High-frequency behaviour of corner singularities in Helmholtz problems

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Abstract. We analyze the singular behaviour of the Helmholtz equation set in a non-convex polygon. Classically, the solution of the problem is split into a regular part and one singular function for each re-entrant corner. The originality of our work is that the "amplitude" of the singular parts is bounded explicitly in terms of frequency. We show that for high frequency problem, the "dominant" part of the solution is the regular part. As an application, we derive sharp error estimates for finite-element discretizations. These error estimates show that the "pollution effect" is not changed by the presence of singularities. Furthermore, a consequence of our theory is that locally refined meshes are not needed for high-frequency problems, unless a very accurate solution is required. These results are illustrated with numerical examples, that are in accordance with the developed theory.

1. Introduction

Exterior Helmholtz problems play a fundamental role in scattering inverse problems [10]. The aim of such problems is to recover the shape of a scatterer using the propagation of waves. Electromagnetic (and/or acoustic and/or elastic) waves are emitted to "illuminate" the scatterer, and the waves reflected by the scatterer are recorded. The strategy is then to reconstruct the shape of the scatterer, based on the scattered wave field. This imaging technique is used in a wide range of applications including acoustics, radar detection, and medical imaging [8].

When the scatterer is known, the scattered waves can be computed by solving the (exterior) Helmholtz problem. For arbitrary geometries, there is no analytical solution available, so that the equation needs to be discretized. Popular approaches to carry out such a discretization include the finite difference method (FDM) [33], the finite element method (FEM) [9, 18] and the boundary element method (BEM) [11, 29].

Obviously, in many applications, the scatterer is unknown. However, in the reconstruction process, many solves the Helmholtz problem are usually required. As a result, the main computational cost of the reconstruction generally resides in the resolution of direct wave propagation problems. Hence, developing performant discretization methods for the Helmholtz problem is of paramount importance to solve the inverse scattering problem.

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Wave propagation problems arising in scattering applications are set in the exterior of the scatterer. An inhomogeneous Dirichlet, Neumann or Robin-type boundary condition is prescribed on the boundary of the scatterer to modelize the illuminating wave, while the Sommerfeld condition is imposed "at infinity" to prevent non-physical ingoing waves and close the problem [10].

If a domain-based method (FDM or FEM) is used to discretized the problem, the scatterer is surrounded by a bounded computational domain Ω_0. Dedicated techniques, such as absorbing boundary conditions or perfectly matched layers are then used to approximate the Sommerfeld condition on the exterior boundary of Ω_0 [7, 12, 14].

Scattering wave propagation problems thus naturally take place in non-convex domains. As a result, when the scatterer is not smooth, the solution is singular close to its corners. In this paper, we will consider polygonal scatterers, and analyze the singular behaviour of the solution around the vertices of the scatterer. This study is especially important, as the presence of singularities strongly impact the performance of numerical methods. Indeed, in the general case, the lack of regularity of the solution is numerically traduced by a decreased convergence rate, unless especially refined (graded) meshes are used [5, 26].

For the sake of simplicity, the acoustic Helmholtz operator $-k^2 - \Delta$ will be consider in this work. Then, the singularities of the Helmholtz problem have the same form as the ones of the Laplace operator $-\Delta$. It allows us to use the vast literature on the subject (see for instance [16, 25]). However, though the theory of singularities is very well established for the Laplace operator, an essential feature is missing: the behaviour of the singularities with respect to the wavenumber k.

Indeed, numerical methods are very sensitive to the wavenumber. In particular, solving high frequency problems accurately is a very computationally expensive task. This is linked to the fact that as the frequency increases, the Helmholtz operator has additional negative eigen-values, so that it is challenging to ensure the stability of discrete numerical schemes. Intuitively, the solution is more oscillatory at higher frequencies, and these oscillations are tricky to numerically capture.

In this paper, we focus on FEM discretizations. In this context, the difficulty to solve at high frequencies is usually called the "pollution effect": for high frequency problems, unless the mesh is heavily refined, there is a gap between the interpolation error, and the error of the finite element solution. It means that the solution obtained by the finite-element procedure is much less accurate than the best representation of the continuous solution in the finite element space.

The "gap" between the interpolation and finite element errors is called the pollution error. On cartesian grid based meshes, the pollution error can be computed thanks to dispersion analysis [3, 21]. Furthermore, there exists an asymptotic range $h \leq h_0$ in which this gap disappears, and the finite element solution is almost as accurate as the interpolant. Of course, the behaviour of $h_0 = h_0(k)$ with respect to the wavenumber is a key to analyze the performance of the finite element method. When the domain is smooth, this analysis has been carried out for Lagrange finite elements of arbitrary order [23, 24], and it is known that $h_0 \simeq k^{-1-1/p}$.

Of course, the regularity of the solution and the dependence of the Sobolev norms of the solution on k play a central role in the above-mentioned analysis. As a result, it is not obvious how the singularities of the solution can affect the pollution effect, in a domain with corners. For instance, a scattering problem with re-entrant corners is discretized using a "plane wave" method in [4]. Therein, the authors propose a convergence analysis, and characterize the asymptotic range $h \leq h_0(k)$ in which the pollution effect vanishes. Because of the singularities, they obtain the condition that $h \lesssim k^{-5/2}$, which is more restrictive than the condition $h \lesssim k^{-2}$ known for $\mathcal{P}_1$ elements in smooth domains.

In this paper, we carefully analyze the corner singularities of a 2D Helmholtz problem. We focus on the acoustic sound-soft scatterer problem, so that a Dirichlet boundary condition is imposed on the scatterer, and an absorbing boundary condition is used to approximation the Sommerfeld condition.

Following [16, 25], we propose a splitting of the solution into a regular part in $H^2(\Omega)$ and one singular function for each corner of the scatterer. Our main achievement is a precise description of the "amplitude" of the singularities depending on the frequency. We also show that the regular part of the solution behaves as the solution of a Helmholtz problem in a smooth domain. These results are derived using slight modifications of arguments used in an other context in [17]. To the best of our knowledge, our results are new, and we prove that our bounds are sharp.

Furthermore, we take advantage of the above-mentioned splitting to derive sharp error estimates for $\mathcal{P}_1$ finite element discretizations of the problem. In particular, we prove that the asymptotic range (and thus the pollution effect) is not affected by the presence of singularities. The newly introduced splitting being the key to improve the error estimates given in [4].

We illustrate these results with numerical experiments. For $\mathcal{P}_1$ elements, the numerical results are in agreement with the theory. Furthermore, we numerically investigate higher order discretizations. Our main observation is that, for high frequencies, the singularities only have a small impact on the numerical schemes. In particular, unless a very accurate solution is needed, the use of graded meshes is not required.

Our work is outlined as follow. In Section 2, we state more formally the scattering problem we consider and recall basic stability estimates in the $H^1(\Omega)$ norm. Sections 3 and 4 are dedicated to the analysis of the singularities of the problem. The particular case of a disc sector featuring one singular corner at the origin is first analyzed in Section 3. This result is then applied, by localization, to analyze the case of a general polygonal scatterer in Section 4. We provide stability conditions and error estimates for finite element discretizations in Section 5, and Section 6 is devoted to numerical experiments.

2. The setting

In this work, the model application is the acoustic scattering by a convex polygon $K \subset \mathbb{R}^2$. As we intend to discretize the problem with a finite element method, we assume that K is embedded into another convex polygon $\Omega_0 \subset \mathbb{R}^2$. Then, the acoustic pressure $v : \Omega \to \mathbb{C}$

is solution to

$$\left\{\begin{array}{rcll} -k^2 v - \Delta v &=& 0 & \text{in } \Omega, \\ \nabla v \cdot \mathbf{n} - ikv &=& 0 & \text{on } \partial\Omega_0, \\ v &=& g & \text{on } \partial K, \end{array}\right. \tag{2.1}$$

where $\Omega = \mathrm{Int}(\Omega_0 \setminus K)$, k is the wavenumber and $g : \partial K \to \mathbb{C}$ is the illuminating wave. In most applications, we actually have $g(\mathbf{x}) = e^{ikd\cdot\mathbf{x}}$, where $d \in \mathcal{S}_2$ is a unit vector giving the direction of propagation of the illuminating plane wave.

Since we are especially interested in the high-frequency behaviour of the solution, we will assume that $k \geq k_0$ for some fixed $k_0 > 0$.

For each $g \in H^{1/2}(\partial K)$, we can find a lifting $\eta \in H^1(\Omega)$ such that

$$\left\{\begin{array}{rcll} -\Delta \eta &=& 0 & \text{in } \Omega, \\ \nabla \eta \cdot \mathbf{n} - ik\eta &=& 0 & \text{on } \partial\Omega_0, \\ \eta &=& g & \text{on } \partial K. \end{array}\right.$$

Hence, looking for the difference $v - \eta$, we recast Problem (2.1) in a problem with a volumic source term, which will be more convenient for our analysis (see [4]). Also, it will be convenient to consider domains for which the Dirichlet and the absorbing boundary conditions are not imposed on two disconnected components of the boundary. As a result, our model problem will be to find $u : \Omega \to \mathbb{C}$ satisfying

$$\left\{\begin{array}{rcll} -k^2 u - \Delta u &=& f & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} - iku &=& 0 & \text{on } \Gamma_0, \\ u &=& 0 & \text{on } \Gamma_1, \end{array}\right. \tag{2.2}$$

where $k \geq k_0 > 0$ is the wavenumber, $f : \Omega \to \mathbb{C}$ is a (known) load term and Γ_0, Γ_1 are two open subsets of the boundary $\partial\Omega$ of Ω such that $\bar{\Gamma}_0 \cup \bar{\Gamma}_1 = \partial\Omega$. Also, $\mathbf{n}$ denotes the unit vector normal to $\partial\Omega$ pointing outside Ω.

Classically, assuming that $f \in L^2(\Omega)$, we recast (2.2) into the variational problem that consists in looking for $u \in H^1_{\Gamma_1}(\Omega)$ solution to

$$B(u, v) = (f, v), \quad \forall v \in H^1_{\Gamma_1}(\Omega), \tag{2.3}$$

where

$$B(u, v) = -k^2(u, v) - ik\langle u, v\rangle_{\Gamma_0} + (\nabla u, \nabla v),$$

and

$$H^1_{\Gamma_1}(\Omega) = \{v \in H^1(\Omega) : \gamma_0 v = 0 \text{ on } \Gamma_1\},$$

γ_0 being the trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$.

Problem (2.3) has been extensively studied, and we can state Theorem 2.1 from [19].

Theorem 2.1. *For all $k \geq k_0$ and $f \in L^2(\Omega)$, there exits a unique solution $u \in H^1_{\Gamma_1}(\Omega)$ to problem (2.3). Furthermore, if $u \in H^{3/2+\epsilon}(\Omega)$ for some $\epsilon > 0$ and if there exists a point $\mathbf{x}_0 \in \mathbb{R}^2$ such that $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}) \geq \gamma_0 > 0$ for all $\mathbf{x} \in \Gamma_0$ and $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Gamma_1$, then we have*

$$k\|u\|_{0,\Omega} + |u|_{1,\Omega} \leq C(\Omega, k_0)\|f\|_{0,\Omega}. \tag{2.4}$$

When the domain Ω is convex, or smooth, one easily obtain a bound for the $H^2(\Omega)$ norm of the solution from (2.4) by applying a shift theorem for the Laplace operator. However, the domain we consider do not have such shifting properties. The analysis of higher order derivatives of the solution must therefore be carried out carefully, by explicitly analyzing the corner singularities.

3. The case of a disc sector

We consider a disc sector $D_{\omega,R} \subset \mathbb{R}^2$ of radius R and opening ω. For the sake of simplicity, we assume that $\pi < \omega < 2\pi$, so that a singularity occurs at the origin for the solution of problem (2.2). The boundary is split into two parts Γ_0 and Γ_1 corresponding respectively to the circular and straight portions of $\partial D_{\omega,R}$. Hence, the domain is defined by

$$D_{\omega,R} = \{(r\cos\theta, r\sin\theta) \quad | \quad 0 < r < R, \quad 0 < \theta < \omega \quad \},$$

and the boundary is specified as $\Gamma_0 = S_{\omega,R}$ and $\Gamma_1 = \overline{I_{0,R}} \cup \overline{I_{\omega,R}}$, where

$$S_{\omega,R} = \{(R\cos\theta, R\sin\theta) \quad | \quad 0 < \theta < \omega\}$$

and

$$I_{0,R} = \{(r,0) \quad | \quad 0 < r < R\}, \quad I_{\omega,R} = \{(r\cos\omega, r\sin\omega) \quad | \quad 0 < r < R\}.$$

We observe that because the two angles where Γ_0 and Γ_1 connect are $\pm\pi/2$, the solution to the problem lies in $H^{1+\alpha-\epsilon}(\Omega)$ with $\alpha = \pi/\omega > 1/2$ and any $\epsilon > 0$ (see below). Hence, setting $\mathbf{x}_0 = 0$ in Theorem 2.1, we obtain a stability result for (2.3). Theorem 2.1 shows that the solution belongs to $H^1(D_{\omega,R})$, and its norm is explicitly bounded with respect to k.

When the domain Ω is regular, it can further be shown that $u \in H^2(\Omega)$ and the semi-norm $|u|_{2,\Omega}$ is explicitly controlled in terms of k. Here, we consider a domain $D_{\omega,R}$ with a re-entrant corner, so that in general, the solution presents a singularity at the origin. Indeed, as mentioned before the solution belongs to $H^{1+\alpha-\epsilon}(D_{\omega,R})$ for any $\epsilon > 0$, but not to $H^{1+\alpha}(D_{\omega,R})$. In particular this solution does not belong to $H^2(D_{\omega,R})$ since $\alpha \in (\frac{1}{2}, 1)$.

Hereafter, we propose a splitting of the solution u into a regular part belonging to $\tilde{u}_R \in H^2(D_{\omega,R})$ and a singular part $S \in H^{1+\alpha-\epsilon}(D_{\omega,R})$. We show that $|\tilde{u}_R|_{2,D_{\omega,R}}$ behaves as $|u|_{2,D_{\omega,R}}$ when the domain is regular. Furthermore, we provide a novel estimate for the semi-norm $|S|_{1+\alpha-\epsilon}$.

In this section, we require some properties linked to Bessel functions that are established in Appendix A and listed below.

