Numerical methods for immersed FSI with thin-walled structures

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Abstract: The numerical simulation of a thin-walled structure immersed in an incompressible fluid can be addressed by various methods. In this paper, three of them are considered: the Arbitrary Lagrangian-Eulerian (ALE) method, the Fictitious Domain/Lagrange multipliers (FD) method and the Nitsche-XFEM method. Taking ALE as a reference, the advantages and limitations of FD and Nitsche-XFEM are carefully discussed on three benchmark test cases which have been chosen to be representative of typical difficulties encountered in valves or living cells simulations.

Key-words: fluid-structure interaction, incompressible fluid, immersed thin-walled structure, unfitted meshes, fitted meshes, fictitious domain method, XFEM method, Nitsche method, ALE method.

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Méthodes numériques pour l'IFS avec des structures minces immergés

Résumé : La simulation numérique d'une structure mince immergée dans un fluide incompressible peut être abordée par différentes méthodes. Dans cet article, trois d'entre elles sont considérées: la méthode Arbitrairement Lagrangienne Eulérienne (ALE), la méthode de domaine fictif avec multiplicateurs de Lagrange (FD) et la méthode Nitsche-XFEM. En prenant la méthode ALE comme référence, les avantages et les limites des méthodes FD et Nitsche-XFEM sont soigneusement discutés sur trois cas tests qui ont été choisis pour être représentatifs des difficultés typiques rencontrées dans les simulations de valves ou de cellules.

Mots-clés : interaction fluide-structure, fluide incompressible, structure mince immergée, maillages non compatibles, maillages compatibles, méthode de domaine fictif, méthode XFEM, méthode de Nitsche, méthode ALE.

1 Introduction

The numerical simulation of the mechanical interaction between an incompressible viscous fluid and an immersed thin-walled elastic structure is an essential ingredient in the mathematical modeling of many living systems. As examples, we can mention the opening and closing dynamics of heart valves, the wings of a bird interacting with the air or the fins of a fish moving in water (see, e.g., [34, 38, 48]).

The numerical methods for this type of fluid-structure interaction (FSI) problems are generally tailored to the amount of solid displacement within the fluid. Problems with low or moderate interface deflections can be successfully simulated using fitted (moving) mesh techniques, based on an arbitrary Lagrangian-Eulerian (ALE) description of the fluid. However, for problems involving large structural displacements (with solids that might come into contact or that might break up) the ALE formalism becomes cumbersome. A favored alternative is the combination of an Eulerian formalism in the fluid with an unfitted mesh discretization, in which the fluid-structure interface deforms independently of a background fluid mesh. Among these approaches, we can mention the Immersed Boundary method (IB) (see, e.g., [40, 9]), and methods with Lagrange multipliers, that will be called Fictitious Domain methods (FD) for simplicity (see, e.g., [28, 4, 20, 3, 10, 11]), which keep a Lagrangian formalism for the solid, and the methodologies based on a fully Eulerian formalism for the fluid (see, e.g., [19, 43]).

In general, these unfitted mesh approaches are known to be inaccurate in space, because the discrete treatment of the interface conditions and/or because the fluid spatial discretization do/does not allow for (weak and strong) discontinuities across the interface. Mesh adaptation mitigates these issues (see, e.g., [30]), but it does not cure the problem. The extended-FEM (XFEM) method, which combines a cut-FEM methodology with a local enrichment (see, e.g., [54, 26, 45]), overcomes these issues but at the price of introducing additional unknowns (Lagrange multipliers) and a degradation of robustness (with respect to the interface cuts). These difficulties have been recently circumvented by the Nitsche-XFEM method reported in [2, 13], which combines a Nitsche's treatment of the interface coupling with overlapping meshes and a suitable stabilization on the interfacial zone. The superior accuracy properties of cut-FEM approaches comes however at a price: these methods demand a much more involved computer implementation and require a specific evaluation of the interface intersections, which can be computationally challenging (notably in 3D, for arbitrary unstructured meshes).

Unfitted mesh methods for FSI are rarely compared between them in the literature. It is hence unclear to what extent cut-FEM methodologies are appealing in practice, with respect to more standard IB or FD methods. The first contribution of this paper consists in providing insight into this problem. To this purpose, we compare and validate some of the aforementioned unfitted mesh methods in a series of 2D-FSI benchmarks with dynamics interfaces undergoing large deflections. We consider an archetypal sample of unfitted mesh methods: the FD method as implemented in [21], an enhanced variant from [34] and the Nitsche-XFEM method introduced in [2]. The numerical approximation provided by an ALE fitted mesh approach is used as reference for the comparisons. The second contribution of this work has to do with the above referenced unfitted Nitsche-XFEM method, which is here formulated and validated, for the first time, in a fully non-linear framework, involving a general thin-wall solid model for large displacements and rotations. The numerical results reported in [2] were limited to moderate (normal) displacements using a simple string model.

The rest of the paper is organized as follows. Section 2 presents the different mathematical models and formulations considered through the paper. The numerical methods are described in Section 3. Section 4 presents the comparisons of the results obtained with the three FSI benchmarks tests. Finally, a summary of the main conclusions is given in Section 5.

2 Problem setting and mathematical formulations

In this section we discuss the mathematical models considered through the paper. The fluid is modeled by the incompressible Navier-Stokes equations in the Eulerian or in the ALE (Arbitrary Lagrangian-Eulerian) formalisms. The thin-walled solid is described by a non-linear beam model in Lagrangian form.

2.1 Geometric configurations and notation

We consider a fluid-structure interaction problem coupling an incompressible viscous fluid with an immersed thin-walled solid. Let $\Sigma \subset \mathbb{R}^2$ be the reference configuration of the solid mid-surface. The current position of the interface, denoted by $\Sigma(t)$, is given in terms of the deformation map $\phi : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$ as $\Sigma(t) = \phi(\Sigma, t)$, with $\phi \stackrel{\text{def}}{=} \mathbf{I}_{\Sigma \times \mathbb{R}^+} + \mathbf{d}$ where $\mathbf{d}$ denotes the displacement of the solid. The structure is supposed to move within a fixed domain $\Omega \subset \mathbb{R}^2$ with boundary

Figure 1: The computational domain Ω .

$\Gamma \stackrel{\text{def}}{=} \partial\Omega$ (see Figure 1). The fluid occupies the time-dependent domain

$$\Omega(t) \stackrel{\text{def}}{=} \Omega \setminus \Sigma(t) \subset \mathbb{R}^2,$$

with its boundary being partitioned as $\partial\Omega(t) = \Sigma(t) \cup \Gamma$. The interface $\Sigma(t)$ is assumed to be oriented by a unit surface normal vector field denoted by $\mathbf{n}_\Sigma$. We can hence define positive and negative sides on $\Sigma(t)$, with respective unit normals $\mathbf{n}_1 \stackrel{\text{def}}{=} \mathbf{n}_\Sigma$ and $\mathbf{n}_2 \stackrel{\text{def}}{=} -\mathbf{n}_\Sigma$. For a given continuous field f defined in Ω (possibly discontinuous across the interface), we define its sided-restrictions to $\Sigma(t)$, denoted by f_1 and f_2 , as

$$f_i(\mathbf{x}) \stackrel{\text{def}}{=} \lim_{\xi \rightarrow 0^+} f(\mathbf{x} + \xi \mathbf{n}_i), \quad i = 1, 2,$$

for all $\mathbf{x} \in \Sigma(t)$, and the following jump and average operators across $\Sigma(t)$:

$$[[f]] \stackrel{\text{def}}{=} f^+ - f^-, \quad [[f\mathbf{n}]] \stackrel{\text{def}}{=} f_1 \mathbf{n}_1 + f_2 \mathbf{n}_2, \quad \{f\} \stackrel{\text{def}}{=} \frac{1}{2}(f^+ + f^-).$$

Finally, in order to ease the presentation, we introduce the following notation $\phi_t \stackrel{\text{def}}{=} \phi(\cdot, t)$, for all field ϕ defined in $\Omega \times \mathbb{R}^+$.

2.2 Solid model

We consider a non-linear Reissner–Mindlin beam model with a reference configuration given by the mid-surface Σ . The beam kinematics are described by the displacement of the mid-surface $\mathbf{d}$

and by a field of director vectors $\mathbf{a}$ in the deformed configuration. These unit vectors correspond to the material lines in the reference configuration. We recall that Reissner–Mindlin kinematics assume that such material lines, originally aligned with $\mathbf{n}_\Sigma$, remain straight and do not stretch during the deformation (see, e.g., [6, 17]).

