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(*Received*)

Submitted to the special issue: The Theory and Applications of Piecewise Smooth Dynamics

In this article we study the higher-order Moreau's sweeping process introduced in [1], in the case where an exogenous time-varying function $u(\cdot)$ is present in both the linear dynamics and in the unilateral constraints. First we show that the well-posedness results obtained in [1] for the autonomous case, extend to the non-autonomous case when $u(\cdot)$ is analytic, after a suitable state transformation is done. Stability issues are discussed. The complexity of such nonsmooth non-autonomous dynamical systems is illustrated in a particular case named the higher-order bouncing ball, where trajectories with accumulations of jumps are exhibited. Examples from mechanics and circuits illustrate some of the results. The link with complementarity dynamical systems and with switching differential-algebraic equations is made.

Key Words: 34A60,34A38,34H15,93D05,93C30

1 Introduction

The sweeping processes are a well-known class of differential inclusions, introduced by J.J. Moreau in [44, 45, 46], and which has had numerous extensions since then (see *e.g.* [4, 22, 23, 25, 35, 26, 39, 59] and references therein). The so-called *higher-order sweeping process* (HOSP) was introduced in [1]. The primary objective of the HOSP is to settle a dynamical formalism which provides a mathematical framework for a state or state-control unilaterally constrained dynamical system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + Eu(t) \\ 0 \leqslant w(t) = Cx(t) + Fu(t) \end{cases} \tag{1.1}$$

where $x(t) \in I\!R^n$ is the state, $u : I\!R_+ \mapsto I\!R^p$ is an input or exogenous disturbance, $\lambda(t) \in I\!R^m$ is a Lagrange multiplier, $w(t) \in I\!R^m$ is an output signal, $A \in I\!R^{n\times n}$, $B \in I\!R^{n\times m}$, $C \in I\!R^{m\times n}$, $E \in I\!R^{n\times p}$, $F \in I\!R^{m\times p}$. These systems are square because λ and w

necessarily have the same dimension, λ being a Lagrange multiplier associated with the constraint $w(t) \geqslant 0$ for all $t \geqslant 0$. In Control Theory language, there is an equal number of inputs and outputs. The complete analysis of the HOSP (including time-discretization) is done in the autonomous case (*i.e.* $u(\cdot) = 0$ in (1.1)) in [1] by embedding (1.1) into a specific *Distribution Differential Inclusion* (DDI) that is an extension of the second order sweeping process which is taylored to Lagrangian systems. The applications may be found in optimal control with state inequality constraints [16] as well as in feedback control of circuits [1, Section 6]. The analysis is also close to viability studies [6, 34] if one thinks of λ as an exogenous input (and not as a Lagrange multiplier associated with the inequality constraint). Within this framework the term $B\lambda + Eu(\cdot)$ may be considered as an input whose distributional part is λ while its function part is $u(\cdot)$.
The HOSP formalism indicates how to design λ to assure the positive invariance of the moving polyhedral set $\Phi_u \stackrel{\Delta}{=} \{x \in I\!R^n \mid Cx + Fu(t) \geqslant 0\}$. Consequently it is of interest to characterize in a precise way the nature of the solutions. The functional framework for the autonomous HOSP is carefully introduced in [1], where solutions are a subclass of Schwartz distributions constructed from functions of local special bounded variation. Whether or not the solutions possess accumulations of state jumps is an important feature. In the case $w(t) = Cx(t) + D\lambda(t)$ for some $\mathbf{P}$-matrix D and $u(\cdot) \equiv 0$, the results in [57] apply. Solutions are time continuous and Zenoness then corresponds to switching modes in the Linear Complementarity Problem (LCP): $0 \leqslant w(t) \perp \lambda(t) \geqslant 0$. Criteria that imply non-Zeno solutions are provided in [57]. In [19] a subclass of (1.1) is analyzed for absolutely continuous and of local bounded variation $u(\cdot)$, under the passivity condition that there exists $P = P^T > 0$ such that $PB = C^T$. The system is transformed into a first-order perturbed sweeping process [59], and solutions are either locally absolutely continuous, or of local bounded variation (in this case the dynamics is a Measure Differential Inclusion (MDI)). In this article it is shown with specific input functions $u(\cdot)$ that left-accumulations of state jumps (*i.e.* accumulations on the left of some time) may occur in the non-autonomous HOSP. The basic idea is to consider systems which are "higher order bouncing balls". More specifically, one may view them as a chain of integrators with a constant input, and a specific "impact law" which acts on the derivatives of the constrained signal $w(\cdot)$. In the mechanical bouncing ball system, $w(\cdot)$ is the continuous position of the ball whereas $\dot{w}(\cdot)$ is its discontinuous velocity.

This article is organized as follows. Section 2 is devoted to the analysis of a state transformation which allows to recast (1.1) into a suitable canonical form for the subsequent existence and uniqueness of solutions analysis. Section 3 recalls the HOSP framework in which (1.1) is embedded. The well-posedness of the non-autonomous HOSP is studied in section 4. The organization of the overall proof follows closely the one in [1, Section 4]. However since many proofs are the same, and since the well-posedness proof in [1] is quite long (about thirty pages), only the proofs which differ from those in [1] are presented. In section 5 the relationships between the HOSP and complementarity systems, as well as switching DAEs, are explained. In Section 6 we study a particular case of the non-autonomous HOSP (named the higher-order bouncing ball for obvious analogy with Mechanics) and we show that the so-called restitution coefficients play a crucial role in the dynamical behaviour. Section 7 deals with the existence of equilibria, stability and positive invariance. Conclusions end the article in section 8, and some auxiliary mathematics are in the Appendix.

Notations and definitions: The indicator function of a set $\Phi \subset I\!R^n$ is defined as $\psi_\Phi(x) = 0$ if $x \in \Phi$, $\psi_\Phi(x) = +\infty$ if $x \notin \Phi$. When Φ is closed, nonempty and convex, so is $\Psi_\Phi(\cdot)$ and its subdifferential $\partial\psi_\Phi : I\!R^n \rightrightarrows I\!R^n$ is a set-valued mapping, equal to the normal cone $N_\Phi(x)$ to Φ at x. We have $N_\Phi(x) = \{0\}$ if x is in the interior of Φ. A matrix M is positive definite ($M > 0$) if and only if $x^T M x > 0$ for all $x \neq 0$. The projection of a vector $x \in I\!R^n$ on Φ, in the metric defined by $M = M^T > 0$, is denoted as $\text{proj}_M[\Phi; x]$. The lexicographical inequality: $(x_1, x_2, ..., x_n) \succcurlyeq 0$ means that either all $x_i = 0$, or the first non zero $x_i > 0$. Let n and r be integers. The $n \times n$ identity matrix is I_n, $0^n = (0, 0, ..., 0)^T \in I\!R^{n\times 1}$, $0_n = (0, 0, ..., 0) \in I\!R^{1\times n}$, $0_{n\times r} \in I\!R^{n\times r}$ is the zero $n \times r$ matrix. Let M be a matrix with n rows, then $M_r \in I\!R^{r\times n}$ denote the first r rows of M, M_{n-r} its last $n - r$ rows. For a matrix M, $M_{i\bullet}$ is its ith row, $M_{\bullet i}$ is its ith column. A square matrix $M \in I\!R^{n\times n}$ is a Stieltjes matrix if [24, Definition 3.11.1] [12, Definition 2.5, p.141]: $M = M^T$, $M_{ij} \geqslant 0$ for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $i \neq j$, and M is a P-matrix (hence it is positive definite since it is symmetric). If M is a Stieltjes matrix, then M^{-1} is symmetric non-negative, *i.e.* the entries $(M^{-1})_{ij}$ are non-negative for all i, j. For a square matrix M, $\lambda_{\max}(M)$ denotes its largest eigenvalue, $\lambda_{\min}(M)$ denotes its smallest eignevalue.

2 State transformation into a canonical form

The analysis of the autonomous HOSP with $u(\cdot) = 0$ is made in [1] from a specific state-space representation which allows to settle a suitable functional set of potential solutions which are Schwartz' distributions. In this section we show how to extend this canonical transformed dynamics for (1.1), which will be useful for the well-posedness analysis. Let $m = 1$ and let the transfer function $C(sI_n - A)^{-1}B \neq 0$, $s \in \mathbb{C}$. Then there exists $1 \leqslant r \leqslant n$ that is the relative degree between w and λ. In other words $CA^{i-1}B = 0$ for all $1 \leqslant i \leqslant r - 1$ and the scalar $CA^{r-1}B \neq 0$. Let us assume that $u(\cdot)$ is r-times differentiable, and let us denote $\mathcal{U}(t) = (u(t)^T, \dot{u}(t)^T, ..., u^{(r-1)}(t)^T)^T \in I\!R^{rp}$, and $\mathcal{W}(t) = (u(t)^T, \dot{u}(t)^T, ..., u^{(r)}(t)^T)^T \in I\!R^{(r+1)p}$. Let us perform the extended state transformation[1]

$$z = Wx + T\mathcal{U} \tag{2.1}$$

with $z = \begin{pmatrix} \bar{z} \\ \xi \end{pmatrix}$, $\bar{z} = (z_1, z_2, ..., z_r)^T$, $\xi \in I\!R^{n-r}$, and where

$$z_i(t) = CA^{i-1}x(t) + \sum_{j=0}^{i-2} CA^j E u^{(i-2-j)}(t) + Fu^{(i-1)}, \tag{2.2}$$

with $2 \leqslant i \leqslant r$, $z_1(t) = w(t) = Cx(t) + Fu$. Notice that $\dot{z}_i = z_{i+1}$, $1 \leqslant i \leqslant r - 1$. Due to the existence of a relative degree between w and λ, there exists a matrix $W \in I\!R^{n\times n}$ which is full-rank and such that [54]:

[1] "Extended" refers here to the fact that this transformation involves both the state and the exogenous term.

