On Optimal Control Problems for Generalized Homogeneous Systems

Andrey Polyakov

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Abstract

The paper deals with optimal control problems for plants modeled by nonlinear ordinary differential equations with homogeneous (in a generalized sense) right hand sides. The classical tools of optimal control theory, namely, Dynamic Programming and Maximum Principle are refined for generalized homogeneous control systems.

1 Introduction

An object (e.g. an operator or a vector field) is homogeneous in a generalized sense if it is symmetric with respect to a certain family of transformations (called dilations) [1], [2], [3], [4]. On the one hand, a lot of well-known models of mathematical physics are homogeneous in a generalized sense [4], e.g. heat, wave, Burgers, Navier-Stocks, Saint-Venant and Korteweg-de-Vries equations as well as Fast Diffusion equation. On the other hand, homogeneity is one of the desirable properties for nonlinear control system, since it allows stability to imply robustness of the system [5], [6], [7] and simplifies the time-constrained stabilization providing the finite-time stability [8], [9] to the closed-loop control system with negative homogeneity degree [10], [11], [12], [13].

Optimal control design is one of classical problems of mathematical control theory [14], [15]. Geometry of a system is important for optimal control design [16], [17]. Being a kind of a symmetry, the homogeneity is expected to simplify analysis and design optimal control systems.

The optimal regulation problem for a class of homogeneous control systems is considered in [**?**], where some sampled-time control is designed. The proposed feedback comes out of the solution of an infinite horizon optimization problem in discrete time. This paper refines Dynamic Programming and Maximum Principle under the assumption that the control system is modeled by homogeneous ordinary differential equation.

Notation:

- $\mathbb{R}$ is the field of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$; $\mathbb{N}$ is the set of natural numbers;

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- $\operatorname{rank}(M)$ denotes the rank of the matrix $M \in \mathbb{R}^{n\times m}$;
- $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix;
- $\operatorname{diag}\{\lambda_i\}_{i=1}^n$ denotes the diagonal $n \times n$ matrix with the elements $\lambda_i \in \mathbb{R}$ on the main diagonal;
- the inequality $P > 0$ for $P \in \mathbb{R}^{n\times n}$ means that P is positive definite symmetric matrix.
- let $W^{1,1}([0,T),\mathbb{R}^n)$ be the Sobolev space of absolutely continuous functions $(0,T) \to \mathbb{R}^n$, where $T \leq +\infty$, and the set $W^{1,1}_{loc}((0,T),\mathbb{R}^n)$ consists of functions which restriction to any interval $(0,T')$ with $0 < T' < T$ belongs to $W^{1,1}((0,T'),\mathbb{R}^n)$;
- let $L^\infty((0,T),\mathbb{R}^m)$ be the space of uniformly essentially bounded functions, where $T \leq +\infty$;
- let $\|\cdot\|$ be a norm in $\mathbb{R}^n$ and $\|\cdot\|_{\mathbb{A}}$ be the matrix norm induced by $\|\cdot\|$, i.e. $\|A\|_{\mathbb{A}} = \sup_{x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|}$ if $A \in \mathbb{R}^{n\times n}$; $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{W^{1,1}_{loc}}$
- $\mathcal{B}(\varepsilon) = \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$ is the ball of the radius $\varepsilon > 0$.
- $S = \{u \in \mathbb{R}^n : \|u\| = 1\}$ is the unit sphere.

2 Problem Statement and Basic Assumptions

Let us consider a control system modeled by the ordinary differential equation

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in (0,T), \tag{1}$$

$$x(0) = x_0 \backslash \{\mathbf{0}\}, \tag{2}$$

where $T \leq +\infty$ is the time horizon, $x(t) \in \mathbb{R}^n$ is the vector of system states, $u(t) \in \mathbb{R}^m$ is the vector of control inputs, the vector-valued function $f : \mathbb{R}^n\backslash\{\mathbf{0}\} \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be at least continuous. We study the optimal control problem (OPC)

$$J(x,u,T) = \int_0^T L(x(\sigma), u(\sigma))d\sigma \quad \to \quad \inf_{x,u} \tag{3}$$

subject to (1), (2), the *control constraint* given by

$$u(t) \in \mathcal{U}, \quad t \in (0,T), \tag{4}$$

and the *terminal set* described as follows

$$\lim_{t\to T} g_i(x(t)) \leq 0, \quad i = 1,2,...,l, \tag{5}$$

where the set $\mathcal{U} \subseteq \mathbb{R}^m$ is compact and the functions $g_i : \mathbb{R}^n\backslash\{\mathbf{0}\} \to \mathbb{R}$ and the Lagrangian $L : \mathbb{R}^n\backslash\{\mathbf{0}\} \times \mathbb{R}^n \to \mathbb{R}_+$ are assumed to be continuous. The time *horizon* T can be fixed or non-fixed, finite $T < +\infty$ or infinite $T = +\infty$.

Below we assume the considered OCP is *homogeneous* in a generalized sense. Homogeneity is a property of an object (e.g. function or vector field) to be symmetric (in a certain sense) with respect to a group of transformations (called *dilations*). The generalized homogeneity [4], [18] deals with linear transformations (*linear dilations*) given below.

Definition 1 ([4]) *A map $\mathbf{d} : \mathbb{R} \to \mathbb{R}^{n\times n}$ is called* ***dilation*** *in $\mathbb{R}^n$ if it satisfies*

- ***Group property:***
$$\mathbf{d}(0)=I_n \quad \text{and} \quad \mathbf{d}(t+s)=\mathbf{d}(t)\mathbf{d}(s), \quad t,s\in\mathbb{R};$$
- ***Continuity property:*** *$\mathbf{d}$ is a continuous map, i.e.*
$$\forall t>0, \forall\varepsilon>0, \exists\delta>0 : |s-t|<\delta \Rightarrow \|\mathbf{d}(s)-\mathbf{d}(t)\|_{\mathbb{A}} \leq \varepsilon;$$
- ***Limit property:***
$$\lim_{s\to-\infty} \|\mathbf{d}(s)u\|=0 \text{ and } \lim_{s\to+\infty} \|\mathbf{d}(s)u\|=+\infty$$
uniformly on ***the unit sphere*** $S=\{u\in\mathbb{R}^n : \|u\|=1\}$.

Obviously, the dilation $\mathbf{d}$ is a continuous group of invertible linear maps $\mathbf{d}(s)$ such that $\mathbf{d}(-s)=[\mathbf{d}(s)]^{-1}$.

Definition 2 *A vector field $g : \Omega \to \mathbb{R}^n$ (a function $h : \mathbb{R}^n \to \mathbb{R}$) is said to be $\mathbf{d}$-homogeneous of degree $\nu\in\mathbb{R}$ if $\mathbf{d}(s)\Omega \subseteq \Omega$ for $s\in\mathbb{R}$ and*

$$g(\mathbf{d}(s)z)=e^{\nu s}\mathbf{d}(s)g(z), \quad \forall z\in\Omega, \quad \forall s\in\mathbb{R}. \tag{6}$$

$$\textit{(resp.}\quad h(\mathbf{d}(s)z)=e^{\nu s}h(z), \quad \forall z\in\Omega, \quad \forall s\in\mathbb{R}.\ \textit{)}$$

More details about standard, weighted and geometric homogeneity of nonlinear systems can be found in [1], [19], [3], [20], [13], and references therein, where a lot of examples of homogeneous control systems are also studied.

We study the considered optimal control problem under assumption that the vector-field f and Lagrangian L are homogeneous (for the control u treated as a parameter).