In this framework, the equilibrium of the solid is described by the following non-linear problem: find the solid mid-surface displacement $\mathbf{d} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$, the solid mid-surface velocity $\dot{\mathbf{d}} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$ and the director vector $\mathbf{a} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$ with unit length, $|\mathbf{a}| = 1$, such that

$$\begin{cases} \rho^s \epsilon \partial_t \dot{\mathbf{d}} + \mathbf{L}_d(\mathbf{d}, \mathbf{a}) = \mathbf{T} & \text{on } \Sigma, \\ \mathbf{L}_a(\mathbf{d}, \mathbf{a}) = \mathbf{0} & \text{on } \Sigma, \\ \dot{\mathbf{d}} = \partial_t \mathbf{d} & \text{on } \Sigma, \end{cases}$$

hold for all $t > 0$. Here, ρ^s stands for the solid density, $\mathbf{T}$ denotes a given source term and the abstract surface operators $\mathbf{L}_d$ and $\mathbf{L}_a$ describe the elastic behaviour of the beam, with $\mathbf{L}_a(\mathbf{d}, \mathbf{a}) = \mathbf{0}$ representing, in particular, the bending moments and shear stresses equilibrium (rotational inertia effects are neglected).

2.3 Coupled problem with Eulerian formalism in the fluid

The first considered non-linear coupled problem reads as follow: find the fluid velocity and pressure $\mathbf{u} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$, $p : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$, the solid displacement and velocity $\mathbf{d} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$, $\dot{\mathbf{d}} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$ and the unit director vector $\mathbf{a} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$ such that

$$\begin{cases} \rho^f \partial_t \mathbf{u} + \rho^f \mathbf{u} \cdot \nabla \mathbf{u} - \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}, p) = \mathbf{0} & \text{in } \Omega(t), \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega(t), \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma, \end{cases} \quad (1)$$

$$\begin{cases} \rho^s \epsilon \partial_t \dot{\mathbf{d}} + \mathbf{L}_d(\mathbf{d}, \mathbf{a}) = \mathbf{T} & \text{on } \Sigma, \\ \mathbf{L}_a(\mathbf{d}, \mathbf{a}) = \mathbf{0} & \text{on } \Sigma, \\ \dot{\mathbf{d}} = \partial_t \mathbf{d} & \text{on } \Sigma, \end{cases} \quad (2)$$

$$\begin{cases} \phi = \mathbf{I}_{\Sigma \times \mathbb{R}^+} + \mathbf{d}, & \Sigma(t) = \phi(\Sigma, t), & \Omega(t) = \Omega \setminus \Sigma(t), \\ \mathbf{u} = \dot{\mathbf{d}} \circ \phi_t^{-1} & \text{on } \Sigma(t), \\ \int_{\Sigma} \mathbf{T} \cdot \mathbf{y} = - \int_{\Sigma(t)} \llbracket \boldsymbol{\sigma}(\mathbf{u}, p) \mathbf{n} \rrbracket \cdot \mathbf{y} \circ \phi_t^{-1}, \end{cases} \quad (3)$$

for all smooth test function $\mathbf{y} : \Sigma \rightarrow \mathbb{R}^2$. Here, the symbol ρ^f denotes the fluid density and the fluid Cauchy stress tensor is given by

$$\boldsymbol{\sigma}(\mathbf{u}, p) \stackrel{\text{def}}{=} 2\mu \boldsymbol{\epsilon}(\mathbf{u}) - p \mathbf{I}, \quad \boldsymbol{\epsilon}(\mathbf{u}) \stackrel{\text{def}}{=} \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T),$$

where μ denotes the fluid dynamic viscosity

The three relations in (3) enforce, respectively, the geometric, the kinematic and the dynamic interface coupling. Note that we fully identify the solid mid-surface with the fluid–structure interface, by neglecting all the beam thickness effects in the interface coupling. This is a commonly used simplification when coupling thin-walled and thick-walled media (see, e.g., [18]). Note that the coupled problem (1)–(3) has an inherent mixed Eulerian–Lagrangian nature, which is a direct consequence of the presence of the maps ϕ_t and ϕ_t^{-1} in the interface coupling (3). This introduces additional (geometrical) non-linearities.

2.4 Coupled problem with ALE formalism in the fluid

The moving control volume $\Omega(t)$ is assumed to be parametrizable as $\Omega(t) = \mathcal{A}(\Omega, t)$, where $\mathcal{A} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$ is a one-to-one mapping given by the relation $\mathcal{A} = I_{\Omega \times \mathbb{R}^+} + \mathbf{d}^f$, in terms of the fluid domain displacement $\mathbf{d}^f$ (see Figure 2).

Figure 2: The ALE map.

The geometrical compatibility between the fluid and solid domains can hence be guaranteed through a relation of the type $\mathbf{d}^f = \mathcal{L}(\mathbf{d})$, where $\mathcal{L}(\mathbf{d})$ represents a suitable lifting of $\mathbf{d}$ from Σ to Ω which vanishes on Γ . Note that $\mathbf{d}^f$ can be arbitrarily defined in $\Omega \setminus \Sigma$. By a simple use of the chain rule, we have the following relation:

$$\partial_t \mathbf{u} = \partial_t|_{\mathcal{A}} \mathbf{u} - \mathbf{w} \cdot \nabla \mathbf{u}, \quad (4)$$

where $\partial_t|_{\mathcal{A}}$ denotes the ALE time derivative and $\mathbf{w} \stackrel{\text{def}}{=} \partial_t \mathcal{A} = \partial_t \mathbf{d}^f$ the fluid domain velocity. The fundamental motivation of (4) is that it facilitates the time discretization of terms in $\Omega(t)$ involving quantities originally defined in Ω which are transported to $\Omega(t)$ through $\mathcal{A}$ (see Section 3.1).

Using the above relations, the coupled problem (1)–(3) admits the following equivalent formulation: find the fluid domain displacement $\mathbf{d}^f : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$, the fluid velocity $\mathbf{u} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$, the fluid pressure $p : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}$, the solid mid-surface displacement $\mathbf{d} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$, the solid mid-surface velocity $\dot{\mathbf{d}} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$ and the unit director vector $\mathbf{a} : \Sigma \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$, such that

$$\begin{cases} \rho^f \partial_t|_{\mathcal{A}} \mathbf{u} + \rho^f (\mathbf{u} - \mathbf{w}) \cdot \nabla \mathbf{u} - \text{div} \boldsymbol{\sigma}(\mathbf{u}, p) = \mathbf{0} & \text{in } \Omega(t), \\ \text{div} \mathbf{u} = 0 & \text{in } \Omega(t), \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma, \end{cases} \quad (5)$$

$$\begin{cases} \rho^s \epsilon \partial_t \dot{\mathbf{d}} + \mathbf{L}_a(\mathbf{d}, \mathbf{a}) = \mathbf{T} & \text{on } \Sigma, \\ \mathbf{L}_a(\mathbf{d}, \mathbf{a}) = \mathbf{0} & \text{on } \Sigma, \\ \dot{\mathbf{d}} = \partial_t \mathbf{d} & \text{on } \Sigma, \end{cases} \quad (6)$$

$$\begin{cases} \mathbf{d}^f = \mathcal{L}(\mathbf{d}), \quad \mathbf{w} = \partial_t \mathbf{d}^f, \quad \mathcal{A} = I_{\Omega \times \mathbb{R}^+} + \mathbf{d}^f, \quad \Omega(t) = \mathcal{A}(\Omega, t), \\ \mathbf{u} = \dot{\mathbf{d}} \circ \mathcal{A}_t^{-1} & \text{on } \Sigma(t), \\ \int_{\Sigma} \mathbf{T} \cdot \mathbf{y} = - \int_{\Sigma(t)} \llbracket \boldsymbol{\sigma}(\mathbf{u}, p) \mathbf{n} \rrbracket \cdot \mathbf{y} \circ \mathcal{A}_t^{-1}, \end{cases} \quad (7)$$

for all smooth function $\mathbf{y} : \Sigma \rightarrow \mathbb{R}^2$.

3 Numerical methods

Though the coupled problems (1)–(3) and (5)–(7) are equivalent at the continuous level, their discretization generally leads to different numerical approaches.

In what follows, $\tau > 0$ denotes the time-step length, $t_n \stackrel{\text{def}}{=} n\tau$ for $n \in \mathbb{N}$, and $\partial_\tau x^n \stackrel{\text{def}}{=} \frac{1}{\tau}(x^n - x^{n-1})$ stands for the first-order backward difference. We will also make use of the standard Sobolev spaces $H^m(\omega)$ ($m \geq 0$). The closed subspaces $H_\Gamma^1(\omega)$, of functions in $H^1(\omega)$ with zero trace on Γ , and $L_0^2(\omega)$, of functions in $L^2(\omega)$ with zero mean in ω , will also be used. The scalar product in $L^2(\omega)$ is denoted by $(\cdot, \cdot)_\omega$. We consider $\mathbf{V} \stackrel{\text{def}}{=} [H_\Gamma^1(\Omega)]^2$ and $Q \stackrel{\text{def}}{=} L_0^2(\Omega)$ as the fluid velocity and pressure functional spaces, respectively. The space of admissible displacements is denoted by $\mathbf{Y} \subset [H^1(\Sigma)]^2$. The trial set for the unitary director vectors is denoted by $\mathbf{S}$ and its variations are taken in the space $\Theta \subset H^1(\Sigma)$.