$$WB = \begin{pmatrix} 0^{r-1} \\ CA^{r-1}B \\ 0^{n-r} \end{pmatrix} \in I\!R^n, \quad CW^{-1} = \begin{pmatrix} 1 & 0_{n-1} \end{pmatrix} \in I\!R^{1\times n} \tag{2.3}$$

$$WAW^{-1} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0_{n-r} \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & \dots & 0 & 1 & 0_{n-r} \\ d_1 & d_2 & d_3 & \dots & d_r & d_\xi^T \\ B_\xi & 0^{n-r} & 0^{n-r} & \dots & 0^{n-r} & A_\xi \end{pmatrix}, \tag{2.4}$$

where $A_\xi \in I\!R^{(n-r)\times(n-r)}, B_\xi \in I\!R^{(n-r)\times 1}$, and $(d^T, d_\xi^T) = (CA^rW^{-1})^T$ with $d = (d_1, ..., d_r)^T$. Moreover from the definition of the variables z_i in (2.2) we have:

$$T = \begin{bmatrix} F & 0_p & 0_p & \dots & \dots & 0_p & 0_p \\ CE & F & 0_p & \dots & 0_p & 0_p & 0_p \\ CAE & CE & F & \dots & 0_p & 0_p & 0_p \\ \vdots & & & & & \vdots & \vdots \\ \vdots & & & & & F & 0_p \\ CA^{r-2}E & CA^{r-3}E & \dots & \dots & CAE & CE & F \\ 0_{(n-r)\times p} & 0_{(n-r)\times p} & \dots & \dots & \dots & 0_{(n-r)\times p} & 0_{(n-r)\times p} \end{bmatrix} \in I\!R^{n\times rp}. \tag{2.5}$$

The matrix W being full-rank, the state transformation is bijective and $x = W^{-1}(z - T\mathcal{U})$. This allows one to transform (1.1) into the following canonical form (which differs from similar forms [55, Equation (9.91)] that would be useless for the well-posedness study that is done in the following):

$$\begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = z_3(t) \\ \vdots \\ \dot{z}_{r-1}(t) = z_r(t) \\ \dot{z}_r(t) = CA^rW^{-1}z(t) - CA^rW^{-1}T\mathcal{U}(t) + CA^{r-1}B\lambda + \sum_{i=0}^{r-1} CA^iEu^{(r-1-i)}(t) + Fu^{(r)} \\ \dot{\xi}(t) = A_\xi\xi(t) + B_\xi z_1(t) + G_\xi\mathcal{U}(t) \\ \\ 0 \leqslant w(t) = z_1(t) \end{cases} \tag{2.6}$$

for some matrix $G_\xi \in I\!R^{(n-r)\times rp}$ such that $G_\xi\mathcal{U}(t) = -W_{n-r}AW^{-1}T\mathcal{U}(t) + W_{n-r}Eu(t) + T_{n-r}\dot{\mathcal{U}}(t)$. This stems from the fact that $\dot{\xi} = W_{n-r}\dot{x} + T_{n-r}\dot{\mathcal{U}}$. Noting that the last $n-r$ lines of T satisfy $T_{n-r} = 0$ we obtain $\dot{\xi} = W_{n-r}AW^{-1}z - W_{n-r}AW^{-1}T\mathcal{U} + W_{n-r}B\lambda + W_{n-r}Eu$, and using (2.3) and (2.4) yields the result (in particular $W_{n-r}B = 0_{(n-r)\times m}$). In Systems and Control the $\xi-$dynamics is called the *zero dynamics*. As we shall see later, the zero-dynamics plays a crucial role in the system's behaviour when trajectories evolve on the boundary of Φ_u. Notice that adding the term $T\mathcal{U}$ in the state transformation is necessary to obtain the chain of integrators in (2.6), but the zero-dynamics depends in general on $u(\cdot)$ and its derivatives. Let us denote $\sum_{i=0}^{r-1} CA^i E u^{(r-1-i)}(t) + F u^{(r)} - CA^r W^{-1} T\mathcal{U}(t) = \bar{G}\mathcal{W}(t)$ for a suitable constant row vector $\bar{G} \in I\!R^{1\times(r+1)}$. One can then rewrite the $r-$th line of (2.6) as

$$\dot{z}_r(t) = CA^r W^{-1} z(t) + CA^{r-1}B\lambda + \bar{G}\mathcal{W}(t) \;\; (= d^T \bar{z} + d_\xi^T \xi + CA^{r-1}B\lambda + \bar{G}\mathcal{W}(t)), \quad (2.7)$$

where $\bar{G} \in I\!R^{1\times(r+1)p}$, and the dynamics in (2.6) more compactly as

$$\dot{z}(t) = WAW^{-1}z(t) + WB\lambda(t) + H\mathcal{W}(t) \quad (2.8)$$

where $H \in I\!R^{n\times(r+1)p}$ is a suitable matrix obtained from $\bar{G}\mathcal{W}$ and $W_\xi \mathcal{U}$ by grouping the terms of equal derivation index in $\mathcal{U}$ and $\mathcal{W}$. Obviously we may also calculate (2.8) from

$$\dot{z}(t) = WAW^{-1}z(t) + WB\lambda(t) + WEu(t) + T\dot{\mathcal{U}}(t) - WAW^{-1}T\mathcal{U}(t) \quad (2.9)$$

so that $H\mathcal{W} = WEu + T\dot{\mathcal{U}} - WAW^{-1}T\mathcal{U}$. The expression in (2.7) will be used later on. Notice that one may have $T = 0$ but this does not mean that the transformed dynamics is independent of $u(\cdot)$. We still assume that $m = 1$ and that $e_i \geqslant 0$, $1 \leqslant i \leqslant r$.

Example 2.1

$$\left\{\begin{array}{l} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) + u(t) \\ \dot{x}_3(t) = \lambda(t) \\ 0 \leqslant w(t) = x_2(t) + u(t) \end{array}\right. \iff \left\{\begin{array}{l} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = \lambda(t) + \dot{u}(t) + \ddot{u}(t) \\ \dot{\xi}(t) = z_1(t) - u(t) \\ 0 \leqslant z_1(t) \end{array}\right. \quad (2.10)$$

In this example one has $n = 3$, $r = 2$, $p = 1$, $W = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$, $G_\xi = (-1, 0)$.

Example 2.2 *Let us consider the system with* $n = 4$, $p = 1$,

$$A = \begin{pmatrix} 2 & 7 & 3-2\alpha & -2\beta+2 \\ -1 & -3 & -1+\alpha & \beta-1 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix},$$

where $\alpha, \beta \in I\!R$, $F = 0$ *and* $E = (1\ 1\ 1\ 1)^T$. *We have* $CB = 0$ *and* $CAB = -7$, *hence* $r = 2$. *The transformed dynamics may be obtained from (2.9) as:*

$$\left\{\begin{array}{l} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = \lambda(t) - z_1(t) - z_2(t) + \alpha\xi_1(t) + \beta\xi_2(t) + (5-2\alpha)u(t) + 3\dot{u}(t) \\ \dot{\xi}_1(t) = \xi_2(t) + u(t) + 2\dot{u}(t) \\ \dot{\xi}_2(t) = \xi_1(t) + z_1(t) + 3u(t) \\ 0 \leqslant w(t) = z_1(t) \end{array}\right. \quad (2.11)$$

In this example one has $W = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $T = (0\ 3\ 2\ 0)^T$. *The derivative of* $u(\cdot)$ *enters the* $\xi-$*dynamics, which in turn enters the* $\bar{z}$*-dynamics in* $\dot{z}_2(t)$.

Other examples can be found in section 5.3. Notice that if $CA^r = 0$ and $CA^iE = 0$ for all $0 \leqslant i \leqslant r-1$, then the canonical z-dynamics in (2.6) is the z-dynamics obtained in the autonomous case, except for the zero-dynamics' state $\xi(\cdot)$. For instance in (2.10) one has $CA^2 = 0$. It is worth remarking that the $\xi-$dynamics may enter the $\bar{z}$-dynamics as shown in (2.11). Therefore the well-posedness analysis has to be made from the most general case in (2.6).

3 Embedding into the HOSP

In this section we briefly recall the mathematical tools and formalisms which are necessary to construct the Higher Order Sweeping Process (HOSP), a particular Distribution Differential Inclusion (DDI). First the set of solutions is introduced, then the DDI, the state-jump mapping and the link with complementarity are presented. The complete developments can be found in [1, §2, 3, 4].

3.1 The space of distributional solutions

Let $I = [\alpha, \beta]$, $\alpha \in I\!R$, $\beta \in I\!R \cup \{+\infty\}$, be a real nondegenerate interval. We denote as $\mathcal{T}_n(I)$ the set of distributions of degree $n+1$ which are generated by RCSLBV functions on I (see (C 1) and (C 2) in Appendix C), whose successive derivatives possess an absolutely continuous part (denoted as $[\cdot]$) that is also RCSLBV on I. More precisely the right derivative of the absolutely continuous component $[h]$ of $h(\cdot)$, is denoted as $\hat{h}^{(1)} \triangleq \frac{d^+[h]}{dt}(t) = \lim_{\sigma \to 0^+} \frac{[h](t+\sigma)-[h](t)}{\sigma}$. The set of such functions is denoted as $\mathcal{F}_\infty(I; \mathbb{R}) = \cap_{k \in \mathbb{N}} \mathcal{F}_k(I; \mathbb{R})$, with

$$\mathcal{F}_k(I; \mathbb{R}) = \{h \in \mathcal{F}_{k-1}(I; \mathbb{R}) \mid \hat{h}^{(k)} \triangleq \frac{d^+}{dt}[\hat{h}^{(k-1)}] \in RCSLBV(I; \mathbb{R})\}. \quad (3.1)$$

In particular $\mathcal{F}_0(I; \mathbb{R}) = RCSLBV(I; \mathbb{R})$, and $\mathcal{F}_1(I; I\!R) = \{h \in \mathcal{F}_0(I; I\!R) \mid \hat{h}^{(1)} \in RCSLBV(I; I\!R)\}$. Furthermore $\hat{h}^{(1)} = [\hat{h}^{(1)}] + \mathcal{J}_{\hat{h}^{(1)}}$ (see (C 1)), and $\mathcal{F}_2(I; I\!R) = \{h \in \mathcal{F}_1(I; I\!R) \mid \hat{h}^{(2)} \triangleq \frac{d^+}{dt}[\hat{h}^{(1)}] \in RCSLBV(I; I\!R)\}$.