Assumption 1 *Let $\mathbf{d}$ be a dilation in $\mathbb{R}^n$. The vector-field f, the Lagrangian L and the functions g_i are assumed to be $\mathbf{d}$-homogeneous with respect to the first argument, i.e.*

$$\exists\nu_f\in\mathbb{R} \quad : \quad f(\mathbf{d}(s)x,u) = e^{\nu_f s}\mathbf{d}(s)f(x,u), \tag{7}$$
$$\exists\nu_L\in\mathbb{R} \quad : \quad L(\mathbf{d}(s)x,u) = e^{\nu_L s}L(x,u) \tag{8}$$
$$\exists\nu_i\in\mathbb{R} \quad : \quad g_i(\mathbf{d}(s)x) = e^{\nu_i s}g_i(x), \tag{9}$$

for $s\in\mathbb{R}, x\in\mathbb{R}^n\backslash\{\mathbf{0}\}$ and $u\in\mathbb{R}^m$, $i=1,2,...,l$.

The optimal control problem (1) - (5) that satisfies the given assumption is called **the homogeneous optimal control problem** (HOCP) in the Lagrange form.

Below (see Section IV) we also make Assumption 2 that implies the uniqueness of solution to the system (1) - (2) in forward time for $u\in L^\infty((0,T),\mathbb{R}^m)$ satisfying (4).

The main goal of the paper is to refine the conventional optimal control design tools (namely, dynamic programming and Pontraygin Maximum Principle) for the case of homogeneous evolution equations, homogeneous terminal sets and homogeneous cost function.

3 Preliminaries: Generalized homogeneity

3.1 Monotone dilations

The matrix $G_{\mathbf{d}} \in \mathbb{R}^{n\times n}$ defined as $G_{\mathbf{d}} = \lim_{s\to 0} \frac{\mathbf{d}(s)-I_n}{s}$ is known (see, e.g. [21, Ch. 1]) as the **generator** of the group $\mathbf{d}(s)$. It satisfies the following properties

$$\tfrac{d}{ds}\mathbf{d}(s) = G_{\mathbf{d}}\mathbf{d}(s) = \mathbf{d}(s)G_{\mathbf{d}} \;\text{ and }\; \mathbf{d}(s) = e^{G_{\mathbf{d}}s} := \sum_{i=0}^{+\infty} \tfrac{s^i G_{\mathbf{d}}^i}{i!}.$$

Denote also $\lfloor A \rfloor_{\mathbb{A}} = \inf_{x\in\mathbb{R}^n} \frac{\|Ax\|}{\|x\|}$. Limit property implies

$$\begin{array}{ll} \bullet \|\mathbf{d}(s)\|_{\mathbb{A}} \to \;\; 0 \;\text{ as } s\to-\infty; & \bullet\; \mathbf{d}(s) \neq I_n \text{ if } s\neq 0; \\ \bullet \lfloor \mathbf{d}(s) \rfloor_{\mathbb{A}} \to +\infty \text{ as } s\to+\infty; & \bullet \lfloor G_{\mathbf{d}} \rfloor > 0 \;(\ker G_{\mathbf{d}} = \{\mathbf{0}\}). \end{array}$$

The most popular dilations are *the uniform (or standard) dilation* (L. Euler) : $\mathbf{d}(s) = e^s$, $s\in\mathbb{R}$ and *the weighted dilation* (Zubov 1958, [1]): $\mathbf{d}(s) = \mathrm{diag}\{e^{r_i s}\}$, $s\in\mathbb{R}$ and $r_i > 0$, $i=1,...,n$. They obviously satisfy Definition 1 with $G_{\mathbf{d}} = I_n$ and $G_{\mathbf{d}} = \mathrm{diag}\{r_i\}$, resp. Geometric dilation [19], [22] is more general since it allows the map $\mathbf{d}(s) : \mathbb{R}^n \to \mathbb{R}^n$ to be nonlinear.

Definition 3 ([18]) *The dilation* $\mathbf{d}$ *is* ***monotone*** *if it is a strong contraction for* $s < 0$, *i.e.* $\|\mathbf{d}(s)\|_{\mathbb{A}} < 1$ *as* $s<0$.

Monotonicity of dilation depends on the norm $\|\cdot\|$ selected in $\mathbb{R}^n$. For example, the dilation $\mathbf{d}(s) = e^s \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix}$ with $G_{\mathbf{d}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is monotone if $\mathbb{R}^2$ is equipped with the weighted norm $\|x\|_P = \sqrt{x^\top P x}$, $P = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} > 0$ and it is non-monotone if, for example, $P = \begin{pmatrix} 1 & 3/4 \\ 3/4 & 1 \end{pmatrix} > 0$. In the latter case, the curve $\{\mathbf{d}(s)u : s \in \mathbb{R}\}$ may cross the unit sphere $\|x\|_P = 1$ in two different points.

Theorem 1 ([18]) *The next four conditions are equivalent*

1. ***1)*** *the dilation* $\mathbf{d}$ *is* ***monotone****;*
2. ***2)*** $\lfloor \mathbf{d}(s) \rfloor_{\mathbb{A}} > 1$ *for* $s > 0$*;*
3. ***3)*** *the continuous function* $\|\mathbf{d}(\cdot)x\| : \mathbb{R}\to\mathbb{R}_+$ *is strictly increasing for any* $x \in S$*;*
4. ***4)*** *for any* $x \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ *there exists a* ***unique pair*** $(s_0, x_0) \in \mathbb{R} \times S$ *such that* $x = \mathbf{d}(s_0)x_0$.

Theorem 1 guarantees the functions $\|\mathbf{d}(\cdot)\|_{\mathbb{A}} : \mathbb{R}\to\mathbb{R}_+$ and $\lfloor \mathbf{d}(\cdot) \rfloor_{\mathbb{A}} : \mathbb{R}\to\mathbb{R}_+$ are also continuous and strictly increasing.

Definition 4 ([18]) *The dilation* $\mathbf{d}$ *is called* ***strictly monotone*** *if*

$$\exists \beta > 0 \quad : \quad \|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\beta s} \text{ for } s \leq 0.$$

The dilation $\mathbf{d}$ considered in the above example is strictly monotone if $\mathbb{R}^2$ equipped with the Euclidean norm.

Theorem 2 ([18]) *Let $\mathbf{d}$ be a dilation in $\mathbb{R}^n$ then*

- *the matrix $-G_{\mathbf{d}}$ is Hurwitz, i.e. all eigenvalues λ_i of $G_{\mathbf{d}}$ are placed in the right complex half-plane;*
- *there exists $P \in \mathbb{R}^{n\times n}$*

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^{\top}P > 0, \quad P = P^{\top} > 0; \tag{10}$$

- *the dilation $\mathbf{d}$ is strictly monotone with respect to the weighted norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ induced by the inner product $\langle u, v\rangle = u^{\top}Pv$ with P satisfying* (10)*:*

$$\begin{array}{l} e^{\alpha s} \leq \lfloor \mathbf{d}(s) \rfloor_{\mathbb{A}} \leq \|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\beta s} \;\; \text{if} \;\; s \leq 0, \\ e^{\beta s} \leq \lfloor \mathbf{d}(s) \rfloor_{\mathbb{A}} \leq \|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\alpha s} \;\; \text{if} \;\; s \geq 0, \end{array} \tag{11}$$

where

$$\begin{array}{l} \alpha = 0.5\lambda_{\max}\left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^{\top}P^{\frac{1}{2}}\right), \\ \beta = 0.5\lambda_{\min}\left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^{\top}P^{\frac{1}{2}}\right). \end{array} \tag{12}$$

Therefore, any dilation $\mathbf{d}$ is strictly monotone if $\mathbb{R}^n$ is equipped with the weighted norm $\|x\| = \sqrt{x^{\top}Px}$ provided that the matrix $P > 0$ satisfies (10)*.*

3.2 Homogeneous norm

Dilation $\mathbf{d}$ introduces a topology in $\mathbb{R}^n$ (spheres and balls [19], [23], [6]) by means of the so-called "homogeneous norm".