3.1 Fitted mesh methods

By using the parametrization of the fluid domain, given by the one-to-one mapping $\mathcal{A}$, we can consider as fluid unknowns

$$\hat{\mathbf{u}} \stackrel{\text{def}}{=} \mathbf{u} \circ \mathcal{A}_t, \quad \hat{p} \stackrel{\text{def}}{=} p \circ \mathcal{A}_t,$$

instead of $\mathbf{u} = \hat{\mathbf{u}} \circ \mathcal{A}_t^{-1}$ and $p = \hat{p} \circ \mathcal{A}_t^{-1}$. The coupled problem (5)–(7) hence admits the following variational formulation (see, e.g., [24]): for $t > 0$, find $(\hat{\mathbf{u}}, \hat{p}, \mathbf{d}, \mathbf{a}) \in \mathbf{V} \times Q \times \mathbf{Y} \times \mathbf{S}$, with $\dot{\mathbf{d}} = \partial_t \mathbf{d}$ and $\hat{\mathbf{u}}|_\Sigma = \dot{\mathbf{d}}$, such that the geometric compatibility (7)₁ holds and

$$\begin{cases} \rho^f \frac{d}{dt} (\mathbf{u}, \mathbf{v})_{\Omega(t)} - \rho^f ((\text{div} \mathbf{w}) \mathbf{u}, \mathbf{v})_{\Omega(t)} + a_{\Omega(t)}^f (\mathbf{u} - \mathbf{w}; (\mathbf{u}, p), (\mathbf{v}, q)) \\ + \rho^s \epsilon(\partial_\tau \dot{\mathbf{d}}, \mathbf{y})_\Sigma + a^s((\mathbf{d}, \mathbf{a}), (\mathbf{y}, \theta)) = 0 \end{cases} \quad (8)$$

for all $(\hat{\mathbf{v}}, \hat{q}, \mathbf{y}, \theta) \in \mathbf{V} \times Q \times \mathbf{Y} \times \Theta$ with $\hat{\mathbf{v}}|_\Sigma = \mathbf{y}$. Here, the trilinear fluid form is defined by

$$a_{\Omega(t)}^f(\mathbf{z}; (\mathbf{u}, p), (\mathbf{v}, q)) \stackrel{\text{def}}{=} c_{\Omega(t)}(\mathbf{z}, \mathbf{u}, \mathbf{v}) + a_{\Omega(t)}((\mathbf{u}, p), (\mathbf{v}, q)),$$

with

$$\begin{aligned} a_{\Omega(t)}((\mathbf{u}, p), (\mathbf{v}, q)) &\stackrel{\text{def}}{=} 2\mu(\epsilon(\mathbf{u}), \epsilon(\mathbf{v}))_{\Omega(t)} - (p, \text{div} \mathbf{v})_{\Omega(t)} + (q, \text{div} \mathbf{u})_{\Omega(t)}, \\ c_{\Omega(t)}(\mathbf{z}, \mathbf{u}, \mathbf{v}) &\stackrel{\text{def}}{=} \rho^f(\mathbf{z} \cdot \nabla \mathbf{u}, \mathbf{v})_{\Omega(t)}. \end{aligned}$$

The term a^s represents the weak form of the solid operators $\mathbf{L}_d$ and $\mathbf{L}_a$.

The coupled variational problem (8) is particularly well adapted to the discretization with fitted meshes. We hence consider a family of triangulations of $\{\mathcal{T}_h^f\}_{0 < h < 1}$ of the reference fluid domain Ω which are fitted to the family of triangulations $\{\mathcal{T}_h^s\}_{0 < h < 1}$ of Σ . In other words, the fluid and solid meshes match at the interface. The discrete spaces $\mathbf{V}_h \subset \mathbf{V}$, $\mathbf{Y}_h \subset \mathbf{Y}$ and $\Theta_h \subset \Theta$ are the standard Lagrange finite element space of continuous piece-wise affine functions, so that we have $\mathbf{Y}_h = \mathbf{V}_h|_\Sigma$. The pressure space $Q_h \subset Q$ is also made of piece-wise affine functions which are continuous on each sub-domain Ω_i , $i = 1, 2$, but which can be discontinuous across Σ .

The geometric coupling condition (7)₁ is treated in an explicit fashion as follows. For a given discrete displacement $\mathbf{d}_h^{n-1} \in \mathbf{Y}_h$ at time level $n - 1$, we define the ALE map $\mathcal{A}_h^n$ as

$$\mathbf{d}_h^{f,n} = \mathcal{L}_h(\mathbf{d}_h^{n-1}), \quad \mathcal{A}_h^n = \mathbf{I}_\Omega + \mathbf{d}_h^{f,n}.$$

Algorithm 1 Fitted mesh ALE method.For $n \geq 1$,

1. Fluid mesh update:

$$\mathbf{d}_h^{f,n} = \mathcal{L}_h(\mathbf{d}_h^{n-1}), \quad \mathbf{w}_h^n = \partial_\tau \mathbf{d}_h^{f,n}, \quad \mathcal{A}_h^n = \mathbf{I}_\Omega + \mathbf{d}_h^{f,n}, \quad \Omega_h^n = \mathcal{A}_h^n(\Omega).$$

2. Find $(\hat{\mathbf{u}}_h^n, \hat{p}_h^n, \mathbf{d}_h^n, \mathbf{a}_h^n) \in \mathbf{V}_h \times Q_h \times \mathbf{Y}_h \times \mathbf{S}_h$, with $\dot{\mathbf{d}}_h = \partial_\tau \mathbf{d}_h^n$ and $\hat{\mathbf{u}}_h^n|_\Sigma = \dot{\mathbf{d}}_h^n$, such that

$$\begin{cases} \frac{\rho^f}{\tau} \left[(\mathbf{u}_h^n, \mathbf{v}_h)_{\Omega_h^n} - (\mathbf{u}_h^{n-1}, \mathbf{v}_h)_{\Omega_h^{n-1}} \right] - \rho^f ((\operatorname{div} \mathbf{w}_h^n) \mathbf{u}_h^n, \mathbf{v}_h)_{\Omega_h^n} \\ + a_{\Omega_h^n, h}^f(\mathbf{u}_h^{n-1} - \mathbf{w}_h^n; (\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) \\ + \rho^s \epsilon (\partial_\tau \dot{\mathbf{d}}_h^n, \mathbf{y}_h)_\Sigma + a_h^s((\mathbf{d}_h^n, \mathbf{a}_h^n), (\mathbf{y}_h, \theta_h)) = 0, \end{cases} \quad (9)$$

for all $(\hat{\mathbf{v}}_h, \hat{q}_h, \mathbf{y}_h, \theta_h) \in \mathbf{V}_h \times Q_h \times \mathbf{Y}_h \times \Theta_h$ with $\mathbf{v}_h|_\Sigma = \mathbf{y}_h$.

The resulting numerical method is detailed in Algorithm 1. Here, we have used the standard notation

$$a_{\Omega_h^n, h}^f(\mathbf{z}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) \stackrel{\text{def}}{=} c_h^n(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}_h) + a_{\Omega_h^n}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) + s_{\Omega_h^n, h}(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}_h),$$

with the convective trilinear form given by

$$c_h^n(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}_h) \stackrel{\text{def}}{=} c_{\Omega_h^n}(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}_h) + \frac{\rho^f}{2} ((\operatorname{div} \mathbf{z}_h) \mathbf{u}_h, \mathbf{v}_h)_{\Omega_h^n}$$

and the form $s_{\Omega_h^n, h}$ corresponds to the SUPG/PSPG and grad-div stabilizations given by (see, e.g., [47]):

$$\begin{aligned} s_{\Omega_h^n, h}(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}_h) &\stackrel{\text{def}}{=} \sum_{K \in \mathcal{T}_h^n} \int_K \frac{\lambda_C h^2}{\delta_h} \operatorname{div} \mathbf{u}_h \operatorname{div} \mathbf{v}_h \\ &+ \sum_{K \in \mathcal{T}_h^n} \int_K \delta_h (\rho^f (\mathbf{z}_h \cdot \nabla) \mathbf{u}_h + \nabla p_h) \cdot (\rho^f (\mathbf{z}_h \cdot \nabla) \mathbf{v}_h + \nabla q_h), \\ \delta_h &\stackrel{\text{def}}{=} \lambda_M \left(\rho^f \sqrt{\frac{4}{\tau^2} + \frac{16\mu^2}{h^4(\rho^f)^2} + \frac{4|\mathbf{z}_h|^2}{h^2}} \right)^{-1}, \end{aligned} \quad (10)$$

with $\lambda_M, \lambda_C > 0$ user-defined parameters. At last, the solid term a_h^s denotes the linear MITC (Mixed Interpolation of Tensorial Components approach) approximation of a^s (see, e.g., [6]), involving 3 degrees of freedom per node in the increments (two displacements and one rotation).