If the distribution $T \in \mathcal{T}_n(I)$ and is generated by a function $F \in \mathcal{F}_\infty(I; \mathbb{R})$, it has a "function" part denoted as $\{T\}(\cdot) = [\hat{F}^{(n)}](\cdot)$, and a "measure" part denoted as $\ll T \gg$ such that $\langle \ll T \gg, \varphi \rangle = \int_{-\infty}^{+\infty} \varphi \; d[\hat{F}^{(n-1)}]$, for all $\varphi \in C_0^\infty(I)$. $\mathbf{D}$ denotes the distributional derivative, and dz denotes the Stieltjes or differential measure generated by a function z of local bounded variation (see Appendix C), while $C_0^\infty(I)$ is the space of real-valued $C^\infty(I)$ mappings with compact support contained in $]\alpha, \beta[$. Thus given $n \in I\!N$, $\mathcal{T}_n(I)$ denotes the set of all Schwartz' distributions such that there exists a function $F \in \mathcal{F}_\infty(I; \mathbb{R})$ such that $T = \mathbf{D}^n F$. We have $\mathcal{T}_0(I) = \mathcal{F}_\infty(I; I\!R)$. Let n be the smallest integer such that $T \in \mathcal{T}_n(I)$, we set the degree of T as:

$$\deg(T) = \begin{cases} n+1 \text{ if } n \geqslant 1 \\ 1 \text{ if } n=0 \text{ and } E_0(\{T\}) \neq \emptyset \\ 0 \text{ if } n=0 \text{ and } E_0(\{T\}) = \emptyset \end{cases} \tag{3.2}$$

where $E_0(f)$ denotes the set of points of discontinuity of the function f. Distributions of degree $n = 0$ are continuous functions in $\mathcal{F}_\infty(I; \mathbb{R})$, those of degree $n = 1$ are discontinuous functions in $\mathcal{F}_\infty(I; \mathbb{R})$. The nth derivative of the Dirac measure, $\delta_t^{(n)}$ $(t \in I)$ is of degree $n + 2$. This concept of solutions is an extension of the case of Nonsmooth Mechanics, where positions are locally absolutely continuous, velocities are RCLBV, accelerations are the differential measures of the velocities [39].

Finally let us define the set $\mathcal{T}_\infty(I) = \cup_{n\in\mathbb{N}}\mathcal{T}_n(I)$: a Schwartz distribution belongs to $\mathcal{T}_\infty(I)$ if there exist $n \in \mathbb{N}$ and $F \in \mathcal{F}_\infty(I; \mathbb{R})$ such that $T = \mathbf{D}^n F$. This set contains Bohl distributions which are used elsewhere for the analysis of Linear Complementarity Systems [30]. Within this functional framework for solutions, the system's state is allowed to be discontinuous with accumulations of discontinuity times, however the set of state-jump times is countable. For examples of functions in $\mathcal{F}_\infty(I; \mathbb{R})$ and distributions in $\mathcal{T}_\infty(I)$, see [1, Examples 1, 2, 3].

3.2 The Distribution Differential Inclusion

Let us first recall that in order to simplify the presentation we shall continue to assume that $m = 1$. In [1, Remarks 3, 7, 20] it is indicated how the material extends to the multivariable (MIMO) case $m \geqslant 2$, when $CA^{r-1}B$ is a Stieltjes matrix. We shall come back on the MIMO case in section 5.2. Let K be a nonempty closed convex subset of $\mathbb{R}$. We denote by $T_K(x)$ the tangent cone of K at $x \in \mathbb{R}$ defined by

$$T_K(x) = \overline{\text{cone}}(K - \{x\}) \tag{3.3}$$

where $\text{cone}(K - \{x\})$ denotes the cone generated by $K - \{x\}$ and $\overline{\text{cone}}(K - \{x\})$ denotes the closure of $\text{cone}(K - \{x\})$, *i.e.* $\overline{\text{cone}}(K - \{x\}) = \overline{\text{cone(K} - \{\text{x}\})}$. The definition in (3.3) allows us to take into account constraints violations (which may occur during the state re-initialization, see Proposition 4.1 below). Note that $T_{\mathbb{R}_+}(x) = \begin{cases} \mathbb{R} & \text{if } x > 0 \\ \mathbb{R}_+ & \text{if } x \leqslant 0 \end{cases}$ and $T_{\mathbb{R}}(x) = \mathbb{R}$. Let us now set $\Phi \triangleq \mathbb{R}_+$. For $z \in \mathbb{R}^r$, we set $Z_i = (z_1, z_2, ..., z_i)$, $(1 \leqslant i \leqslant r)$. By convention, we set $Z_0 = 0$ and $T_\Phi^0(Z_0) = \Phi$, and we define $T_\Phi^1(Z_1) = T_\Phi(z_1)$, $T_\Phi^2(Z_2) = T_{T_\Phi^1(Z_1)}(z_2)$, $T_\Phi^r(Z_r) = T_{T_\Phi^1(Z_{r-1})}(z_r)$ (so that $Z_r = \bar{z}$). To summarize:

$$T_\Phi^i(Z_i) = T_{T_\Phi^{i-1}(Z_{i-1})}(z_i), \text{ for all } 1 \leqslant i \leqslant r. \tag{3.4}$$

It follows that $z_1 > 0 \Rightarrow T_\Phi^i(Z_i) = \mathbb{R}$ and $\partial\psi_{T_\Phi^i(Z_i)}(\cdot) = \{0\}$ for all $1 \leqslant i \leqslant r$. Moreover assume that there exists $j \in \{1, r-1\}$ such that $z_1 \leqslant 0$, ..., $z_j \leqslant 0$, and $z_{j+1} > 0$. Then $T_\Phi^0(Z_0) = T_\Phi^1(Z_1) = ... = T_\Phi^j(Z_j) = \mathbb{R}_+$, and $T_\Phi^{j+1}(Z_{j+1}) = ... = T_\Phi^{r-1}(Z_{r-1}) = \mathbb{R}$. Let us finally remind that $\partial\psi_{\mathbb{R}}(x) = \{0\}$ for all reals x, $\partial\psi_{\mathbb{R}_+} = \{0\}$ if $x > 0$, $\partial\psi_{\mathbb{R}_+}(0) = \mathbb{R}^-$. We now pass to the DDI. Let $T > 0$, $T \in \mathbb{R} \cup \{+\infty\}$ be given and set $I = [0, T[$. Let $z_0^T = (\bar{z}_0^T, \xi_0^T)$ be given in $\mathbb{R}^n$ with $\bar{z}_0 \in \mathbb{R}^r$ and $\xi_0 \in \mathbb{R}^{n-r}$. We also introduce a set

$(e_1, ..., e_r)$ of r real numbers named *restitution coefficients* from an obvious analogy with Mechanics. The choice of these coefficients depends on the application (for instance in Mechanics the restitution applied to the velocity belongs to $[0, 1]$ because of the kinetic energy dissipation and kinematic consistency [18]). Let us denote

$$\zeta_i(t) = \frac{\{z_i\}(t^+) + e_i\{z_i\}(t^-)}{1 + e_i}, \quad 1 \leqslant i \leqslant r. \tag{3.5}$$

The DDI formalism is as follows:

Problem HOSP(z_0;I): Find $z_1, ..., z_r \in \mathcal{T}_\infty(I)$ and $\xi_i \in \mathcal{T}_\infty(I)$ $(1 \leqslant i \leqslant n - r)$, satisfying the distributional differential equations:

$$\begin{cases} \mathbf{D}z_1 - z_2 = 0 \\ \mathbf{D}z_2 - z_3 = 0 \\ \mathbf{D}z_3 - z_4 = 0 \\ \vdots \\ \mathbf{D}z_{r-1} - z_r = 0 \\ \mathbf{D}z_r - CA^rW^{-1}\{z\} - \bar{G}\mathcal{W} = CA^{r-1}B\lambda \\ \mathbf{D}\xi = A_\xi\xi + B_\xi z_1 + G_\xi\mathcal{U}, \end{cases} \tag{3.6}$$

$$\lambda = (CA^{r-1}B)^{-1}\left[\sum_{i=1}^{r-1}\mathbf{D}^{(r-i)} \ll \mathbf{D}z_i - \{z_{i+1}\} \gg\right] + \ll \mathbf{D}z_r - CA^rW^{-1}\{z\} \gg -(CA^{r-1}B)^{-1}\bar{G}\mathcal{W}, \tag{3.7}$$

and satisfying the Measure Differential Inclusion (MDI) on $]0, T[$:

$$\begin{cases} d\{z_1\} - \{z_2\}(t)dt \in -\partial\psi_\Phi(\zeta_1(t)), \\ d\{z_2\} - \{z_3\}(t)dt \in -\partial\psi_{T^1_\Phi(\{Z_1\}(t^-))}(\zeta_2(t)), \\ \vdots \\ d\{z_{r-1}\} - \{z_r\}(t)dt \in -\partial\psi_{T^{r-2}_\Phi(\{Z_{r-2}\}(t^-))}(\zeta_{r-1}(t)), \\ (CA^{r-1}B)^{-1}[d\{z_r\} - CA^rW^{-1}\{z\}(t)dt - \bar{G}\mathcal{W}dt] \in -\partial\psi_{T^{r-1}_\Phi(\{Z_{r-1}\}(t^-))}(\zeta_r(t)), \end{cases} \tag{3.8}$$

and the initial conditions:

$$\begin{cases} \{z_1\}(0^+) - z_1(0^-) \in -\partial\psi_\Phi(\zeta_1(0)), \\ \{z_2\}(0^+) - z_2(0^-) \in -\partial\psi_{T^1_\Phi(Z_1(0^-))}(\zeta_2(0)), \\ \vdots \\ \{z_{r-1}\}(0^+) - z_{r-1}(0^-) \in -\partial\psi_{T^{r-2}_\Phi(Z_{r-2}(0^-))}(\zeta_{r-1}(0)), \\ (CA^{r-1}B)^{-1}[\{z_r\}(0^+) - z_r(0^-)] \in -\partial\psi_{T^{r-1}_\Phi(Z_{r-1}(0^-))}(\zeta_r(0)), \\ \{\xi\}(0^+) = \xi_0. \end{cases} \tag{3.9}$$

The rationale behind the expression of λ in (3.7), is that this is a distribution whose degree depends on which of the state components z_i, $1 \leqslant i \leqslant r$, are discontinuous. The fact that the state-space representation uses a chain of integrators, explains the term between brackets in (3.7). For instance, a jump in z_1 will propagate through the differentiations and induce a distribution of degree 2 in z_2 (a Dirac measure), or degree 3 in z_3, of degree r in z_r, and finally of degree $r + 1$ in λ. For more details see [1, Equ.