Proposition 3.1. *The following properties hold*

$$\int_0^R |J_\alpha(kr)|^2 r dr \leq C(\alpha, R, k_0) k^{-1}, \tag{3.1}$$

$$\int_0^R |H_\alpha^{(2)}(kr)|^2 r dr = C(\alpha, R, k_0) k^{-1} + \mathcal{O}\left(k^{-3}\right), \tag{3.2}$$

$$\frac{Y_\alpha'(kR) + iY_\alpha(kR)}{J_\alpha'(kR) + iJ_\alpha(kR)} = -i + \mathcal{O}\left((kR)^{-3/2}\right), \tag{3.3}$$

$$J_\alpha(\epsilon k)Y_\alpha'(\epsilon k) - J_\alpha'(\epsilon k)Y_\alpha(\epsilon k) = \frac{2}{\epsilon\pi k}, \tag{3.4}$$

for all $k \geq k_0$ and $\epsilon > 0$.

3.1. **Splitting of the solution.** We propose a splitting of the solution u into a regular part $\tilde{u}_R \in H^2(D_{\omega,R})$ and a singular part in $H^{1+\alpha-\epsilon}(D_{\omega,R})$. The singular properties of the Helmholtz operator are strongly linked to the ones of the Laplacian. Hence, our analysis heavily relies on the theory developed by Grisvard [16]. More precisely, we can state that the solution u can be decomposed as

$$u = \tilde{u}_R + \tilde{c}_k(f)\chi(r)r^\alpha \sin(\alpha\theta), \tag{3.5}$$

where $\tilde{u}_R \in H^2(D_{\omega,R})$ is the regular part, $r^\alpha \sin(\alpha\theta) \in H^{1+\alpha-\epsilon}(D_{\omega,R})$ represents the singularity of the solution, $\tilde{c}_k(f) \in \mathbb{C}$ is a constant depending on the data of the problem, and χ is a C^∞ cutoff function that equals 1 in a neighborhood of the origin and 0 close to Γ_0.

Decomposition (3.5) is especially useful when analyzing Laplace problems, as $r^\alpha \sin(\alpha\theta)$ is a harmonic function. Also, this decomposition will be useful when analyzing the approximation properties of finite element spaces. However, it is tricky to directly estimate the constant $\tilde{c}_k(f)$. As a result, we will use the decomposition

$$u = u_R + c_k(f)J_\alpha(kr)\sin(\alpha\theta) \tag{3.6}$$

with $u_R \in H^2(D_{\omega,R})$ and $J_\alpha(kr)\sin(\alpha\theta)$ represents the singularity.

As we detail later, $J_\alpha(kr)$ and r^α have the same behaviour close to the origin, so that both functions can be used to describe the singularity. The advantage of decomposition (3.6) over (3.5) is that the representation of the singularity satisfies

$$\left(-k^2 - \Delta\right)\left(J_\alpha(kr)\sin(\alpha\theta)\right) = 0.$$

As a result, decomposition (3.6) is easier to handle, and we shall use it to estimate $c_k(f)$. We easily recover an estimate for $\tilde{c}_k(f)$ in a "post-processing" fashion.

In Theorem 3.1, we show that the solution u can be decomposed according to (3.5) or (3.6). Furthermore, we give a relation between the constants $c_k(f)$ and $\tilde{c}_k(f)$. Also, in order to simplify the notations, we introduce

$$s(\mathbf{x}) = J_\alpha(kr)\sin(\alpha\theta), \quad \tilde{s}(\mathbf{x}) = \chi(r)r^\alpha\sin(\alpha\theta). \tag{3.7}$$

Theorem 3.1. *For all $k \geq k_0$ and $f \in L^2(D_{\omega,R})$, if $u \in H^1_{\Gamma_1}(D_{\omega,R})$ is solution to (2.3), there exist a function $\tilde{u}_R \in H^1_{\Gamma_1}(D_{\omega,R}) \cap H^2(D_{\omega,R})$ and a constant $\tilde{c}_k(f) \in \mathbb{C}$ such that $u = \tilde{u}_R + \tilde{c}_k(f)\tilde{s}$.*

Furthermore, there exists a function $u_R \in H^1_{\Gamma_1}(D_{\omega,R}) \cap H^2(D_{\omega,R})$ such that $u = u_R + c_k(f)s$, where

$$c_k(f) = 2^\alpha\Gamma(\alpha+1)k^{-\alpha}\tilde{c}_k(f). \tag{3.8}$$

Proof. The existence and uniqueness of $u \in H^1_{\Gamma_1}(D_{\omega,R})$ being established, we can write that u is solution to

$$\left\{ \begin{array}{rcll} -\Delta u &=& \tilde{f} & \text{in } \Omega \\ u &=& 0 & \text{on } \Gamma_1, \\ \nabla u \cdot \mathbf{n} &=& g & \text{on } \Gamma_0, \end{array} \right.$$

where $\tilde{f} = f + k^2 u \in L^2(D_{\omega,R})$, $g = iku \in \tilde{H}^{1/2}(\Gamma_0)$ [1].

As g belongs to $\tilde{H}^{1/2}(\Gamma_0)$, by Theorem 1.5.2.8 of [16], there exists an element $\eta \in H^2(D_{\omega,R})$ such that

$$\gamma_0 \eta = 0, \quad \gamma_0 \left(\nabla \eta \cdot \mathbf{n}\right) = \tilde{g} = iku \text{ on } \Gamma, \tag{3.9}$$

with the estimate

$$\|\eta\|_{2,\Omega} \leq C(\omega, R) \|\tilde{g}\|_{H^{\frac{1}{2}}(\Gamma_0)} = C(\omega, R) k \|u\|_{H^{\frac{1}{2}}(\Gamma_0)},$$

for some positive constant $C(\omega, R)$ that depends only on ω and R. Hence by a trace theorem and estimate (2.4), we deduce that

$$\|\eta\|_{2,D_{\omega,R}} \leq C(\omega, R, k_0) k \|f\|_{0,D_{\omega,R}}. \tag{3.10}$$

Since $\gamma_0 \eta = 0$, it is also clear that we have $\eta \in H^1_0(D_{\omega,R}) \subset H^1_{\Gamma_1}(D_{\omega,R})$.

As a result, we see that $v = u - \eta \in H^1_{\Gamma_1}(D_{\omega,R})$ is solution to

$$\left\{ \begin{array}{rcll} -\Delta v &=& h & \text{in } D_{\omega,R} \\ v &=& 0 & \text{on } \Gamma_1, \\ \nabla v \cdot \mathbf{n} &=& 0 & \text{on } \Gamma_0, \end{array} \right.$$

where $h = f + k^2 u + \Delta \eta \in L^2(D_{\omega,R})$. This allows us to apply Theorem 4.4.3.7 of [16], stating that $v \in \text{span}\{H^2(D_{\omega,R}), \tilde{s}\}$. Hence, there exist a function $v_R \in H^1_{\Gamma_1}(D_{\omega,R}) \cap H^2(D_{\omega,R})$ and a constant $\tilde{c}_k(f) \in \mathbb{C}$ such that $v = v_R + \tilde{c}_k(f)\tilde{s}$. Obviously, we obtain (3.5) by setting $u_R = v_R + \eta$.

Once (3.5) is established, (3.6) and (3.8), directly follows from a careful inspection of the definition of J_α. Indeed, if we isolate the first term in the development of J_α, we see that

$$J_\alpha(kr)\sin(\alpha\theta) = \frac{k^\alpha}{2^\alpha \Gamma(\alpha+1)} r^\alpha \sin(\alpha\theta) + \phi,$$

with $\phi \in H^1_{\Gamma_1}(D_{\omega,R}) \cap H^2(D_{\omega,R})$. □

3.2. **Estimation of $c_k(f)$.** For each $k \geq k_0$, it is clear that the mapping $c_k : L^2(D_{\omega,R}) \to \mathbb{C}$ is continuous, and linear. Then, the Riesz representation theorem implies the existence of a unique $w_k \in L^2(D_{\omega,R})$ such that

$$c_k(f) = (f, w_k), \quad \forall f \in L^2(D_{\omega,R}). \tag{3.11}$$

[1] as usual, for $s > 0$, a function g belongs to $\tilde{H}^s(\Gamma_0)$ if $\tilde{g}$, its extension by zero outside Γ_0, belongs to $H^s(\partial D_{\omega,R})$

Lemma 3.1. *For all $k \geq k_0$, w_k can be characterized as the unique element of $L^2(D_{\omega,R})$ satisfying the following conditions:*

$$-k^2 w_k - \Delta w_k = 0 \text{ in } \mathcal{D}'(D_{\omega,R}), \tag{3.12}$$

$$\nabla w_k \cdot \mathbf{n} + ikw_k = 0 \text{ in } (\tilde{H}^{\frac{3}{2}}(\Gamma_0))', \tag{3.13}$$

$$w_k = 0 \text{ on } \text{ in } (\tilde{H}^{\frac{1}{2}}(\Gamma_1 \setminus \{(0,0)\}))', \tag{3.14}$$

$$\big(-k^2\eta s - \Delta(\eta s), w_k\big) = 1, \tag{3.15}$$

where $\eta \in C^\infty(D_{\omega,R})$ is any function such that $\eta = 1$ in a neighborhood of the origin and $\eta = 0$ in a neighborhood of Γ_0. Further, here and below $\tilde{H}^{\frac{1}{2}}(\Gamma_1 \setminus \{(0,0)\})$ is the set of functions $g \in H^{\frac{1}{2}}(\Gamma_1)$ such that its restriction g_0 (resp. g_ω) to $I_{0,R}$ (resp. $I_{\omega,,R}$) belongs to $\tilde{H}^{\frac{1}{2}}(I_{0,R})$ (resp. $\tilde{H}^{\frac{1}{2}}(I_{\omega,R})$).

Proof. First, let us prove that the function $w_k \in L^2(D_{\omega,R})$ defined at (3.11) satisfies conditions (3.12)-(3.15).

To prove (3.12), consider $\phi \in \mathcal{D}(D_{\omega,R})$, and define $f = -k^2\phi - \Delta\phi$, by definition of f, it is clear that $\phi = \phi_R + c_k(f)s$ with $\phi_R \in H^2(D_{\omega,R})$. But since $\phi \in \mathcal{D}(D_{\omega,R})$, we must have $c_k(f) = 0$. As a result,

$$c_k(-k^2\phi - \Delta\phi) = (-k^2\phi - \Delta\phi, w_k) = 0 \tag{3.16}$$

for all $\phi \in \mathcal{D}(D_{\omega,R})$. One sees that (3.16) is precisely (3.12).

To analyse the boundary conditions satisfied by w_k, we pick an arbitrary function $\phi \in C^\infty(D_{\omega,R})$ such that $\phi = 0$ in a neighborhood of the origin, $\nabla\phi \cdot \mathbf{n} - ik\phi = 0$ on Γ_0 and $\phi = 0$ on Γ_1. Because ϕ is regular near the origin, for the same reason as above, we have

$$c_k(-k^2\phi - \Delta\phi) = (-k^2\phi - \Delta\phi, w_k) = 0. \tag{3.17}$$

The pair (w_k, ϕ) satisfies the assumptions of Corollary 1.38 of [25], this corollary yields

$$\int_{D_{\omega,R}} (w_k\Delta\phi - \Delta w_k\phi)\, dx = \langle\nabla\phi\cdot\mathbf{n}, w_k\rangle_{\Gamma_0} + \langle\nabla\phi\cdot\mathbf{n}, w_k\rangle_{\Gamma_1} - \langle\phi, \nabla w_k\cdot\mathbf{n}\rangle_{\Gamma_0} - \langle\phi, \nabla w_k\cdot\mathbf{n}\rangle_{\Gamma_1}.$$

Then, by (3.12), we obtain

$$\begin{aligned} 0 &= (-k^2\phi - \Delta\phi, w_k) \\ &= \langle\nabla\phi\cdot\mathbf{n}, w_k\rangle - \langle\phi, \nabla w_k\cdot\mathbf{n}\rangle \\ &= \langle\nabla\phi\cdot\mathbf{n}, w_k\rangle_{\Gamma_0} + \langle\nabla\phi\cdot\mathbf{n}, w_k\rangle_{\Gamma_1} - \langle\phi, \nabla w_k\cdot\mathbf{n}\rangle_{\Gamma_0} - \langle\phi, \nabla w_k\cdot\mathbf{n}\rangle_{\Gamma_1} \\ &= ik\langle\phi, w_k\rangle_{\Gamma_0} + \langle\nabla\phi\cdot\mathbf{n}, w_k\rangle_{\Gamma_1} - \langle\phi, \nabla w_k\cdot\mathbf{n}\rangle_{\Gamma_0} \\ &= \langle\nabla\phi\cdot\mathbf{n}, w_k\rangle_{\Gamma_1} - \langle\phi, \nabla w_k\cdot\mathbf{n} + ikw_k\rangle_{\Gamma_0}, \end{aligned}$$

for all $\phi \in C^\infty(D_{\omega,R})$ such that $\phi = 0$ in a neighborhood of the origin and satisfying the boundary conditions of Helmholtz problem (2.2). Since the traces of $v = \phi|_{\Gamma_0}$ and $z = \nabla\phi\cdot\mathbf{n}|_{\Gamma_1}$ runs in a dense subset of $\tilde{H}^{\frac{3}{2}}(\Gamma_0)$ and $\tilde{H}^{\frac{1}{2}}(\Gamma_1 \setminus \{(0,0)\})$, we deduce that

$$\langle z, w_k\rangle_{\Gamma_1} = 0, \quad \forall z \in \tilde{H}^{\frac{1}{2}}(\Gamma_1 \setminus \{(0,0)\}),$$

and

$$\langle v, \nabla w_k \cdot \mathbf{n} + ikw_k \rangle_{\Gamma_0} = 0, \quad \forall v \in \tilde{H}^{\frac{3}{2}}(\Gamma_0).$$

By duality, we obtain (3.13) and (3.14).

Let $\eta \in C^\infty(\overline{D_{\omega,R}})$ be a cutoff function like in (3.15). Clearly, because $\eta = 1$ near the origin, we have

$$\eta s = u_R + s,$$

with $u_R \in H^2(D_{\omega,R})$. Also, $\Delta(\eta s) \in L^2(D_{\omega,R})$ and ηs satisfies the boundary conditions of problem (2.2). As a result, it is clear that

$$c_k \left(-k^2(\eta s) - \Delta(\eta s)\right) = 1,$$

and (3.15) follows by definition (3.11) of w_k.

