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ABSTRACT. Hearing loss may be caused by a number of factors, including: genetics, ageing, exposure to noise, some infections, birth complications, trauma to the ear, and certain medications or toxins. In this study, we propose a new mathematical model of hearing loss resulting from the infectious disease using the ordinary differential equations with the objective to analyze the local and global stability of the model. In addition, we present some numerical simulations in order to validate our theoretical results

Keywords: *viral infection, ordinary differential equations, stability.*

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1 Introduction

Hearing loss may result from genetic causes, complications at birth, certain infectious diseases, chronic ear infections, the use of particular drugs, exposure to excessive noise, and aging, which is represent a major global health problem. According to the World Health Organization (WHO), about 360 million people worldwide have disabling hearing loss, and 32 million of these are children [1].

The are many infections and contagious diseases related to hearing loss such as mumps. Mumps is an enveloped, single-stranded RNA virus belonging to the family paramyxoviridae, and causes an acute infectious disease mainly in children and young adults[2]. Mumps is transmitted through infected respiratory secretions and is highly contagious [3].
The most common clinical manifestations of infection include a flu-like illness and bilateral swelling of the parotid glands.
Mumps infection occasionally induces the potential for complications such as pancreatitis, orchitis, oophoritis, aseptic meningitis, encephalitis and sensorineural hearing loss. Hearing loss due to mumps is thought to be unilateral and profound with rapid onset[4].

To better understand the dynamics of hearing loss, we introduce a new mathematical model by using ordinary differential equations (ODEs). In this paper, we consider the hearing loss model caused by mumps(contagion factor).

This model is described by the following system of ODEs:

$$\begin{aligned} \frac{dH}{dt} &= \Lambda - \mu H - \beta LH \\ \frac{dL}{dt} &= \beta LH - (\mu + \gamma)L \\ \frac{dR}{dt} &= \gamma L - \mu R \end{aligned} \tag{1}$$

The population is divided into three epidemiological classes that are : H(t) is the number of susceptible individuals (normal hearing), L(t) is the number of infected individuals (loss of hearing), and R(t) is the number of removed individuals at time t (recovered hearing).
Where Λ is the recruitment rate of the population, μ is the natural death rate of the population, β is the transmission rate due to social contagion of hearing loss (Mumps),γ is the recovery rate of the infective individuals.
The first two equations in system (1) do not depend on the third equation, and therefore this equation can be omitted without loss of generality. System (1) can be rewritten as;

$$\begin{aligned} \frac{dH}{dt} &= \Lambda - \mu H - \beta LH \\ \frac{dL}{dt} &= \beta LH - (\mu + \gamma)L \end{aligned} \tag{2}$$

The paper is organized as follows. In the next section, Positivity and boundedness of solutions are studied. In Section 3, the basic reproduction number is derived, also the local and the global asymptotic stability of the equilibrium point are analyzed. The numerical result is obtained in Section 4. Lastly, we give a conclusion of our results in Section 5.

2 Positivity and boundedness of solutions

In this section, we will establish the positivity and boundedness of solutions of model (1), which imply that our model is well posed.

Proposition 2.1 *All solutions starting from non-negative initial conditions exist for all $t > 0$ and remain bounded and non-negative.*

Proof.
For the positivity, we show that any solution starting in non-negative orthant $\mathbb{R}^3_+ = \{$(H,L,R) $\in$ $\mathrm{R}^3_+ : H \geq 0,\ L \geq 0,\ R \geq 0\}$. In fact (H(t),L(t),R(t)) $\in \mathrm{R}^3_+$ and we have;

$$\begin{aligned} \frac{dH}{dt} &|_{H=0} = \Lambda \geqslant 0 \\ \frac{dL}{dt} &|_{L=0} = 0 \geqslant 0 \\ \frac{dR}{dt} &|_{R=0} = \gamma L \geqslant 0 \end{aligned} \tag{3}$$