Definition 5 *A continuous function $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \to \mathbb{R}_+$ is said to be $\mathbf{d}$-homogeneous norm if $\|x\|_{\mathbf{d}} \to 0$ as $x \to \mathbf{0}$ and $\|\mathbf{d}(s)x\|_{\mathbf{d}} = e^s\|x\|_{\mathbf{d}} > 0$ for $x \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ and $s \in \mathbb{R}$.*

For monotone dilations the **canonical homogeneous norm** can be defined as follows:

$$\|x\|_{\mathbf{d}} = e^{s_x} \quad \text{with} \quad s_x : \|\mathbf{d}(-s_x)x\| = 1. \tag{13}$$

In [24] such a homogeneous norm was called canonical since it is induced by the canonical norm $\|\cdot\|$ in $\mathbb{R}^n$ and $\|x\|_{\mathbf{d}} = \|x\| = 1$ on the unit sphere S. In this case, obviously,

$$\lfloor \mathbf{d}(\ln\|x\|_{\mathbf{d}}) \rfloor_{\mathbb{A}} \leq \|x\| \leq \|\mathbf{d}(\ln\|x\|_{\mathbf{d}})\|_{\mathbb{A}}.$$

Note also that if $\mathbf{d}(s) = e^s$ then $\|\cdot\|_{\mathbf{d}} = \|\cdot\|$. By default, *below we deal only with the canonical homogeneous norm.*

Let $B_{\mathbf{d}}(r)$ be **homogeneous ball** of the radius $r > 0$, i.e.

$$B_{\mathbf{d}}(r) = \{x \in \mathbb{R}^n : \|x\|_{\mathbf{d}} < r\}.$$

Proposition 1 ([18]) *If $\mathbf{d}$ is a strictly monotone dilation then*

- $\left|\|x_1\|_{\mathbf{d}}^{\beta} - \|x_2\|_{\mathbf{d}}^{\beta}\right| \leq \|x_1 - x_2\|$ *for* $x_1, x_2 \in \mathbb{R}^n\backslash B_{\mathbf{d}}(1)$,

- *the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is Lipschitz continuous outside the origin;*
- *if the norm $\|\cdot\|$ is smooth outside the origin then the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is also smooth outside the origin, $\frac{d\|\mathbf{d}(-s)x\|}{ds}<0$ if $s\in\mathbb{R}$, $x\in\mathbb{R}^n\backslash\{\mathbf{0}\}$ and*

$$\frac{\partial\|x\|_{\mathbf{d}}}{\partial x}=\left.\frac{\|x\|_{\mathbf{d}}\left.\frac{\partial\|z\|}{\partial z}\right|_{z=\mathbf{d}(-s)x}\mathbf{d}(-s)}{\left.\frac{\partial\|z\|}{\partial z}\right|_{z=\mathbf{d}(-s)x}G_{\mathbf{d}}\mathbf{d}(-s)x}\right|_{s=\ln\|x\|_{\mathbf{d}}} \tag{14}$$

3.3 Generalized Homogeneous Functions and Vectors Fields

Homogeneous functions and vector fields (see, Definition 2) have a lot of properties useful for control design and state estimation of both linear and nonlinear plants as well as for analysis of convergence rates [22], [25], [26], [13]. It is worth stressing that essentially non-linear vector-fields and functions may be **d**-homogeneous.

Example 1 *Let the dilation **d** be defined as follows*

$$\mathbf{d}(s)=e^s\begin{pmatrix}1 & 0 & 0\\ 0 & \cos(s) & \sin(s)\\ 0 & -\sin(s) & \cos(s)\end{pmatrix}.$$

It is strictly monotone with respect to the Euclidean norm $\|x\|=\sqrt{x^Tx}$ and $G_{\mathbf{d}}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & -1 & 1\end{pmatrix}$. The vector field $f:\mathbb{R}^3\to\mathbb{R}^3$ defined as

$$f(x)=\begin{pmatrix}x_2^2+x_3^2\\ x_1^2(\cos(\ln|x_1|)+\sin(\ln|x_1|))\\ x_1^2(\cos(\ln|x_1|)-\sin(\ln|x_1|))\end{pmatrix}$$

*and the function $h:\mathbb{R}^3\to\mathbb{R}$ given by $h=x_1^3+(x_2^2+x_3^2)^{\frac{3}{2}}$ are **d**-homogeneous of degree 1 and 3, respectively.*

In the general case, homogeneous functions and vector fields can defined on some open unbounded subsets of $\mathbb{R}^n$, e.g. $h(x_1,x_2)=\frac{1}{[x_1-x_2^2]_+}$, where $x_1,x_2\in\mathbb{R}$, but $[\rho]_+=\rho$ if $\rho>0$ and $[\rho]_+=0$ if $\rho=0$. The domain of such homogeneous function forms a homogeneous cone (set).

Definition 6 *An open nonempty set $\Omega\subset\mathbb{R}^n$ is said to be **d**-homogeneous cone if $\mathbf{d}(s)\Omega\subseteq\Omega$.*

The name"homogeneous cone" is very natural for the set Ω, since each point x belongs to Ω together with the homogeneous curve $\{z: z=\mathbf{d}(s)x, s\in\mathbb{R}\}$. If the dilation is uniform $\mathbf{d}(s)=e^sI_n$ then Ω becomes the conventional cone in $\mathbb{R}^n$.

*Let $\mathbb{F}_{\mathbf{d}}(\Omega)$ (resp. $\mathbb{H}_{\mathbf{d}}(\Omega)$) be the set of **d**-homogeneous vector fields $\Omega\subseteq\mathbb{R}^n\to\mathbb{R}^n$ (resp. functions $\mathbb{R}^n\to\mathbb{R}$), which are continuous on $\Omega\subseteq\mathbb{R}^n\backslash\{\mathbf{0}\}$. Here Ω is **d**-homogeneous cone. Let $\deg_{\mathbf{d}}(h)$ (resp. $\deg_{\mathbf{d}}(f)$) denote the homogeneity degree of $h\in\mathbb{H}_d(\Omega)$ (resp. $f\in\mathbb{F}_{\mathbf{d}}(\Omega)$).*

The homogeneity allows local properties (e.g. smoothness) of vector fields (functions) to be extended globally [1], [2]. Moreover, if the function (resp. vector field) is smooth then homogeneity is inherited by its derivatives.

Corollary 1 *If $h \in \mathbb{H}_{\boldsymbol{d}}(\Omega) \cap C^1(\Omega\backslash\{0\}, \mathbb{R})$ and $f \in \mathbb{F}_{\boldsymbol{d}}(\Omega) \cap C^1(\Omega\backslash\{0\}, \mathbb{R}^n)$ then*

$$e^{\deg(h)s}\frac{\partial h(x)}{\partial x} = \left.\frac{\partial h(x)}{\partial z}\right|_{z=\boldsymbol{d}(s)x}\boldsymbol{d}(s), \tag{15}$$

$$\frac{\partial h(x)}{\partial x}G_{\boldsymbol{d}}x = \deg_{\boldsymbol{d}}(h)\, h(u), \tag{16}$$

$$e^{\deg(f)s}\boldsymbol{d}(s)\frac{\partial f(x)}{\partial x} = \left.\frac{\partial f(z)}{\partial z}\right|_{z=\boldsymbol{d}(s)x}\boldsymbol{d}(s), \tag{17}$$

$$\frac{\partial f(x)}{\partial x}G_{\boldsymbol{d}}x = (\deg_{\boldsymbol{d}}(f)I_n + G_{\boldsymbol{d}})\, f(x), \tag{18}$$

for $x \in \Omega\backslash\{\mathbf{0}\}$ and $s \in \mathbb{R}$.