We now prove the opposite statement that a function v_k satisfying (3.12)-(3.15) is the function w_k defined by (3.11). Indeed if $u \in H^1_{\Gamma_1}(D_{\omega,R})$ is the unique solution of (2.3), then the splitting (3.6) is equivalent to

$$u = u_R^* + c_k(f)\eta s,$$

with $u_R^* \in H^2(D_{\omega,R}) \cap H^1_{\Gamma_1}(D_{\omega,R})$. This splitting and condition (3.15) satisfied by v_k directly yield

$$\begin{aligned} -(f, v_k) &= ((\Delta + k^2)u, v_k) \\ &= ((\Delta + k^2)u_R^*, v_k) - c_k(f). \end{aligned}$$

Then the conclusion follows if we can show that

$$((\Delta + k^2)u_R^*, v_k) = 0. \tag{3.18}$$

But it is not difficult to show that $H^3(D_{\omega,R}) \cap H^1_{\Gamma_1}(D_{\omega,R})$ is dense in $H^2(D_{\omega,R}) \cap H^1_{\Gamma_1}(D_{\omega,R})$, hence (3.18) holds if and only if

$$((\Delta + k^2)w, v_k) = \int_{\Gamma_1} \left(\nabla w \cdot \mathbf{n} v_k - w \nabla v_k \cdot \mathbf{n}\right), \forall w \in H^3(D_{\omega,R}) \cap H^1_{\Gamma_1}(D_{\omega,R}). \tag{3.19}$$

But for $w \in H^3(D_{\omega,R}) \cap H^1_{\Gamma_1}(D_{\omega,R})$, for a cut-off function η as before, we clearly have

$$\eta w \in W^{2,p}(D_{\omega,R}),$$

for any $p > 2$ and furthermore ηw and $\nabla(\eta w)$ is zero at the origin. Since $v_k \in L^q(D_{\omega,R})$ and $\Delta v_k \in L^q(D_{\omega,R})$ with $1 < q < 2$ such that $1/p + 1/q = 1$, we can apply Corollary 1.38 of [25] and find that

$$((\Delta + k^2)(\eta w), v_k) = 0.$$

On the other hand, since the angle between Γ_0 and Γ_1 is equal to $\pi/2$, v_k belongs to H^2 far from the origin and consequently by standard Green's formula, we have

$$((\Delta + k^2)((1 - \eta)w), v_k) = \int_{\Gamma_1} \left(\nabla w \cdot \mathbf{n} v_k - w \nabla v_k \cdot \mathbf{n}\right).$$

The sum of these two last identities proves (3.19). □

As we consider a simple geometry, the analytical expression of w_k is available, as shown in the following theorem.

Theorem 3.2. *We have*

$$w_k(\mathbf{x}) = -\alpha \left\{ Y_\alpha(kr) - \frac{Y_\alpha'(kR) + iY_\alpha(kR)}{J_\alpha'(kR) + iJ_\alpha(kR)} J_\alpha(kr) \right\} \sin(\alpha\theta).$$

Proof. We are going to show that the function w_k defined above satisfies conditions (3.12)-(3.15). For the sake of simplcity, let us write

$$s^\star(\mathbf{x}) = Y_\alpha(kr)\sin(\alpha\theta),$$

so that

$$w_k = -\alpha(s^\star - \frac{Y_\alpha'(kR) + iY_\alpha(kR)}{J_\alpha'(kR) + iJ_\alpha(kR)} s).$$

We observe that by construction, both s and $s^\star$ satisfy the Helmholtz PDE, as a result,

$$-k^2 w_k - \Delta w_k = 0,$$

and (3.12) is satisfied. As w_k clearly belongs to $L^2(D_{\omega,R})$, we deduce that w_k belongs to

$$D(\Delta, L^2(D_{\omega,R})) = \{v \in L^2(D_{\omega,R}) : \Delta v \in L^2(D_{\omega,R})\},$$

hence Theorem 1.37 of [25] gives a meaning of its trace on Γ_1 as element of $(\tilde{H}^{\frac{1}{2}}(\Gamma_1 \setminus \{(0,0)\}))'$. Furthermore, since $w_k \in C^\infty(\mathbb{R}^2 \setminus (0,0))$, it is clear that $w_k = 0$ on $\Gamma_1 \setminus (0,0)$, and (3.14) holds.

Boundary condition (3.13) is also satisfied by construction. Indeed, if $\mathbf{x} = (R\cos\theta, R\sin\theta) \in \Gamma_0$, we have

$$(\nabla s \cdot \mathbf{n} + iks)(\mathbf{x}) = k(J_\alpha'(kR) + iJ_\alpha(kR))\sin(\alpha\theta),$$

and

$$(\nabla s^\star \cdot \mathbf{n} + iks^\star)(\mathbf{x}) = k(Y_\alpha'(kR) + iY_\alpha(kR))\sin(\alpha\theta),$$

so that $\nabla w_k \cdot \mathbf{n} + ikw_k = 0$ on Γ_0.

Hence it remains to show that (3.15) holds. We have

$$-\frac{1}{\alpha}\left(-k^2\eta s - \Delta(\eta s), w_k\right) = \left(-k^2\eta s - \Delta(\eta s), s^\star\right) + \frac{Y_\alpha'(kR) + iY_\alpha(kR)}{J_\alpha'(kR) + iJ_\alpha(kR)}\left(-k^2(\eta s) - \Delta(\eta s), s\right).$$

We are going to show that

$$\left(-k^2(\eta s) - \Delta(\eta s), s\right) = 0, \tag{3.20}$$

and

$$\left(-k^2(\eta s) - \Delta(\eta s), s^\star\right) = -\frac{1}{\alpha}, \tag{3.21}$$

which will conclude the proof. In spirit, the proof relies on simple integration by parts techniques. However, because we manipulate functions with low regularity close to the origin, these integrations by parts have to be done carefully.

The technique is then to substract the ball $B(0,\epsilon)$ from $D_{\omega,R}$, so that all manipulated functions are C^∞ on $D_{\omega,R,\epsilon} = D_{\omega,R} \setminus B(0,\epsilon)$. Then, integration by parts is allowed on $D_{\omega,R,\epsilon}$ and the desired inner products are recovered by letting $\epsilon \to 0$.

The beginning of the proof of (3.20) and (3.21) is the same. Thus, let us set $\mu = s$ or $s^\star$. Because $s, \mu \in C^\infty(D_{\omega,R,\epsilon})$, and $-k^2\mu - \Delta\mu = 0$, double integration by parts yields

$$\int_{D_{\omega,R,\epsilon}} \left(-k^2\eta s - \Delta(\eta s)\right)\mu = \int_{\partial D_{\omega,R,\epsilon}} \left(\nabla(\eta s)\cdot\mathbf{n}\mu - \eta s\nabla\mu\cdot\mathbf{n}\right).$$

Since $\eta s = \nabla(\eta s)\cdot\mathbf{n} = 0$ on Γ_0, $s = \mu = 0$ on $\Gamma_1 \setminus B_\epsilon$, and $\eta = 1$ on $B_{1/2}$, we have

$$\begin{aligned}\int_{D_{\omega,R,\epsilon}} \left(-k^2\eta s - \Delta(\eta s)\right)\mu &= \int_{\partial B_\epsilon} \left(\nabla(\eta s)\cdot\mathbf{n}\mu - \eta s\nabla\mu\cdot\mathbf{n}\right)\\ &= \int_{|\mathbf{x}|=\epsilon, 0<\theta<\omega} \left(\nabla s\cdot\mathbf{n}\mu - s\nabla\mu\cdot\mathbf{n}\right). \qquad (3.22)\end{aligned}$$

Obviously, when $\mu = s$, the right-hand-side of (3.22) vanishes, so that

$$\int_{D_{\omega,R,\epsilon}} \left(-k^2\eta s - \Delta(\eta s)\right) s = 0,$$

and (3.20) follows since

$$\begin{aligned}\left(-k^2(\eta s) - \Delta(\eta s), s\right) &= \int_{D_{\omega,R}} \left(-k^2\eta s - \Delta(\eta s)\right)\overline{s}\\ &= \int_{D_{\omega,R}} \left(-k^2\eta s - \Delta(\eta s)\right) s\\ &= \lim_{\epsilon\to 0}\int_{D_{\omega,R,\epsilon}} \left(-k^2\eta s - \Delta(\eta s)\right) s.\end{aligned}$$

On the other hand, to prove (3.21), since

$$\nabla s\cdot\mathbf{n} = kJ'_\alpha(\epsilon k)\sin(\alpha\theta), \quad \nabla s^\star\cdot\mathbf{n} = kY'_\alpha(\epsilon k)\sin(\alpha\theta), \text{ on } \partial B_\epsilon,$$

by (3.22) with $\mu = s^\star$, we have

$$\int_{D_{\omega,R,\epsilon}} \left(-k^2(\eta s) - \Delta(\eta s)\right) s^\star = \epsilon k\left(J'_\alpha(\epsilon k)Y_\alpha(\epsilon k) - J_\alpha(\epsilon k)Y'_\alpha(\epsilon k)\right)\int_0^\omega \sin^2(\alpha\theta)d\theta$$

Direct computations show that

$$\int_0^\omega \sin^2(\alpha\theta) = \frac{\pi}{2\alpha}.$$

Furthermore, recalling (3.4) from Proposition 3.1 we have

$$J'_\alpha(\epsilon k)Y_\alpha(\epsilon k) - J_\alpha(\epsilon k)Y'_\alpha(\epsilon k) = -\frac{2}{\epsilon\pi k},$$

and we obtain

$$\int_{D_{\omega,R,\epsilon}} \left(-k^2 s - \Delta s\right) s^\star = -\frac{1}{\alpha}.$$

Then, (3.21) holds by letting $\epsilon \to 0$. □

Corollary 3.1. *We have*

$$|w_k(\mathbf{x}) + i\alpha H_\alpha^{(2)}(kr)\sin(\alpha\theta)| \leq C(\omega, R, k_0)k^{-3/2}|J_\alpha(kr)|\sin(\alpha\theta), \tag{3.23}$$

and

$$\|w_k\|_{0,D_{\omega,R}} = C(\omega, R, k_0)\left(k^{-1/2} + \mathcal{O}\left(k^{-3/2}\right)\right). \tag{3.24}$$

Proof. First, recalling (3.3) from Proposition 3.1, we have

$$\frac{Y_\alpha'(kR) + ikY_\alpha(kR)}{J_\alpha'(kR) + ikJ_\alpha(kR)} = i + \mathcal{O}\left((kR)^{-3/2}\right),$$

from which (3.23) follows.

Then, because of (3.23), we have

$$\|w_k\|_{0,D_{\omega,R}}^2 = C(\omega, R, k_0)\left(\int_0^R |H_\alpha^{(2)}(kr)|^2 r dr + \mathcal{O}\left(k^{-3}\right)\int_0^R |J_\alpha(kr)|^2 r dr\right),$$

then, from (3.1) and (3.2), we have

$$\|w_k\|_{0,D_{\omega,R}}^2 = C(\omega, R, k_0)\left(k^{-1} + \mathcal{O}\left(k^{-3}\right)\right),$$

and (3.24) follows. □

We are now ready to establish the main result of this section.

Theorem 3.3. *The estimate*

$$|c_k(f)| \leq C(\omega, R, k_0)k^{-1/2}\|f\|_{0,D_{\omega,R}}, \tag{3.25}$$

holds for all $k \geq k_0$ and $f \in L^2(D_{\omega,R})$.

Furthermore, estimate (3.25) is optimal in the sense that for all $k \geq k_0$, there exists an $f \in L^2(D_{\omega,R})$ such that

$$|c_k(f)| \geq C(\omega, R, k_0)k^{-1/2}\|f\|_{0,D_{\omega,R}}. \tag{3.26}$$

Proof. By definition, for all $k \geq k_0$ and $f \in L^2(D_{\omega,R})$, we have

$$|c_k(f)| = |(f, w_k)| \leq \|f\|_{0,D_{\omega,R}}\|w_k\|_{0,D_{\omega,R}},$$

and

$$|c_k(w_k)| = |(w_k, w_k)| = \|w_k\|_{0,D_{\omega,R}}^2.$$

But, Corollary 3.1 shows that

$$C_1(\omega, R, k_0)k^{-1/2} \leq \|w_k\|_{0,D_{\omega,R}} \leq C_2(\omega, R, k_0)k^{-1/2} \quad \forall k \geq k_0,$$

assuming that k_0 is large enough. As a result, we have (3.25), and (3.26) follows by taking $f = w_k$. □

3.3. **Behaviour of the regular part.** So far, we have isolated the singular part of the solution u and described its behaviour with respect to the frequency. To complete the analysis, we now investigate the regular part $\tilde{u}_R \in H^2(D_{\omega,R})$.

Theorem 3.4. *For all $k \geq k_0$ and $f \in L^2(D_{\omega,R})$, if $u \in H^1_{\Gamma_1}(D_{\omega,R})$ is solution to* (2.3)*, there exist a function $\tilde{u}_R \in H^2(D_{\omega,R})$ and a constant $\tilde{c}_k(f) \in \mathbb{C}$ such that*

$$u = \tilde{u}_R + \tilde{c}_k(f)\tilde{s},$$

and it holds that

$$\|\tilde{u}_R\|_{2,D_{\omega,R}} \leq C(\omega, R, k_0)k\|f\|_{0,D_{\omega,R}}.$$

Proof. We proceed as in Theorem 3.1 and use the lifting $\eta \in H^1_{\Gamma_1}(D_{\omega,R}) \cap H^2(D_{\omega,R})$ satisfying (3.9). Then, we let $v = u - \eta$ so that

$$\begin{cases} -\Delta v &= h & \text{in } D_{\omega,R} \\ \nabla v \cdot \mathbf{n} &= 0 & \text{on } \Gamma_0 \\ v &= 0 & \text{on } \Gamma_1. \end{cases}$$

with $h = \Delta u - \Delta\eta = f + k^2 u - \Delta\eta$.