(32)-(40), Example 7]. In the MDI formalism (3.8), one considers only the measure parts of the distributions in order to give a meaning to the inclusions into normal cones. In view of (3.6) (3.7) and (3.8) it is legitimate to name the HOSP a *distribution differential inclusion.*

The rationale behind the choice for the normal cones to the tangent cones in the right-hand sides of the differential inclusions in (3.8), is that this guarantees, as we will see in the next sections, that any solution of the HOSP satisfies $z_1(t)(= (w(t) = Cx(t) + Fu(t)) \geqslant 0$ for all $t > 0$ (*i.e.* except possibly initially on the left of $t = 0$) even if some of the derivatives of $z_1(\cdot)$ tend to make it leave this admissible domain. One can view this in the n-dimensional state space, as trajectories "grazing" the admissible domain boundary $w = 0$ with a certain degree of tangency (that corresponds to the number of null derivatives of $z_1(\cdot)$): the selections inside the normal cones as defined in (3.8) secure that if the first non-zero derivative has a negative sign on the left of t, then it jumps to a non negative value on the right of t. Actually, the construction of the normal cones sequence imposes a lexicographical inequality on $\{\bar{z}\}(t^+)$, see section 3.3 and Proposition 4.1. The fact that the measures $d\nu_i \triangleq d\{z\}_i - \{z_{i+1}\}dt$, $1 \leqslant i \leqslant r-1$, do not appear in the right-hand side of (2.8) (only λ does, see WB in (2.3)) stems from the fact that these measures are present in the MDI formalism only to take into account state re-initializations (see also (4.3) (4.4) below).

The set-valued functions in the right-hand side of (3.9) naturally extend the second order sweeping process right-hand side (called Moreau's set [18]), hence it is justified to name (3.6)-(3.9) a higher-order sweeping process, though higher-relative-degree sweeping process could be more appropriate.

Remark 3.1 *If z denotes a solution of Problem* $\mathbf{HOSP}(z_0;I)$, *we will write by convention that*

$$\{\bar{z}\}(0^-) = \bar{z}_0,\ \{\xi\}(0^-) = \xi_0.$$

Then the relations in (3.8) formulated on $]0,T[$ together with the initial conditions in (3.9) reduce to the relations in (3.8) formulated on $I = [0,T[$. Moreover, recalling that ξ is here necessarily a continuous function, the last condition in (3.9) reads $\xi(0) = \xi_0$, and with our convention, we see that the last relation in (3.6) together with the initial condition (3.9) reduce to the measure differential equation: $d\xi - (A_\xi \xi(t) + B_\xi z_1(t) + G_\xi \mathcal{U}(t))dt = 0$ on I.

The solutions of the Problem $\mathbf{HOSP}(z_0;I)$ are distributions of a certain degree as the next proposition shows.

Proposition 3.1 *[*1, Proposition 3*] Let $(z_1,...,z_r,\xi) \in (\mathcal{T}_\infty(I))^n$ be a solution of Problem* $\mathbf{HOSP}(z_0;I)$. *Then*

$$deg(z_i) \leqslant i\ (1 \leqslant i \leqslant r),$$

$$z_1 = \{z_1\} \in \mathcal{F}_\infty(I;\mathbb{R}),\ \xi = \{\xi\} \in (\mathcal{F}_\infty(I;\mathbb{R}))^{n-r} \cap (C^0(I;\mathbb{R}))^{n-r}$$

The proof follows the same arguments as in [1, Example 1] and it is intuitively clear from (3.6) since $z_2 = Dz_1$, $z_3 = Dz_2 = D^2 z_1$, *etc*, while z_1 is a time function (possibly discontinuous).

3.3 The state jump mapping

Another way to write the MDI in (3.8) is as follows: find $z_1, ..., z_r, \xi_1, ..., \xi_{n-r} \in \mathcal{F}_\infty(I; I\!R)$ such that

$$\left\{\begin{array}{l} dz_1 = z_2(t)dt + d\nu_1 \\ dz_2 = z_3(t)dt + d\nu_2 \\ \vdots \\ dz_{r-1} = z_r(t)dt + d\nu_{r-1} \\ dz_r = CA^r W^{-1} z(t)dt + \bar{G}\mathcal{W}(t)dt + CA^{r-1}B\; d\nu_r \\ \\ d\xi = (A_\xi \xi(t) + B_\xi z_1(t) + G_\xi \mathcal{U}(t))dt \end{array}\right. \tag{3.10}$$

where $d\nu_i$ denotes Radon measures, dz_i $(1 \leqslant i \leqslant r)$ is the differential measure generated by z_i, and the measures $d\nu_i$ satisfy the inclusions (see (3.8)):

$$d\nu_i \in -\partial \psi_{T_\Phi^{i-1}(\{Z_{i-1}\}(t^-))}(\zeta_i(t)) \quad \text{on } I, \quad (1 \leqslant i \leqslant r), \tag{3.11}$$

Roughly speaking, we retain only the measure part of the DDI, and the multiplier λ is replaced in the MDI formalism by the measure $d\nu_r$. It makes sense then to write inclusions into normal cones as in (3.8) or (3.11) since measures are signed while distributions are not (see [1, Section 2] for the rigorous mathematical meaning of the inclusions in (3.11)). The relationship between λ and $d\nu_r$ is further understood by the fact that outside the atoms of the measures $d\nu_i$, $1 \leqslant i \leqslant r$, each measure $d\nu_i = \chi_i(t)dt$ for some function $\chi_i \in \mathcal{F}_\infty(I; I\!R)$, and $\lambda = \chi_r(t)$.

The following holds [1, Propositions 4 and 5, Remark 15 iii)].

Proposition 3.2 *Let $m = 1$ and $CA^{r-1}B > 0$. Let z be a solution of the Problem* **HOSP**$(z_0; I)$ *in (3.6)-(3.9). One has*

$$\left\{\begin{array}{lll} \{z_i\}(t^+) - \{z_i\}(t^-) \in -\partial \psi_{T_\Phi^{i-1}(\{Z_{i-1}\}(t^-))}(\zeta_i(t)) & \textit{for all} & 1 \leqslant i \leqslant r-1 \\ \\ \{z_r\}(t^+) - \{z_r\}(t^-) \in -(CA^{r-1}B)\, \partial \psi_{T_\Phi^{r-1}(\{Z_{r-1}\}(t^-))}(\zeta_r(t)) & \textit{for} & i = r \end{array}\right. \tag{3.12}$$

if and only if

$$\{z_i\}(t^+) = -e_i \{z_i\}(t^-) + (1 + e_i)\text{proj}\left[T_\Phi^{i-1}(\{Z_{i-1}\}(t^-)); \{z_i\}(t^-)\right]. \tag{3.13}$$

Moreover:

$$d\nu_i(\{t\}) = d\{z_i\}(\{t\}) - \{z_{i+1}\}(t)dt(\{t\}) = d\{z_i\}(\{t\}) = \{z_i\}(t^+) - \{z_i\}(t^-)$$

for all $1 \leqslant i \leqslant r-1$, and

$$\begin{aligned} (CA^{r-1}B)\; d\nu_r(\{t\}) &= d\{z_r\}(\{t\}) - CA^r W^{-1}\{z\}(t)dt(\{t\}) = d\{z_r\}(\{t\}) \\ &= \{z_r\}(t^+) - \{z_r\}(t^-). \end{aligned} \tag{3.14}$$

It follows that if $T_\Phi^i(\{Z_i\}(t^-)) = I\!R_+$ and if $\{z_{i+1}\}(t^-) < 0$, then $\{z_{i+1}\}(t^+) = -e_{i+1}\{z_{i+1}\}(t^-)$ so that sign($\{z_{i+1}\}(t^+)$) =sign(e_{i+1}). We infer that $\{z_{i+1}\}(t^-) \geqslant 0 \Rightarrow e_{i+1} \geqslant 0$: only non-negative coefficients e_i bring the trajectories back in the admissible domain when a grazing trajectory tends to leave it. We therefore *choose* in the following $e_i \geqslant 0$ for all $1 \leqslant e_i \leqslant r$ (we shall see in section 6 that further bounds may be imposed on e_i to better characterize the system's behaviour, see Lemma 6.1). In the case $m = 1$, both expressions in (3.12) are the same as long as $CA^{r-1}B > 0$, however when $m \geqslant 2$ the expression for $i = r$ is crucial.
As alluded to above, in the HOSP formalism the positivity of the multiplier λ, that makes sense only if λ is a measure, is replaced by the positivity of the measure $d\nu_r$, which is the "measure part" of the distribution λ. The definition of $d\nu_r$ is clear from (3.14): it is defined from the discontinuity in the function part of z_r.

Remark 3.2 *The relations given in (3.10) (3.11), formulated on the interval I, have to be interpreted in the following sense: Find nonnegative real-valued Radon measures $d\mu_i$ $(1 \leqslant i \leqslant r)$ relative to which the Lebesgue measure dt and the Stieltjes measure dz_i possess densities $\frac{dt}{d\mu_i}$ and $\frac{dz_i}{d\mu_i}$ respectively such that $d\mu_i$-a.e. $t \in I$:*

$$\frac{dz_i}{d\mu_i}(t) - z_{i+1}(t)\frac{dt}{d\mu_i}(t) \in -\partial\psi_{T_\Phi^{i-1}(Z_{i-1}(t^-))}(\zeta_i(t)), \quad (1 \leqslant i \leqslant r-1) \tag{3.15}$$

and

$$(CA^{r-1}B)^{-1}\left[\frac{dz_r}{d\mu_r}(t) - (CA^rW^{-1}z(t) - \bar{G}\mathcal{W}(t))\frac{dt}{d\mu_r}(t)\right] \in -\partial\psi_{T_\Phi^{r-1}(Z_{r-1}(t^-))}(\zeta_r(t)). \tag{3.16}$$

Also, a convenient way to see the relationships between solutions of the MDI (3.10) (3.11) and solutions of the DDI is as follows [1, Proposition 3]: Let $(w_1, ..., w_r, \xi) \in (\mathcal{F}_\infty(I; I\!R))^n$ be a solution of the MDI such that, for each $1 \leqslant i \leqslant r-1$, the measure $dw_i - w_{i+1}dt$ is atomic. Let $z_1, ..., z_r$ be defined by

$$z_1 \stackrel{\Delta}{=} w_1$$

and

$$z_i \stackrel{\Delta}{=} w_i + \sum_{j=1}^{i-1}\Big(\sum_{t_k \in E_0(w_j)} (w_j(t_k^+) - w_j(t_k^-))\delta_{t_k}^{(i-j-1)}\Big)\ ,\ 2 \leqslant i \leqslant r.$$

where $E_0(w_j)$ is the set of points of discontinuity of the function w_j. Then $(z_1, ..., z_r, \xi) \in (\mathcal{T}_\infty(I))^n$ and is a solution of Problem **HOSP**$(z_0; I)$*. This allows to see that the distributional part of the solutions is a consequence of state jumps, and that the solutions of the DDI and of the MDI differ only at the instants of state re-initialization (which make a countable set due to the fact that solutions of the MDI are in $\mathcal{F}_\infty(I; I\!R)$).*

4 Existence and uniqueness of solutions

The existence and uniqueness of solutions to the autonomous HOSP is shown in [1, §4.4, 4.6]. As alluded to in the introduction, on one hand the complete proof is rather long, on the other hand it happens that the proof of several key results is not changed when we consider the non-autonomous case. Therefore in this well-posedness proof we consider only the results of [1, §4.3, 4.4, §4.6] which need to be modified in order to comply with the non-autonomous case and $e_i > 0$ (only the case $e_i = 0$ is treated in [1]). We still assume that $m = 1$, keeping in mind that all the results still hold in the multivariable case $m \geqslant 2$, provided that the vector relative degree is $\bar{r} = (r, r, ..., r)^T$ and the so-called decoupling matrix $CA^{r-1}B$ is a Stieltjes matrix. We shall give more details on the case $m \geqslant 2$ in section 5.2.

