilon\omega^2)\mathbf{T}\eta + \mathbf{curl}\left(\mu^{-1}\operatorname{curl}\mathbf{T}\eta\right) = 0.$$

Letting $\phi = \mu^{-1}\operatorname{curl}\mathbf{T}\eta$, we see that

$$\boldsymbol{\nabla}^\perp\phi = \mathbf{curl}\phi = \left(\epsilon\omega^2 - i\omega\sigma\right)\mathbf{T}\eta \in \mathbf{L}^2(\Omega).$$

It follows that $\phi \in H^1(K)$ with

$$\begin{aligned}\|\phi\|_{1,K} &\lesssim \|\phi\|_{0,K} + \|\boldsymbol{\nabla}\phi\|_{1,K}\\ &\lesssim \|\mu^{-1}\operatorname{curl}\mathbf{T}\eta\|_{0,K} + \|\left(\epsilon\omega^2 - i\omega\sigma\right)\mathbf{T}\eta\|_{1,K}\\ &\lesssim \|\mathbf{T}\eta\|_{\operatorname{curl},K}.\end{aligned}$$

Since in addition, $\phi|_{\partial K} = \mu|_{\partial K}$ we obtain a global definition for $\phi \in H^1(\Omega)$ with

$$\|\phi\|_{1,\Omega} \lesssim \|\mathbf{T}\eta\|_{\mathbf{V}},$$

but by definition of $\|.\|_\Lambda$ as the quotient norm, we have

$$\|\eta\|_\Lambda \lesssim \|\phi\|_{1,\Omega},$$

and the result follows. □

Corollary 2. *For all $\eta \in \Lambda$, we have*

$$\|\eta\|_\Lambda \lesssim \|\mathbf{T}\eta\|_{\mathbf{V}} \lesssim \|\eta\|_\Lambda.$$

Finally, we established that the sesquilinar form associated with the MHM formulation is coercive over Λ:

Corollary 3. *We have*

$$|b(e^{-i\theta}\eta, \mathbf{T}\eta)| \gtrsim \|\eta\|_\Lambda^2$$

for all $\eta \in \Lambda$.

Proof. Let $\eta \in \Lambda$, by definition of $\mathbf{T}$, we have

$$b(e^{-i\theta}\eta, \mathbf{T}\eta) = -a(\mathbf{T}\left(e^{-i\theta}\eta\right), \mathbf{T}\eta) = a(\mathbf{T}\eta, e^{i\theta}\mathbf{T}\eta),$$

and it follows that

$$|b(e^{-i\theta}\eta, \mathbf{T}\eta)| = |a(\mathbf{T}\eta, e^{i\theta}\mathbf{T}\eta)| \gtrsim \|\mathbf{T}\eta\|_{\mathbf{V}}^2.$$

Then, the conclusion follows from Corollary (2). □

Theorem 2. *For all $\eta \in \Lambda$ and $\mathbf{h} \in \mathbf{H}(\operatorname{div}, \Omega)$, we have $\mathbf{T}\eta, \hat{\mathbf{T}}\mathbf{h} \in \mathbf{H}(\operatorname{div}, \Omega)$, and*

$$\operatorname{div}\left(\left(i\omega\sigma - \epsilon\omega^2\right)\mathbf{T}\eta\right) = 0, \quad \operatorname{div}\left(\left(i\omega\sigma - \epsilon\omega^2\right)\hat{\mathbf{T}}\mathbf{h}\right) = \operatorname{div}\mathbf{h}.$$

Proof. Let $\eta \in \Lambda$. For all $\phi \in H^1_0(\Omega)$, since $\operatorname{curl} \boldsymbol{\nabla}\phi = 0$, we have

$$\begin{aligned}\int_\Omega \left(i\omega\sigma - \epsilon\omega^2\right) \mathbf{T}\eta \cdot \overline{\boldsymbol{\nabla}\phi} &= \sum_{K\in\mathcal{T}_H} \int_K \left(i\omega\sigma - \epsilon\omega^2\right) \mathbf{T}\eta \cdot \overline{\boldsymbol{\nabla}\phi} \\ &= \sum_{K\in\mathcal{T}_H} \int_K \left(i\omega\sigma - \epsilon\omega^2\right) \mathbf{T}\eta \cdot \overline{\boldsymbol{\nabla}\phi} + \mu^{-1} \operatorname{curl} \mathbf{T}\eta \operatorname{curl} \overline{\boldsymbol{\nabla}\phi} \\ &= a(\mathbf{T}\eta, \boldsymbol{\nabla}\phi).\end{aligned}$$