The proofs of the identities (15) and (16) are given in [18]. The identities (17) and (18) can be proven similarly.

The next corollary shows that homogeneity degree specifies some properties of homogeneous functions.

Corollary 2 *If $h : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a **d**-homogeneous function and $\sup\limits_{x\in S\cap\Omega} |h(x)| < +\infty$ then*

a) for $\deg(h) > 0$ one has $h(x) \to 0$ as $x \to \mathbf{0}$ and h is radially unbounded[1] *provided that $h(x) \neq 0$ on $S \cap \Omega$;*

b) for $\deg(h) = 0$ the function h is bounded in Ω and continuity of h at the origin implies that $h(x) \equiv const$;

c) for $\deg(h) < 0$ one has that h is discontinuous at the origin, unbounded in any neighborhood of the origin and $|h(x)| \to 0$ as $x \to \infty$.

Proof. Since the homogeneous norm $\|\cdot\|$ is continuous at the origin, then using the homogeneous identity $h(x) = h(\mathbf{d}(\ln\|x\|_{\mathbf{d}})z) = \|x\|_{\mathbf{d}}^{\deg(h)} h(z)$ with $z = \mathbf{d}(-\ln\|x\|)x \in S \cap \Omega$ we trivially complete the proof. ■

The similar results can be obtained for **d**-homogeneous vector fields.

Corollary 3 *Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a **d**-homogeneous vector field and $\sup\limits_{x\in S\cap\Omega} \|f(x)\| < +\infty$. If the numbers $\alpha, \beta \in \mathbb{R}_+$ defined by (12) for some $P > 0$ satisfying (10) are such that*

a) $\deg(f) + \beta > 0$ then f is continuous at the origin, $f(x) \to \mathbf{0}$ as $\|x\| \to \mathbf{0}$ and $\|f\|$ is radially unbounded if $\|f(x)\| \neq 0$ on $S \cap \Omega$;

b) $\deg(f) + \beta = 0$ (resp. $\deg(f) + \alpha = 0$) then f is bounded on $B_{\boldsymbol{d}}(r) \cap \Omega$ (resp. on $\Omega\backslash B_{\boldsymbol{d}}(r)$) for any fixed $r > 0$;

c) $\deg(f) + \alpha = 0$ then f is bounded on $\Omega\backslash B_{\boldsymbol{d}}(r)$ for any fixed $r > 0$;

[1] A function h (resp. a vector field f) is radially unbounded if $x \to \infty$ implies $\|h(x)\| \to +\infty$ (resp. $|f(x)| \to +\infty$).

d) $\deg(f) + \beta = \deg(f) + \alpha = 0$ *then* f *is globally bounded on* $\mathbb{R}^n$*;*

e) $\deg(f) + \alpha < 0$ *then* f *is discontinuous at the origin, unbounded in any neighbourhood of the origin and* $\|f(x)\| \to 0$ *as* $x \to \infty$*;*

Proof. Similarly for the vector field f we derive $f(x) = \|x\|_{\mathbf{d}}^{\nu}\mathbf{d}(\ln\|x\|_{\mathbf{d}})f(z)$ and $\|x\|_{\mathbf{d}}^{\nu}\lfloor\mathbf{d}(\ln\|x\|_{\mathbf{d}})\rfloor_{\mathbb{A}}\|f(z)\| \leq \|f(x)\| \leq \|x\|_{\mathbf{d}}^{\nu}\|\mathbf{d}(\ln\|x\|_{\mathbf{d}})\|_{\mathbb{A}}\|f(z)\|$. Hence, using Theorem 2 for $\|u\| < 1$ we have $\|x\|_{\mathbf{d}}^{\nu+\alpha}\|f(z)\| \leq \|f(x)\| \leq \|x\|_{\mathbf{d}}^{\nu+\beta}\|f(z)\|$ and $\|x\|_{\mathbf{d}}^{\nu+\beta}\|f(z)\| \leq \|f(x)\| \leq \|x\|_{\mathbf{d}}^{\nu+\alpha}\|f(z)\|$ if $\|x\| > 1$. ■

3.4 Homogeneous approximation

Local homogeneity and homogeneous approximations has been studied in [1], [6], [23] in order to cover a wider class of non-linear systems.

Definition 7 *Let a function* $h : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *be continuous on* ***d****-homogeneous cone* Ω*. A* ***d****-homogeneous function* $h_0 : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *(resp.* $h_\infty : \Omega \subset \mathbb{R}^n \to \mathbb{R}$*) of degree* ν *is said to be homogeneous approximation of* h *at* 0 *(resp. at* ∞*) if*

$$\lim_{s\to-\infty}\frac{h(\boldsymbol{d}(s)x)}{e^{\nu s}} = h_0(x) \quad \left(\lim_{s\to+\infty}\frac{h(\boldsymbol{d}(s)x)}{e^{\nu s}} = h_\infty(x)\right) \tag{19}$$

In [6] these limits are assumed to be uniform on the unit sphere S in $\mathbb{R}^n$. In our case, such a condition cannot be posed since the functions h, h_0, h_∞ may be unbounded on $S \cap \Omega$.

4 Homogeneous Optimal Control Problem

4.1 Admissible solutions and maximal horizon

Homogeneity (symmetry) of ODE implies symmetry of its solutions. The similar facts can be proved for optimal/admissible pairs of HOCP.

Definition 8 *A pair* $(x,u) \in W^{1,1}_{loc}((0,T),\mathbb{R}^n) \times L^\infty((0,T),\mathbb{R}^m)$ *is said to be* ***admissible*** *for HOCP with* $T \leq +\infty$ *and* $x_0 \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ *if it is satisfies* (1) *almost everywhere on* $(0,T)$*,* (2)*,* (4)*,* (5) *and* $J(x,u,T) < +\infty$*.*

Let the set of admissible pairs (**the admissible set**) of HOCP with $T \leq +\infty$ and $x_0 \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ be denoted as

$$\mathcal{P}(x_0,T) \subset W^{1,1}_{loc}((0,T),\mathbb{R}^n) \times L^\infty((0,T),\mathbb{R}^m). \tag{20}$$

The next theorem discovers homogeneous relations between admissible sets.

Theorem 3 *If one has* $(x,u) \in \mathcal{P}(x_0,T)$ *then*

$$(x_s,u_s) \in \mathcal{P}(\boldsymbol{d}(s)x_0, e^{-\nu_f s}T),$$

$$J(x_s, u_s, e^{-\nu_f s}T) = e^{(\nu_L-\nu_f)s}J(x, u, T),$$

for any $s \in \mathbb{R}$, where

$$x_s(t) = \boldsymbol{d}(s)x(e^{\nu_f s}t),\ u_s(t) = u(e^{\nu_f s}t),\ t \in (0, e^{-\nu_f s}T).$$