Definition 4.1 Let $0 \leqslant a < b \leqslant T \leqslant +\infty$ be given. We say that a solution $z \in (\mathcal{T}_\infty([0,T[))^n$ of Problem $\mathbf{HOSP}(z_0;[0,T[)$ is regular on $[a,b[$ if for each $t \in [a,b[$, there exists a right neighborhood $[t, t+\sigma[$ $(\sigma > 0)$ such that the restriction of $\{z\}$ to $[t, t+\sigma[$ is analytic.

As we shall see regular solutions do not hamper the existence of Zeno behaviour with possible left accumulations of state jump times. Let us state the following fundamental assumption on the input $u(\cdot)$.

Assumption 4.1 *The function $u : I\!R_+ \to I\!R^p$ is bounded and analytic, and its first r derivatives are bounded.*

This assumption entails that $\mathcal{U}$ and $\mathcal{W}$ are bounded and analytic, since all the derivatives $u^{(i)}(\cdot)$, $i \geqslant 1$, are analytic. Assumption 4.1 can be relaxed, see Remark 4.1. Then we have the following extension of [1, Theorem 2], which is pivotal for the proof of global existence in Corollary 4.2 below:

Theorem 4.2 *Let Assumption 4.1 hold. Let us set*

$$\Lambda \triangleq |||WAW^{-1}||| = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{\|WAW^{-1}x\|}{\|x\|}, \tag{4.1}$$

and

$$\alpha \triangleq \sup_{s \in [0,T]} \|H\mathcal{W}(s)\|^2 < +\infty. \tag{4.2}$$

Suppose that $CA^{r-1}B > 0$. Let z be a solution of Problem $\mathbf{HOSP}(z_0;[0,T[)$. Then

i) $\|\{z\}(t)\| \leqslant \left(\|z_0\|^2 e^{(2\Lambda+1)t} + \alpha t + \frac{\alpha}{2}(2\Lambda+1)t + \frac{\alpha}{2\Lambda+1}\left\{e^{(2\Lambda+1)t} - 1\right\}\right)^{\frac{1}{2}} \triangleq z_{\max}(t)$, *for all $t \in I$.*

ii) *If $T < +\infty$ then* $\mathrm{var}(\{z\}, I) < +\infty$.

iii) *If $T < +\infty$ then, for all $1 \leqslant i \leqslant n$, $\{z_i\}(T^-)$ exists and is finite.*

Proof: i) Using (2.8) we may write

$$dz = WAW^{-1}\{z\}(t)dt + Nd\nu + H\mathcal{W}(t)dt \tag{4.3}$$

where

$$d\nu \overset{\Delta}{=} (d\nu_1, ..., d\nu_{r-1}, d\nu_r, 0_{n-r})^T$$

and N is the $n \times n$ diagonal matrix:

$$N = \begin{pmatrix} I_{r-1} & 0^{r-1} & 0_{(r-1)\times(n-r)} \\ 0_{r-1} & CA^{r-1}B & 0_{n-r} \\ 0_{(n-r)\times(r-1)} & 0^{n-r} & I_{n-r} \end{pmatrix}, \tag{4.4}$$

which is not equal to WB in (2.3) because of the presence of the measures $d\nu_1$,...,$d\nu_{r-1}$. Thus

$$(\{z\}^+)^T d\{z\} = (\{z\}^+)^T (WAW^{-1}\{z\}^+)dt + (\{z\}^+)^T H\mathcal{W}dt(\{z\}^+)^T N d\nu.$$

where $\{z\}^+$ denotes the function defined by $\{z\}^+(t) = \{z\}(t^+) = \lim_{s\to t, s>t}\{z\}(s)$ for all $t \in [0, T[$. Here

$$(\{z\}^+)^T N d\nu = \sum_{i=1}^{r-1} \{z_i\}^+ \frac{d\nu_i}{d\mu_i} d\mu_i + CA^{r-1}B \frac{d\nu_r}{d\mu_r} d\mu_r$$

for some non-negative real-valued Radon measures $d\mu_i$ $(1 \leqslant i \leqslant r)$. We have $(\{z\}^+)^T d\nu \equiv 0$ on $[0, T[$ since $\{z_i\}(t^+)\frac{d\nu_i}{d\mu_i}(t) = 0$ for $d\mu_i$-a.e. $t \in [0, T[$. Consequently

$$(\{z\}^+)^T d\{z\} = (\{z\}^+)^T (WAW^{-1}\{z\}(t) + H\mathcal{W}(t))dt.$$

Then using Moreau's rule for BV functions: $d(u^T u) = (u^+ + u^-)^T du \leqslant 2(u^+)^T du$ [39] and recalling that $\{z\}(\cdot)$ is right-continuous, we get

$$d(\{z\}^T\{z\}) \leqslant 2(\{z\})^T (WAW^{-1}\{z\} + \{z\})^T H\mathcal{W}(t))dt.$$

We arrive at

$$d(\{z\}^T\{z\})([0, t]) \leqslant 2\int_0^t (\{z\}(s))^T (WAW^{-1}\{z\})(s)ds + \int_0^t (\{z\}(s))^T H\mathcal{W}(s)ds \tag{4.5}$$

for any $t \in [0, T[$. Now using (4.1) and (4.2) and $2\int_0^t \{z\}(s))^T H\mathcal{W}(s)ds \leqslant \int_0^t \|\{z\}(s)\|^2 ds + \int_0^t \|H\mathcal{W}(s)(s)\|^2 ds$ one obtains:

$$\|\{z\}(t)\|^2 \leqslant \|z_0\|^2 + (2\Lambda + 1)\int_0^t \|\{z\}(s)\|^2 ds + \alpha t. \tag{4.6}$$

Applying Lemma E.1 with $x(t) = \|\{z\}(t)\|^2$, $f(t) = \|z_0\|^2 + \alpha t$, $g(t) = 1$, $y(t) = 2\Lambda + 1$, η the Lebesgue measure, the result follows. The details of the calculations are given in Appendix B.

ii), iii) The proofs follow step by step the proofs of [1, Theorem 2, (ii), (iii)] with slight modifications of the upperbounds using **i)**, and are omitted. ■

It is noteworthy that the results of Theorem 4.2 do not involve the restitution coefficients e_i, $1 \leqslant i \leqslant r$. As alluded to in section 3.2, the structure of the measure differential equation in (4.3) (4.4) does not exactly match with the structure of the ODE in (2.8), because $N \neq WB$: this allows to take the measures $d\nu_i$, $1 \leqslant i \leqslant r - 1$, into account in the dynamics.

Remark 4.1 *The Gronwall's Lemma E.1 is central in the proof of existence of solutions. The boundedness hypothesis in Assumption 4.1 can be modified as long as an upperbound of $\|H\mathcal{W}(\cdot)\|$ can be used to compute a value $z_{\max}(\cdot)$. Suppose for instance that $\sup_{s\in[0,T]} \|H\mathcal{W}(s)\|^2 \leqslant \alpha e^{\beta T}$ for some $\alpha > 0$, $\beta \in \mathbb{R}$. Inequality (4.6) becomes*

$$\|\{z\}(t)\|^2 \leqslant \|z_0\|^2 + \frac{\alpha^2}{2\beta}\left(e^{2\beta t} - 1\right) + (2\Lambda + 1)\int_0^t \|\{z\}(s)\|^2 ds \tag{4.7}$$

so that

$$z_{\max}^2(t) = \|z_0\|^2 - \frac{\alpha^2}{2\beta} + \frac{\alpha^2}{2\beta}e^{2\beta t} + \frac{\|z_0\|^2 - \frac{\alpha^2}{2\beta}}{2\Lambda + 1}\left(e^{2\Lambda+1)t} - 1\right) + \frac{\alpha^2}{2\beta(2\beta - 2\Lambda - 1)}\left(e^{2\beta t} - e^{(2\Lambda+1)t}\right).$$

This allows us to relax the boundedness of the derivatives of $u(\cdot)$ in Assumption 4.1, as:

Assumption 4.1' *The function $u : \mathbb{R}_+ \to \mathbb{R}^p$ is bounded analytic, and $\sup_{s\in[0,T]} \|H\mathcal{W}(s)\|^2 \leqslant \alpha e^{\beta T}$ for some $\alpha > 0$, $\beta \in \mathbb{R}$.*

The next proposition extends [1, Propositions 6 and 7, Remark 16 ii)] to the case $e_i \geqslant 0$, its proof follows from the results in Proposition 3.2 and the definition of the normal cones in the right-hand side of (3.8).

Proposition 4.1 *Suppose that $CA^{r-1}B > 0$ and let z be a solution of Problem* **HOSP**$(z_0; [0, T[)$. *Let $t \in I$ be given. Then:*

i) If $\{z_1\}(t^-) > 0$ then for all $1 \leqslant i \leqslant r$, we have $\{z_i\}(t^+) = \{z_i\}(t^-)$.

ii) If for some $1 \leqslant j \leqslant r$, $\{z_i\}(t^-) \leqslant 0$ for all $1 \leqslant i \leqslant j$ and $\{z_{j+1}\}(t^-) > 0$, then $1 \leqslant i \leqslant j \Rightarrow \{z_i\}(t^+) \geqslant 0$, and $j + 1 \leqslant i \leqslant r \Rightarrow \{z_i\}(t^+) = \{z_i\}(t^-)$.

iii) If $\{z_i\}(t^-) \leqslant 0$ for all $1 \leqslant i \leqslant r$ then, for all $1 \leqslant i \leqslant r$, we have $\{z_i\}(t^+) = -e_i\{z_i\}(t^-)$.

iv) $\{\bar{z}\}(t^+) \succeq 0$ for all $t \in I$.

v) If $\{\bar{z}\}(t^-) \succeq 0$ then $\{\bar{z}\}(t^+) = \{\bar{z}\}(t^-)$.