Then, since $\boldsymbol{\nabla}\phi \in \mathbf{H}_0(\operatorname{curl}, \Omega)$, using Lemma 1 and by definition of the operator $\mathbf{T}$, we obtain

$$a(\mathbf{T}\eta, \boldsymbol{\nabla}\phi) = -b(\eta, \boldsymbol{\nabla}\phi) = 0,$$

so that

$$\int_\Omega \left(i\omega\sigma - \epsilon\omega^2\right) \mathbf{T}\eta \cdot \overline{\boldsymbol{\nabla}\phi} = 0, \quad \forall \phi \in H^1_0(\Omega),$$

which means that $(i\omega\sigma - \epsilon\omega^2)\mathbf{T}\eta \in \mathbf{H}(\operatorname{div}, \Omega)$, and

$$\operatorname{div}\left((i\omega\sigma - \epsilon\omega^2)\mathbf{T}\eta\right) = 0.$$

Now, if $\mathbf{h} \in \mathbf{H}(\operatorname{div}, \Omega)$, the same arguments than above show that

$$\int_\Omega \left(i\omega\sigma - \epsilon\omega^2\right) \hat{\mathbf{T}}\mathbf{h} \cdot \overline{\boldsymbol{\nabla}\phi} = a(\hat{\mathbf{T}}\mathbf{h}, \boldsymbol{\nabla}\phi) = \int_\Omega \mathbf{h} \cdot \overline{\boldsymbol{\nabla}\phi}, \quad \forall \phi \in H^1_0(\Omega).$$

It follows that $(i\omega\sigma - \epsilon\omega^2)\hat{\mathbf{T}}\mathbf{h} \in \mathbf{H}(\operatorname{div}, \Omega)$ with

$$\operatorname{div}\left((i\omega\sigma - \epsilon\omega^2)\hat{\mathbf{T}}\mathbf{h}\right) = \operatorname{div}\mathbf{h}.$$

□

4.2. **The MHM formulation.**

Theorem 3. *For each $\mathbf{f} \in \mathbf{L}^2(\Omega)$, there exists a unique $\lambda \in \Lambda$ such that*

$$b(\eta, \mathbf{T}\lambda) = -b(\eta, \hat{\mathbf{T}}\mathbf{f}) \tag{10}$$

for all $\eta \in \Lambda$. In addition, if we define

$$\mathbf{u} = \mathbf{T}\lambda + \hat{\mathbf{T}}\mathbf{f} \in \mathbf{V},$$

then the pair $(\mathbf{u}, \lambda)$ is solution to (7).

Proof. The existence and uniqueness of $\eta \in \Lambda$ follows from Corollary 3 and Lax-Milgram lemma. Then, we let

$$\mathbf{u} = \mathbf{T}\lambda + \hat{\mathbf{T}}\mathbf{f} \in \mathbf{V}.$$

From (10), we see that

$$b(\eta, \mathbf{u}) = 0, \quad \forall \eta \in \Lambda.$$

On the other hand, by linearity, we have

$$a(\mathbf{u}, \mathbf{v}) = a(\mathbf{T}\lambda, \mathbf{v}) + a(\hat{\mathbf{T}}\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V},$$

and by definition of the operators $\mathbf{T}$ and $\hat{\mathbf{T}}$, we see that

$$a(\mathbf{u},\mathbf{v}) + b(\lambda,\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V},$$

so that the pair $(\mathbf{u},\lambda)$ is solution to (7). □

5. Discrete MHM problem

We obtain a discrete version of (10) by introducing a finite dimensional subspace $\Lambda_H \subset \Lambda$. Here, we consider spaces Λ_H that are made of piecewise polynomials. First we introduce the space $\mathcal{L}_H$ of Lagrange finite elements of degree k:

$$\mathcal{L}_H^k = \left\{ v_H \in H^1(\Omega) \mid v_H|_K \in \mathcal{P}_k(K) \quad \forall K \in \mathcal{T}_H \right\}.$$

Then, the space Λ_H is obtained by restrictions of Lagrange finite elements onto the $\mathcal{E}_H$:

$$\Lambda_H^k = \left\{ \lambda_H \in \Lambda \mid \exists v_H \in \mathcal{L}_H^k;\ \lambda_H = v_H|_{\varepsilon_H} \right\}.$$

Theorem 4. *For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, there exists a unique $\lambda_H \in \Lambda_H^k$ such that*

$$b(\eta_H, \mathbf{T}\lambda_H) = -b(\eta_H, \hat{\mathbf{T}}\mathbf{f}), \quad \forall \eta_H \in \Lambda_H. \tag{11}$$