But, $v = v_R + \tilde{c}_k(f)\tilde{s}$ with $v_R \in H^2(D_{\omega,R})$, and it follows that

$$\begin{cases} -\Delta v_R &= \tilde{h} & \text{in } D_{\omega,R} \\ \nabla v_R \cdot \mathbf{n} &= 0 & \text{on } \Gamma_0 \\ v_R &= 0 & \text{on } \Gamma_1. \end{cases}$$

with $\tilde{h} = f + k^2 u - \Delta\eta - \tilde{c}_k(f)\Delta\tilde{s} \in L^2(D_{\omega,R})$. Then we see that v_R is solution to a Laplace problem with mixed boundary condition. Furthermore, since we have $v_R \in H^2(D_{\omega,R})$, we can apply the *a priori* bound derived by Grisvard in Theorem 4.3.1.4 of [16]:

$$\|v_R\|_{2,D_{\omega,R}} \leq C(\omega, R)\left(\|\tilde{h}\|_{0,D_{\omega,R}} + \|v_R\|_{0,D_{\omega,R}}\right).$$

By applying the Poincaré inequality, we further see that

$$\|v_R\|_{2,D_{\omega,R}} \leq C(\omega, R)\left(\|\tilde{h}\|_{0,D_{\omega,R}} + |v_R|_{1,D_{\omega,R}}\right) \leq C(\omega, R)\|\tilde{h}\|_{0,D_{\omega,R}},$$

and it only remains to estimate the $L^2(D_{\omega,R})$-norm of $\tilde{h}$. But, we have

$$\|\tilde{h}\|_{0,D_{\omega,R}} \leq \|f\|_{0,D_{\omega,R}} + k^2\|u\|_{0,D_{\omega,R}} + \|\Delta\eta\|_{0,D_{\omega,R}} + |\tilde{c}_k(f)|\|\Delta\tilde{s}\|_{0,D_{\omega,R}}.$$

Also, one trivially has

$$\|\Delta\tilde{s}\|_{0,D_{\omega,R}} \leq C(\omega, R),$$

further by (2.4) and (3.10), one deduces that

$$\|u\|_{0,D_{\omega,R}} \leq C(\omega, R, k_0)k^{-1}\|f\|_{0,D_{\omega,R}}, \quad \|\Delta\eta\|_{0,D_{\omega,R}} \leq C(\omega, R, k_0)k\|f\|_{0,D_{\omega,R}}.$$

Hence, by (3.25), we finally get

$$\|\tilde{h}\|_{0,D_{\omega,R}} \leq C(\omega, R, k_0)\left(1 + k^{\alpha-1/2} + k\right)\|f\|_{0,D_{\omega,R}},$$

and the result follows. Indeed, since $\alpha < 1$, we have $1 + k^{\alpha-1/2} + k \leq C(k_0)k$. □

As a conclusion, we summarize the key features of the presented splitting. First, the regular part $\tilde{u}_R \in H^2(D_{\omega,R})$ has the standard behaviour of the Helmholtz solution in the sense that $|\tilde{u}_R|_{2,D_{\omega,R}} \leq Ck$. Second, we are able to isolate the singularity of u, that is represented by the function $S = c_k(f)\tilde{s}$ which only belongs to $H^{1+\alpha-\epsilon}(D_{\omega,R})$ (for all $\epsilon > 0$). Furthermore, the behaviour of S is controlled as $|S|_{1+\alpha-\epsilon} \leq Ck^{\alpha-1/2}|\tilde{s}|_{1+\alpha-\epsilon} = Ck^{\alpha-1/2}$.

The crucial observation is that the norm of the regular part grows faster for increasing frequencies than the one of the singular part. As a result, we can expect the regular part to be "dominant" in some sense for high frequency problems. In Section 5, we show that this observation has important consequences on numerical methods. Roughly speaking, the singular part (and the low convergence rate) are "invisible" until an asymptotic regime (for small mesh size) is reached. A particularly important consequence is that the "pollution effect", which is the main source of numerical error, is not affected by the singularity.

4. Scattering by a convex polygon

We now consider the problem of scattering by a convex polygon K. For the sake of simplicity, we assume that K is closed and we denote by Γ_1 its boundary. Instead of considering the exterior problem in $\mathbb{R}^2 \setminus K$ we enclose K in an open convex polygon Ω_0 with boundary Γ_0. Then, we consider problem (2.2) on the domain $\Omega = \Omega_0 \setminus K$.

Since the boundary Γ_0 and Γ_1 are disconnected, singularities can only happen at vertices of K. Furthermore, since K is a polygon, the solution is in $H^{3/2+\epsilon}(\Omega)$ for some $\epsilon > 0$. Thus, we can apply Theorem 2.1 by using any point $\mathbf{x}_0 \in \operatorname{Int} K$. Hence, the $H^1(\Omega)$ norm of the solution to (2.3) is explicitly bounded for all $k \geq k_0$ and $f \in L^2(\Omega)$.

Because Ω is not convex, singularities can happen at the vertices of K, and the solution does not belong to $H^2(\Omega)$. However, we will prove that it admits a splitting

$$u = u_R + \sum_{j=1}^{N} S_j,$$

where $u_R \in H^2(\Omega)$, N is the number of vertices of K and S_j is a singular function associated with the jth corner of K. Furthermore, we estimate the norms of u_R and S_j in the same fashion than in Section 3.

In contrast to the case of a disc sector which feature one singular point, each vertex of K is a singular point in the case consider here. From the analysis point of view, the main difference is that here, one must localize the functions representing the singularities (see the definition of $\tilde{s}_j$ below (4.1)).

4.1. Notations.

We denote by $\{\mathbf{x}_j\}_{j=1}^N$ the set of corners of K. To each corner $\mathbf{x}_j$ we associate a polar coordinates system (r_j, θ_j). We denote by ω_j the angle corresponding to $\mathbf{x}_j$ and set $\alpha_j = \pi/\omega_j$, $\alpha = \min_j \alpha_j$. We denote by L the length of the smallest edge of K.

In the following, we denote by $\chi \in C^\infty(\mathbb{R})$ a cutoff function so that $\chi(\rho) = 1$ if $\rho < L/3$ and $\chi(\rho) = 0$ if $\rho > 2L/3$.

To each corner $\mathbf{x}_j$, we associate the singular function $\tilde{s}_j$ defined by

$$\tilde{s}_j(\mathbf{x}) = \chi_j(\mathbf{x}) r_j^{\alpha_j} \sin(\alpha_j\theta_j), \tag{4.1}$$

where $\chi_j(\mathbf{x}) = \chi(r_j(\mathbf{x}))$.

We further write $D_j = \Omega \cap B(\mathbf{x}_j, L)$. One sees that D_j is a disc sector centered at $\mathbf{x}_j$, of opening ω_j and radius L. As a result, D_j is obtained from $D(\omega_j, L)$ by rotation and translation. Hence, we are able to apply results from Section 2 when localizing the analysis in D_j.

4.2. **Splitting of the solution.** By a localization argument, we give a splitting of the solution into a regular part in $H^2(\Omega)$ and N singular functions, associated with each corner of K.

Theorem 4.1. *For all $k \geq k_0$ and $f \in L^2(\Omega)$, if $u \in H^1_{\Gamma_1}(\Omega)$ is solution to (2.3), there exist a function $\tilde{u}_R \in H^1_{\Gamma_1}(\Omega) \cap H^2(\Omega)$ and constants $c^j_k(f) \in \mathbb{C}$ such that*

$$u = \tilde{u}_R + \sum_{j=1}^{N} \tilde{c}^j_k(f)\tilde{s}_j,$$

and it holds that

$$|\tilde{c}^j_k(f)| \leq C(\Omega, k_0)k^{\alpha_j - \frac{1}{2}}\|f\|_{0,\Omega}, \tag{4.2}$$

for all $j = 1, \ldots, N$, while

$$|\tilde{u}_R|_{2,\Omega} \leq C(\Omega, k_0)k\|f\|_{0,\Omega}. \tag{4.3}$$

Proof. The proof heavily relies on a localization argument and the results of the previous section. Indeed for all $j = 1, \ldots, N$, we set

$$u_j = \chi_j u.$$

Up to an isometric change of coordinates, we see that u_j belongs to $H^1_{\Gamma_1}(D_{\omega_j,L})$, and is the variational solution of the problem (2.3) in $D_{\omega_j,L}$ with data $f_j = \chi_j f - 2\nabla\chi_j \cdot \nabla u - u\Delta\chi_j$, namely

$$\left\{\begin{array}{rcll} -k^2 u_j - \Delta u_j &=& f_j & \text{in } D_{\omega_j,L}, \\ \nabla u_j \cdot \mathbf{n} - iku_j &=& 0 & \text{on } S_{\omega_j,L}, \\ u_j &=& 0 & \text{on } I_{0,L} \cup I_{\omega_j,L}. \end{array}\right. \tag{4.4}$$

First let us notice that the estimate (2.4) yields

$$\|f_j\|_{0,D_j} \leq C(\Omega, k_0)\|f\|_{0,\Omega}. \tag{4.5}$$

Hence applying Theorem 3.1 to problem (4.4) one gets the splitting

$$u_j = \tilde{u}_{R,j} + \tilde{c}^j_k(f_j)\tilde{s}_j, \tag{4.6}$$

with $\tilde{u}_{R,j} \in H^2(\Omega_j)$ and $\tilde{c}^j_k(f_j) \in \mathbb{C}$. Furthermore with the help of Theorems 3.3 and 3.4 (and the estimate (4.5)), one has

$$|\tilde{c}^j_k(f_j)| \leq C(\omega_j, L, k_0)k^{\alpha_j - \frac{1}{2}}\|f_j\|_{0,\Omega_j} \leq C(\Omega, k_0)k^{\alpha_j - \frac{1}{2}}\|f\|_{0,\Omega}, \tag{4.7}$$

and

$$\|\tilde{u}_{R,j}\|_{2,\Omega_j} \leq C(\omega_j, L, k_0)k\|f_j\|_{0,\Omega_j} \leq C(\Omega, k_0)k\|f\|_{0,\Omega}. \tag{4.8}$$

Finally setting $\chi = 1 - \sum_{j=1}^{N} \chi_j$, we can look at $U = \chi u$ as the solution of (2.3) in $\mathcal{O}$ with data $F = \chi f - 2\nabla\chi \cdot \nabla u - u\Delta\chi$, where $\mathcal{O}$ is a smooth domain corresponding to Ω

where we have rounded the corners of Ω (without loss of generality, we can assume that the boundary of $\mathcal{O}$ has two connected components Γ_0 and $\Gamma_{1,s}$), namely

$$\left\{ \begin{array}{rcll} -k^2U - \Delta U & = & F & \text{in } \mathcal{O}, \\ \nabla U \cdot \mathbf{n} - ikU & = & 0 & \text{on } \Gamma_0, \\ U & = & 0 & \text{on } \Gamma_{1,s}. \end{array} \right. \tag{4.9}$$

Again by (2.4) one has

$$\|F\|_{0,\mathcal{O}} \leq C(\Omega, k_0)\|f\|_{0,\Omega}.$$

By standard regularity results, one has $U \in H^2(\mathcal{O})$ with

$$\|U\|_{2,\mathcal{O}} \leq C(\mathcal{O}) \left(\|U\|_{1,\mathcal{O}} + \|\Delta U\|_{0,\mathcal{O}}\right),$$

and applying (2.4) to the system (4.9), we deduce that

$$\begin{aligned} \|U\|_{2,\mathcal{O}} &\leq C(\mathcal{O}, k_0) \left(\|F\|_{0,\Omega} + k^2\|U\|_{0,\Omega}\right) \\ &\leq C(\mathcal{O}, k_0)(1+k)\|F\|_{0,\mathcal{O}} \\ &\leq C(\Omega, k_0)k\|f\|_{0,\Omega}. \end{aligned} \tag{4.10}$$

Since $u = U + \sum_{j=1}^N u_j$, the conclusion follows from splitting (4.6), the regularity $U \in H^2(\mathcal{O})$ and estimates (4.7)-(4.10). □

5. Frequency explicit stability of finite element discretizations

5.1. **The finite element space.** In this section, we investigate the discretization of problem (2.3) by linear finite elements. We consider meshes $\mathcal{T}_h$ made of triangles K satisfying

$$\operatorname{diam}(K) \leq h, \quad \operatorname{diam}(K) \leq \gamma\rho(K),$$

where γ is a constant independent of h, and

$$\operatorname{diam}(K) = \sup_{x,y\in K} |x-y|, \quad \rho(K) = \sup\left\{r > 0 \mid \exists x \in K;\ B(x,r) \subset K\right\}.$$

The solution $u \in H^1_{\Gamma_1}(\Omega)$ to problem (2.3) is then approximated by a function $u_h \in V_h$ satisfying

$$B(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{5.1}$$

where

$$V_h = \left\{v_h \in H^1_{\Gamma_1}(\Omega) \mid v_h|_K \in \mathcal{P}_1(K);\ \forall K \in \mathcal{T}_h\right\}$$

is the space of Lagrange linear elements build on $\mathcal{T}_h$. For more detail on the construction of V_h and its properties, we refer the reader to [9].

In this section, we will consider meshes such that

$$kh \leq 1. \tag{5.2}$$

This assumption is natural and means that the number of elements par wavelength is bounded from below. As we shall see, more restrictive conditions on h must be imposed to ensure that the finite element error remains bounded independently of k, so that we can assume (5.2) without loss of generality.

In order to simplify notations, we introduce the $k-$dependent norm

$$|||v|||^2 = k^2\|v\|_{0,\Omega}^2 + |v|_{1,\Omega}^2, \quad \forall v \in H^1_{\Gamma_1}(\Omega).$$

The $|||.|||$ norm is equivalent to the standard $H^1(\Omega)$ norm (with a constant obviously depending on k). This norm turns out to be the "natural" one to analyze the problem, since in view of stability estimate (2.4), the L^2 and H^1 terms of $|||u|||$ are "balanced" when u is a solution of the Helmholtz problem.

In the following, we denote by $\mathcal{I}_h$ the "quasi-interpolation" operator of Scott & Zhang [31]. We have $\mathcal{I}_h : H^1_{\Gamma_1}(\Omega) \to V_h$, and it holds that (see Theorem 4.1 of [31])

$$|v - \mathcal{I}_h v|_{l,\Omega} \leq C(\Omega,\gamma)h^{1-l}|v|_{1,\Omega} \quad (l = 0, 1). \tag{5.3}$$

Furthermore, if $v \in H^1_0(\Omega) \cap H^2(\Omega)$, it holds that

$$|v - \mathcal{I}_h v|_{l,\Omega} \leq C(\Omega,\gamma)h^{2-l}|v|_{2,\Omega} \quad (l = 0, 1). \tag{5.4}$$

5.2. **Preliminary discussion.** It is well known that the main source of error in numerical discretizations of high frequency problems is the dispersion. This is known as the "pollution effect": unless the mesh is heavily refined, the finite-element solution is not quasi-optimal, it is "polluted".