The only differences with [1, Propositions 6] are that in item ii) $\{z_i\}(t^+) \geqslant 0$ replaces $\{z_i\}(t^+) = 0$, and in item iii) $\{z_i\}(t^+) = -e_i\{z_i\}(t^-)$ replaces $\{z_i\}(t^+) = 0$, due to possibly non zero coefficients e_i. Notice that the definition of the tangent cones for non-positive arguments appears to be necessary here (consider for instance the case in item iii)). Proposition 4.1 can be used to determine when a state jump in $\bar{z}$ is necessary or not. Proposition 3.2 shows how the jumps are calculated in (3.13).

Let us now proceed with the extension of [1, Lemma 4]. Recall that we write $z^T = (\bar{z}^T \;\; \xi^T)$.

Lemma 4.1 *Let assumption 4.1 hold. Let $0 < T \leqslant +\infty$ be given. Suppose that $e_i \geqslant 0$, $1 \leqslant i \leqslant r$, $CA^{r-1}B > 0$ and $\bar{z}_0 \succeq 0, \bar{z}_0 \neq 0$. If a solution z of Problem* **HOSP**$(z_0; [0, T[)$ *exists, then there exists $0 < \eta \leqslant T$ such that $z \equiv \{z\}$ is analytic on $[0, \eta[$ and $z_1(t) > 0$, for all $t \in]0, \eta[$.*

Proof: The first part of the proof follows very closely the proof of [1, Lemma 4], which continues to hold for $e_i > 0,\ 1 \leqslant i \leqslant r$. The only slight modification is that, recalling that $(CA^{r-1}B)^{-1} \neq 0$, then on $[0, \eta[,\ z \equiv \{z\}$ [2] is continuous and using (2.8) it is a solution of the ODE (which replaces equation (86) in [1]):

$$\dot{z}(t) = WAW^{-1}z(t) + H\mathcal{W}(t),\ z(0) = z_0 \tag{4.8}$$

As a consequence $z(t) = We^{At}W^{-1}z_0 + W\int_0^t e^{A(t-s)}W^{-1}H\mathcal{W}(s)ds$, for all $t \in [0, \eta[$ and the result follows using Assumption 4.1. ■

The next Theorems 4.3 and 4.4 prove the local existence and uniqueness, as well as the bounded variation, of regular solutions, with two different initial conditions on the state variable $\bar{z}$. The next result is the extension of [1, Theorem 3].

Theorem 4.3 *Let assumption 4.1 hold. Suppose that $CA^{r-1}B > 0$. If $\bar{z}_0 \succeq 0, \bar{z}_0 \neq 0$, then:*

i) *(Local existence) There exists $T > 0$ such that Problem* $\mathbf{HOSP}(z_0; [0, T[)$ *has at least one regular solution $z \equiv \{z\}$ given by*

$$z(t) = We^{At}W^{-1}z_0 + W\int_0^t e^{A(t-s)}W^{-1}H\mathcal{W}(s)ds,\ \textit{for all}\, t \in [0, T[; \tag{4.9}$$

ii) $z_1(t) > 0$*, for all* $t \in]0, T[$*;*

iii) $\|z(t)\| \leqslant z_{\max}(t)$*, for all* $t \in [0, T[$*;*

iv) $\mathrm{var}(z, [0, t]) \leqslant ae^{\frac{2\Lambda+1}{2}t} + bt^{\frac{3}{2}} + ct + a$*, for all $t \in [0, T[$, and for some positive constants a, b, c;*

v) *(Local uniqueness) If $z^1(\cdot)$ is a solution of problem* $\mathbf{HOSP}(z_0;\ [0, T_1[)$ $(0 < T_1 \leqslant +\infty)$ *and $z^2(\cdot)$ is a solution of Problem* $\mathbf{HOSP}(z_0; [0, T_2[)$ $(0 < T_2 \leqslant +\infty)$ *then there exists $T \in]0, \min\{T^1, T^2\}]$ such that $z^1 \equiv \{z^1\}$ on $[0, T[$, $z^2 \equiv \{z^2\}$ on $[0, T[$ and $\{z^1\}_{|[0,T[} \equiv \{z^2\}_{|[0,T[}$.*

Proof: The unique solution of the ODE in (4.8) on $[0, +\infty[$ given by

$$z(t) = e^{WAW^{-1}t}z_0 + \int_0^t e^{WAW^{-1}(t-s)}H\mathcal{W}(s)ds, \text{for all } t \geqslant 0,$$

which can be rewritten as in (4.9) using the matrix exponential definition. Here $z(\cdot)$ is analytic on $I\!R_+$. Let α be such that $\bar{z}_{0,\alpha} > 0$ and $\bar{z}_{0,k} = 0$ for all $1 \leqslant k \leqslant \alpha - 1$ (if $\alpha \neq 1$). We claim that there exists $T > 0$ such that

$$z_1(t) > 0,\ \text{for all}\, t \in]0, T[. \tag{4.10}$$

If $\alpha = 1$ then the result is clear. If $\alpha \geqslant 2$ then there exists $T > 0$ such that $z_\alpha(t) > 0$, for all $t \in]0, T[$ and by integrating the chain of integrators in the second relations in (4.8), we finally obtain that (4.10) holds.

[2] That is, $z(t) = \{z\}(t)$ for all $t \in [0, \eta[$.

Condition (4.10) entails that $N_{\mathbb{R}^+}(z_1(t)) = \{0\}$ and $T^i_\Phi(Z_i(t)) = \mathbb{R}$ for all $1 \leqslant i \leqslant r-1$ and for all $t \in]0,T[$. It results that z satisfies the relations in (3.6)-(3.8) (3.9) on $[0,T[$ (hence the necessity of $CA^{r-1}B > 0$). Thus parts **i)** and **ii)** are proved.

This solution satisfies the ODE in (4.8) and thus one can apply the result **i)** of Theorem 4.2 to obtain **iii)**. Let $0 \leqslant t < T$ be given. For all $0 \leqslant s \leqslant t$, we have

$$\|\dot{z}(s)\| = \|WAW^{-1}z(s)\| + \|H\mathcal{W}\| \leqslant \Lambda z_{\max}(s) + \sqrt{\alpha}s.$$

Thus

$$\mathrm{var}(z,[0,t]) \leqslant \Lambda \int_0^t z_{\max}(s)ds + \sqrt{\alpha}t.$$

From the expression of $z_{\max}$ in **i)** of Theorem 4.2 and since $\sqrt{x-y} \leqslant \sqrt{x+y} \leqslant \sqrt{x}+\sqrt{y}$ for any positive reals x and y with $x-y \geqslant 0$, we may define:

$$z_{\max}(t) \stackrel{\Delta}{=} \left(\alpha_1 e^{(2\Lambda+1)t} + \alpha_2 t - \alpha_3\right)^{\frac{1}{2}} \leqslant \sqrt{\alpha_1} e^{\frac{(2\Lambda+1)}{2}t} + \sqrt{\alpha_2 t} + \sqrt{\alpha_3},$$

with $\alpha_1 = \|z_0\|^2 + \frac{\alpha}{2\Lambda+1}$, $\alpha_2 = \alpha + \frac{\alpha}{2}(2\Lambda+1)$, $\alpha_3 = \frac{\alpha}{2\Lambda+1}$. Then by integrating $\int_0^t z_{\max}(s)ds$ and upper-bounding we get the expression in **iv)** with

$$a = 2\Lambda\frac{\sqrt{\alpha_1}}{2\Lambda+1},\ b = \frac{2}{3}\sqrt{\alpha_2},\ c = \sqrt{\alpha_3} + \sqrt{\alpha}.$$

Item **v)** is proved as [1, Theorem 3 **v)**] using Lemma 4.1 and the fact that $z \equiv \{z\}$ is locally a solution of the ODE (4.8). ■

The main results of [1] are in Theorem 4 and Corollary 1 on local existence and uniqueness of regular solutions. Their extensions are given now.

Theorem 4.4 *Let Assumption 4.1 hold. Assume that $CA^{r-1}B > 0$ and define $\beta \stackrel{\Delta}{=} \sup_{t\in[0,T]} ||G_\xi \mathcal{U}(t)|| < +\infty$. If $\bar{z}_0 = 0$, then:*

i) *(Local existence) There exists $T > 0$ such that Problem* **HOSP**$(z_0; [0,T[)$ *has at least one regular solution $z \equiv \{z\}$;*

ii) $z_1(t) \geqslant 0$*, for all $t \in [0,T[$;*

iii) $\|z(t)\| \leqslant \max(z_{\max}(t), e^{\Lambda t}(||z_0|| + \beta t))$*, for all $t \in [0,T[$, where $z_{\max}(t)$ is in* **i)** *of Theorem 4.2;*

iv) $\mathrm{var}(z,[0,t]) \leqslant \max(ae^{\frac{2\Lambda+1}{2}t} + bt^{\frac{3}{2}} + ct + a, ||z_0||e^{\Lambda t} + \Lambda\beta t^2 e^{\Lambda t} + \Lambda\beta t)$*, for all $t \in [0,T[$, where a, b and c are as in Theorem 4.2;*

v) *(Local uniqueness in the class of regular solutions) If z^1 is a regular solution of Problem* **HOSP**$(z_0; [0,T_1[)$ *$(0 < T_1 \leqslant +\infty)$ and z^2 is a regular solution of Problem* **HOSP**$(z_0; [0,T_2[)$ *$(0 < T_2 \leqslant +\infty)$ then there exists $T \in]0, \min\{T^1, T^2\}]$ such that $z^1 \equiv \{z^1\}$ on $[0,T[$, $z^2 \equiv \{z^2\}$ on $[0,T[$ and $\{z^1\}_{|[0,T[} \equiv \{z^2\}_{|[0,T[}$.*

Proof: It happens once again that the proof follows closely the proof of [1, Theorem 4]. Let us consider (2.6) with (2.4) which in particular implies $CA^rW^{-1}z =$

$d^T\bar{z} + d_\xi^T\xi$ (see (2.8)). First of all let us notice that when $z_1 \equiv 0$ then $\xi(t) = e^{A_\xi t}\xi_0 + e^{A_\xi t}\int_0^t e^{-A_\xi s}G_\xi\mathcal{U}(s)ds$. Let us denote $F(t) = \int_0^t e^{-A_\xi s}G_\xi\mathcal{U}(s)ds$. The function $F(\cdot)$ is analytic since $\mathcal{U}(\cdot)$ has analytic components. From the Leibniz formula one obtains

$$\left(e^{A_\xi \cdot}F(\cdot)\right)^{(k)}(0) = \sum_{i=0}^{k-1} C_k^i A_\xi^i \sum_{j=0}^{k-i-1}(-1)^j C_{k-i}^j A_\xi^j G_\xi \mathcal{U}^{(k-i-j)}(0). \tag{4.11}$$

The calculations are made in section D in the Appendix. Let us denote the right-hand side of (4.11) as $L_k(\mathcal{W}(0))$. Considering (2.8) we are now left with two main cases: either **(a)** $d_\xi^T A_\xi^k\xi_0 + d_\xi^T L_k(\mathcal{U}(0)) + \bar{G}\mathcal{W}^{(k)}(0) = 0$, for all $k \in \mathbb{N}$ or **(b)** there exists $\alpha \in \mathbb{N}$ such that $d_\xi^T A_\xi^\alpha\xi_0 + d_\xi^T L_\alpha(\mathcal{W}(0)) + \bar{G}\mathcal{W}^{(\alpha)}(0) \neq 0$ and (if $\alpha \neq 0$) $d_\xi^T A_\xi^k\xi_0 + d_\xi^T L_k(\mathcal{U}(0)) + \bar{G}\mathcal{W}^{(k)}(0) = 0$, for all $0 \leqslant k \leqslant \alpha - 1$.