In addition, if $\lambda \in \Lambda$ solve (10), we have

$$\|\lambda - \lambda_H\|_\Lambda \lesssim \inf_{\eta_H \in \Lambda_H} \|\lambda - \eta_H\|_\Lambda.$$

It remains to analyze the approximation properties of the space Λ_H in the $\|.\|_\Lambda$ norm to obtain an error estimate. To this end, we introduce an interpolation operator. Assume that $\eta \in \Lambda$ is such that $\eta = v|_{\varepsilon_H}$ for some $v \in H^2(\mathcal{T}_H)$. Then $\pi_H^k \eta \in \Lambda_H^k$ is defined as

$$\pi_H^k \eta = (\mathcal{I}_H^k v)|_{\varepsilon_H},$$

where $\mathcal{I}_H^k$ is the Lagrange interpolant of v. We have:

Lemma 3. *Let $\eta \in \Lambda$ such that $\eta = v|_{\varepsilon_H}$ for some $v \in H^{k+1}(\mathcal{T}_H)$. Then we have*

$$\|\eta - \pi_H \eta\|_\Lambda \lesssim H^k |v|_{k+1,\mathcal{T}_H}. \tag{12}$$

Proof. Let $\eta \in \Lambda$ such that $\eta = v|_{\varepsilon_H}$ with $v \in H^2(\mathcal{T}_H)$. For all element $K \in \mathcal{T}_H$, we have

$$\|\eta - \pi_H \eta\|_{1/2,\partial K} = \inf_{w \in H^1(K)} \left\{ \|w\|_{1,K} \mid w|_{\partial K} = \eta - \pi_H \eta \right\}.$$

We observer that the function $\tilde{w} = v - \mathcal{I}hv \in H^1(K)$ satisfies $\tilde{w}|_{\partial K} = \eta - \pi_H \eta$. As a result,

$$\|\eta - \pi_H \eta\|_{1/2,\partial K} \leq \|\tilde{w}\|_{1,K} = \|v - \mathcal{I}_h v\|_{1,K}.$$

Then, standard interpolation properties ensure that

$$\|\eta - \pi_H \eta\| \lesssim H^k |v|_{k+1,K},$$

and we obtain (12) by summation over $K \in \mathcal{T}_H$. □

Corollary 4. *Assume the solution* $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ *to* (2) *is such that* $\mu^{-1} \operatorname{curl} \mathbf{u} \in H^{k+1}(\mathcal{T}_H)$. *Then, we have*

$$\|\lambda - \lambda_H\|_\Lambda \lesssim H^k |\operatorname{curl} \mathbf{u}|_{k+1,\mathcal{T}_H},$$

and

$$\|\mathbf{u} - \mathbf{u}_H\|_{\mathrm{curl},\mathcal{T}_H} \lesssim H^k |\operatorname{curl} \mathbf{u}|_{k+1,\mathcal{T}_H},$$

where $\lambda = \operatorname{curl} \mathbf{u}|_{\mathcal{E}_H}$, $\lambda_H \in \Lambda_H$ *is the solution to* (11), *and* $\mathbf{u}_H = \mathbf{T}\lambda_H + \hat{\mathbf{T}}\mathbf{f}$.

6. Representation of the shape functions

In the following, we focus on the discrete spaces Λ_H^1 and Λ_H^2. For the case of Λ_H^1, we consider a basis $\{\psi^j\}$ that consists of one shape function ψ^j for each vertex in the mesh $\mathcal{T}_H$. In each element $K \in \mathcal{T}_H$ the basis functions ψ^j admit local expressions. These local expressions are associated with the vertices of $K = ((i-1)h, ih) \times ((j-1)h, jh)$, and have definitions

$$\begin{cases} \hat\lambda^1(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) &=& (1-\hat{\mathbf{x}}_1)(1-\hat{\mathbf{x}}_2) \\ \hat\lambda^2(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) &=& (1-\hat{\mathbf{x}}_1)\hat{\mathbf{x}}_2 \\ \hat\lambda^3(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) &=& \hat{\mathbf{x}}_1\hat{\mathbf{x}}_2 \\ \hat\lambda^4(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) &=& \hat{\mathbf{x}}_1(1-\hat{\mathbf{x}}_2) \end{cases}$$

where

$$\hat{\mathbf{x}}_1 = \frac{\mathbf{x}_1 - (i-1)h}{h}, \quad \hat{\mathbf{x}}_2 = \frac{\mathbf{x}_2 - (j-1)h}{h}.$$