This proves the positivity of solutions.
Now, we prove that the solutions are bounded. We define T(t)=H(t)+L(t).
By non-negativity of the solution, it follows that;

$$\begin{aligned} \frac{dT(t)}{dt} &= \Lambda - \mu H - (\mu + \gamma) L \\ \frac{dT(t)}{dt} &\leqslant \Lambda - \mu (H + L) \end{aligned}$$

We deduce that;

$$T(t) \leqslant \frac{\Lambda}{\mu} + (H(0) + L(0)) e^{-\mu t} \leqslant \frac{\Lambda}{\mu}$$

Hence, H(t) and L(t) are bounded.
Next, we show the boundedness of R(t).

$$\begin{aligned} \frac{dR}{dt} &= \gamma L - \mu R \\ \frac{dR}{dt} &\leqslant \frac{\gamma \Lambda}{\mu} - \mu R \end{aligned}$$

Then,

$$R(t) \leqslant \frac{\Lambda \gamma}{\mu^2}$$

Finally, all the solutions are bounded.

3 Stability analysis for Hearing loss

Notice that the system (1) has a basic reproductive number of $R_0 = \dfrac{\beta\Lambda}{\mu(\mu+\gamma)}$

R_0 represents the average number of secondary infections caused by an infective individual introduced into a group of susceptible.

By equalizing to zero the rights members of the system (1), we find two equilibrium points that exists for above model:

1. Disease-free Equilibrium Point $E_f = (\dfrac{\Lambda}{\mu}, 0, 0)$.

2. Endemic equilibrium point $E^* = (H^*, L^*, R^*)$ where;

$$H^* = \frac{(\mu+\gamma)}{\beta}$$
$$L^* = \frac{\mu(\beta\frac{\Lambda}{\mu} - \mu - \gamma)}{\beta(\mu+\gamma)}$$
$$R^* = \frac{\gamma(\beta\frac{\Lambda}{\mu} - \mu - \gamma)}{\beta(\mu+\gamma)}$$

The endemic equilibrium point exist only when $\beta\dfrac{\Lambda}{\mu} > \gamma + \mu$ i.e the infection rate must be greater than the death rate of the infected individuals or $R_0 > 1$.

3.1 Local stability of equilibria

The following theorems discusses the local stability of the equilibrium point.

Theorem 3.1

1. *If $R_0 < 1$, then the disease-free equilibrium, E_f is locally asymptotically stable.*

2. *If $R_0 > 1$, E_f is unstable.*

Proof.

The Jacobian matrix evaluated in the disease-free equilibrium $E_f = (\dfrac{\Lambda}{\mu}, 0, 0)$ is given by

$$J(E_f) = \begin{pmatrix} -\mu & -\beta\frac{\Lambda}{\mu} & 0 \\ 0 & \beta\frac{\Lambda}{\mu} - (\gamma+\mu) & 0 \\ 0 & \gamma & -\mu \end{pmatrix} \tag{4}$$

whose eigenvalues are $\lambda_1 = -\Lambda$ and $\lambda_2 = \beta\dfrac{\Lambda}{\mu} - \gamma - \mu$ $\lambda_1 = -\mu < 0$ and $\lambda_2 = \beta\dfrac{\Lambda}{\mu} - \gamma - \mu < 0$ then $R_0 < 1$ and therefore E_f is locally asymptotically stable.

The disease-free equilibrium point is unstable if $\beta\frac{\Lambda}{\mu} - \gamma - \mu > 0$ which translate into $R_0 > 1$. Now, we focus on local stability of the endemic infection equilibrium E^*.