Proof. Obviously, if $(x, u) \in W^{1,1}((0, T), \mathbb{R}^n) \times L^\infty((0, T), \mathbb{R}^m)$ then $(x_s, u_s) \in W^{1,1}((0, e^{-\nu_f s}T), \mathbb{R}^n) \times L^\infty((0, e^{-\nu_f s}T), \mathbb{R}^m)$. In addition, $x_s(0) = \mathbf{d}(s)x_0$ and $g_i(x_s(e^{-\nu_f s}T)) = g_i(\mathbf{d}(s)x(T)) = e^{\nu_i s}g_i(x(T)) \le 0$. Let us show that (x_s, u_s) satisfies (1) almost everywhere on $(0, e^{-\nu_f s})$. If $t \in (0, e^{-\nu_f s}T)$ then $\tau = e^{\nu_f s}t \in (0, T)$ and $\frac{d}{dt}x_s(t) = \mathbf{d}(s)\frac{d}{dt}x\left(e^{\nu_f s}t\right) = e^{\nu_f s}\mathbf{d}(s)\frac{d}{d\tau}x(\tau) = e^{\nu_f s}\mathbf{d}(s)f(x(\tau), u(\tau)) = e^{\nu_f s}\mathbf{d}(s)f(x\left(e^{\nu_f s}t\right), u\left(e^{\nu_f s}t\right)) = f(\mathbf{d}(s)x\left(e^{\nu_f s}\right), u\left(e^{\nu_f s}t\right)) = f(x_s(t), u_s(t))$ almost everywhere on $(0, e^{-\nu_f s}T)$. Similarly, due to homogeneity of L we derive $L(x_s(t), u_s(t)) = L(\mathbf{d}(s)x\left(e^{\nu_f s}t\right), u\left(e^{\nu_f s}t\right)) = e^{\nu_L s}L(x\left(e^{\nu_f s}t\right), u\left(e^{\nu_f s}t\right)) = e^{\nu_L s}L(x_s(t), u_s(t))$ and

$$J(x_s, u_s, e^{-\nu_f s}T) = e^{\nu_L s}\int_0^{T/e^{\nu_f s}} L(x\left(e^{\nu_f s}t\right), u\left(e^{\nu_f s}t\right))dt =$$
$$e^{(\nu_L-\nu_f)s}\int_0^T L(x\left(\tau\right), u\left(\tau\right))d\tau = e^{(\nu_L-\nu_f)s}J(x, u, T).$$

The obtained relation completes the proof. ■

Continuity of the vector-field f and the Lagrangian L on $\mathbb{R}^n\backslash\{\mathbf{0}\} \times \mathbb{R}^m$ imply that for any $x_0 \in \mathbb{R}^n\backslash\{\mathbf{0}\}$ there exists $T > 0$ such that the admissible set $\mathcal{P}(x_0, T)$ is non-empty.

Let us introduce the function $\chi : \mathbb{R}^n\backslash\{\mathbf{0}\} \to \mathbb{R}_+$

$$\chi(x_0) = \inf_{\mathcal{P}(x_0,T)\neq\emptyset} 1/T \tag{21}$$

that characterizes an **admissible horizon** of HOCP in the following way:

- if $\chi(x_0) > 0$ then $\mathcal{P}(x_0, T) \neq \emptyset$ for $T \in (0, 1/\chi(x_0))$ and $\mathcal{P}\left(x_0, \varepsilon + 1/\chi(x_0)\right) = \emptyset$ for any $\varepsilon > 0$;
- $\chi(x_0) = 0$ implies $\mathcal{P}(x_0, T) \neq \emptyset$ for any $T > 0$.

Corollary 4 *The function χ is **d**-homogeneous of degree ν_f and*

1) if $\chi(x_0) = 0$ for all $x_0 \in S$ then $\chi \equiv 0$, where S is a unit sphere in $\mathbb{R}^n$;

2) if $\nu_f > 0$ then $\chi(x_0) \to 0$ as $\|x_0\| \to 0$;

3) if $\nu_f < 0$ then $\chi(x_0) \to 0$ as $\|x_0\| \to \infty$;

4) if $\nu_f = 0$ then $\chi(x_0)$ is bounded on $\mathbb{R}^n\backslash\{\mathbf{0}\}$;

5) if $0 \le \nu_f + \beta \le 2\beta - \alpha$ and $\nu_L \ge \nu_f$ then $\chi \equiv 0$ and the set $\mathcal{P}(x_0, T)$ is bounded for any $x_0 \in \mathbb{R}^n$ and any $T < +\infty$, where $\alpha, \beta \in \mathbb{R}$ are defined by (12) *for some $P > 0$ satisfying* (10).

Proof. Homogeneity of the function χ immediately follows from Theorem 3. The claims 1)–4) are straightforward consequences of Corollary 2. Let us prove the claim 5) using the same corollary. Due to homogeneity one has

$$\|f(x,u)\| \leq \|x\|_{\mathbf{d}}^{\nu_f}\|\mathbf{d}(\ln\|x\|_{\mathbf{d}})\|_{\mathcal{A}}\|f(z,u)\|$$

with $z=\mathbf{d}(-\ln\|x\|)x \in S$ and

$$\|\mathbf{d}(\ln\|x\|_{\mathbf{d}})\|_{\mathcal{A}} \leq \|x\|^{\beta}$$

if $\|x\|\leq 1$, but

$$\|\mathbf{d}(\ln\|x\|_{\mathbf{d}})\|_{\mathcal{A}} \leq \|x\|^{\alpha}$$

if $\|x\|\geq 1$. Taking into account $\nu_f+\alpha\leq\beta$ and $\|x\|_{\mathbf{d}}^{\beta}\leq\|x\|$ for $\|x\|\geq 1$ we derive $\|f(x,u)\|\leq C_2\|x\|$ if $\|x\|\geq 1$, where $C_2>0$. On the other hand, $\nu_f+\beta\geq 0$ implies that $\|f(x,u)\|\leq C_1$ if $\|x\|\leq 1$, where $C_1>0$. Therefore, for any $u\in L^{\infty}((0,+\infty),\mathbb{R}^n): u(t)\in\mathcal{U}$ we have $\|f(x,u)\|\leq C_1+C_2\|x\|$ and the system (1) is forward complete. Finally, taking into account $\nu_L-\nu_f>0$ and homogeneity of the Lagrangian we conclude $L(0,u)=0$ and continuity of L with respect to the first variable. So, $P(x_0,T)$ is bounded for any $x_0\in\mathbb{R}^n$ and any $T\in(0,+\infty)$. ■

The function ξ defined above may be discontinuous in the general case. However, for the HOCP without terminal constraints (i.e. $g_i\equiv 0, i=1,..,l$) the continuity of χ on $\mathbb{R}^n\backslash\{\mathbf{0}\}$ is granted by the continuous dependence of solution x and functional J on parameters $x_0\in\mathbb{R}^n\backslash\{\mathbf{0}\}$ and $T\in(0,+\infty)$.

Homogeneous systems with positive degree ($\nu_f>0$) may have solutions that blow up in a finite time, e.g. $\dot{x}(t)=x^3(t)u(t)$. In the latter case the HOCP may remain meaningful, e.g. if $\nu_L<0$ then $L(x,u)\to 0$ as $\|x\|\to+\infty$ and an infinum of the cost functional J may correspond to fastest blow up of the solution.

Definition 9 *An admissible pair $(x^*,u^*)\in P(x_0,T)$ is said to be **optimal** if*

$$J(x^*,u^*,T)\leq J(x,u,T)$$

for all admissible pairs $(x,u)\in\mathcal{P}(x_0,T)$.

In general, the optimal pair may be non-unique. Let $\mathcal{O}(x_0,T)$ denote **the set of optimal pairs**. Obviously, $\mathcal{O}(x_0,T)\subset P(x_0,T)$ and the next corollary immediately follows from Theorem 3.