In the case of the space Λ_H^2, we add to the basis of Λ_H^1 one shape function for each edge of $\mathcal{T}_H$. These shape functions also admits local expressions in each element $K \in \mathcal{T}_H$, that are associated with the edges of $K = ((i-1)h, ih) \times ((j-1)h, jh))$:

$$\begin{cases} \hat\lambda^5(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) &=& \mathbf{x}_1(1-\mathbf{x}_1)(1-\hat{\mathbf{x}}_2) \\ \hat\lambda^6(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) &=& \mathbf{x}_1\mathbf{x}_2(1-\hat{\mathbf{x}}_2) \\ \hat\lambda^7(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) &=& (1-\mathbf{x}_1)(1-\mathbf{x}_1)(1-\hat{\mathbf{x}}_2) \\ \hat\lambda^8(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2) &=& (1-\mathbf{x}_1)\mathbf{x}_2(1-\hat{\mathbf{x}}_2) \end{cases}$$

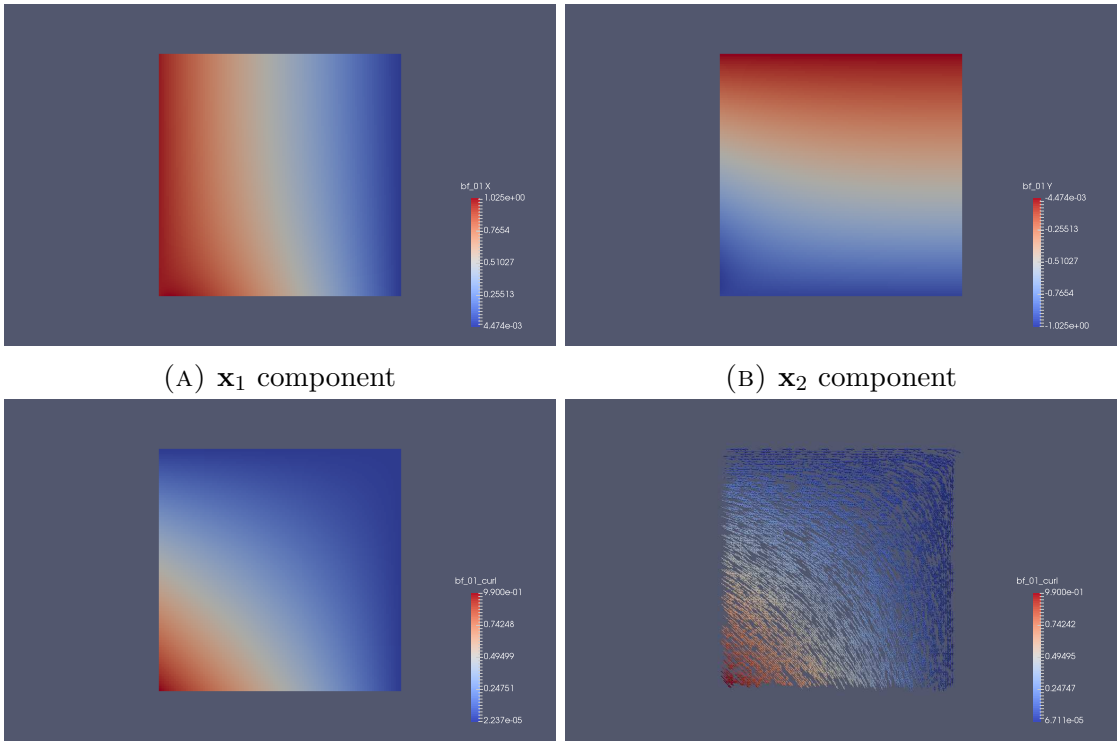
where

$$\hat{\mathbf{x}}_1 = \frac{\mathbf{x}_1 - (i-1)h}{h}, \quad \hat{\mathbf{x}}_2 = \frac{\mathbf{x}_2 - (j-1)h}{h}.$$

In order to better illustrate the shape functions, we plot $(\mathbf{T}\psi^j)$ for two different meshes. First, we consider a very simple mesh made of a single square $K = (0,1)^2$. In that case, since we only have one element, the shape functions ψ^j exactly coincide with their local expressions $\hat\lambda^j$. The nodal shape function $\mathbf{T}\hat\lambda^1$ is represented on Figure 1. Similarly, we depict on Figure 2 the edge shape function $\mathbf{T}\hat\lambda^5$.

Then, we represent on Figures 3 and 4 a nodal and an edge shape function in the case of a 2×2 mesh.