Theorem 3.2

1. *If $R_0 > 1$, E^* is locally asymptotically stable.*

2. *If $R_0 < 1$, then the endemic equilibrium E^* does not exist.*

Proof. By substituting the endemic equilibrium $E^* = (H^*, L^*, R^*)$ in the Jacobien matrix of the system (8)

$$J(E^*) = \begin{pmatrix} -\mu(\frac{\beta\frac{\Lambda}{\mu} - \gamma - \mu}{\gamma + \mu} + 1) & -(\gamma + \mu) & 0 \\ \mu(\frac{\beta\frac{\Lambda}{\mu} - \gamma - \mu}{\gamma + \mu}) & 0 & 0 \\ 0 & \gamma & -\mu \end{pmatrix} \tag{5}$$

The characteristic equation of the endemic equilibria point is given by

$$(-\mu - \lambda)(\lambda^2 + \frac{\beta\lambda}{\gamma + \mu} + \mu(\beta N - \gamma - \mu)) = 0 \tag{6}$$

The disease persists when $L^* > 0$ then $R_0 > 1$

Clearly when $R_0 > 1$, both $\frac{\mu\beta\frac{\Lambda}{\mu}}{\mu + \gamma} > 0$ and $\mu(\beta\frac{\Lambda}{\mu} - \gamma - \mu) > 0$, then all roots of the characteristic equation have negative real parts. Consequently E^* is locally asymptotically stable.

3.2 Global stability of equilibria

The following theorem discusses the global stability of the endemic equilibrium.

Theorem 3.3 *The endemic equilibrium E^* of the system (1) is globally asymptotically stable E^*.*

Proof. Consider the following Lyapunov functional;

$$W(t) = H_\varepsilon^*(t)\phi(\frac{H(t)}{H_\varepsilon^*(t)}) + L_\varepsilon^*(t)\phi(\frac{L(t)}{L_\varepsilon^*(t)}) \tag{7}$$

Where $\phi(t) = x - 1 - \ln(x)$ $x \in \mathbb{R}^+$. Obviously, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ attains its global minimum at x = 1 and $\phi(1) = 0$. To simplify the presentation, we shall use the following notation: H = H(t) and

L=L(t).

$$W(t) = H^*(\frac{H(t)}{H^*_\varepsilon} - 1 - \ln(\frac{H(t)}{H^*_\varepsilon})) + L^*_\varepsilon(\frac{L(t)}{L^*_\varepsilon} - 1 - \ln(\frac{L(t)}{L^*_\varepsilon}))$$
$$= \dot{H}(1 - \frac{H^*_\varepsilon}{H}) + \dot{L}(1 - \frac{L^*_\varepsilon}{L})$$
$$= (\Lambda - \mu H - \beta LH)(1 - \frac{H^*}{H}) + (1 - \frac{L^*}{L})([\beta LH - (\gamma + \mu)L)$$

Note that $\Lambda = \mu H^* + (\mu + \gamma)L^*$ and $(\beta L^* H^* = (\mu + \gamma)L^*$
Hence,

$$\dot{W}(t) = (1 - \frac{H^*_\varepsilon}{H})(\mu H^* + (\mu + \gamma)L^* - \mu H - \beta LH + (1 - \frac{I^*}{I})(\beta LH - (\mu + \gamma)L)$$

$$\dot{W}(t) = \mu(1 - \frac{H^*}{H})(H^* - H) + (\mu + \gamma)L^*(1 - \frac{H^*}{H}) - (1 - \frac{L^*}{L})\beta LH + (1 - \frac{L^*}{L})\beta LH$$
$$- (\mu + \gamma)L(1 - \frac{L^*}{L})$$
$$= \mu(H^* - H)(1 - \frac{H^*}{H}) + (\mu + \gamma)L^*(1 - \frac{H^*}{H} + \beta\frac{H^*}{L}(\mu + \gamma)L^*) + (\mu + \gamma)L^*(1 - \frac{L}{L^*} - \frac{\beta LH}{(\mu + \gamma)L})$$
$$= \mu(H^* - H)(1 - \frac{H^*}{H}) + (\mu + \gamma)L^*(-1 - \frac{L}{L^*} + \frac{(\mu + \gamma)L}{\beta H^* L} + \frac{\beta H^* L}{(\mu + \gamma)L^*})$$
$$- (\mu + \gamma)(\phi(\frac{H^*}{H}) + \ln(\frac{H^*}{H}) + \phi(\frac{(\mu + \gamma)L}{H^*(\beta L + \varepsilon)} + \ln(\frac{(\mu + \gamma)L}{\beta H^* L} + \phi(\frac{\beta LH}{(\mu + \gamma)L}) + \ln(\frac{\beta LH}{(\mu + \gamma)L})$$