Corollary 5 *If one has $(x^*,u^*)\in\mathcal{O}(x_0,T)$ then $(x^*_s,u^*_s)\in\mathcal{O}(\boldsymbol{d}(s)x_0,e^{-\nu_f s}T)$ for any $s\in\mathbb{R}$, where*

$$x^*_s(t)=\boldsymbol{d}(s)x^*(e^{\nu_f s}t),\ u^*_s(t)=u^*(e^{\nu_f s}t),\ t\in(0,e^{-\nu_f s}T).$$

The system (1), (2) may have non-unique solution in a forward time for a given x_0. So, an optimal control u^* may generate a family of solutions

$$X_{x_0,u^*}\subset W^{1,1}_{loc}((0,T),\mathbb{R}^n)$$

and, probably, just one of them forms an optimal pair $(x^*,u^*)\in\mathcal{O}: J(x^*,u^*,T)\leq J(x,u^*,T)$ for all $x\in X_{x_0,u^*}$. The results proven above hold for such a *weak* statement of HOCP. Definitely, additional assumption about uniqueness of solutions to (1) in forward time makes them correct in a *strong* (conventional) sense.

Assumption 2 *For any* $x_0 \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ *it holds: if* $(x_1, u_1) \in \mathcal{P}(x_0, T)$ *and* $(x_2, u_2) \in \mathcal{P}(x_0, T)$ *then the identity* $\|u_1 - u_2\|_{L^\infty} = 0$ *implies* $\|x_1 - x_2\|_{W^{1,1}((0,T'),\mathbb{R}^n)} = 0$ *for all* $T' \in (0, T)$.

Under this assumption the specific relations between optimal/admissible solutions to HOCP with different optimization horizons imply homogeneity of necessary and/or sufficient optimality conditions. In fact, to fulfill Assumption 2 is it sufficient to consider the regular case, i.e. $f, L, g_i \in C^1(\mathbb{R}^n \backslash \{\mathbf{0}\})$. By default below we assume that Assumption 2 holds.

4.2 Homogeneous Bellman Function

The cost functional J is additive, so HOCP, obviously, satisfies the Bellman principle of optimality (see e.g. [27]) saying that *"any tail of optimal trajectory is optimal too"*. The sufficient condition of optimality can be derived using on the so-called **Bellman (value) function** $V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$, which in our case can be defined as follows

$$V(x_0, T) = J(x^*, u^*, T),$$

where (x^*, u^*) is an optimal pair for HOCP with a given $x_0 \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and a given $T \leq \frac{1}{\chi(x_0)}$. For time invariant systems (as we consider) Bellman function usually depends only of the space argument $x \in \mathbb{R}^n$ (see e.g. [27]). We have added the second argument to V in order to study also its dependence on the optimization horizon. Obviously, that $V(x_0, T) \to 0$ as $T \to 0^+$ for any fixed $x_0 \in \mathbb{R}^n \backslash \{\mathbf{0}\}$.

Theorem 4 *Let the function* ξ *defined by* (21) *be continuous on* $\mathbb{R}^n \backslash \{\mathbf{0}\}$. *Then the set* $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}_+$ *given by*

$$\mathcal{D} = \left\{(x_0, T) : x_0 \in \mathbb{R}^n \backslash \{\mathbf{0}\} \text{ and } 0 < T < \tfrac{1}{\chi(x_0)}\right\}$$

is nonempty, open and connected and the Bellman function V *is continuous on* $\mathcal{D}$. *Moreover, if* V *is continuously differentiable on* $\mathcal{D}$ *then* V *satisfies the Hamilton-Jacobi-Bellman(HJB) equation of the form*

$$\min_{u \in \mathcal{U}} \left(\frac{\partial V}{\partial x} f(x, u) + L(x, u)\right) = \frac{\partial V}{\partial T}, \quad (x, T) \in \mathcal{D} \tag{22}$$

and the identity $V(\boldsymbol{d}(s)x, e^{-\nu_f s}T) = e^{(\nu_L - \nu_f)s} V(x, T)$ *holds for* $(x, T) \in \mathcal{D} \backslash \{\mathbf{0}\}$.

Proof. The set $\mathcal{D}$ is non-empty and open by construction. Let us show that it is connected due to continuity and homogeneity of the function χ defined by (21). Indeed, if $(x_1, T_1) \in \mathcal{D}$ and $(x_2, T_2) \in \mathcal{D}$ then the homogeneous curves $\Gamma_1 = \{(x, T) : x = \mathbf{d}(s)x_1, T = e^{-\nu s}T_1, s \in \mathbb{R}\}$ and $\Gamma_2 = \{(x, T) : x = \mathbf{d}(s)x_2, T = e^{-\nu s}T_2, s \in \mathbb{R}\}$ belong to $\mathcal{D}$. Since $\mathbf{d}$ is a dilation then there exists $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ such that $\mathbf{d}(s_1)x_1 \in S$ and $\mathbf{d}(s_2)x_2 \in S$, where S is a unit sphere. This means that any point from $\mathcal{D}$ can be connected by continuous curve with the cylinder $\Pi = S \times (0, +\infty)$. Let $(\tilde{x}_1, \tilde{T}_1) \in \mathcal{D} \cap \Pi$ and $(\tilde{x}_2, \tilde{T}_2) \in \mathcal{D} \cap \Pi$ are the corresponding crossing points of Γ_1 and Γ_2, respectively. Since the function χ is non-negative and continuous on $\mathbb{R}^n \backslash \{\mathbf{0}\}$

then there exists a continuous function $\tilde{T} : S \to \mathbb{R}_+$ such that $0 < \tilde{T}(x) < 1/\chi(x)$ for $x \in S$ and $T(\tilde{x}_1) = \tilde{T}_1$, $T(x_2) = \tilde{T}_2$. Let $\{x = \phi(s) : s \in [0,1]\} \in S$ be an arbitrary continuous curve that connects the points $\tilde{x}_1 \in S$ and $\tilde{x}_2 \in S$. In this case, the continuous curve $\Gamma = \{(x,T) : x = \phi(s), T = \tilde{T}(\phi(s)), s \in [0,1]\} \in \mathcal{D} \cap \Pi$ connects the points $\tilde{x}_1, \tilde{x}_2$, i.e. the set $\mathcal{D}$ is connected.
The rest part of the proof can be done in the standard way (see, e.g. [27, Theorem 22.18]) taking into account that the conventional Bellman function $\tilde{V} : (0,T) \times \mathbb{R}^n \to \mathbb{R}_+$ in our case admits the representation $\tilde{V}(t,x) = V(x, T-t)$. Homogeneity of V follows from the identity $J(\tilde{x},\tilde{u}) = e^{(\nu_L-\nu_f)s}J(x,u)$ given by Theorem 3. ■

More constructive representation of the homogeneous Bellman function V can be obtained in the case of non-zero degree of homogeneity of the vector field f.

Corollary 6 *If $\nu_f \neq 0$ then under conditions of Theorem 4 the Bellman function V admits the representation*

$$V(x,T) = T^{\frac{\nu_f-\nu_L}{\nu_f}} \Phi\left(\boldsymbol{d}\left(\frac{\ln T}{\nu_f}\right)x\right) \tag{23}$$

where $\Phi : \Omega_1 \to \mathbb{R}_+$ is a non-negative continuously differentiable function defined on $\Omega_1 = \{x \in \mathbb{R}^n \backslash \{\mathbf{0}\} : \chi(x) < 1\}$ as a solution to the modified HJB equation

$$\min_{u\in\mathcal{U}}\left(\frac{\partial\Phi}{\partial y}f(y,u) + L(y,u)\right) = \frac{(\nu_f-\nu_L)\Phi + \frac{\partial\Phi}{\partial y}G_{\boldsymbol{d}}y}{\nu_f}. \tag{24}$$

Moreover, for $x \in \Omega_0 = \{x \in \mathbb{R}^n \backslash \{\mathbf{0}\} : \chi(x) = 0\}$ one has

$$\begin{array}{lll} \lim\limits_{T\to+\infty} V(x,T) = \Phi_\infty(x) & \text{if} & \nu_f > 0 \\ \lim\limits_{T\to+\infty} V(x,T) = \Phi_0(x) & \text{if} & \nu_f < 0, \end{array} \tag{25}$$

where the functions $\Phi_0 : \Omega_0 \to \mathbb{R}_+$ and $\Phi_\infty : \Omega_0 \to \mathbb{R}_+$ are homogeneous approximations of the function Φ at zero and at infinity, respectively.