Remark 3.1. In (9), the fluid integrals in the current configurations involve the composition with the corresponding discrete ALE map. For instance, the second term is given by

$$(\mathbf{u}_h^{n-1}, \mathbf{v}_h)_{\Omega_h^n} = (\hat{\mathbf{u}}_h^{n-1} \circ (\mathcal{A}_h^n)^{-1}, \hat{\mathbf{v}}_h \circ (\mathcal{A}_h^n)^{-1})_{\Omega_h^n}.$$

The main drawback of Algorithm 1 comes from the motion of the fluid domain mesh in step 1. Indeed, the viability and efficiency of the whole numerical method strongly depends on

the capability of the discrete lifting operator $\mathcal{L}_h$ to deliver a correct deformed fluid mesh which maintains a reasonable quality. This constraint often becomes a challenging problem in the case of large interface deflections (see, e.g., [46, 51, 1]). In this paper, we consider the non-linear lifting operator introduced in [36] which is simply an incremental variant of the approach reported in [46].

3.2 Unfitted mesh methods

The coupled problem (1)–(3) is often discretized with an unfitted mesh method to circumvent the above mentioned limitation of Algorithm 1. In this paper, we consider two different unfitted methods which are discussed in the next two paragraphs.

3.2.1 Fictitious domain method

The basic idea of the so-called Fictitious Domain method, is to consider the kinematic condition (3)₂ as a constraint in the fluid problem which is enforced through Lagrange multipliers, viz.,

$$b(\boldsymbol{\mu}, \mathbf{u} \circ \boldsymbol{\phi}) = b(\boldsymbol{\mu}, \dot{\mathbf{d}}), \quad \forall \boldsymbol{\mu} \in \boldsymbol{\Lambda},$$

where $\boldsymbol{\Lambda}$ denotes the Lagrange multiplier space and $b : \boldsymbol{\Lambda} \times [H^{\frac{1}{2}}(\Sigma)]^2 \rightarrow \mathbb{R}$ is a continuous bilinear form, such that $b(\boldsymbol{\mu}, \mathbf{z}) = 0$ for all $\boldsymbol{\mu} \in \boldsymbol{\Lambda}$ yields $\mathbf{z} = \mathbf{0}$ on Σ . For instance, we can take $\boldsymbol{\Lambda} = ([H^{\frac{1}{2}}(\Sigma)]^2)'$ and $b(\boldsymbol{\mu}, \mathbf{z}) = \langle \boldsymbol{\mu}, \mathbf{z} \rangle$, where $\langle \cdot, \cdot \rangle$ represents the duality pairing between $([H^{\frac{1}{2}}(\Sigma)]^2)'$ and $[H^{\frac{1}{2}}(\Sigma)]^2$ (see, e.g., [39, 11]).

Hence, in weak form the coupled problem (1)–(3) can be formulated as follows: for $t > 0$, find $(\mathbf{u}, p, \boldsymbol{\lambda}, \mathbf{d}, \mathbf{a}) \in \mathbf{V} \times Q \times \boldsymbol{\Lambda} \times \mathbf{Y} \times \mathbf{S}$, with $\dot{\mathbf{d}} = \partial_t \mathbf{d}$ such that the geometric compatibility (3)₁ holds and

$$\begin{cases} \rho^f (\partial_t \mathbf{u}, \mathbf{v})_{\Omega} + a_{\Omega}^f(\mathbf{u}; (\mathbf{u}, p), (\mathbf{v}, q)) + \rho^s \epsilon(\partial_t \dot{\mathbf{d}}, \mathbf{y})_{\Sigma} + a^s((\mathbf{d}, \mathbf{a}), (\mathbf{y}, \theta)) \\ + b(\boldsymbol{\lambda}, \mathbf{v} \circ \boldsymbol{\phi}_t - \mathbf{y}) - b(\boldsymbol{\mu}, \mathbf{u} \circ \boldsymbol{\phi}_t - \dot{\mathbf{d}}) = 0, \end{cases} \quad (11)$$

for all $(\mathbf{v}, q, \boldsymbol{\mu}, \mathbf{y}, \theta) \in \mathbf{V} \times Q \times \boldsymbol{\Lambda} \times \mathbf{Y} \times \Theta$.

Algorithm 2 Fictitious domain method (from [21]).

For $n \geq 1$,

1. Interface update:

$$\boldsymbol{\phi}_h^n = \mathbf{I}_{\Sigma} + \mathbf{d}_h^{n-1}, \quad \Sigma_h^n = \boldsymbol{\phi}_h^n(\Sigma), \quad \Omega_h^n = \Omega \setminus \Sigma_h^n.$$

2. Find $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\lambda}_h^n, \mathbf{d}_h^n, \mathbf{a}_h^n) \in \mathbf{V}_h \times Q_h \times \boldsymbol{\Lambda}_h \times \mathbf{Y}_h \times \mathbf{S}_h$, with $\dot{\mathbf{d}}_h = \partial_{\tau} \mathbf{d}_h^n$, such that

$$\begin{cases} \rho^f (\partial_{\tau} \mathbf{u}_h^n, \mathbf{v}_h)_{\Omega} + a_{\Omega, h}^f(\mathbf{u}_h^{n-1}; (\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) \\ + \rho^s \epsilon(\partial_{\tau} \dot{\mathbf{d}}_h^n, \mathbf{y}_h)_{\Sigma} + a^s((\mathbf{d}_h^n, \mathbf{a}_h^n), (\mathbf{y}_h, \theta_h)) \\ + b(\boldsymbol{\lambda}_h^n, \mathbf{v}_h \circ \boldsymbol{\phi}_h^n - \mathbf{w}_h) - b(\boldsymbol{\mu}_h, \mathbf{u}_h^n \circ \boldsymbol{\phi}_h^n - \dot{\mathbf{d}}_h^n) = 0, \end{cases} \quad (12)$$

for all $(\mathbf{v}_h, q_h, \boldsymbol{\mu}_h, \mathbf{y}_h, \theta_h) \in \mathbf{V}_h \times Q_h \times \boldsymbol{\Lambda}_h \times \mathbf{Y}_h \times \Theta_h$.

At the discrete level, the fluid velocity is searched for in the same space $\mathbf{V}_h$ as in the fitted mesh formulation of Section 3.1. Nevertheless, the fluid mesh is now fixed and the pressure space

Q_h is made of globally continuous functions. For the solid, we consider the same discretization as in Section 3.1. As approximation space for the Lagrange multipliers, we take

$$\Lambda_h = \left\{ \mu_h = \sum_{i=1}^{N_h^s} \mu_i \delta_{x_i^s} \mid \mu_i \in \mathbb{R}^2, \quad i = 1, \dots, N_h^s \right\}, \quad (13)$$

where $\{x_i^s\}_{i=1}^{N_h^s}$ denotes the points of the triangulation $\mathcal{T}_h^s$ and $\delta_{x_i^s}$ stands for the Dirac's measure at point x_i^s . This non-conforming approximation was used for example in [28, 21] and analyzed in [23]. The discrete method is hence obtained from the approximation of the weak form (11), as detailed in Algorithm 2. Note that the geometric compatibility condition (3)₁ is treated explicitly via step 1.

From (13), the discrete treatment of the kinematic constraint in (12), namely,

$$b(\mu_h, u_h^n \circ \phi_h^n - \dot{d}_h^n) = 0 \quad \forall u_h \in \Lambda_h,$$

simply reads:

$$u_h \circ \phi_h(x_i^s) - \dot{d}_h(x_i^s) = 0, \quad \forall i = 1, \dots, N_h^s. \quad (14)$$

The present computer implementation of this method is performed by penalization (omitted in the concise description of Algorithm 2 for the sake of clarity). Let $\varepsilon > 0$ be a given (non-dimensionless) parameter, the kinematic coupling condition (14) is hence replaced by:

$$u_h \circ \phi_h(x_i^s) - \dot{d}_h(x_i^s) = \varepsilon \lambda_i, \quad \forall i = 1, \dots, N_h^s. \quad (15)$$

Owing to the choice of Λ_h , the elimination of the Lagrange multipliers obtained with the penalization can be done without affecting the sparse pattern of the matrix of the fluid problem. This property makes the approach very convenient from a computer implementation point of view. The influence of the penalty parameter ε on the accuracy of the method will be discussed in Section 4, devoted to the numerical experiments. The choice here is to enforce relation (15) at every time step, but let us mention an interesting alternative which consists in introducing a time dynamics for the Lagrange multiplier, in the spirit of what is done for the pressure in pseudo-compressibility methods. This approach is adopted in [34], and further analyzed in [35] in a scalar case.