Then, we obtain the following equation:

$$\dot{W}(t) = \mu(H^* - H)(1 - \frac{H^*}{H}) + (\mu + \gamma)L^*(-1 - \frac{L}{L^*} + \frac{(\mu + \gamma)L}{\beta H^* L} + \frac{\beta H^* L}{(\mu + \gamma)L^*}) - (\mu + \gamma)L^*[\phi(\frac{H^*}{H})$$
$$+ \phi(\frac{(\mu + \gamma)L}{\beta H^* L} + \phi(\frac{\beta LH}{(\mu + \gamma)L})] - 1 + \frac{L}{L^*} + \frac{(\mu + \gamma)L}{\beta H^* L} + \frac{\beta H^* L}{(\mu + \gamma)L^*} \quad (8)$$

From (7), we have:

$$-1 + \frac{L}{L^*} + \frac{(\mu + \gamma)L}{\beta H^* L} + \frac{\beta H^* L}{(\mu + \gamma)L^*} = -1 - \frac{L}{L^*} + \frac{f(H, L^*)}{f(H, L)} + \frac{L}{L^*}\frac{f(H, L)}{f(H, L^*)}$$

Where $f(H, L) = \beta H$

$$= -1 - \frac{L}{L^*}\frac{\beta H L^*}{\beta H L} + \frac{\beta H L}{\beta H L^*}$$
$$= -1 + \frac{L}{L^*}[-1 + \frac{(\beta H L^*)^2 L + L^*}{(}\beta H L)^2 L(\beta H L)(\beta H L^*)]$$
$$= -1 + \frac{L^*(\beta L)^2 + L(\beta L^*)^2 - (\beta L)(\beta L^*}{\beta L^* L \beta L^*}$$
$$= \frac{-(L + L^*)\beta L \beta L^* + L^*(\beta L)^2 + (\beta L^*)^2 L}{L^* \beta L \beta L^*}$$
$$= 0$$

Since,

$$\begin{cases} \mu(H^* - H)(1 - \dfrac{H^*}{H}) \leqslant 0 \\ (\mu+\gamma)L^*(-1 - \dfrac{L}{L^*} - \dfrac{(\mu+\gamma)L}{H^*}(\beta L) + \dfrac{H^*\beta L}{(\mu+\gamma)L^*}) \leqslant 0 \\ \text{and } -(\mu+\gamma)L^*[\phi(\dfrac{H^*}{H}) + \phi(\dfrac{(\mu+\gamma)L}{H^*\beta L} + \phi(\dfrac{\beta LH}{(\mu+\gamma)L})] \leqslant 0 \end{cases}$$

We have $\dot{W}(t) \leqslant 0$. Thus, E^* is stable, and $\dot{W} = 0$ if and only if $H = H^*$ and $L = L^*$. From LaSalle invariance principle [5], we conclude that E^* is globally asymptotically stable.

4 Numerical simulations

In this section we show the numerical simulations and the graphs of system (1) to illustrate the different result obtained for each of the two cases $\varepsilon > 0$ and $\varepsilon = 0$ previously analyzed.