To simplify the discussion, let us denote by

$$\eta = \frac{|||u - \mathcal{I}_h u|||}{\|f\|_{0,\Omega}}$$

the best approximation error. In a smooth domain, it is known [23, 24] that the parameter η is bounded as

$$\eta \lesssim kh,$$

for linear Lagrange elements. Our main achievement is to establish that

$$\eta \lesssim k^{-1/2}k^\alpha h^\alpha + kh, \tag{5.5}$$

in a domain presenting re-entrant corners.

For 1D problems, the behaviour of the finite element solution and the pollution effect have been precisely analysed, see for instance [20, 21]. It is shown that if there are sufficiently many discretization points per wavelength (i.e. $\eta = kh$ is small enough), then

$$|||u - u_h||| \lesssim \left(\eta + k\eta^2\right)\|f\|_{0,\Omega} \simeq \left(kh + k^3h^2\right)\|f\|_{0,\Omega}. \tag{5.6}$$

The pollution effect is clearly depicted on (5.6), where the pollution term $k\eta^2$ is added to the best approximation error η. For large k, the pollution term k^3h^2 is dominant unless h is sufficiently small. This is called the "pre-asymptotic range", where the pollution error is the largest. On the other hand, the "asymptotic range" is achieved when $k\eta \leq C_0$ is small enough. Then, the finite element solution is quasi-optimal since $k\eta^2 \leq C_0\eta$.

If we use bound (5.5) for non-convex domains into (5.6), we obtain

$$\begin{aligned} |||u-u_h||| &\lesssim \left(\eta + k\eta^2\right)\|f\|_{0,\Omega} \\ &\lesssim \left(k^{-1/2}k^\alpha h^\alpha + kh + k^{2\alpha}h^{2\alpha} + k^3h^2\right)\|f\|_{0,\Omega}. \\ &\lesssim \left(k^{-1/2}k^\alpha h^\alpha + kh + k^3h^2\right)\|f\|_{0,\Omega}. \end{aligned}$$

Hence, we see that the presence of singularities is translated by the term $k^{-1/2}k^\alpha h^\alpha$. It is crucial to observe that this term is only significant in an asymptotic range where h is fine. More precisely, the term $k^{-1/2}k^\alpha h^\alpha$ is dominant only when $k^{-1/2}k^\alpha h^\alpha \leq kh$, which corresponds to $h \leq k^{\frac{\alpha-3/2}{1-\alpha}} \leq k^{-2}$. Also, we see that the pollution term k^3h^2 is not affected by the presence of singularities. We thus conclude that the dispersive behaviour of the finite element scheme remains unchanged in the presence of singularities. Furthermore, unless a highly accurate solution is required, the problem can be solved without using special techniques to "resolve" the singularities.

Unfortunately, the authors are not aware of a proof of (5.6) for general meshes in 2D. Nevertheless, in the following, we are able to give two interesting results.

First, we give asymptotic error estimates that are based on the so-called Shatz argument [30]. The methodology can be found, for instance, in [13] or [22]. Classically, we have that if $k\eta$ is small enough, the finite element solution is optimal and it holds that $|||u-u_h||| \leq \eta\|f\|_{0,\Omega}$. We show that the condition that $k\eta$ is small is satisfied as soon as k^2h is small. Hence, the presence of singularities does not change the asymptotic range.

Second, though we are not able to prove a general pre-asymptotic error estimate like in [20], we can derive weaker version thanks to a method recently introduced in [15]. The method relies on the introduction of a special elliptic projection. When applied in a smooth domain, it implies that if $kh\eta$ is small enough, then (5.6) holds. Hence, it provides the same error estimate, but the condition $kh\eta \leq C$ is imposed on the mesh step. When linear elements are consider, this condition is equivalent to $k^3h^2 \leq C$, so that we obtain an optimal bound on the error. For higher p however, this condition is not optimal. Hereafter, we adapt this method to the case of domains with singular points. Unfortunately, the elliptic projection used in [15] is affected by the singularities. As a result, we can show that (5.6) holds, but only if $kh^\alpha\eta$ is small enough (which is more restrictive than the original condition $kh\eta \leq C$ for regular domains).

5.3. **Interpolation of singularities.** Before deriving our main results, we present an interpolation result for the singularity functions. The analysis is subtle as $s_j \in H^{1+s}(\Omega)$ holds for $s < \alpha$, but not in the limiting case $s = \alpha$. As a result, direct approximation results in Sobolev spaces do not provide interpolation error estimates in $\mathcal{O}(h^\alpha)$. We will use a regularity result from [6] involving Besov spaces giving the desired estimates.

Lemma 5.1. *For $l = 0$ or 1, we have*

$$|\tilde{s}_j - \mathcal{I}_h\tilde{s}_j|_{l,\Omega} \leq C(\Omega,\gamma)h^{1-l+\alpha_j}. \tag{5.7}$$

Proof. If we write $s_j(\mathbf{x}) = r_j^{\alpha_j}\sin(\alpha_j\theta_j)$ and $\tilde{s}_j = \chi_j s_j$, we see that $\Delta s_j = 0$. Since $\operatorname{supp} s_j \subset D_j$, we observe that $\tilde{s}_j$ is solution to

$$\left\{\begin{array}{rcll} -\Delta\tilde{s}_j &=& g_j & \text{in } \Omega \\ \tilde{s}_j &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{5.8}$$

where

$$g_j = -\nabla\chi\cdot\nabla s_j - \Delta\chi s_j.$$

We observe that $\nabla\chi$ and $\Delta\chi$ are supported in $B(\mathbf{x}_j, 2L/3)\setminus B(\mathbf{x}_j, L/3)$. Since s_j is smooth on that set, we clearly have $g_j \in L^2(\Omega)$ and $\|g_j\|_{0,D_j} \leq C(\Omega)$.

The main ingredient of the proof is then Theorem 4.1 of [6], which give a regularity result for the solution of Laplace problem (5.8) in the Besov space

$$B^{1+\alpha} = [H^2(\Omega)\cap H_0^1(\Omega), H_0^1(\Omega)]_{1-\alpha,\infty}$$

obtained by real interplation. In particular, we can state that

$$\|s_j\|_{B^{1+\alpha}} \leq C(\Omega)\|g_j\|_{0,D_j}.$$

Since $H_0^1(\Omega) \subset H_{\Gamma_1}^1(\Omega)$, (5.3) and (5.4) hold for all $v \in H_0^1(\Omega)$. Hence the linear operator $T = (\operatorname{Id} - \mathcal{I}_h)$ is linear and bounded from $H_0^1(\Omega) \to H^l(\Omega)$ with norm $M_0 = C(\Omega,\gamma)h^{1-l}$ and from $H^2(\Omega)\cap H_0^1(\Omega) \to H^l(\Omega)$ with norm $M_1 = C(\Omega,\gamma)h^{2-l}$. Since $B^{1+\alpha}$ is an exact interpolation space (see, for instance [2]), we have that T is bounded from $B^{1+\alpha}$ to $H^l(\Omega)$ with norm $M_\alpha = M_0^\alpha M_1^{1-\alpha} = C(\Omega,\gamma)h^{1-l+\alpha_j}$. It follows that

$$\|v - \mathcal{I}_h v\|_{l,\Omega} \leq C(\Omega,\gamma)h^{1-l+\alpha_j}\|v\|_{B^{1+\alpha}},$$

for all $v \in B^{1+\alpha}$, and the result follows by taking $v = \tilde{s}_j$. □

5.4. **Asymptotic error estimate.** We start by deriving an asymptotic error estimate. The first step consists in estimating the best approximation error. The right-hand-side of estimate (5.9) contains the quantity η introduced above and we conclude that here

$$\eta \leq C(\Omega, k_0, \gamma)\left(k^{-1/2}h^\alpha k^\alpha + kh\right).$$

Lemma 5.2. *For $\phi \in L^2(\Omega)$, define $u_\phi \in H_{\Gamma_1}^1(\Omega)$ as the solution to*

$$B(u_\phi, v) = (\phi, v), \quad \forall v \in H_{\Gamma_1}^1(\Omega).$$

Then we have

$$|||u_\phi - \mathcal{I}_h u_\phi||| \leq C(\Omega, k_0, \gamma)\left(k^{-1/2}k^\alpha h^\alpha + kh\right)\|\phi\|_{0,\Omega}. \tag{5.9}$$

Furthermore, estimate (5.9) *also holds for the function $u_\phi^\star$ defined as the unique element of $H_{\Gamma_1}^1(\Omega)$ solution to*

$$B(v, u_\phi^\star) = (v, \phi), \quad \forall v \in H_{\Gamma_1}^1(\Omega).$$

Proof. We recall that we have the decomposition

$$u_\phi = \tilde{u}_R + \sum_{j=1}^N \tilde{c}_k^j(\phi)\tilde{s}_j$$

where $\tilde{s}_j = \chi(r_j) r^{\alpha_j} \sin(\alpha_j \theta_j)$, $\tilde{u}_R \in H^2(\Omega) \cap H^1_{\Gamma_1}(\Omega)$. Hence, since

$$u_\phi - \mathcal{I}_h u_\phi = (\tilde{u}_R - \mathcal{I}_h \tilde{u}_R) + \sum_{j=1}^N \tilde{c}_k^j(\phi) \left(\tilde{s}_j - \mathcal{I}_h \tilde{s}_j\right),$$

we have

$$k^{1-l} |u_\phi - \mathcal{I}_h u_\phi|_{l,\Omega} \leq k^{1-l} \left(|\tilde{u}_R - \mathcal{I}_h \tilde{u}_R|_{l,\Omega} + \sum_{j=1}^N |\tilde{c}_k^j(\phi)| |\tilde{s}_j - \mathcal{I}_h \tilde{s}_j|_{l,\Omega} \right),$$

for $l = 0, 1$. Recalling that $kh \leq 1$, we obtain from (5.4) and (4.3) that

$$\begin{aligned} k^{1-l} |\tilde{u}_R - \mathcal{I}_h \tilde{u}_R|_{l,\Omega} &\leq C(\Omega, \gamma) k^{1-l} h^{2-l} |\tilde{u}_R|_{2,\Omega} \\ &\leq C(\Omega, k_0, \gamma) k^{2-l} h^{2-l} \|\phi\|_{0,\Omega}. \\ &\leq C(\Omega, k_0, \gamma) kh \|\phi\|_{0,\Omega}, \end{aligned} \tag{5.10}$$

On the other hand, recalling (4.2) and (5.7), we have

$$\begin{aligned} k^{1-l} \sum_{j=1}^N |\tilde{c}_k^j(\phi)| |\tilde{s}_j - \mathcal{I}_h \tilde{s}_j|_{l,\Omega} &\leq C(\Omega, k_0, \gamma) k^{1-l} \sum_{j=1}^N k^{\alpha_j - 1/2} h^{1-l+\alpha_j} \|\phi\|_{0,\Omega} \\ &\leq C(\Omega, k_0, \gamma) k^{-1/2} k^{1-l} h^{1-l} \sum_{j=1}^N k^{\alpha_j} h^{\alpha_j} \|\phi\|_{0,\Omega}. \end{aligned}$$

Since $kh \leq 1$, we have $k^{1-l} h^{1-l} \leq 1$ and $k^{\alpha_j} h^{\alpha_j} \leq k^\alpha h^\alpha$ for $j = 1, \dots, N$.

$$\begin{aligned} k^{1-l} \sum_{j=1}^N |\tilde{c}_k^j(\phi)| |\tilde{s}_j - \mathcal{I}_h \tilde{s}_j|_{l,\Omega} &\leq C(\Omega, k_0, \gamma) k^{-1/2} \sum_{j=1}^N k^\alpha h^\alpha \|\phi\|_{0,\Omega} \\ &\leq C(\Omega, k_0, \gamma) N k^{-1/2} k^\alpha h^\alpha \|\phi\|_{0,\Omega} \\ &\leq C(\Omega, k_0, \gamma) k^{-1/2} k^\alpha h^\alpha \|\phi\|_{0,\Omega}. \end{aligned} \tag{5.11}$$

Then (5.9) follows from (5.10) and (5.11). □

Thanks to the estimates of the best approximation error derived in Lemma 5.2, we obtain an asymptotic error estimate by applying the Shatz argument [30, 13, 22]. A crucial observation is that the asymptotic range is defined by the condition that $k^2 h$ is small enough. This condition is the same than in the case of a smooth domain. Then, in error estimate (5.13), the term $k^{-1/2} k^\alpha h^\alpha$ is added in comparison to the case of a smooth domain. As we discussed above, for high frequencies, this term is less important than the usual term kh unless h is very small.

We also compare our results to the literature. In [4], the analysis of a "plane wave" numerical method is analyzed. The authors consider meshes made of squares, so that re-entrant corners of angle $3\pi/2$ ($\alpha = 2/3$) are allowed. The $H^{5/3}$ norm of the continuous solution is estimated without considering the singularities explicitly. As a result, the obtained asymptotic error estimate only holds under the condition that $k^{5/2} h$ is small enough.

In contrast, our asymptotic error estimate holds under the less restrictive condition that k^2h is small.

Theorem 5.1. *Assume that k^2h is small enough, then problem* (5.1) *admits a unique solution $u_h \in V_h$. Furthermore, the finite-element solution u_h is quasi-optimal:*

$$|||u - u_h||| \leq C(\Omega, k_0, \gamma)|||u - \mathcal{I}_h u|||, \tag{5.12}$$

and the error estimate

$$|||u - u_h||| \leq C(\Omega, k_0, \gamma)\left(k^{-1/2}k^\alpha h^\alpha + kh\right)\|f\|_{0,\Omega} \tag{5.13}$$

holds.