Case **(a)**. Let us check that $z^T \equiv (\bar{z}^T, \xi^T)$ with

$$\bar{z}(t) = 0^r,\ \xi(t) = e^{A_\xi t}\xi_0 + e^{A_\xi t}F(t)$$

is a regular solution of Problem **HOSP**$(z_0; [0, +\infty[)$. Indeed, for sufficiently small $t > 0$ the we have

$$d_\xi^T\xi(t) + \bar{G}\mathcal{W}(t) = \sum_{i=0}^{+\infty}\left(d_\xi^T A_\xi^i\xi_0 + d_\xi^T L_i(\mathcal{U}(0)) + \bar{G}\mathcal{W}^{(i)}(0)\right)\frac{t^k}{k!} = 0,$$

so that for all $t \geqslant 0$:

$$\dot{z}_i(t) - z_{i+1}(t) = 0,\ 1 \leqslant i \leqslant r-1$$

$$(CA^{r-1}B)^{-1}(\dot{z}_r(t) - d^T\bar{z}(t) - d_\xi^T\xi(t) - \bar{G}\mathcal{W}(t)) = -(CA^{r-1}B)^{-1}(d_\xi^T\xi(t) - \bar{G}\mathcal{W}(t)) = 0, \tag{4.12}$$

and

$$\dot{\xi}(t) = A_\xi\xi(t) + W_\xi\mathcal{U}(t)\ (= A_\xi\xi(t) + B_\xi z_1(t) + G_\xi\mathcal{U}(t)) \quad \text{for all } t \geqslant 0.$$

Case **(b)**. The proof in this case is split into two sub-cases: **(b-1)** $d_\xi^T A_\xi^\alpha\xi_0 + d_\xi^T L_\alpha(\mathcal{W}(0)) + \bar{G}\mathcal{W}^{(\alpha)}(0) < 0$ and (if $\alpha \neq 0$) $d_\xi^T A_\xi^k\xi_0 + d_\xi^T L_k(\mathcal{U}(0)) + \bar{G}\mathcal{W}^{(k)}(0) = 0$, for all $0 \leqslant k \leqslant \alpha - 1$, and **(b-2)** $d_\xi^T A_\xi^\alpha\xi_0 + d_\xi^T L_\alpha(\mathcal{W}(0)) + \bar{G}\mathcal{W}^{(\alpha)}(0) > 0$ and (if $\alpha \neq 0$) $d_\xi^T A_\xi^k\xi_0 + d_\xi^T L_k(\mathcal{U}(0)) + \bar{G}\mathcal{W}^{(k)}(0) = 0$, for all $0 \leqslant k \leqslant \alpha - 1$. The proof of both cases is quite similar to [1, pp.180-181] taking into account the modifications due to the presence of $\mathcal{U}$ and $\mathcal{W}$ as for the proof of case **(a)**, and we omit it here. In both cases it can be shown that there exists $\eta > 0$ such that a solution can be exhibited on $[0, \eta[$. This proves **i)** and **ii)**.

Roughly speaking, in case **(b-2)** the solution detaches from the constraint boundary and $z(t)$ is given as in (4.8), its upperbound and the upperbound on its variation are as in Theorem 4.2. In case **(b-1)**, thanks to the condition $CA^{r-1}B > 0$ and replacing $= 0$ by $\geqslant 0$ in the second equality of (4.12), there exists at least one solution which remains stuck on the boundary $z_1 = 0$ so that $\bar{z}(t) = 0^r$ and $\xi(t) = e^{A_\xi t}\xi_0 + e^{A_\xi t}\int_0^t e^{-A_\xi s}G_\xi\mathcal{U}(s)ds$ is a local solution. Thus $||z(t)|| = ||\xi(t)|| \leqslant e^{\Lambda t}(||\xi_0|| + \beta t)$ where Λ is in (4.1). We have also $\mathrm{var}(z, [0,t]) = \mathrm{var}(\xi, [0,t]) \leqslant \int_0^t ||\dot{\xi}(s)||ds$. From $||\dot{\xi}(s)|| \leqslant \Lambda e^{\Lambda t}(||\xi_0|| + \beta t) = \Lambda e^{\Lambda t}(||z_0|| + \beta t)$ it follows that $\mathrm{var}(z, [0,t]) \leqslant ||\xi_0||e^{\Lambda t} + \Lambda\beta t^2 e^{\Lambda t} + \Lambda\beta t = ||z_0||e^{\Lambda t} + \Lambda\beta t^2 e^{\Lambda t} + \Lambda\beta t$. This proves **iii)** and **iv)**.

The last part of the proof consist in proving item **v)**. The only difference with the proof of [1, Theorem 4, **v)**] is that one starts with (4.3), hence the first two equalities in [1, p.182] are changed to $dz = WAW^{-1}\{z\}dt + Nd\nu + H\mathcal{W}dt$ and $dz_a = WAW^{-1}\{z_a\}dt + Nd\nu_a + H\mathcal{W}dt$, respectively. Thus the subsequent developments of the proof of item **v)** in [1, p.182] remain unchanged. ■

The same comments as in Remark 4.1 hold for the calculation of the upperbound β, where Assumption 4.1 could be relaxed. Theorems 4.2 and 4.4 deal with $\bar{z}_0 \succeq 0$ and $\bar{z}_0 = 0$, respectively. In case $\bar{z}_0 \preceq 0$ (that is, $-\bar{z}_0 \succeq 0$), then a state jump applies that brings the system on the right of $t = 0$ to one of the foregoing cases, provided that $e_i \geqslant 0$, $1 \leqslant e_i \leqslant r$ (see Proposition 4.1 iii)). We infer that the following result is true, which is the counterpart of [1, Corollary 1] and whose proof follows from Theorems 4.3 and 4.4:

Corollary 4.1 *(Local existence and uniqueness in the class of regular solutions) Let Assumption 4.1 hold, assume that* $CA^{r-1}B > 0$ *and* $e_i \geqslant 0$, $1 \leqslant e_i \leqslant r$. *Then an* $LBV([0,T[;\mathbb{R}^n)$ *solution to (3.6) (3.7) in the sense of Definition 4.1 exists for any initial condition* $z(0) = z_0$ *and for some* $T > 0$, *and it is unique in the sense that if* z^1 *and* z^2 *are regular solutions of* $\mathbf{HOSP}(z_0;[0,T_1[)$ *and* $\mathbf{HOSP}(z_0;[0,T_2[)$, *respectively, with* $0 < T_1 \leqslant +\infty$, $0 < T_2 \leqslant +\infty$, *then there exists* $T \in]0, \min\{T_1,T_2\}]$ *such that* $\langle z^1, \varphi\rangle = \langle z^2, \varphi\rangle$ *for all for all* $\varphi \in C_0^\infty([0,T[;\mathbb{R}^n)^3$. *Moreover we have:* **i)** $z \equiv \{z\}$ *on* $]0,T[$, **ii)** $z_1 \equiv \{z_1\}$ *on* $[0,T[$, **iii)** $z_i = \{z_i\} + \sum_{j=1}^{i-1}(z'_{0,j} - z_{0,j})\delta_0^{(i-j-1)}$ *for* $2 \leqslant i \leqslant r$, *where* $z_{0,j} = z_j(0^-)$, $z'_{0,j} = z_j(0^+)$, *and* $z_1 = \{z_1\} \geqslant 0$ *on* $[0,T]$, **iv)** $||\{z\}(t)|| \leqslant \max(z_{\max}(t), e^{\Lambda t}(||z_0 + \beta t))$, *for all* $t \in [0,T[$, *where* $z_{\max}(t)$ *is in* **i)** *of Theorem 4.2,* **v)** $\mathrm{var}(z,[0,t]) \leqslant \max(ae^{\frac{2\Lambda+1}{2}t} + bt^{\frac{3}{2}} + ct + a, ||z_0||e^{\Lambda t} + \Lambda\beta t^2 e^{\Lambda t} + \Lambda\beta t)$, *for all* $t \in [0,T[$, *where* a, b *and* c *are as in Theorem 4.2.*

Finally, the proof of global well-posedness in the class of regular solutions is provided in the next result that corresponds to [1, Corollary 2]. It shows the coherency of the whole well-posedness proof, consequently its proof is given.

Corollary 4.2 *(Global existence and uniqueness in the class of regular solutions) Let* $CA^{r-1}B > 0$ *and Assumption 4.1 hold. For each* $z_0 \in \mathbb{R}^n$, *Problem* $\mathbf{HOSP}(z_0;[0,+\infty[)$ *has at least one regular solution* z *such that:* **i)** $z_1 \equiv \{z_1\} \geqslant 0$ *on* $[0,+\infty[$, **ii)** $||\{z\}(t)|| \leqslant \max(z_{\max}(t), e^{\Lambda t}(||z_0 + \beta t))$ *for all* $t \in [0,+\infty[$, **iii)** *If* z^* *is a regular solution of problem* $\mathbf{HOSP}(z_0;[0,T^*[)$, $0 < T^* \leqslant +\infty$, *then* $\langle z^*, \varphi\rangle = \langle z, \varphi\rangle$ *for all* $\varphi \in C_0^\infty([0,T^*[;\mathbb{R}^n)$.