Proof. The formula (23) follows from the identity

$$V(\mathbf{d}(s)x, e^{-\nu_f s}T) = e^{(\nu_L-\nu_f)s}V(x,T)$$

if $\Phi(y) = V(y,1)$, $y = \mathbf{d}(s)x$ and $s = \frac{\ln T}{\nu_f}$. The modified HJB equation (24) is obtained from (22) using the identity

$$\frac{\partial V}{\partial x}G_{\mathbf{d}}x - \nu_f T\frac{\partial V}{\partial T} = (\nu_L - \nu_f)V,$$

which holds due to (16). Finally, the identities (25) immediately follow from the definition of homogeneous approximations. ■

Example 2 *For $f(x,u) = -ux^{1/3}$, $L(x,u) = x^{2/3}$ and $\mathcal{U} = [0,1]$ the optimal control problem* (1) - (4) *is an HOCP with $\boldsymbol{d}(s) = e^s$, $\nu_f = -2/3$ and $\nu_L = 2/3$. Its Bellman function V can be found analytically*

$$V(x,T) = \begin{cases} 3x^{4/3}/4 & \text{if } 3x^{2/3}/2 \le T, \\ Tx^{2/3} - T^2/3 & \text{if } 3x^{2/3}/2 > T. \end{cases}$$

Obviously, it admits the representation (23) *with*

$$\Phi(y) = \begin{cases} 3/4y^{4/3} & \text{if } y \le (2/3)^{3/2}, \\ y^{2/3} - 1/3 & \text{if } y > (2/3)^{3/2}. \end{cases}$$

Note that $\chi \equiv 0$ *(i.e. any horizon is admissible), so* $\Omega_1 = \mathbb{R}^n \setminus \{\mathbf{0}\}$. *The function* Φ *is continuously differentiable on* Ω_1 *and it satisfies the modified HJB equation* (22). *Moreover,* $V(x, +\infty) = \Phi_0(x) = 3x^{4/3}/4$, *where* Φ_0 *is the homogeneous approximation of* Φ *at zero.*

Remark 1 *The Bellman function* $V : \Omega_0 \subset \mathbb{R}^n \to \mathbb{R}_+$ *of HOCP on infinite horizon* ($T = +\infty$) *is* ***d****-homogeneous, i.e.* $V(\boldsymbol{d}(s)x) = e^{(\nu_L - \nu_f)s} V(x)$, $x \in \Omega_0$ *and* $s \in \mathbb{R}_+$. *To construct V on the whole domain* Ω *one can be defined on the unit sphere (or its part) only.*

4.3 Homogeneous Maximum Principle

The necessary condition of optimality of HOCP given by Pontryagin Maximum Principle also admits a homogeneous representation. In particular, homogeneity of the vector-field f and Lagrangian L implies certain homogeneity of the Hamiltonian:

$$H(\psi, x, u) = \psi^\top f(x, u) - \mu L(x, u),$$

where $x, \psi \in \mathbb{R}^n, u \in \mathbb{R}^m, \mu \in \mathbb{R}_+$. Indeed, one has

$$H(e^{(\nu_L - \nu_f)s} \mathbf{d}^\top(-s)\psi, \mathbf{d}(s)x, u) = e^{\nu_L s} H(\psi, x, u), \quad s \in \mathbb{R}.$$

Theorem 5 *If the functions* f, L *and* $g = (g_1, g_2, ..., g_l)^\top$ *are differentiable on* $\mathbb{R}^n \setminus \{\mathbf{0}\}$ *and* $(x^*, u^*) \in \mathcal{O}(x_0, T)$ *is an optimal pair with* $x^*(t) \neq 0$ *for* $t \in [0, T]$ *then there exist a number* $\mu \in \mathbb{R}_+$, *a vector* $v \in \mathbb{R}^l_+$ *and a function* $\psi^* : [0, T] \to \mathbb{R}^n$ *such that*

a) $\|\psi^*(T)\| + \mu + \|v\| > 0$ *and* $|g(x^*(T))|^\top v = 0$*;*

b) *the tuple* (ψ^*, x^*, u^*) *solves the boundary value problem*

$$\begin{aligned} &\begin{pmatrix} \dot\psi^*(t) \\ \dot x^*(t) \end{pmatrix} = F(\psi^*(t), x^*(t), u^*(t)), \quad F = \begin{pmatrix} -\left(\frac{\partial H}{\partial x}\right)^\top \\ \left(\frac{\partial H}{\partial \psi}\right)^\top \end{pmatrix}, \\ &x^*(0) = x_0, \quad \psi^*(T) = -\left(\frac{\partial g(x(T))}{\partial x}\right)^\top v, \end{aligned} \tag{26}$$

where the vector field $F : \mathbb{R}^n \times \mathbb{R}^n \setminus \{\mathbf{0}\} \times \mathbb{R}^m \to \mathbb{R}^{2n}$ *satisfies the homogeneous relation*

$$F(\tilde{\boldsymbol{d}}(s)z, u) = e^{\nu_f s} \tilde{\boldsymbol{d}}(s) F(z, u), \;\; z \in \mathbb{R}^{2n}, u \in \mathbb{R}^m, s \in \mathbb{R}, \tag{27}$$

for the uniformly continuous group $\tilde{\boldsymbol{d}}$ *given by*

$$\tilde{\boldsymbol{d}}(s) = \begin{pmatrix} e^{(\nu_L - \nu_f)s} \boldsymbol{d}^\top(-s) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{d}(s) \end{pmatrix};$$

c) $H(\varphi^*(t), x^*(t), u^*(t)) = \max\limits_{u \in \mathcal{U}} H(\varphi^*(t), x^*(t), u) = C$ *for all* $t \in [0, T]$, *where* $C \in \mathbb{R}$ *is a constant;*

d) $\frac{d\left[(\psi^*(t))^\top G_{\boldsymbol{d}} x^*(t)\right]}{dt} = -\nu_f \left(\psi^*(t)\right)^\top f(x^*(t), u^*(t)) + \mu \nu_L L(x^*(t), u^*(t))$ *for all* $t \in (0, T)$.

Proof. Since $x^*(t) \neq \mathbf{0}$ for all $t \in [0,T]$ then the proof of the properties **a**)-**c**) can be completed using conventional arguments based on concept of Lagrange multipliers (see, e.g. [27, Theorem 22.11, Corollaries 22.4 and 22.5]). The only identity (27) must be proven. Since $\left(\frac{\partial H}{\partial \psi}\right)^\top = f$ is **d**-homogeneous vector field of degree ν_f then we just need to study the term

$$\left(\frac{\partial H}{\partial x}\right)^\top = \left(\frac{\partial f}{\partial x}\right)^\top \psi - \mu \left(\frac{\partial L}{\partial x}\right)^\top .$$

The required homogeneous relation follows from the identities

$$e^{\nu_L s}\frac{\partial L(x,u)}{\partial z} = \left.\frac{\partial L(y,u)}{\partial y}\right|_{y=\mathbf{d}(s)x}\mathbf{d}(s)$$

and $e^{\nu_f s}\mathbf{d}(s)\frac{\partial f(x,u)}{\partial z}$
$= \left.\frac{\partial f(y,u)}{\partial y}\right|_{y=\mathbf{d}(s)x}\mathbf{d}(s)$ proved in Corollary 1.