The main sources of inaccuracies in Algorithm 2 come from the choice of the Lagrange multiplier space (13) and from the continuous nature of the pressure approximation across the interface. While the impact of the former can be mitigated via mesh refinement, the latter is known to yield major mass conservation issues across the interface. It is worth noting that similar poor mass conservation was encountered with the original immersed boundary method using finite difference approximations in space (see, e.g., [41, 5]). In the variational setting of Algorithm 2, the use of alternative spatial approximations, based on globally discontinuous pressures and higher order polynomials for the velocities, has been investigated in [4, 8, 7, 10] with the purpose of overcoming this issue. In this paper, we follow the approach considered in [34] (see also [25, 15]), with which the original (low-order) velocity/pressure approximation is preserved by simply modifying the SUPG/PSPG and grad-div stabilizations in a neighborhood ω_h^n of the interface Σ_h^n (typically 2 layers of fluid elements in each side). More precisely, in (10) we take

$$\lambda_C = 1 \quad \text{in } \Omega, \quad \lambda_M = \begin{cases} 1 & \text{in } \Omega \setminus \omega_h^n, \\ \varepsilon_M & \text{in } \omega_h^n, \end{cases}$$

with $0 < \varepsilon_M \ll 1$ a user-defined parameter. Thus, the idea of the method consists in boosting the grad-div stabilization while reducing the SUPG/PSPG stabilization near the interface.

The first improves mass conservation while the second limits the impact of the local residual inconsistencies.

3.2.2 Nitsche-XFEM method

As in Algorithm 2, the geometric compatibility condition (3)₁ is treated in an explicit fashion and we consider the same discretization of the solid. For the construction of the fluid approximation spaces, we introduce two family of meshes $\{\mathcal{T}_{h,i}^n\}_{0 < h < 1}$, $i = 1, 2$, where each $\mathcal{T}_{h,i}^n$ covers the i -th fluid region defined by Σ_h^n . Each mesh $\mathcal{T}_{h,i}^n$ is fitted to the exterior boundary Γ but not to Σ_h^n (see Figure 3). Furthermore for every element $K \in \mathcal{T}_{h,1}^n \cap \mathcal{T}_{h,2}^n$ we assume that $K \cap \Sigma_h^n \neq \emptyset$. Note

Figure 3: One dimensional illustration of the construction of the discrete spaces $X_{h,i}^n$.

that $\mathcal{T}_{h,1}^n \cup \mathcal{T}_{h,2}^n$ is a triangulation of the whole fluid domain Ω . We denote by $\Omega_{h,i}^n$ the domain covered by $\mathcal{T}_{h,i}^n$, viz.,

$$\Omega_{h,i}^n \stackrel{\text{def}}{=} \text{int} \left(\bigcup_{K \in \mathcal{T}_{h,i}^n} K \right).$$

We can hence introduce the following time level dependent spaces of continuous piecewise affine functions:

$$X_{h,i}^n \stackrel{\text{def}}{=} \left\{ v_h \in C^0(\overline{\Omega_{h,i}^n}) / v_h|_K \in \mathbb{P}_1(K), \quad \forall K \in \mathcal{T}_{h,i}^n \right\},$$

for $i = 1, 2$. For the approximation of the fluid velocity and pressure we will consider the product spaces

$$\mathbf{V}_h^n \stackrel{\text{def}}{=} ([X_{h,1}^n]^2 \times [X_{h,2}^n]^2) \cap \mathbf{V}, \quad Q_h^n \stackrel{\text{def}}{=} (X_{h,1}^n \times X_{h,2}^n) \cap Q,$$

respectively. Note that interfacial (strong and weak) discontinuities are now incorporated at the discrete level. Indeed, the functions of these spaces are continuous in the fluid domain $\Omega_h^n \stackrel{\text{def}}{=} \Omega \setminus \Sigma_h^n$, but discontinuous across the moving interface Σ_h^n (see Figure 3).

In this framework, the kinematic/dynamic fluid-solid coupling (3)_{2,3} can be enforced in a consistent fashion (and Lagrange multipliers free) through a fluid-sided Nitsche's mortaring (see [2, 13]). The resulting numerical method is detailed in Algorithm 3.

Here, $\gamma > 0$ denotes the Nitsche's penalty parameter and the fluid trilinear form $a_h^{f,n}$ is given by the following expression

$$\begin{aligned} a_h^{f,n}(\mathbf{z}_h; (\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) &\stackrel{\text{def}}{=} c_h^n(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}_h) + a_{\Omega_h^n}((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) \\ &\quad + s_{v,h}^n(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}_h) + s_{p,h}^n(\mathbf{z}_h; p_h, q_h) + g_h^n(\mathbf{u}_h, \mathbf{v}_h). \end{aligned} \quad (17)$$

Algorithm 3 Nitsche-XFEM method (from [2]).For $n \geq 1$,

1. Interface update:

$$\phi_h^n = \mathbf{I}_\Sigma + \mathbf{d}_h^{n-1}, \quad \Sigma_h^n = \phi_h^n(\Sigma), \quad \Omega_h^n = \Omega \setminus \Sigma_h^n.$$

2. Find
- $(\mathbf{u}_h^n, p_h^n, \mathbf{d}_h^n, \mathbf{a}_h^n) \in \mathbf{V}_h^n \times Q_h^n \times \mathbf{Y}_h \times \mathbf{S}_h$
- , with
- $\dot{\mathbf{d}}_h = \partial_\tau \mathbf{d}_h^n$
- and such that

$$\left\{ \begin{aligned} & \rho^f (\partial_\tau \mathbf{u}_h^n, \mathbf{v}_h)_{\Omega_h^n} + a_h^{f,n}(\mathbf{u}_h^{n-1}; (\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) \\ & + \rho^s \epsilon (\partial_\tau \dot{\mathbf{d}}_h^n, \mathbf{y}_h)_\Sigma + a_h^s((\mathbf{d}_h^n, \mathbf{a}_h^n), (\mathbf{y}_h, \theta_h)) \\ & - \sum_{i=1}^2 (\boldsymbol{\sigma}(\mathbf{u}_{h,i}^n, p_{h,i}^n) \mathbf{n}_i, \mathbf{v}_{h,i} - \mathbf{y}_h)_{\Sigma_h^n} \\ & - \sum_{i=1}^2 (\mathbf{u}_{h,i}^n - \dot{\mathbf{d}}_h^n, \boldsymbol{\sigma}(\mathbf{v}_{h,i}, -q_{h,i}) \mathbf{n}_i)_{\Sigma_h^n} \\ & + \frac{\gamma \mu}{h} \sum_{i=1}^2 (\mathbf{u}_{h,i}^n - \dot{\mathbf{d}}_h^n, \mathbf{v}_{h,i} - \mathbf{y}_h)_{\Sigma_h^n} = 0, \end{aligned} \right. \quad (16)$$

for all $(\mathbf{v}_h, q_h, \mathbf{y}_h, \theta_h) \in \mathbf{V}_h^n \times Q_h^n \times \mathbf{Y}_h \times \Theta_h$.

In this expression, the convective term is defined as

$$\begin{aligned} c_h^n(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}_h) &\stackrel{\text{def}}{=} c_{\Omega_h^n}(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}_h) + \frac{\rho^f}{2} ((\operatorname{div} \mathbf{z}_h) \mathbf{u}_h, \mathbf{v}_h)_{\Omega_h^n} \\ &\quad - \rho^f (\llbracket \mathbf{z}_h \rrbracket \cdot \mathbf{n} \llbracket \mathbf{u}_h \rrbracket, \llbracket \mathbf{v}_h \rrbracket)_{\Sigma_h^n} - \frac{\rho^f}{2} (\llbracket \mathbf{z}_h \rrbracket \cdot \mathbf{n}, \llbracket \mathbf{u}_h \rrbracket \cdot \mathbf{v}_h)_{\Sigma_h^n}, \end{aligned}$$

where the last two (consistent) terms are introduced for stability purposes to guarantee that $c_h^n(\mathbf{z}_h; \mathbf{v}_h, \mathbf{v}_h) = 0$.The terms in $s_{v,h}^n$ and $s_{p,h}^n$ in (17) correspond to the continuous interior penalty (CIP) stabilization (see [14]) given by:

$$\begin{aligned} s_{v,h}^n(\mathbf{z}_h; \mathbf{u}_h, \mathbf{v}_h) &\stackrel{\text{def}}{=} \gamma_v h^2 \sum_{i=1}^2 \sum_{F \in \mathcal{F}_{h,i}^n} \xi_F(\mathbf{z}_h) \|\mathbf{z}_h \cdot \mathbf{n}\|_{L^\infty(F)} (\llbracket \nabla \mathbf{u}_h \rrbracket_F, \llbracket \nabla \mathbf{v}_h \rrbracket_F)_F, \\ s_{p,h}^n(\mathbf{z}_h; p_h, q_h) &\stackrel{\text{def}}{=} \gamma_p h^2 \sum_{i=1}^2 \sum_{F \in \mathcal{F}_{h,i}^n} \frac{\xi_F(\mathbf{z}_h)}{\|\mathbf{z}_h\|_{L^\infty(F)}} (\llbracket \nabla p_h \rrbracket_F, \llbracket \nabla q_h \rrbracket_F)_F, \end{aligned}$$

where $\mathcal{F}_{h,i}^n$ denotes the set of interior edges of $\mathcal{T}_{h,i}^n$, the symbol $\llbracket \cdot \rrbracket_F$ stands for the jump operator across the edge F ,

$$\xi_F(\mathbf{z}_h) \stackrel{\text{def}}{=} \min\{1, \operatorname{Re}_F(\mathbf{z}_h)\}, \quad \operatorname{Re}_F(\mathbf{z}_h) \stackrel{\text{def}}{=} \frac{\rho^f \|\mathbf{z}_h\|_{L^\infty(F)} h}{\mu},$$

and $\gamma_p, \gamma_v > 0$ are user-defined parameters. The main reason of using the CIP stabilization (instead of (10)) is that it is strongly consistent in the whole computational domain $\Omega_{h,1}^n \times \Omega_{h,2}^n$,

while (10) it is strongly consistent only in the physical domain Ω_h^n due to its residual-based nature (see [2, Section 3.2.1] for a discussion).