We remark on Figures 3 and 4 that indeed, $T\lambda^j \in \mathbf{H}(\mathrm{div}, \Omega)$ but $T\lambda^j \notin \mathbf{H}(\mathrm{curl}, \Omega)$.



(A) $\mathbf{x}_1$ component (B) $\mathbf{x}_2$ component

(C) curl (D) vectorial representation

FIGURE 1. $\hat{\mathbf{T}}\lambda^1$

7. Numerical experiments

7.1. **Convergence of linear elements.** We consider the problem

$$\begin{cases} \mathbf{u} + \mathbf{curl}\,\operatorname{curl}\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega \\ \mathbf{u}\times\mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega = (0,1)^2$ and

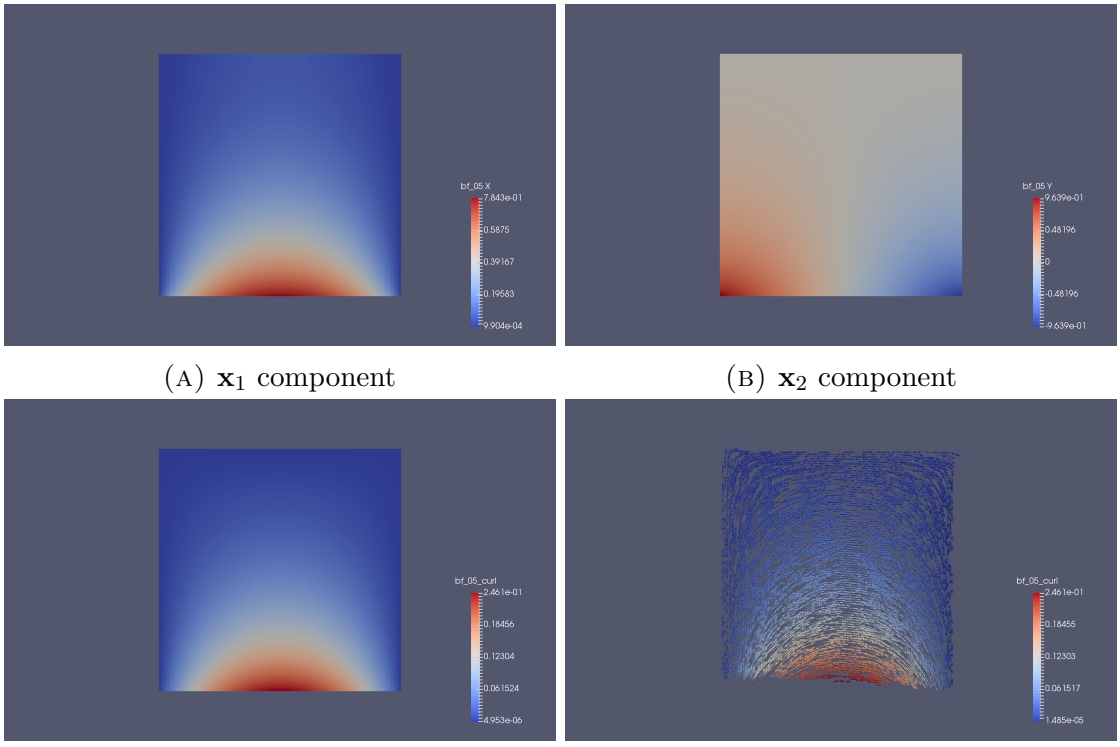
$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \sin(\pi\mathbf{x}_1)\sin(\pi\mathbf{x}_2) + \pi^2\sin(\pi\mathbf{x}_1)\sin(\pi\mathbf{x}_2) \\ \pi^2\cos(\pi\mathbf{x}_1)\cos(\pi\mathbf{x}_2) \end{pmatrix},$$

whose solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \sin(\pi\mathbf{x}_1)\sin(\pi\mathbf{x}_2) \\ 0 \end{pmatrix}.$$

We represent the convergence curves of

$$\|\mathbf{u}-\mathbf{u}_H\|_{0,\Omega} \text{ and } \left(\sum_{K\in\mathcal{T}_H} \|\operatorname{curl}(\mathbf{u}-\mathbf{u}_H)\|_{0,K}^2\right)^{1/2}$$



(A) $\mathbf{x}_1$ component (B) $\mathbf{x}_2$ component

(C) curl (D) vectorial representation

FIGURE 2. $\hat{\mathbf{T}}\lambda^5$

on Figure 5 for the space Λ_H^1. We also compare the accuracy proposed by first-order Nédélec's edge elements on the same mesh $\mathcal{T}_H$ on Figure 5.