Proof. The proof uses the standard Shatz argument. Let $u_h \in V_h$ be any solution to (5.1). We introduce $\xi \in H^1_{\Gamma_1}(\Omega)$ solution to $B(v, \xi) = (v, u - u_h)$, so that

$$\|u - u_h\|^2_{0,\Omega} = B(u - u_h, \xi) = B(u - u_h, \xi - \mathcal{I}_h\xi).$$

By definition of ξ, recalling (5.9), we have

$$|||\xi - \mathcal{I}_h\xi||| \leq C(\Omega, k_0, \gamma)\left(kh + k^{-1/2}k^\alpha h^\alpha\right)\|u - u_h\|_{0,\Omega},$$

and therefore

$$\begin{aligned}
\|u - u_h\|^2_{0,\Omega} &= B(u - u_h, \xi - \mathcal{I}_h\xi) \\
&\leq C(\Omega)|||u - u_h||||||\xi - \mathcal{I}_h\xi||| \\
&\leq C(\Omega, k_0, \gamma)\left(kh + k^{-1/2}k^\alpha h^\alpha\right)|||u - u_h||||\|u - u_h\|_{0,\Omega}.
\end{aligned}$$

It follows that

$$\|u - u_h\|_{0,\Omega} \leq C(\Omega, k_0, \gamma)\left(kh + k^{-1/2}k^\alpha h^\alpha\right)|||u - u_h|||. \tag{5.14}$$

Now, we write that

$$\begin{aligned}
|||u - u_h|||^2 &= \operatorname{Re} B(u - u_h, u - u_h) + 2k^2\|u - u_h\|^2_{0,\Omega} \\
&= \operatorname{Re} B(u - u_h, u - \mathcal{I}_h u) + 2k^2\|u - u_h\|^2_{0,\Omega} \\
&\leq C(\Omega, k_0, \gamma)\left\{|||u - u_h|||.|||u - \mathcal{I}_h u||| + k^2\left(kh + k^{-1/2}k^\alpha h^\alpha\right)^2|||u - u_h|||^2\right\},
\end{aligned}$$

and dividing by $|||u - u_h|||$, we obtain

$$\left\{1 - C(\Omega, k_0, \gamma)\left(k^2h + k^{1/2}k^\alpha h^\alpha\right)^2\right\}|||u - u_h||| \leq C(\Omega, k_0, \gamma)|||u - \mathcal{I}_h u|||. \tag{5.15}$$

Recalling that $kh \leq 1$, since $\alpha \geq 1/2$, we have $k^{1/2}h^\alpha k^\alpha \leq k^2h$. Hence, assuming that k^2h is small enough, we have

$$C(\Omega, k_0, \gamma)\left(k^2h + k^{1/2}k^\alpha h^\alpha\right)^2 \leq \frac{1}{2},$$

and (5.12) follows from (5.15). Finally, (5.13) follows from (5.12) and (5.9).

The uniqueness of u_h is a direct consequence of (5.13), and existence follows, since u_h is defined as the solution of a finite-dimensional square linear system. □

5.5. **Pre-asymptotic error estimate.** In the following, we derive a pre-asymptotic error estimate using the elliptic projection introduced in [15]. The projection $\mathcal{P}_h u$ is introduced in Lemma 5.3 where we derived error estimates for $u - \mathcal{P}_h u$. We emphasize that because of the singularities, L^2 error estimate (5.16) is different from the case of a smooth domain. This is the reason why our pre-asymptotic error estimate is only valid under the condition that $k^3 h^{1+\alpha}$ is small enough.

Lemma 5.3. *For $u, v \in H^1_{\Gamma_1}(\Omega)$, we define*

$$a(u, v) = -ik\langle u, v\rangle_{\Gamma_0} + (\nabla u, \nabla v),$$

as well as

$$|u|_\star = \sqrt{|a(u, u)|},$$

so that $B(u, v) = -k^2(u, v) + a(u, v)$. Then, we have

$$|a(u, v)| \leq |u|_\star |v|_\star, \forall u, v \in H^1_{\Gamma_1}(\Omega).$$

Furthermore, for each $u \in H^1_{\Gamma_1}(\Omega)$, we define its projection $\mathcal{P}_h u \in V_h$ as the unique solution to

$$a(\mathcal{P}_h u, v_h) = a(u, v_h), \quad \forall v_h \in V_h.$$

If $u_\phi \in H^1_{\Gamma_1}(\Omega)$ solve $B(u_\phi, v) = (\phi, v)$ for all $v \in H^1_{\Gamma_1}(\Omega)$ for some $\phi \in L^2(\Omega)$, then we have

$$k^2 \|u_\phi - \mathcal{P}_h u_\phi\|_{0,\Omega} \leq C(\Omega, k_0, \gamma) \left(k^3 h^{1+\alpha} + k^{3/2+\alpha} h^{2\alpha}\right) \|\phi\|_{0,\Omega}, \tag{5.16}$$

and

$$|u_\phi - \mathcal{P}_h u_\phi|_\star \leq C(\Omega, k_0, \gamma) \left(k^{-1/2} k^\alpha h^\alpha + kh\right) \|\phi\|_{0,\Omega}. \tag{5.17}$$

Proof. By using Poincaré inequality, it is clear that the map

$$u \to |u|_\star = \sqrt{|a(u, u)|} = \sqrt{k|u|^2_{0,\Gamma_0} + |u|^2_{1,\Omega}}$$

is a norm on $H^1_{\Gamma_1}(\Omega)$, equivalent to the usual $H^1(\Omega)$ norm (with a constant of equivalence depending on k). As a result, a is a coercive and continuous sesquilinear form, and the existence and uniqueness of $\mathcal{P}_h u$ follow.

Furthermore, the multiplicative trace inequality shows that

$$|u|_\star \leq C(\Omega) |||u|||.$$

As a result, Céa's Lemma gives

$$|u_\phi - \mathcal{P}_h u_\phi|_\star \leq C(\Omega) |||u_\phi - \mathcal{I}_h u_\phi|||,$$

and we conclude that (5.17) holds with the help of (5.9).

We establish (5.16) using an Aubin-Nitsche trick. We introduce $\xi \in H^1_{\Gamma_1}(\Omega)$ solution to $a(v, \xi) = (v, u_\phi - \mathcal{P}_h u_\phi)$ for all $v \in H^1_{\Gamma_1}(\Omega)$. The existence and uniqueness of ξ follows from the properties of a, and we have

$$\begin{aligned}
\|u_\phi - \mathcal{P}_h u_\phi\|_{0,\Omega}^2 &= a(u_\phi - \mathcal{P}_h u_\phi, \xi) \\
&= a(u_\phi - \mathcal{P}_h u_\phi, \xi - \mathcal{P}_h \xi) \\
&\leq C(\Omega)|u_\phi - \mathcal{P}_h u_\phi|_\star |\xi - \mathcal{I}_h \xi|_\star \\
&\leq C(\Omega)|u_\phi - \mathcal{P}_h u_\phi|_\star |||\xi - \mathcal{I}_h \xi||| \\
&\leq C(\Omega) h^\alpha |u_\phi - \mathcal{P}_h u_\phi|_\star \|u_\phi - \mathcal{P}_h u_\phi\|_{0,\Omega},
\end{aligned}$$

so that

$$k\|u_\phi - \mathcal{P}_h u_\phi\|_{0,\Omega} \leq C(\Omega) k h^\alpha |u_\phi - \mathcal{P}_h u_\phi|_\star \leq C(\Omega) k h^\alpha |||u_\phi - \mathcal{I}_h u_\phi|||,$$

and (5.16) follows from (5.9). □

The elliptic projection and its approximation properties being introduced in Lemma 5.3, we can follow [15] to produce a preasymptotic error estimate in Theorem 5.2.

Theorem 5.2. *Assume that $k^3 h^{1+\alpha}$ is small enough, then there exists a unique solution $u_h \in V_h$ to problem* (5.1) *and it holds that*

$$|||u - u_h||| \leq C(\Omega, k, \gamma)\left(k^{-1/2} k^\alpha h^\alpha + kh + k^3 h^2\right)\|f\|_{0,\Omega}. \tag{5.18}$$

Proof. The proof relies on an Aubin-Nitsche type argument. We thus introduce $\xi \in H^1_{\Gamma_1}(\Omega)$ solution to $B(v,\xi) = (v, u - u_h)$ for all $v \in H^1_{\Gamma_1}(\Omega)$. Then, we have

$$\|u - u_h\|_{0,\Omega}^2 = B(u - u_h, \xi) = B(u - u_h, \xi - \mathcal{P}_h \xi).$$

Thanks to the properties of $\mathcal{P}_h$, we have

$$\begin{aligned}
B(u - u_h, \xi - \mathcal{P}_h \xi) &= -k^2(u - u_h, \xi - \mathcal{P}_h \xi) + a(u - u_h, \xi - \mathcal{P}_h \xi) \\
&= -k^2(u - u_h, \xi - \mathcal{P}_h \xi) + a(u - \mathcal{I}_h u, \xi - \mathcal{P}_h \xi),
\end{aligned}$$

so that

$$\begin{aligned}
\|u - u_h\|_{0,\Omega}^2 &\leq k^2\|u - u_h\|_{0,\Omega}\|\xi - \mathcal{P}_h \xi\|_{0,\Omega} + |a(u - \mathcal{I}_h u, \xi - \mathcal{P}_h \xi)| \\
&\leq k^2\|u - u_h\|_{0,\Omega}\|\xi - \mathcal{P}_h \xi\|_{0,\Omega} + |u - \mathcal{I}_h u|_\star |\xi - \mathcal{P}_h \xi|_\star.
\end{aligned}$$

As Lemmas 5.2 and 5.3 yield

$$k^2\|\xi - \mathcal{P}_h \xi\|_{0,\Omega} \leq C(\Omega, k_0, \gamma)\left(k^3 h^{1+\alpha} + k^{3/2+\alpha} h^{2\alpha}\right)\|u - u_h\|_{0,\Omega},$$

$$|||\xi - \mathcal{P}_h \xi||| \leq C(\Omega, k_0, \gamma)\left(kh + h^{-1/2} k^\alpha h^\alpha\right)\|u - u_h\|_{0,\Omega},$$

and

$$|u - \mathcal{I}_h u|_\star \leq C(\Omega)|||u - \mathcal{I}_h u||| \leq C(\Omega, k_0, \gamma)\left(kh + h^{-1/2} k^\alpha h^\alpha\right)\|f\|_{0,\Omega},$$

it follows that

$$\begin{aligned}
\|u - u_h\|_{0,\Omega}^2 \leq C(\Omega, k_0, \gamma)\Big\{&\left(k^3 h^{1+\alpha} + k^{3/2+\alpha} h^{2\alpha}\right)\|u - u_h\|_{0,\Omega}^2 \\
&+ \left(kh + h^{-1/2} k^\alpha h^\alpha\right)^2 \|f\|_{0,\Omega}\|u - u_h\|_{0,\Omega}\Big\},
\end{aligned}$$

and

$$\left\{1 - C(\Omega, k_0, \gamma)\left(k^3 h^{1+\alpha} + (k^{3+2\alpha} h^{4\alpha})^{1/2}\right)\right\} \|u - u_h\|_{0,\Omega} \leq C(\Omega, k_0, \gamma)\left(kh + k^{\alpha-1/2} h^{\alpha}\right)^2 \|f\|_{0,\Omega}.$$

We see that

$$k^{3+2\alpha} h^{4\alpha} = h^{\beta}\left(k^3 h^{1+\alpha}\right)^{(3+2\alpha)/3},$$

with

$$\beta = \left(\frac{12\alpha}{3+2\alpha} - 1 - \alpha\right)\frac{3+2\alpha}{3} > 0,$$

for $1/2 \leq \alpha \leq 1$. Hence, assuming that $k^3 h^{1+\alpha}$ is small enough, we have

$$C(\Omega, k_0, \gamma)\left(k^3 h^{1+\alpha} + (k^{3+2\alpha} h^{4\alpha})^{1/2}\right) \leq \frac{1}{2},$$

and

$$\|u - u_h\|_{0,\Omega} \leq C(\Omega, k_0, \gamma)\left(kh + k^{-1/2} k^{\alpha} h^{\alpha}\right)^2 \|f\|_{0,\Omega}.$$

Since

$$k(kh + k^{-1/2} k^{\alpha} h^{\alpha})^2 \leq 2\left(k^3 h^2 + k^{2\alpha} h^{2\alpha}\right) \leq 2\left(k^3 h^2 + kh\right),$$

we obtain

$$k\|u - u_h\|_{0,\Omega} \leq C(\Omega, k_0, \gamma)\left(k^3 h^2 + kh\right) \|f\|_{0,\Omega}.$$

Finally, we have

$$\begin{aligned}
|||u - u_h|||^2 &= 2k^2 \|u - u_h\|_{0,\Omega}^2 + \operatorname{Re} B(u - u_h, u - u_h) \\
&= 2k^2 \|u - u_h\|_{0,\Omega}^2 + \operatorname{Re} B(u - u_h, u - \mathcal{I}_h u) \\
&\leq 2k^2 \|u - u_h\|_{0,\Omega}^2 + C(\Omega, k_0) |||u - u_h|||.|||u - \mathcal{I}_h u|||,
\end{aligned}$$

and using the algebraic inequality, we obtain

$$|||u - u_h||| \leq C(\Omega, k_0)\left(k\|u - u_h\|_{0,\Omega} + |||u - \mathcal{I}_h u|||\right).$$

Then, the result follows since

$$|||u - \mathcal{I}_h u||| \leq C(\Omega, k_0, \gamma)\left(kh + k^{-1/2} k^{\alpha} h^{\alpha}\right) \|f\|_{0,\Omega}.$$

□

Error estimate (5.18) is called pre-asymptotic because it is valid in the range $k^3 h^{1+\alpha} \leq C$ which (in general) is larger than the asymptotic range $k^2 h \leq C$. In error estimate (5.18), we see that the pollution term $k^3 h^2$ is added to the best approximation error.

The validity range of (5.18) depends on α. The authors believe this is not sharp, and the dependence on α is due to the particular proof techniques. In the limit case $\alpha = 1/2$, the condition $k^3 h^{1+\alpha} \leq C$ is equivalent to $k^2 h \leq C$, so that the result is equivalent to asymptotic error estimate of Theorem 5.1. On the other hand, in the limit case $\alpha = 1$, $k^3 h^{1+\alpha} = k^3 h^2$ and we recover the usual validity condition of smooth domains [15]. In the general case where $1/2 < \alpha < 1$, we obtain a pre-asymptotic error estimate valid under a condition less restrictive than the quasi-optimality condition $k^2 h \leq C$, but more restrictive than the validity condition $k^3 h^2 \leq C$ of smooth domains.