At last, the term g_h^n in (17) corresponds to the so-called ghost-penalty stabilization operator which guarantees the robustness of the method with respect to the way the interface Σ_h^n cuts the fluid elements. It is given by the following expression (see [12]):

$$g_h^n(\mathbf{u}_h, \mathbf{v}_h) \stackrel{\text{def}}{=} \gamma_g \mu h \sum_{i=1}^2 \sum_{F \in \mathcal{F}_{h,i}^{n,\Sigma}} (\llbracket \nabla \mathbf{u}_{h,i} \rrbracket_F, \llbracket \nabla \mathbf{v}_{h,i} \rrbracket_F)_F,$$

where $\mathcal{F}_{h,i}^{n,\Sigma}$ denotes the set of interior edges of the elements of $\mathcal{T}_{h,i}^n$ intersected by Σ_h^n and $\gamma_g > 0$ is a user-defined parameter.

It should be noted that, for consistency, all the fluid bulk terms in (16) are evaluated only in the physical zones of $\Omega_{h,i}^n$ (i.e., the integrals are evaluated over cut-elements). Furthermore, the interface integrals on Σ_h^n involve the products of fields defined on different meshes. Hence, in comparison with Algorithm 2, Algorithm 3 requires a non-standard computer implementation and a specific evaluation of all the interface intersections. We refer to [2, Section 2.1.4] (and the references therein) for a critical review on this point.

4 Numerical experiments

The purpose of this section is to illustrate and compare, via a series of 2D-numerical examples, the accuracy of the unfitted methods described in Section 3.2. To this purpose, we consider as reference solution the one provided by the fitted mesh method (Algorithm 1) using a highly refined space-time grid. The accuracy of the unfitted mesh methods will be then evaluated by comparing the results obtained with three successive levels of mesh refinement in space and in time.

Through this section, the acronyms ALE, FD and NXFEM respectively refer to Algorithms 1, 2 and 3. Moreover, the variant of Algorithm 2 discussed in Section 3.2.1, with enhanced interfacial mass conservation, will be referred to as FD_{stab} . Table 1 summarizes the different user-defined parameters involved in each method, with some prescribed values. For the FD and FD_{stab} meth-

ALE	FD	FD_{stab}	NXFEM
$\lambda_C = 0$	$\lambda_C = 0$	$\lambda_C = 1$	$\gamma = 10^2$
$\lambda_M = 1$	$\lambda_M = 1$	$\lambda_M = \begin{cases} 1 & \text{in } \Omega \setminus \omega_h^n \\ \varepsilon_M & \text{in } \omega_h^n \end{cases}$	$\gamma_v = \gamma_p = 10^{-2}$
	ε	ε	$\gamma_g = 1$

Table 1: Free parameters of the numerical methods.

ods, the choice of the penalty parameter ε and (for the latter) of the stabilization parameter ε_M requires a specific care which will be discussed in the next paragraphs.

4.1 Example 1. Open valve

As first example, we consider the heart-valve-inspired benchmark problem proposed in [34, 27, 31, 52]. Due to the symmetric properties of the problem, only half of the domain is simulated with appropriate symmetry boundary conditions. The problem consists of one cantilevered elastic

Figure 4: Geometric configuration and external boundary conditions.

beam immersed in a 2D-channel filled with an incompressible Newtonian fluid, as shown in Figure 4.

The geometry of the fluid domain is given by $\Omega = [0, 8] \times [0, 0.805]$. The reference configuration of the solid is the segment with endpoints $(2, 0)$ and $(2, 0.7)$ (see Figure 4). The physical parameters for the fluid are $\rho^f = 100$ and $\mu = 10$. For the solid, we have $\rho^s = 100$, $\epsilon = 0.0212$, with Young's modulus $E = 5.6 \times 10^7$ and Poisson's ratio $\nu = 0.4$. All the units are given in the CGS units system.

As regards the boundary conditions for the fluid, a no-slip boundary condition is enforced on Γ_{bot} while a symmetry boundary condition is imposed on Γ_{top} . Zero traction is enforced on the lateral boundary Γ_{out} . The velocity is prescribed on Γ_{in} , as a half parabolic profile whose maximum amplitude is defined by a positive time-dependent function $u_{\text{max}}(t)$ given by

$$u_{\text{max}}(t) = 5 \times (0.805)^2 \times (\sin(2\pi t) + 1.1).$$

The beam is fully clamped at its bottom endpoint. The fluid and the solid are initially at rest.

4.1.1 Spatial and temporal discretization

A reference solution has been generated using Algorithm 1 with a high space-time grid resolution. The solid mesh is made of 59 segments and the body-fitted fluid mesh of 48 706 quadrilaterals which are refined around the interface. The initial configuration of the fitted meshes is depicted in Figure 5a. Here, the fluid velocity and pressure fields are approximated using $\mathbb{Q}_1$ finite elements. The time step is $\tau = 10^{-3}$ and the final time is $t = 3$, corresponding to around 3 full cycles of oscillations of the structure. A snapshot of the resulting deformed mesh at time $t = 0.5$ is shown in Figure 6a.

The first level of refinement for the unfitted meshes, which we refer to as M_1 , divides the fluid domain into 4 096 triangular elements and the structure domain into 32 segments. The corresponding time step is $\tau = 10^{-2}$. The two subsequent space-time grids, respectively noted M_2 and M_3 , are uniform refinements of M_1 with a factor of, respectively, 2 and 4 along the spatial and temporal directions. Hence, M_2 is made of 16 384 triangular elements and M_3 of 65 536 triangular elements. Their corresponding structure domains are respectively divided into 64 and 128 segments. The time steps are $\tau = 5 \times 10^{-3}$ in M_2 and $\tau = 2.5 \times 10^{-3}$ in M_3 . For illustration purposes, M_1 , M_2 and M_3 at $t = 0$ are shown in Figures 5b–5d and at $t = 0.5$ in Figures 6b–6d (obtained with the NXFEM method).

4.1.2 Comparison of results

The values of the penalty parameter ε for the FD method are equal to 10^{-5} with M_1 , 10^{-6} with M_2 and 10^{-6} with M_3 . The motivation of these choices will be discussed in Section 4.1.3 below.

Figure 5: Zoom on the fluid and solid meshes at time $t = 0$. Figure 6: Zoom on the fluid and solid meshes at time $t = 0.5$.

For this example, the results obtained with the FD_{stab} variant method are very close to those obtained with the FD method and, hence, are omitted.

Figures 7 and 8 present snapshots of the interface location and fluid velocity magnitude near the leaflet at $t = 0.5$ for the three levels of refinement and for the NXFEM and FD methods, respectively. Overall, a good agreement of the velocity field with the ALE method is already observed for the coarsest grid M_1 . Note that the correct computation of the induced vortex after the leaflet requires more refinement and it is better captured by the NXFEM method.

The results for the pressure are given in Figures 9 and 10, respectively for the NXFEM and FD methods. The NXFEM method gets the correct shape of the pressure field, even with the coarsest grid resolution. After refinement, the agreement with the ALE method is very good in both approaches. For the FD method, comparisons between M_1 and M_3 clearly indicates that

Figure 7: Snapshots of the fluid velocity magnitude at $t = 0.5$ obtained with NXFEM.

Figure 8: Snapshots of the fluid velocity magnitude at $t = 0.5$ obtained with FD.

mesh refinement improves the results.

Figure 11 reports a comparison of the time history of the x -solid displacement at the upper leaflet tip obtained with the NXFEM and FD methods. The NXFEM method shows better agreement with the ALE method. Nevertheless, the overall agreement of the FD methods is satisfying for the M_2 and M_3 refinement levels.

Similar observations can be made from Figure 12, which reports the time history of the y -solid displacement of the upper leaflet tip. Note that the convergence of the FD method towards the reference solution is clearly visible, but slower than for the NXFEM method.

Figure 9: Snapshots of the fluid elevated pressure at $t = 0.5$ obtained with NXFEM. Figure 10: Snapshots of the fluid elevated pressure at $t = 0.5$ obtained with FD.

Figure 11: Time history of the x -displacement of the upper leaflet tip. Figure 12: Time history of the y -displacement of the upper leaflet tip.