Figure 5 depicts that the MHM and FE methods provide the same linear convergence rate in both $\mathbf{L}^2(\Omega)$ and $\mathbf{H}(\mathrm{curl}, \mathcal{T}_H)$ norms. MHM solution is less accurate than the FE solution in terms of $\mathbf{L}^2(\Omega)$ error. On the other hand, the MHM solution is more accurate in terms of the error on the curl.

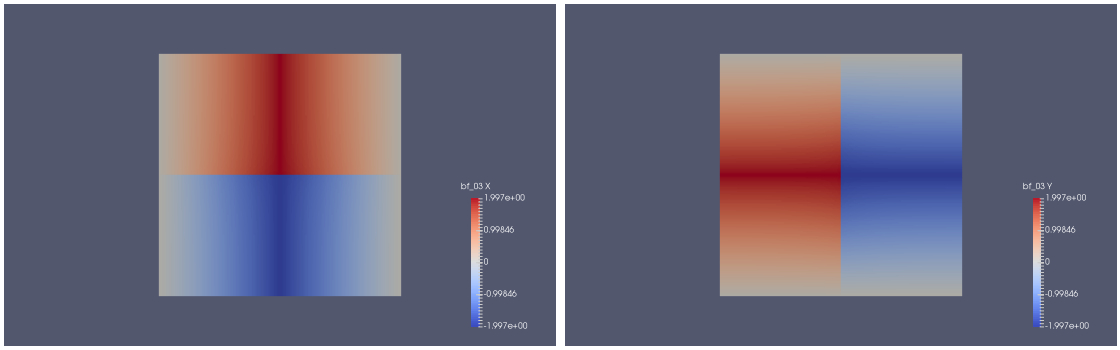
7.2. **Convergence of quadratic elements.** We present the convergence curves for the MHM method equiped with the Λ_H^2 space on Figure 6. In this case, the right-hand-side and solution we consider are

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \sin(5\pi\mathbf{x}_1)\sin(5\pi\mathbf{x}_2) + 25\pi^2\sin(5\pi\mathbf{x}_1)\sin(5\pi\mathbf{x}_2) \\ 25\pi^2\cos(5\pi\mathbf{x}_1)\cos(5\pi\mathbf{x}_2) \end{pmatrix},$$

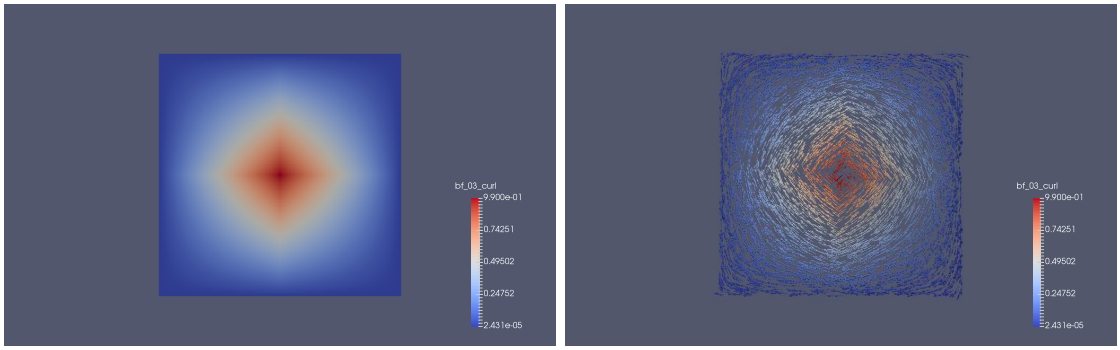
whose solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \sin(5\pi\mathbf{x}_1)\sin(5\pi\mathbf{x}_2) \\ 0 \end{pmatrix}.$$

We observe that in this case, the convergence rates are quadratic.