6. Numerical examples

We present here two experiments to illustrate our analysis. Our aim is to investigate if the condition

$$k^{2p+1}h^{2p} \leq C \tag{6.1}$$

is sufficient to ensure that the $hp-$finite element error remains bounded independently of k. This condition is known to be optimal in the case of smooth non-trapping domains. A proof is available for 1D problems [21], and 2D and 3D problems with cartesian grids have been analyzed using dispersion analysis [3].

In the analysis presented above, we "almost" show that condition (6.1) is sufficient for the case of linear elements. Indeed, error estimate (5.18) clearly shows that the finite element error is clearly bounded independently of k as soon as k^3h^2 is bounded. Unfortunately, we are only able to prove (5.18) under the more restrictive condition that $k^3h^{1+\alpha}$ is small enough.

For each experiment, we start by selecting an initial wavenumber k_0. We empirically find a mesh size h_0 so that the relative L^2 finite element error is about 5%, when solving problem (5.1) for the wavenumber k_0. Then, we validate condition (6.1) by solving problem (5.1) for increasing values of k, the mesh size h being constraint by $k^{2p+1}h^{2p} = k_0^{2p+1}h_0^{2p}$, and checking that the error remains bounded.

In the two experiments, the relative $L^2(\Omega)$ error

$$\frac{\|u - u_h\|_{0,\Omega}}{\|u\|_{0,\Omega}}$$

is measured.

6.1. **Analytical solution.** We investigate a test-case with an analytical solution. The choice of the solution is guided by the analysis of Section 2. Hence, the test-case is designed so that the solution is

$$\phi(\mathbf{x}) = k^{-1/2}J_\alpha(kr)\sin(\alpha\theta).$$

In order to avoid curve elements, we consider a square with a re-entrant corner at the origin, namely the domain of computation is defined by

$$\Omega_\alpha = \{\mathbf{x} = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid |\mathbf{x}_1| \leq 1,\ |\mathbf{x}_2| \leq 1,\ 0 \leq \theta \leq \frac{\pi}{\alpha}\},$$

with $\frac{1}{2} < \alpha < 1$. The boundary of Ω_α is split up as

$$\Gamma_0 = \{\mathbf{x} \in \partial\Omega \mid |\mathbf{x}_1| = 1 \text{ or } |\mathbf{x}_2| = 1\}, \quad \Gamma_1 = \{(r\cos\theta, r\sin\theta) \in \partial\Omega \mid \theta = 0 \text{ or } \theta = \frac{\pi}{\alpha}\}.$$

Then, we solve

$$\left\{\begin{array}{rcll} -k^2u - \Delta u &=& 0 & \text{in } \Omega \\ \nabla u \cdot \mathbf{n} - iku &=& g & \text{on } \Gamma_0 \\ u &=& 0 & \text{on } \Gamma_1, \end{array}\right.$$

with $g = \nabla\phi \cdot \mathbf{n} - ik\phi$. Because $|J_\alpha(kr)| \leq C(\Omega, k_0)k^{-1/2}$ on Γ_1 when k is large, we see that $\|g\|_{L^2(\Omega)} \leq C(\Omega, k_0)$ for all $k \geq k_0$. Obviously the solution u of this problem is $u = \phi$ and presents a singularity near the origin.

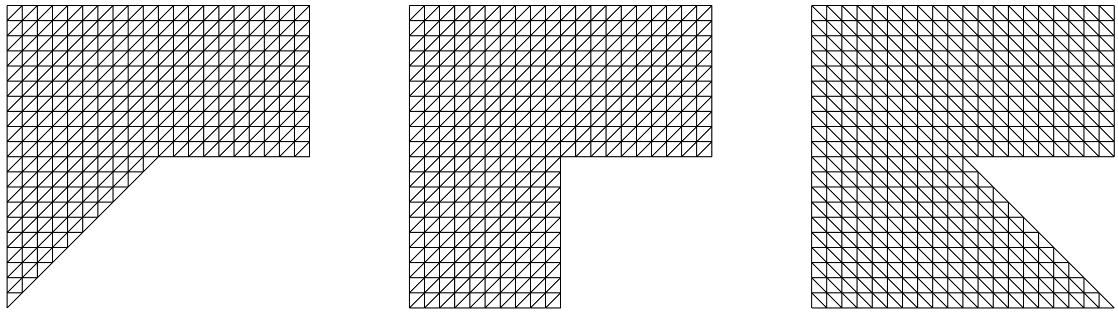

FIGURE 1. Structured meshes

We solve the problem for three different values of α: 4/5, 4/6, 4/7. These values correspond to domains that are easily meshed, so that we can used structured meshes to solve the problem, as shown in Figure 1. These meshes are not refined near the singularity.

As we explained before, we solve the problem for different values of k by starting with an initial guess (k_0, h_0) (fixed heuristically) and then impose the mesh size for higher values of k so that $k^{2p+1}h^{2p} = k_0^{2p+1}h_0^{2p}$. We employ three different values of p: 1, 2 and 6 and for this experiment, the heuristically determined values of (k_0, h_0) are given by $(3\pi, 1/50)$,$(14\pi, 1/50)$ and $(18\pi, 1/10)$ for $p = 1, 2$ and 6.

We present the dependence of the relative $L^2(\Omega)$ error with respect to k on Figures 2, 3 and 4. As shown there, the error is clearly bounded independently of k under the condition that $k^{2p+1}h^{2p} \leq C$. As observed above, for the case $p = 1$, this is almost consistent with the pre-asymptotic error estimate derived in Theorem 5.2.

Figures 2-4 also show that the error is more important for smaller values of α. This is not surprising, since in this case the solution is more singular and furthermore, the domain of computation is wider. However, the error is only increased by a constant factor that is about 3 between the largest and smallest considered values of α. In particular, as predicted by our analysis for the linear case, the stability of the scheme is not affected by the value of α.

6.2. **Scattering by a triangle.** The problem of scattering we consider reads

$$\left\{ \begin{array}{rcll} -k^2 u - \Delta u &=& 0 & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} - iku &=& 0 & \text{on } \Gamma_0, \\ u &=& e_\phi & \text{on } \Gamma_1, \end{array} \right.$$

where $e_\phi(\mathbf{x}) = \exp(ik\nu \cdot \mathbf{x})$, with $\nu = (\cos\phi, \sin\phi)$ and $\phi = \pi/3$. The numerical solutions obtained for different frequencies are depicted on Figure 5.

Instead of computing a lifting for e_ϕ, we weakly impose the inhomogeneous Dirichlet condition with a penalization method [27, 28]. Hence, we modify the sesquilinear form B as

$$B_h(u_h, v_h) = B(u_h, v_h) + \int_{\Gamma_1} \nabla u_h \cdot \mathbf{n}\overline{v_h} + \int_{\Gamma_1} u_h \nabla\overline{v_h} \cdot \mathbf{n} + \frac{p^2}{h} \int_{\Gamma_1} u_h \overline{v_h},$$

and we solve

$$B_h(u_h, v_h) = (f, v_h) + \frac{p^2}{h} \int_{\Gamma_1} e_\phi \overline{v_h}, \quad \forall v_h \in V_h^p. \tag{6.2}$$

Though problem (6.2) is not directly covered by our analysis, similar error estimates can be obtained with slight modifications of our arguments.

The domain of computations as well as the used meshes are depicted at Figure 6. The meshes are obtained using the software `triangle` [32]. The mesh size is imposed as an area condition ($|K| \leq h^2/2$) and the meshes satisfy a minimal angle condition of 33 degrees. We also produce "refined" meshes by forcing the mesh to include three additional points. Each of the three points is placed at a distance $h/1000$ of one vertex of the triangle. In that way, the local mesh size at the singular points is 1000 times finer than the global mesh size in refined meshes.

Following our methodology, we impose the condition that $k^{2p+1}h^{2p} = k_0^{2p+1}h_0^{2p+1}$. We use three different values of p: 1, 3 and 4, and the associated couple (k_0, h_0) are $(4\pi, 1/50)$, $(14\pi, 1/10)$ and $(18\pi, 1/10)$. In order to evaluate the $L^2(\Omega)$-norm error a "reference solution" is computed with a $p+1$ finite element method on the same mesh. Then, each solution is evaluated onto a 1024×1024 grid, and the $L^2(\Omega)$ error is computed as the l^2-norm of this discrete vector.

We present the results on Figures 7, 8 and 9. The error is bounded independently of the frequency for both uniform and refined meshes. For low frequency simulations, refined meshes improve the precision of the finite element method (up to a factor 3). However, we see that this improvement is greatly reduced for higher frequencies. This is in agreement with our analysis, where we pointed out that the singular part of the solution is "less important" for high frequencies.

7. Conclusion

In this work, we have analyzed the acoustic Helmholtz problem set in a domain $\Omega = \Omega_0 \setminus K$, where K is a convex polygon, and Ω_0 is a surrounding computational domain. A Dirichlet condition is imposed on ∂K and the Sommerfeld radiation condition is approximated using an absorbing boundary condition on $\partial\Omega_0$. Thus, the considered problem modelizes diffraction by sound-soft scatterer.

Since the computational domain Ω is not convex, the solution might become singular close to each corner of the scatterer K. We have proposed a splitting of the solution of the Helmholtz problem with a regular part in $H^2(\Omega)$ and one singular function for each corner of K. The regularity as well as the high frequency behaviour of each component of the splitting have been rigorously analyzed. Our main conclusion is that in some sense, as the frequency increases, the "amplitude" of the singularities vanishes before the amplitude of the regular part.

We have taken advantage of this splitting to derive sharp error estimates for finite element discretizations. The different behaviours of the regular and singular parts in terms of frequency is visible in these error estimates. The main conclusion is that if the frequency is high, numerical discretizations do not "see" the singularities unless the mesh size is "small".

Numerical experiments have been presented to illustrate the above-mentioned error estimates. In smooth non-trapping domains, it is known that the condition $h^{2p+1}h^{2p} \leq C$ is optimal to ensure that the finite element error remains bounded independently of the frequency. We have numerically investigated if this condition is also sufficient for the case of non convex domains with re-entrant corners. We conclude that this condition is indeed sufficient. Furthermore, we have analyzed the dependence of the error with respect to the singular exponent α, and we conclude that if the error does increase when α gets closer to $1/2$, this increase never exceeds one order of magnitude.

As a final note, we would like to point out that we have focused on the acoustic scattering problem mostly for the sake of simplicity. The main ideas presented in this paper might be also applied for electromagnetic and/or elastic wave propagation, as well as other types of propagation media. In particular, when seismic waves are modelized using the elastic Helmholtz equation, singularities happen at points where physical interfaces cross, which could be analyzed with the methodology introduced in this paper. Also, though we have focused on finite element discretization, the frequency explicit analysis of the singularities should also help to analyze other discretization strategy, including boundary element methods.

Future works will be guided towards edge and corner singularities of 3D scattering problems, as well as the analysis of other wave operators in 2D.

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Appendix A. Bessel functions

Here, $\nu \in (1/2, 1)$ is an arbitrary real number. Bessel functions of first and second kind are defined by

$$J_{\pm\nu}(\rho) = \left(\frac{z}{2}\right)^{\pm\nu} \sum_{l=0}^{+\infty} \frac{1}{l!\Gamma(\pm\nu + l + 1)} \left(-\frac{z^2}{4}\right)^l$$

and

$$Y_\nu(\rho) = \frac{J_\nu(\rho)\cos(\nu\pi) - J_{-\nu}(\rho)}{\sin(\nu\pi)}.$$

Here after, we list well-known properties of Bessel functions that can be bound in Chapter 9 of [1].

For $0 < \rho \leq 1$, and $\nu > 0$, it holds that

$$|J_\nu(\rho)| \leq \frac{1}{\Gamma(\nu+1)} \left(\frac{\rho}{2}\right)^\nu$$

$$|Y_\nu(\rho)| \leq \frac{1}{\Gamma(\nu+1)} \left(\frac{\rho}{2}\right)^{-\nu}$$

The following expansions hold for large ρ:

$$J_\nu(\rho) = \sqrt{\frac{2}{\pi\rho}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(\rho^{-3/2}).$$

$$J_\nu'(\rho) = -\sqrt{\frac{2}{\pi\rho}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(\rho^{-3/2}).$$

$$Y_\nu(\rho) = \sqrt{\frac{2}{\pi\rho}} \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(\rho^{-3/2}).$$

$$Y_\nu'(\rho) = \sqrt{\frac{2}{\pi\rho}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(\rho^{-3/2}).$$

With the above properties, one easily shows:

Lemma A.1. *For all $\nu \in (1/2, 1)$, there exists a constant $C(\nu)$ such that*

$$|J_\nu(\rho)| \leq C(\nu)\rho^\nu, \quad |Y_\nu(\rho)| \leq C(\nu)\rho^{-\nu},$$

for all $\rho \in (0, 1)$, and

$$|J_\nu(\rho)| \leq C(\nu)\rho^{-1/2}, \quad |Y_\nu(\rho)| \leq C(\nu)\rho^{-1/2},$$

for all $\rho \geq 1$.

We are now ready to prove Proposition 3.1.

Proof of Proposition 3.1. First, (3.1) directly follows from Lemma A.1. Indeed, we have

$$\begin{aligned}
k^2 \int_0^R |J_\alpha(kr)|^2 r dr &= \int_0^{kR} |J_\alpha(\rho)|^2 \rho d\rho \\
&= \int_0^1 |J_\alpha(\rho)|^2 \rho d\rho + \int_1^{kR} |J_\alpha(\rho)|^2 \rho d\rho \\
&\leq C(\alpha) \left(\int_0^1 |\rho^{-\alpha}|^2 \rho d\rho + \int_1^{kR} |\rho^{-1/2}|^2 \rho d\rho \right) \\
&\leq C(\alpha) \left(\frac{1}{2-2\alpha} + kR - 1 \right) \\
&\leq C(\alpha) \left\{ R + \left(\frac{1}{2-2\alpha} - 1 \right) k_0^{-1} \right\} k.
\end{aligned}$$

The upper bound of (3.2) is proved exactly as above, with J_α replaced by $H_\alpha^{(2)}$. In order to establish the lower bound, we first write that

$$\int_0^{kR} |H_\alpha^{(2)}(\rho)|^2 \rho d\rho \geq \int_{kR/2}^{kR} |H_\alpha^{(2)}(\rho)|^2 \rho d\rho.$$

Then, we have

$$H_\alpha^{(2)}(\rho) = \sqrt{\frac{2}{\pi\rho}} \exp\left\{ -i \left(\rho - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right\} + \mathcal{O}\left(\rho^{-3/2}\right)$$

so that

$$|H_\alpha^{(2)}(\rho)|^2 \geq \frac{2}{\pi\rho} - M(\alpha, R, k_0)\rho^{-3}$$

for all $\rho \geq kR/2$.