4.1.3 Choice of the penalty parameters for the fictitious domain methods

For the FD method introduced in Algorithm 2, the critical parameter to fix is the penalization coefficient ε . The choice of the values of penalty parameters is usually delicate and arbitrary (see, e.g., [53, 42]). Here, we present the workflow we carried on to determine a range of admissible values for the penalization coefficient ε . Our approach is similar to the one used in [53]: a given simulation has been run with a large range of values for the penalty parameter ε . The value is initially 10^{-1} and is reduced until the convergence is lost due to ill-conditioning problems of the system. A physical value, characterizing the current case, is then computed with respect to the time. This convergence study is done on for each level of refinement.

The physical value of interest here is the displacement magnitude of the immersed upper leaflet tip. The set of values used for the ε are

$$\varepsilon = \begin{cases} \{10^{-i}, i = 1 \dots 6\}, & \text{for } M_1, \\ \{10^{-i}, i = 1 \dots 7\}, & \text{for } M_2, \\ \{10^{-i}, i = 1 \dots 7\}, & \text{for } M_3, \end{cases}$$

and the corresponding results for the levels of refinement M_1 , M_2 and M_3 are respectively shown in Figures 13a–13c

As expected, all the curves have the same behavior, independently of the penalization or the refinement of the meshes. The penalty parameter ε has an impact on the amplitude of the displacement: when the penalization is stronger, the different curves convergence towards the same value. As the penalization of the immersed structure appears in the fluid problem under

Figure 13: Convergence study of the penalty parameter ε .

the form $\frac{1}{\varepsilon}$, a small penalty parameter may induce ill-conditioning of the system and then, low convergence rate. This is why we assumed that, once a good over-all convergence has been achieved and the results are independent of the values of ε , it is no more useful to further reduce it.

4.2 Example 2. Closed valve

As second example, we consider a benchmark problem mimicking the behavior of a closed valve (see, e.g., [34, 49]). The main difficulty here are the correct evaluation of the pressure jump and the conservation of mass across the valve.

Figure 14: Geometric configuration and external boundary conditions.

The problem consists of an elastic beam clamped on its endpoints and immersed in a 2D-channel filled with an incompressible Newtonian fluid, as shown in Figure 4. The geometry of the fluid domain is given by $\Omega = [0, 4] \times [0, 1]$. The reference configuration for the solid is defined as the segment with endpoints $(2, 0)$ and $(2, 1)$ (see Figure 14). The physical parameters for the fluid are $\rho^f = 100$ and $\mu = 10$. For the solid, we have $\rho^s = 100$, $\epsilon = 0.0212$, with Young's modulus $E = 5.6 \times 10^7$ and Poisson's ratio $\nu = 0.4$. All the units are given in the CGS units system.

As regards the boundary conditions for the fluid, a no-slip boundary condition is enforced on Γ_{bot} and Γ_{top} . Zero traction is enforced on the lateral boundary Γ_{out} , while on Γ_{in} traction is imposed in terms of the following time-dependent pressure

$$p_{\text{in}}(t) = 3 \times 10^5 \times \tanh(10 \times t).$$

The structure is fully clamped at its endpoints A_0 and A_1 . The fluid and the solid are initially at rest. As the prescribed pressure builds up, the solid starts to bend and elongate. After this

brief transition phase, the system reaches a steady state with a pressure jump across the interface. The fluid and solid velocities vanish and the pressure is a piecewise constant function taking the value 3×10^5 and zero on each side of the interface, respectively.

4.2.1 Spatial and temporal discretization

A reference solution has been generated using Algorithm 1 with a high space-time grid resolution. The solid mesh is made of 80 segments and the body-fitted fluid mesh of 25 600 quadrilaterals. The initial configuration of the fitted meshes is depicted in Figure 15a.

Figure 15: Zoom on the fluid and solid meshes at time $t = 0$.

Here, the fluid velocity and pressure fields are approximated using $\mathbb{Q}_1$ finite elements. The time step is $\tau = 10^{-3}$ and the final time is $t = 3$, corresponding to the time required by the structure to reach its steady state. A snapshot of the resulting deformed mesh at time $t = 0.125$ is shown in Figure 16a.

The first level of refinement for the unfitted meshes, M_1 , divides the fluid domain into 1 034 triangular elements and the structure domain into 25 segments. The corresponding time step is $\tau = 2 \times 10^{-3}$. As in the previous example, the two subsequent space-time grids are uniform refinements of M_1 : M_2 is made of 4 370 triangular elements and M_3 of 17 954 triangular elements. Their corresponding structure domains are respectively divided into 50 and 100 segments. The time steps are $\tau = 10^{-3}$ in M_2 and $\tau = 5 \times 10^{-4}$ in M_3 . For illustration purposes, M_1 , M_2 and M_3 at $t = 0$ are shown in Figures 15b–15d and at $t = 0.125$ in Figures 16b–16d (obtained with the NXFEM method).

4.2.2 Comparison of results

For all the refinement levels, we have taken $\varepsilon = 10^{-5}$ in the FD and FD_{stab} methods. In the latter, the stabilization parameter is set to $\varepsilon_M = 10^{-4}$. The motivation of these choices will be

 Figure 16: Zoom on the fluid and solid meshes at time $t = 0.125$.

discussed in Section 4.2.3 below.

 Figure 17: Snapshots of the fluid velocity magnitude at $t = 0.125$ obtained with NXFEM.

Figures 17–19 present, respectively for the NXFEM, FD and FD_{stab} methods, the snapshots

Figure 18: Snapshots of the fluid velocity magnitude at $t = 0.125$ obtained with FD.

Figure 19: Snapshots of the fluid velocity magnitude at $t = 0.125$ obtained with FD_{stab} .

of the interface location and the fluid velocity magnitude near the leaflet at $t = 0.125$ for the three levels of refinement. The NXFEM method retrieves the negligible velocity field of the reference ALE solution in all the grid refinement levels. The color palette has been kept the same for the

Figure 20: Snapshots of the fluid elevated pressure at $t = 0.125$ obtained with NXFEM.

Figure 21: Snapshots of the fluid elevated pressure at $t = 0.125$ obtained with FD.

three figures in order to highlight the loss of mass featured by the FD method. In Figure 18, no matter what level of refinement is used, a significant spurious flow is observed through the channel. This phenomenon is an important limitation of the FD method with continuous pressure approximations (see, e.g., [4, 34]), which guarantees global but not local mass conservation (i.e.,

Figure 22: Snapshots of the fluid elevated pressure at $t = 0.125$ obtained with FD_{stab} .

Figure 23: Time history of the x -displacement of the valve mid-point.

on each side of the interface). Figure 19 shows, on the contrary, that the FD_{stab} variant is able to significantly reduce this leak across the interface, which highlights the benefits of the modified SUPG/PSPG and grad-div stabilizations proposed in [34].

The results for the pressure are reported in Figures 20–22, respectively for the NXFEM, FD and FD_{stab} methods. Once more, the NXFEM method gets the correct pressure jump, particularly with the grid refinement levels M_2 and M_3 . On the opposite, the FD method struggles to correctly compute the pressure jump, which is largely under-estimated compared to the reference solution. The FD_{stab} method shows a very good correction of the pressure jump across the valve compared to the FD method. Moreover, the fact that the prediction of the pressure jump improves with the refinement is much more striking in Figure 22.

Figure 23 reports a comparison of the time history of the x -solid displacement of the valve mid-point obtained with the NXFEM, FD and FD_{stab} methods. Similar observations can be made from these results. The NXFEM and FD_{stab} methods shows better agreement with the

reference solution than with the FD method. One particular point of interest is the computation of the dynamics of the valve, characterized by the oscillations of the x -value around the reference steady state. The FD method is clearly unable to compute the initial dynamics of the valve, no matter what level of refinement is used. On the contrary, the NXFEM and FD_{stab} predict the dynamics of the closed and show convergence to the reference after grid refinement.

4.2.3 Choice of the penalty parameters for the fictitious domain methods

As explained in Subsection 4.1.3, we carried on the same convergence study with respect to the penalty parameter ε , appearing in the FD method. The physical value of interest here is the x -displacement of the valve mid-point. For each level of refinement, the case has been run several times with a large panel of values for ε . The same behaviors as the ones described in Section 4.1.3 have also been observed here. The chosen values of ε are based on the previous mentioned criteria of convergence.

For the sake of illustration, the interfacial zone in which the modified FD_{stab} acts, is depicted at $t = 0.125$ in Figures 24a–24c, respectively for the meshes M_1 , M_2 and M_3 . In order to choose

Figure 24: The neighborhood ω_h^n at time $t = 0.125$.

the values of ε_M , the same kind of convergence study as the one described in Subsection 4.1.3 for ε has been carried on. For each case M_1 , M_2 and M_3 , ε is equal to the value given by the convergence study and the case is run several times with a large range of values for the user-defined parameter ε_M . The set of values used for ε_M are:

$$\varepsilon_M = \begin{cases} \{10^{-i}, i = 0 \dots 4\}, & \text{for } M_1, \\ \{10^{-i}, i = 0 \dots 4\}, & \text{for } M_2, \\ \{10^{-i}, i = 0 \dots 4\}, & \text{for } M_3. \end{cases}$$

The lowest value, which do not compromise the convergence of the simulation, is chosen.