(A) $\mathbf{x}_1$ component (B) $\mathbf{x}_2$ component



(C) curl (D) vectorial representation

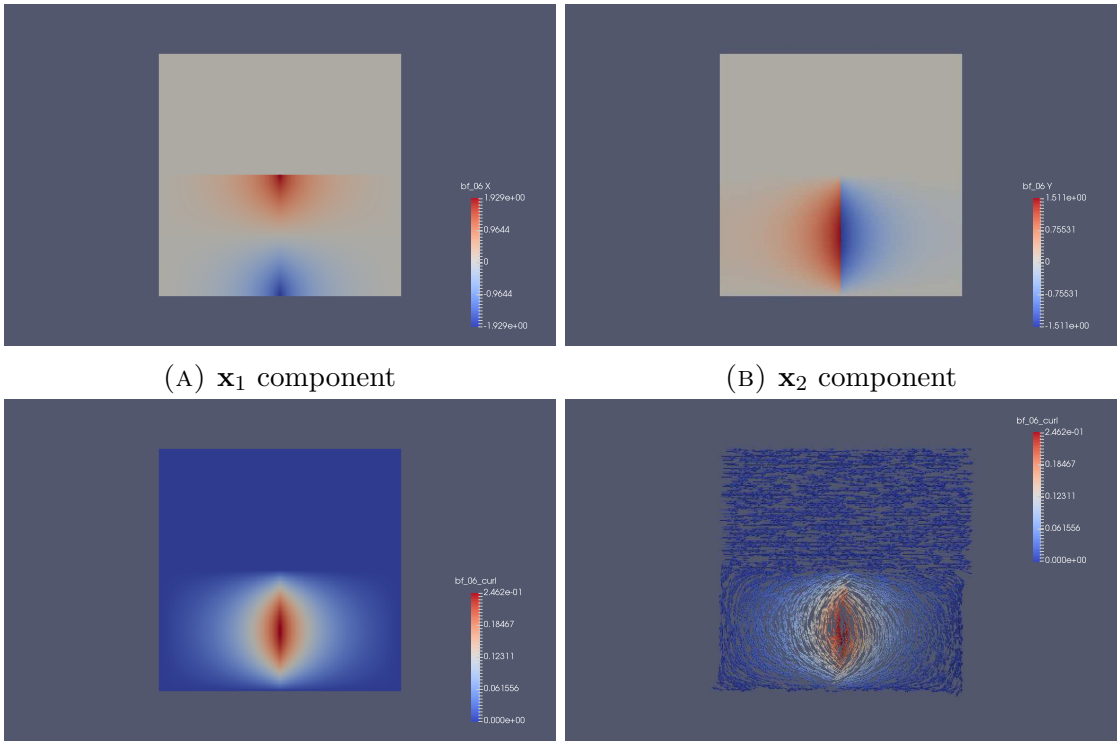
FIGURE 3. Nodal shape function $T\psi^3$

7.3. **A remark concerning tangential continuity.** On Figure 7 we show the $\mathbf{x}_1$ component of $\mathbf{u}_H$ on a 16×16 mesh when the analytical solution is

$$\mathbf{u} = \begin{pmatrix} \sin(\pi \mathbf{x}_1)\sin(\pi \mathbf{x}_2) \\ 0 \end{pmatrix}.$$

We can see there that the discrete solution obtained with linear elements exhibits important tangential jumps across the edges of $\mathcal{T}_H$. Also, we observe some "oscillations". This might be the reason why we observed that the error in $\mathbf{L}^2(\Omega)$ norm is more important for MHM that for FE discretization in the previous section. The effect we just mentioned for the linear discretization is much less important, however, for the quadratic discretization.

As shown at Figure 8, the approximation of the curl is continuous, and does not suffer of the aforementioned effect, even for the linear discretization.