As a result, we have

$$\begin{aligned}
\int_{kR/2}^{kR} |H_\alpha^{(2)}(\rho)|^2 \rho d\rho &\geq \frac{2}{\pi} \int_{kR/2}^{kR} d\rho - M(\alpha, k_0) \int_{kR/2}^{kR} \rho^{-2} d\rho \\
&\geq \frac{kR}{\pi} - M(\alpha, k_0)(kR)^{-1} \\
&\geq C(\alpha, R, k_0)k,
\end{aligned}$$

assuming that k_0 is sufficiently high.

We now prove (3.3). We start by writing

$$\kappa = \rho - \frac{\nu\pi}{2} - \frac{\pi}{4},$$

so that

$$\begin{aligned} J_\nu'(\rho) + iJ_\nu(\rho) &= \sqrt{\frac{2}{\pi\rho}}\left(-\sin\kappa + i\cos\kappa\right) + \mathcal{O}\left(\rho^{-3/2}\right) \\ &= i\sqrt{\frac{2}{\pi\rho}}\left(\cos\kappa + i\sin\kappa\right) + \mathcal{O}\left(\rho^{-3/2}\right) \\ &= i\sqrt{\frac{2}{\pi\rho}}e^{i\kappa} + \mathcal{O}\left(\rho^{-3/2}\right), \end{aligned}$$

and

$$\begin{aligned} Y_\nu'(\rho) + iY_\nu(\rho) &= \sqrt{\frac{2}{\pi\rho}}\left(\cos\kappa + i\sin\kappa\right) + \mathcal{O}\left(\rho^{-3/2}\right) \\ &= \sqrt{\frac{2}{\pi\rho}}e^{-i\kappa} + \mathcal{O}\left(\rho^{-3/2}\right). \end{aligned}$$

Then, we have

$$\frac{Y_\nu'(\rho) + iY_\nu(\rho)}{J_\nu'(\rho) + iJ_\nu(\rho)} = \frac{1}{i} + \mathcal{O}\left(\rho^{-3/2}\right),$$

and the result follows.

Finally, (3.4) is just the Wronskian of J_α and Y_α that is given by

$$J_\nu(\rho)Y_\nu'(\rho) - J_\nu'(\rho)Y_\nu(\rho) = \frac{2}{\pi\rho},$$

for all $\rho > 0$. □

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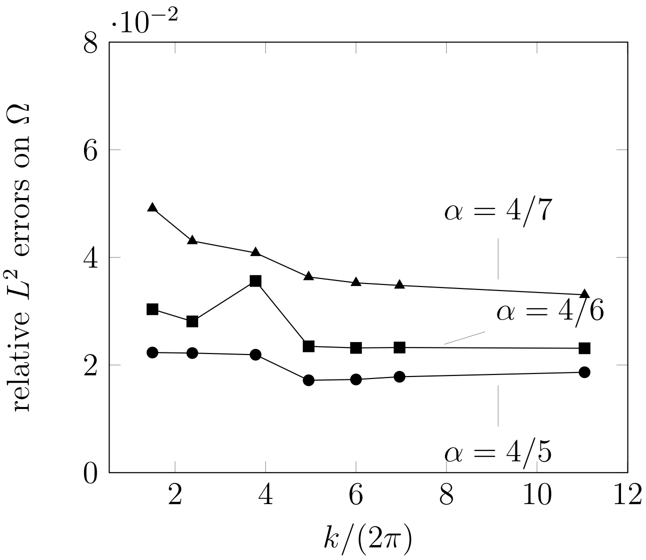

FIGURE 2. $\mathcal{P}_1$ elements, $k^3h^2 = C$

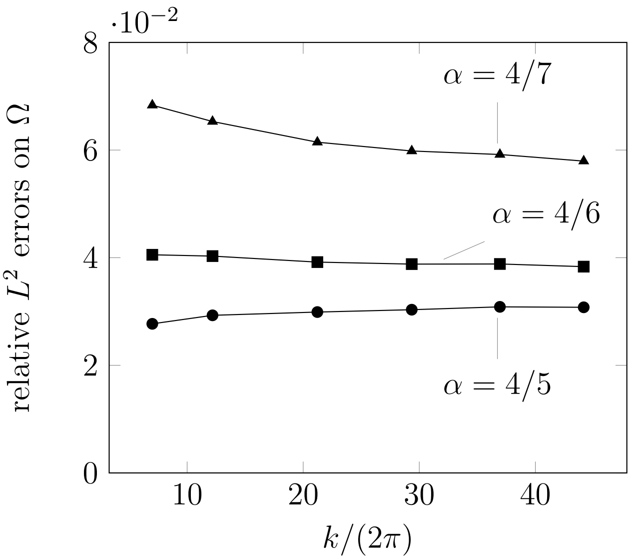

FIGURE 3. $\mathcal{P}_2$ elements, $k^5h^4 = C$

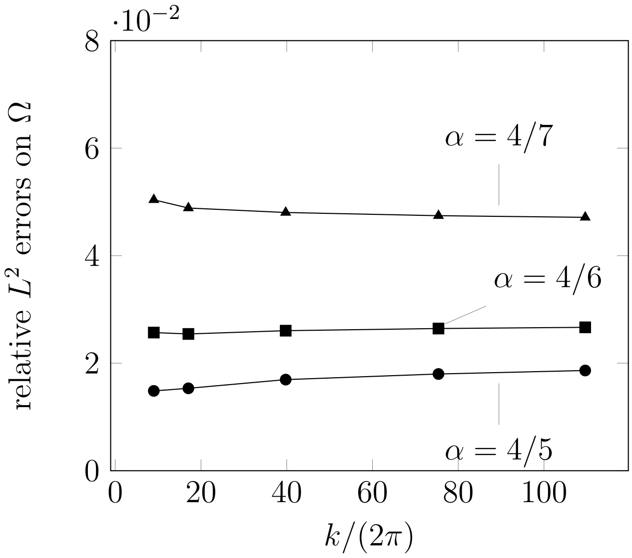

FIGURE 4. $\mathcal{P}_6$ elements, $k^{13}h^{12} = C$

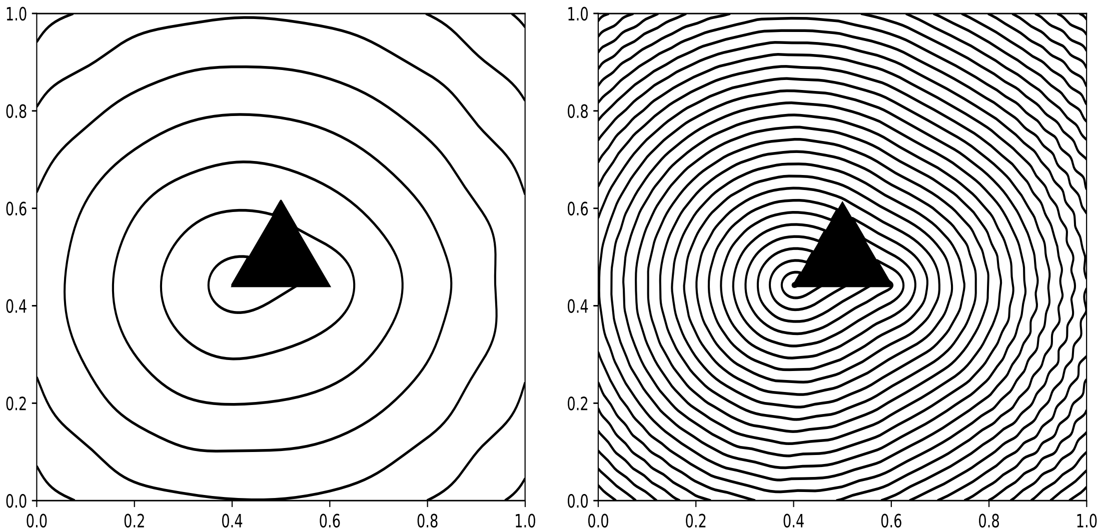

FIGURE 5. Zero-level sets of the real part of the solution of the scattering problem for $k = 10\pi$ (left) and 20π (right)

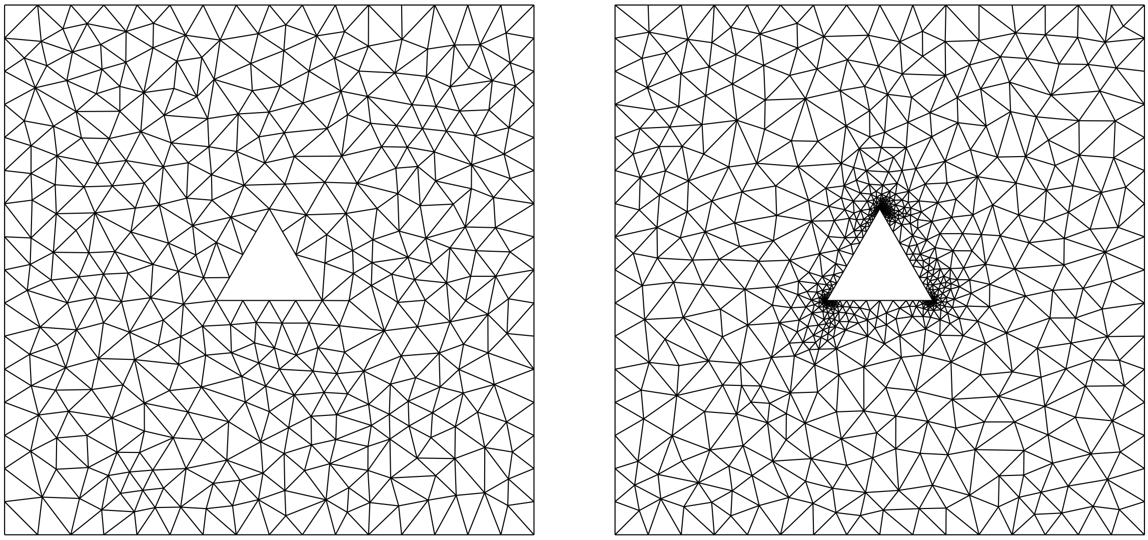

FIGURE 6. Uniform (left) and refined (right) meshes for the scattering problem

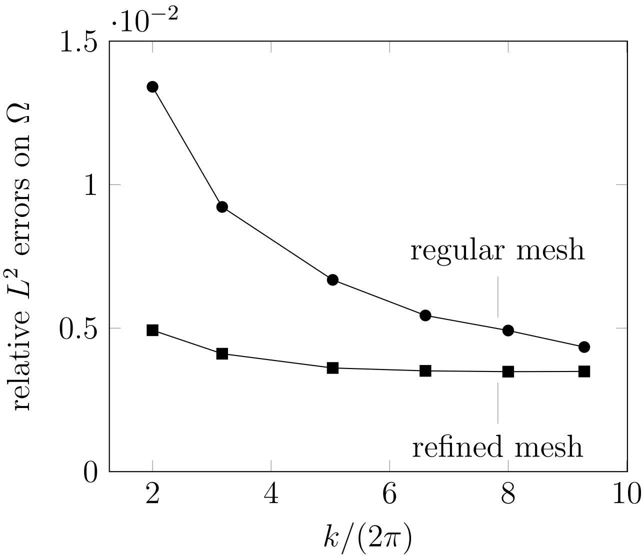

FIGURE 7. $\mathcal{P}_1$ elements, $k^3h^2 = C$

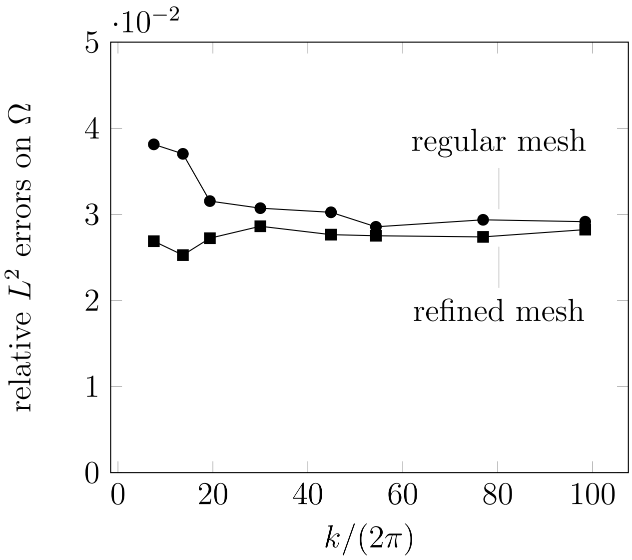

FIGURE 8. $\mathcal{P}_3$ elements, $k^7h^6 = C$

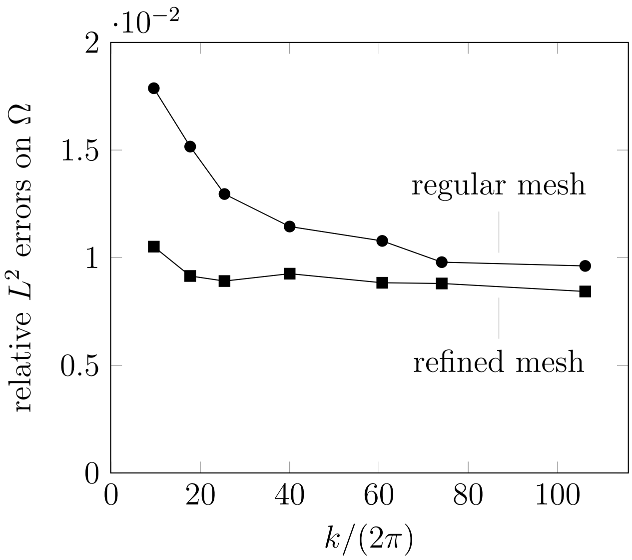

FIGURE 9. $\mathcal{P}_4$ elements, $k^9h^8 = C$