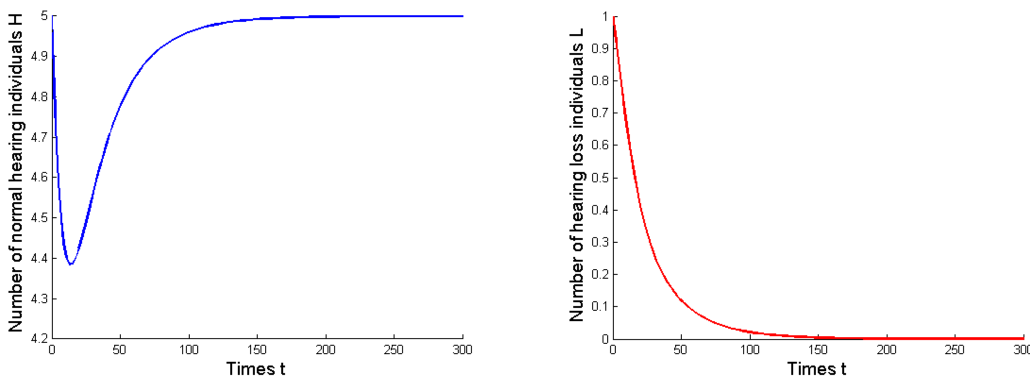

Figure 1: Normal hearing and loss of hearing individuals as function of time in the case of $R_0 < 1$.

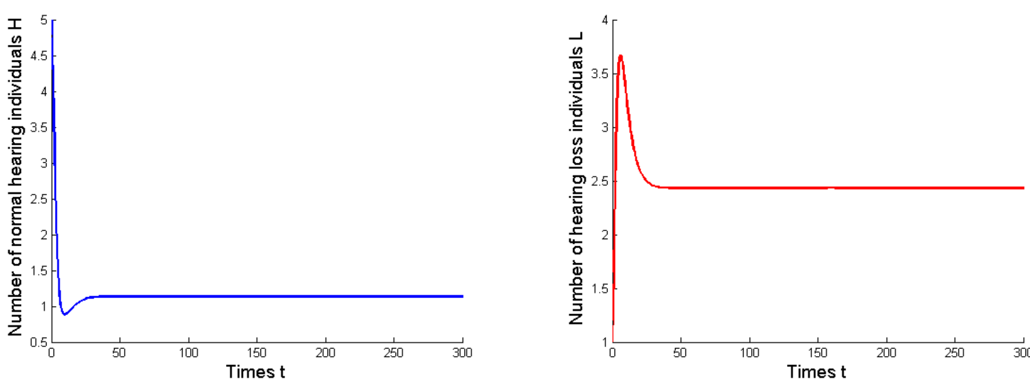

Figure 2: Normal hearing and loss of hearing individuals as function of time in the case of $R_0 > 1$.

We choose the following data set of system (1) as follows: $\Lambda = 0.5$, $\mu = 0.1$, and $\gamma = \frac{1}{17}$. By calculation, we have β=0.1433 for $R_0 > 1$ already defined for Mumps disease [6]. Therefore, according to Theorem 3.2 has a unique endemic equilibrium, the solution of system (1) is persistent in mean, this result is illustrated in Figure 1
In this case, the system (8) has a disease free equilibrium $E_f = (5.0)$ which is globally asymptotically stable when $R_0 < 1$. We have $R_0 = 4.4 > 1$ which satisfy Theorem 3.6, then the endemic $E_* = (1.1364, 2.4327)$ is globally asymptotically stable (see Figure 2).

5 Conclusion

In this paper, we have presented a mathematical model of hearing loss based on a nonlinear system of differential equations.
We analysis the hearing loss resulting from contagious factor due to Mumps disease.

By analysis the model, we have proved the existence, positivity and the boundedness of solutions of the problem, which implies that the model is well posed.
We have shown in the case of that the disease free equilibrium is globally asymptotically stable if the basic reproductive number $R_0 < 1$ and the endemic point is globally asymptotically stable when $R_0 > 1$, which means that the disease persists in the population.
The next stage of our research project is to analysis the model in the case of the hearing loss model caused by mumps(contagion factor) and noise exposure (social factor).

Conflict of Interests

The authors declares that there is no conflict of interest regarding the publication of this paper.

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