(A) $\mathbf{x}_1$ component (B) $\mathbf{x}_2$ component

(C) curl (D) vectorial representation

FIGURE 4. Edge shape function $T\psi^6$

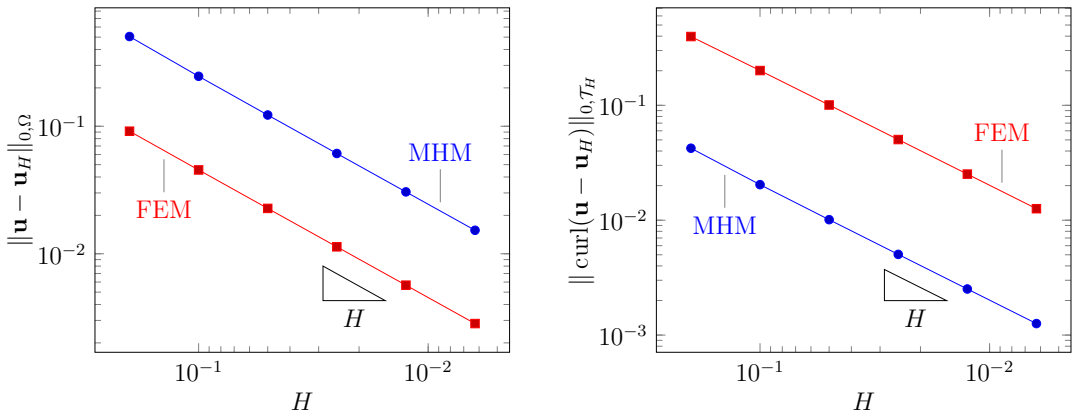


FIGURE 5. Convergence curves for first-order elements

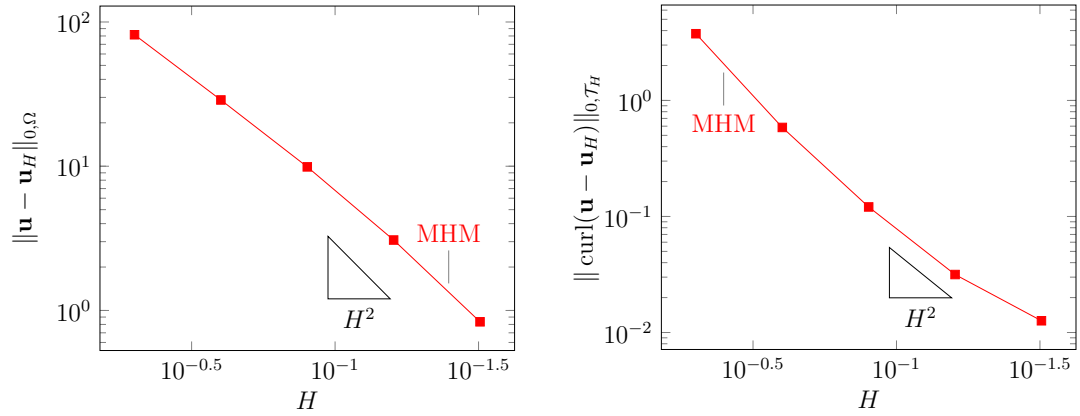
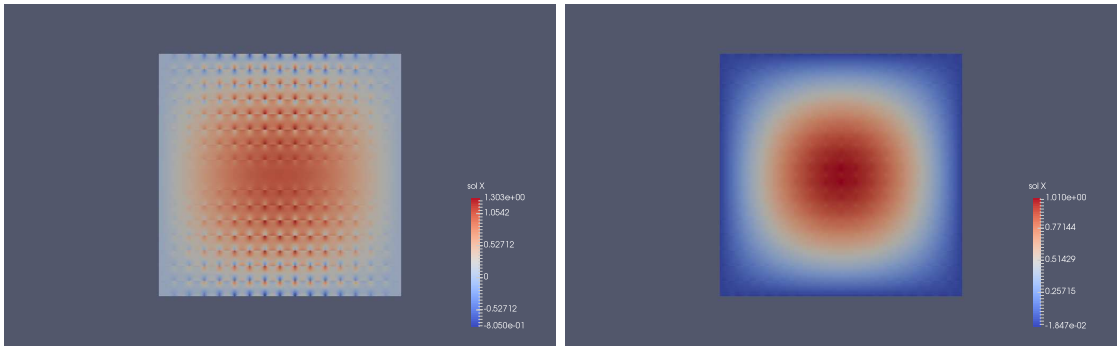
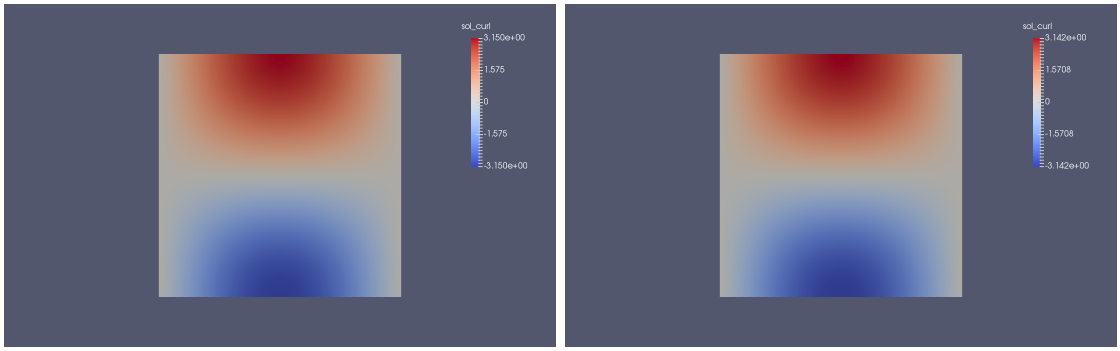


FIGURE 6. Convergence curves for second-order elements



(A) Λ_H^1 space (B) Λ_H^2 space

FIGURE 7. $\mathbf{x}_1$ component of $\mathbf{u}_H$ on a 16×16 mesh



(A) Λ_H^1 space (B) Λ_H^2 space

FIGURE 8. $\operatorname{curl} \mathbf{u}_H$ on a 16×16 mesh