Finally, let us prove the claim **d**). One has

$$\frac{d\left[(\psi^*(t))^\top G_{\mathbf{d}} x^*(t)\right]}{dt} = (\dot{\psi}^*(t))^\top G_{\mathbf{d}} x^*(t) + (\psi^*(t))^\top G_{\mathbf{d}} \dot{x}^*(t) =$$

$$-(\psi^*(t))^\top(\nu_f I_n + G_{\mathbf{d}})f(x^*(t),u^*(t)) + \mu\nu_L L(x^*(t),u^*(t)) +$$
$$+(\psi^*(t))^\top G_{\mathbf{d}} f(x^*(t),u^*(t)),$$

where the identities (16), (18) are utilized on the last step. ■

The presented theorem holds for both fixed and non-fixed horizon (see, e.g. [27, Theorem 22.13]). Evidently, the restriction $x^*(t) \neq 0$ can be omitted in some cases, e.g. if f, L and g can be smoothly prolonged to the origin.

The property **d**) is the specific feature of the homogeneous systems. If $\nu_f = \nu_L = \nu$ then the right-hand side of the identity given by **d**) form the Hamiltonian multiplied by $-\nu$, i.e $\frac{d\left[(\psi^*(t))^\top G_{\mathbf{d}} x^*(t)\right]}{dt} = -\nu H(\psi^*(t), x^*(t), u^*(t))$.

Corollary 7 *If the time horizon T is non-fixed and $\nu_f = \nu_L$ then under conditions of Theorem 5 one has $(\phi^*(t))^\top G_{\mathbf{d}} x^*(t) = \mathrm{const}$ for $t \in [0,T]$.*

This claim immediately follows from the property **d**) proven in Theorem 5 and the identity $H(\varphi^*(t), x^*(t), u^*(t)) = 0$, $t \in [0,T]$ that holds for the case of non-fixed horizon (see, e.g. [27, Corollary 22.6]).

5 Conclusions

In this paper the problem of optimal control design for nonlinear plant is considered under assumption that the plant model is described by ordinary differential equation with a generalized homogeneous right-hand side. The conventional theorems of Dynamic Programming and Maximum Principle are refined under assumption cost function and the terminal constraints are also homogeneous in a generalized sense. There are several features of homogeneity, which can be useful for optimal control design.

- Homogeneity reduces the dimension of the control problem since optimal solutions for initial states x_0 belonging to the unit sphere uniquely define optimal solutions for all other initial conditions. This feature may essentially simplify the computational algorithms for optimal control design based on gradient methods/Maximum Principle.

- Homogeneity of the Bellman function implies interrelation between optimal solutions with different time horizon. This may allow to design optimal control for infinite time horizon ($T = +\infty$) using an optimal control solution constructed for a finite time horizon ($T < +\infty$). Such a property may be useful for model predictive control (MPC).

Two problems mentioned above are promising directions of future research of homogeneous control problems.

References

[1] V. Zubov, "On systems of ordinary differential equations with generalized homogenous right-hand sides," *Izvestia vuzov. Mathematica.*, vol. 1, pp. 80–88, 1958 (in Russian).

[2] H. Hermes, "Nilpotent approximations of control systems and distributions," *SIAM Journal of Control and Optimization*, vol. 24, no. 4, p. 731, 1986.

[3] L. Rosier, "Homogeneous Lyapunov function for homogeneous continuous vector field," *Systems & Control Letters*, vol. 19, pp. 467–473, 1992.

[4] A. Polyakov, D. Efimov, E. Fridman, and W. Perruquetti, "On homogeneous distributed parameters equations," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3657–3662, 2016.

[5] E. Ryan, "Universal stabilization of a class of nonlinear systems with homogeneous vector fields," *Systems & Control Letters*, vol. 26, pp. 177–184, 1995.

[6] V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous Approximation, Recursive Observer Design, and Output Feedback," *SIAM Journal of Control and Optimization*, vol. 47, no. 4, pp. 1814–1850, 2008.

[7] E. Bernuau, A. Polyakov, D. Efimov, and W. Perruquetti, "Verification of ISS, iISS and IOSS properties applying weighted homogeneity," *System & Control Letters*, vol. 62, no. 12, pp. 1159–1167, 2013.

[8] E. Roxin, "On finite stability in control systems," *Rendiconti del Circolo Matematico di Palermo*, vol. 15, pp. 273–283, 1966.

[9] S. Bhat and D. Bernstein, "Finite time stability of continuous autonomous systems," *SIAM J. Control Optim.*, vol. 38, no. 3, pp. 751–766, 2000.

[10] J.-M. Coron and L. Praly, "Adding an integrator for the stabilization problem," *Systems & Control Letters*, vol. 17, no. 2, pp. 89–104, 1991.

[11] A. Levant, "Homogeneity approach to high-order sliding mode design," *Automatica*, vol. 41, no. 5, pp. 823–830, 2005.

[12] Y. Orlov, "Finite time stability and robust control synthesis of uncertain switched systems," *SIAM Journal of Control and Optimization*, vol. 43, no. 4, pp. 1253–1271, 2005.

[13] W. Perruquetti, T. Floquet, and E. Moulay, "Finite-time observers: application to secure communication," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 356–360, 2008.

[14] V. G. Boltyanskii, R. V. Gamkrelidze, and L. S. Pontryagin, "Towards a theory of optimal processes," *Doklady Academii Nauk SSSR (in Russian)*, vol. 110, no. 1, pp. 7–10, 1956.

[15] R. Bellman, "The theory of dynamic programming," *Bull. Amer. Math. Soc.*, vol. 60, no. 6, pp. 503–515, 1954.

[16] J. Baillieul and J. C. Willems, Eds., *Mathemtical Control Theory*. Springer-Verlag New York, 1999.

[17] A. Agrachev and Y. Sachkov, *Control Theory from the Geometric Viewpoint*. Springer, 2004.

[18] A. Polyakov, "Quadratic-like stabilizability of homogeneous systems," in *Conference on Decision and Control*, (https://hal.inria.fr/hal-01525252), 2017.

[19] M. Kawski, "Geometric homogeneity and stabilization," in *Proc. IFAC Nonlinear Control Symposium*, A. Krener and D. Mayne, Eds., Lake Tahoe, CA, 1995, pp. 164–169.

[20] H. Hermes, "Homogeneous feedback controls for homogeneous systems," *Systems & Control Letters*, vol. 24, pp. 7–11, 1995.

[21] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, 1983.

[22] L. Rosier, "Etude de quelques problemes de stabilization," *PhD Thesis, Ecole Normale Superieure de Cachan (France)*, 1993.

[23] D. Efimov and W. Perruquetti, "Oscillations conditions in homogenous systems," in *Proc. IFAC NOLCOS Symp.*, 2010, pp. 1379–1384.

[24] A. Polyakov, J.-M. Coron, and L. Rosier, "On finite-time stabilization of evolution equations: A homogeneous approach," in *Conference on Decision and Control*, 2016, pp. 3143–3148.

[25] L. Praly, "Generalized weighted homogeneity and state dependent time scale for linear controllable systems," in *Conference on Decision and Control*, 1997, pp. 4342–4347.

[26] S. P. Bhat and D. S. Bernstein, "Geometric homogeneity with applications to finite-time stability," *Mathematics of Control, Signals and Systems*, vol. 17, pp. 101–127, 2005.

[27] A. Poznyak, *Advanced Mathematical Tools for Automatic Control Engineers. Volume 1: Deterministic Technique*. Elsevier, 2008.