4.3 Example 3. Vesicle in lid-driven cavity flow

As last example, we consider the lid-driven cavity benchmark problem (inspired from, e.g., [44, 50, 29, 22]) with an immersed elastic circle, as shown in Figure 25.

The geometry of the fluid domain is given by $\Omega = [0, 0] \times [1, 1]$. The reference solid configuration is a circle of radius $R = 0.2$ centred at $(0.6, 0.5)$ (see Figure 25). The physical parameters for the fluid are $\rho^f = 100$ and $\mu = 10$. For the solid, we have $\rho^s = 100$, $\epsilon = 0.0212$, with Young's modulus $E = 5.6 \times 10^3$ and Poisson's ratio $\nu = 0.4$. All the units are given in the CGS units system.

Figure 25: Geometric configuration and external boundary conditions.

As regards the boundary conditions for the fluid, a no-slip boundary condition is enforced on Γ_{bot} and Γ_{wall} . Zero traction is enforced on the lateral boundaries Γ_{in} and Γ_{out} . The velocity is prescribed on Γ_{top} , as a positive velocity whose amplitude is constant and equal to 1. Both the fluid and the solid are initially at rest.

4.3.1 Spatial and temporal discretization

The reference ALE solution has been generated using a solid mesh made of 200 segments and a body-fitted fluid mesh of 46 078 triangles. The initial configuration of the fitted meshes is depicted in Figure 26a. The time step is $\tau = 2.5 \times 10^{-3}$ and the final time is $t = 10$, approximately corresponding to a full turn of the vesicle. A snapshot of the resulting deformed mesh at time $t = 6$ is shown in Figure 27a. We consider that $t = 6$ is the upper limit where we can still refer to the ALE solution as the reference (close to the break-down of the mesh motion procedure).

(a) Fitted meshes - Reference.

(b) Unfitted meshes - M_1 . (c) Unfitted meshes - M_2 . (d) Unfitted meshes - M_3 .Figure 26: Zoom on the fluid and solid meshes at time $t = 0$.

Figure 27: Zoom on the fluid and solid meshes at time $t = 6$.

The first level of refinement for the unfitted meshes, M_1 , divides the fluid domain into 800 triangular elements and the structure domain into 40 segments. The corresponding time step is $\tau = 10^{-2}$. As in two previous test cases, the two subsequent space-time grids, denoted by M_2 and M_3 respectively, are uniform refinements of M_1 . Thus, M_2 is made of 3 200 triangular elements and M_3 of 12 800 triangular elements. Their corresponding structure domains are divided into 80 and 160 segments respectively. The time steps are $\tau = 5 \times 10^{-3}$ in M_2 and $\tau = 2.5 \times 10^{-3}$ in M_3 . For illustration purposes, M_1 , M_2 and M_3 at $t = 0$ are shown in Figures 26b–26d and at $t = 6$ in Figures 27b–27d (obtained with the NXFEM method).

4.3.2 Comparison of results

The penalty parameter ε for the FD and FD_{stab} methods is $\varepsilon = 10^{-4}$ for all levels of refinement. As regards the stabilization parameter ε_M , we have $\varepsilon_M = 1$ with M_1 , $\varepsilon_M = 10^{-1}$ with M_2 and $\varepsilon_M = 10^{-4}$ with M_3 . These optimal choices will be discussed in Section 4.3.3 below.

Figures 28–30 present the snapshots of the fluid velocity magnitude near the vesicle and its location, at $t = 6$, for the three levels of refinement and for the NXFEM, FD and FD_{stab} methods, respectively. Note that some slight oscillations in the isolines of the reference ALE solution are visible, which can be related to the distortion of the deformed fluid mesh (see Figure 27a). We can clearly observe that the NXFEM method is able to predict the correct location of the interface even with the coarsest approximation (see Figure 28b). The situation is much more delicate for the FD method which, at all grid levels, shows major mass losses across the interface. In other words, the area of the vesicle significantly evolves during the simulation. This unphysical phenomenon is corrected by the FD_{stab} after grid refinement (see Figure 30d). This issue is explicitly measured in Figure 31, which reports the time history of the relative vesicle area for the three unfitted mesh methods. As expected, the area of the vesicle obtained with ALE method is practically constant over time. Note that the reference simulation breaks down at $t = 6.0275$, due to the stretching of fluid elements around the interface. Area changes are negligible with the NXFEM method and, after reasonable grid refinement, with the FD_{stab} method.

Some pressures snapshots are given in Figures 32–34, respectively for the NXFEM, FD and

Figure 28: Snapshots of the fluid velocity magnitude at $t = 6$ obtained with NXFEM. Figure 29: Snapshots of the fluid velocity magnitude at $t = 6$ obtained with FD.

FD_{stab} methods. Note that the FD method does not predict any significant pressure jump

Figure 30: Snapshots of the fluid velocity magnitude at $t = 6$ obtained with FD_{stab} .

Figure 31: Time history of the relative area of the vesicle.

between the inner vesicle and the rest of the cavity, no matter what level of refinement is used. On the contrary, the NXFEM method is able to predict it and it also improves after refinement. A similar behavior can be observed from M_2 with the FD_{stab} variant. Particularly noticeable is the very good agreement between the NXFEM and FD_{stab} approximations with the reference solution, for the highest level of refinement.

Similar observations can be made from Figure 35, which reports the time history of the trajectory of the bottom of the vesicle. Again, the FD method presents shifted trajectories compared to the other methods, consequence of the continuous pressure approximation.

Figure 32: Snapshots of the fluid elevated pressure at $t = 6$ obtained with NXFEM.

Figure 33: Snapshots of the fluid elevated pressure at $t = 6$ obtained with FD.

Figure 34: Snapshots of the fluid elevated pressure at $t = 6$ obtained with FD_{stab} .

Figure 35: Time history of the trajectory of the bottom of the vesicle.

4.3.3 Choice of the penalty parameters for the fictitious domain methods

As explained in Subsection 4.1.3, we carried on the same convergence study with respect to the penalty parameter ε , appearing in the FD method. The physical value of interest here is the displacement magnitude of the bottom of the vesicle. For each level of refinement, the case has been run several times with a large panel of values for ε . The same behaviors as the ones described in Section 4.1.3 and Section 4.2.3 have also been observed here. The chosen values of ε are based on the previous mentioned criteria of convergence.

In order to choose the values of ε_M , we carried on the same kind of convergence study as the one described in Subsection 4.2.3. The possible values taken by this parameter are dependent

on the size of the area where the stabilization is modified and, hence, of the level of refinement. This explains why ε_M is lower for more refined space-time grids.

5 Conclusions

Different unfitted mesh methods for immersed FSI have been compared and validated in a series of 2D-benchmarks involving fully non-linear models and large interface deflections. All the methods discussed are based on low-order first-order Lagrange spatial approximations, both in the fluid and in the solid. *A priori*, the main strengths and weaknesses of the methods can be summarized as follows:

- The FD methods based on Dirac’s Lagrange multipliers and globally continuous fluid approximations are very appealing in practice, particularly for 3D applications, since their computer implementation demands relatively simple modifications on existing fluid solvers. The price to pay is the accuracy issues and the condition number degradation, due to penalty terms;
- The Nitsche-XFEM method is based on recent mathematically sound results (fully consistent treatment of the interface coupling, optimal error estimates). However, in practice, this method requires major intrusive modifications on existent fluid solvers, due in particular to the dynamic nature of the cut-FEM methodology, that can be challenging in 3D with non-structured meshes (a recent study in this direction can be found in [55], see also [37, 32]).

The detailed comparisons of Section 4 showed that, for a given mesh refinement, the best accuracy is obtained with the Nitsche-XFEM method. Another salient result of the present study is that, whenever present, the inaccuracies of the FD method are mainly driven by artificial interfacial mass losses, rather than by the sub-optimal treatment of the interface coupling, notably in the case of closed domains. This observation is supported by the very good results obtained with the FD_{stab} variant and a reasonable space-time grid refinement. The main limitation of this approach is the degradation of the stiffness matrix conditioning induced by the grad-div penalty near the interface.

These ill-conditioning issues can be circumvented by considering either divergence-conforming approximations (see [33, 16] within the immersogeometric framework) or globally discontinuous pressures with higher order polynomials for the velocities (see [4, 8, 7, 10] within the finite element framework). We hence believe that an interesting extension of this work could be the development of alternative fictitious domain methods with enhanced interfacial mass conservation which preserve, for convenience, the low-order nature of the finite element fluid approximations. In particular, we will explore the use of $\mathbb{P}_1/\mathbb{P}_0$ stabilized approximations with a pressure stabilization coefficient which vanishes within a h -dependent vicinity of the interface, in order to remove the inconsistency of the pressure jumps.

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