Proof: Theorems 4.2, 4.3 and 4.4 are used to prove Corollary 4.2, as well as Corollary 4.1 which is itself a direct consequence of Theorems 4.3 and 4.4. We reproduce here the complete proof which is the same as the proof of [1, Corollary 1]. From Corollary 4.1 we infer the existence of $T \in]0,+\infty[$ such that Problem $\mathbf{HOSP}(z_0;[0,T[)$ has a regular solution z such that $z \equiv \{z\}$ on $]0,T[$ and $z_1 \geqslant 0$ on $[0,T[$. From Theorem 4.2 **i)** and **iii)**, the limit $\{z\}(T^-) = \lim_{t\to T, t<T}\{z\}(t)$ exists and it is finite. We may then apply a state re-initialization such that $\{z\}(T^+) = z^+$ where z^+ is uniquely defined by

[3] The set of $C^\infty([0,T[)$ mappings with compact support.

$z_i^+ = -e_i\{z_i\}(t^-) + (1+e_i)\text{proj}\left[T_\Phi^{i-1}(\{Z_{i-1}\}(t^-)); \{z_i\}(t^-)\right]$ for all $1 \leqslant i \leqslant r$, and $z_i^+ = \{z_i\}(T^-)$ for all $r+1 \leqslant i \leqslant n$ (the zero dynamics state is continuous). Using Corollary 4.1, z can be prolongated as a regular solution of Problem $\mathbf{HOSP}(z_0; [0, T_1[)$ with $T_1 > T$, such that $z_1 \geqslant 0$ on $[0, T_1[$. Now let us denote as $H_{\max}$ the supremum of all $H > T$ such that a prolongation of z as a regular solution of Problem $\mathbf{HOSP}(z_0; [0, H[)$ exists. Let us assume that $H_{\max} < +\infty$. As above, from Theorem 4.2, the limit $z(H_{\max}^-)$ exists and is finite. Thus a prolongation exists from Corollary 4.1, as a regular solution of Problem $\mathbf{HOSP}(z_0; [0, H_{\max} + \epsilon[)$ for some $\epsilon > 0$, which is a contradiction. The existence of a global regular solution satisfying **i)** follows. Item **ii)** is a consequence of Theorems 4.3 and 4.4. Let us now prove item **iii)**. Suppose that there exists $s \in [0, T^*[$ such that $\{z\}(s) \neq \{z^*\}(s)$. We have by hypothesis that $\{z\}(0) = \{z^*\}(0) = z_0$. Thus $s > 0$. Let us define $\tau = \inf\{s \in]0, T[\mid \{z\}(s) \neq \{z^*\}(s)\}$. If $\tau > 0$ then necessarily $\{z\}(h) = \{z^*\}(h)$ for all $h \in [0, \tau[$ and thus $\{z\}(\tau^-) = \{z^*\}(\tau^-)$ (in case $\tau = 0$ we have also $\{z\}(\tau^-) = \{z^*\}(\tau^-) = z_0$). Consequently $\{z\}(\tau^+) = \{z^*\}(\tau^+)$ and by Theorems 4.3 and 4.4, the existence of a $\delta > 0$ such that $\{z\}(s) = \{z^*\}(s)$ for all $s \in [\tau, \tau + \delta[$ is guaranteed, hence a contradiction with the definition of τ. Thus there does not exist $s \in [0, T^*[$ such that $\{z\}(s) \neq \{z^*\}(s)$. The proof of item **iii)** is done. ■

It is clear that the class of regular solutions prevents the existence of right accumulations of state jumps[4]. It is however not excluded that other classes of solutions exist. It is noteworthy that the well-posedness proof relies on Assumption 4.1, or its relaxed version Assumpion 1'. In view of the way the global existence has been shown (by concatenation of local solutions), we can suppose that $u(\cdot)$ satisfies these assumptions but is only piecewise analytic, *i.e.* it is smooth on $\mathbb{R}_+$ and analytic on intervals $]\tau_i, \tau_{i+1}[$, with $\mathbb{R}_+ = \cup_{i \geqslant 0}]\tau_i, \tau_{i+1}[$, $\tau_{i+1} \geqslant \tau_i + \gamma$ for some $\gamma > 0$. Existence and uniqueness of a regular solution can be proved on each $]\tau_i, \tau_{i+1}[$, and the global solution is constructed by concatenating the solutions. We can therefore state:

Assumption 4.1" The function $u : \mathbb{R}_+ \to \mathbb{R}^p$ is a smooth piecewise analytic and bounded function, such that $\sup_{s \in [0,T]} \|H\mathcal{W}(s)\|^2 \leqslant \alpha e^{\beta T}$ for some $\alpha > 0$, $\beta \in \mathbb{R}$.

Remark 4.2 *As alluded to in the introduction, the case where $w = Cx + Fu(t) + D\lambda$ where D is a P-matrix (hence invertible), is embedded in the sweeping process by simply setting the complementarity $0 \leqslant w(t) \perp \lambda(t) \geqslant 0$ which is a linear complementarity system with unique solution $\lambda(t)$ that is Lipschitz continuous in u and x. We may view this case as the relative degree zero HOSP. The canonical transformation is then trivial where x plays the role of the zero dynamics state vector ξ, and we obtain $\dot{\xi} = (A - BD^{-1}C)\xi + BD^{-1}z_1 + (E - BD^{-1}F)u$, $z_1 = C\xi + D\lambda + Fu$ (compare with (2.6)). Notice that if we set $u = Kx + L\lambda + v$ in (1.1), then the closed-loop system becomes:*

$$\begin{cases} \dot{x} = (A + EK)x + (B + EL)\lambda + Ev \\ 0 \leqslant w = (C + FK)x + FL\lambda + Fv \end{cases} \tag{4.13}$$

If there exists L such that FL is a P-matrix, the system (4.13) embedded into linear complementarity systems is well-posed as an ODE with Lipschitz vector field. If there

[4] That is, accumulations which start on the right of their accumulation point.

exists L such that $FL = 0$*,* K *and* $P = P^T > 0$ *such that* $P(B+EL) = (C+FK)^T$*, then the results in [*19*] can be used to prove the well-posedness. This is closed to the results in [*21*] which require that* $(A+EK, B+EL, C+FK)$ *be dissipative. As is known, in this case the total index of the system's transfer matrix is one [*31, Definition 3.5, Theorem 3.14*], which implies a relative degree zero or one in case w is scalar. The relative degree two case is also much studied in the framework of frictionless Lagrangian systems [*25, 39*]. In all these cases the existence of solutions is proved. Analycity of the data has been shown to be crucial for uniqueness in [*9, 51, 52*].*

5 The link with complementarity systems and switching DAEs

There is a close link between the DDI in (3.6)-(3.8) and complementarity systems. This stems from the MDI in (3.10). We may write each measure $d\nu_i$ as

$$d\nu_i = \chi_i(t)dt + d\mathcal{J}_i, \tag{5.1}$$

where $\chi_i \in \mathcal{F}_\infty(I; I\!R)$ and $d\mathcal{J}_i$ is an atomic measure with countable set of atoms generated by a right-continuous jump function $\mathcal{J}_i$.

Theorem 5.1 *[*1, Theorem 1, Remark 14*] Let* $m = 1$*, and* z *be a solution of Problem* **HOSP**$(z_0;I)$*. Then, for each* $t \in I$*, we have:*

$$0 \leqslant z_1(t^+) \perp d\nu_r(\{t\}) \geqslant 0. \tag{5.2}$$

The measure $d\nu_i$ *is atomic, consequently* $\chi_i(t) = 0$ a.e. $t \in I$*, for all* $1 \leqslant i \leqslant r-1$*, while* $\chi_r(t) \in -\partial\psi_{T_\Phi^{r-1}(\{Z_{r-1}\}(t^-))}(\{z_r\}(t^+))$, a.e. $t \in I$*. Thus:*

$$0 \leqslant z_1(t^+) \perp \chi_r(t) \geqslant 0, \text{ a.e. } t \in I. \tag{5.3}$$

This means that if $z_1(0) \geqslant 0$, then $z_1(t) \geqslant 0$ for all future times. However the left limits of the derivatives of $z_1(\cdot)$ may take wrong signs at any time: the HOSP right-hand side takes care of bringing them back to non-negative values *via* the impact law (3.13), see Proposition 3.2. Here we recover also the case of Mechanics, because (5.3) implies that if $z_1(0) \geqslant 0$, then $z_1(t) \geqslant 0$ for all $t \in I$ and $d\nu_1 = 0$: the function $z_1(\cdot)$ is time continuous, and $z_2(\cdot)$ (the velocity) can have jumps. The fact that measures $d\nu_i$ are atomic for all $1 \leqslant i < r-1$ follows from (3.10).

Using (3.6) (3.7), the original dynamics (1.1) can therefore be written as the complementarity system with distributional solutions:

$$\begin{cases} \mathbf{D}x = A\{x\} + B\lambda + Eu \\ 0 \leqslant w(t) = Cx(t) + Fu(t) \perp \chi_r(t) \geqslant 0 \text{ for all } t \text{ that is not an atom of } d\nu_i,\ 1 \leqslant i \leqslant r. \end{cases} \tag{5.4}$$

This section is devoted also to show that the HOSP may be seen as a system of switching Differential Algebraic Equations (DAEs), where the switches are ruled by complementarity conditions (hence are state dependent), as in Linear Complementarity Systems (LCS) [21, 30]. Other types of switching DAEs have been studied in [61, 38, 49, 58, 60], where switching times are exogenous and do not accumulate in finite time. State jumps are defined in switching DAEs with a projection onto the consistency space (that is a linear space of consistent initial conditions after each switching instant). In the HOSP, the state re-initialization is also defined *via* a suitable projection in (3.13) which guarantees consistency of the right limit $\{\bar{z}\}(t^+)$ at each t that is an atom of the measures $d\nu_i$, $1 \leqslant i \leqslant r$. This projection comes as a natural consequence of the constraints imposed on the measures $d\nu_i$ *via* (3.11). It is noteworthy however that the switching DAEs studied in [61, 38, 49], and the HOSP are different dynamical systems, designed to fulfill different objectives (the fundamental objective of the HOSP being to render the set $\Phi_u = \{x \in I\!R^n \mid Cx + Fu(t) \geqslant 0\}$, invariant; while the objective of switching DAEs in [61, 38, 49, 58, 60] is to analyse systems with equality constraints that switch according to a time-dependent signal). As such both formalisms cannot be treated on the same footing. In particular the HOSP formalism allows for finite accumulations of switching times (state re-initialization times), due to the use of the multiplier λ which takes care of transitions between the system's modes. We also note that the HOSP provides a family of well-defined and unique state re-initialization mappings when a state jump is needed, while the formalism in [61, 38, 49] yields a consistency projector. Finally the HOSP is well-posed under the main assumption that the triplet (A, B, C) has a relative degree and the leading Markov parameter $CA^{r-1}B$ is a Stieltjes matrix. Switching DAEs well-posedness mainly relies on the regularity of all pairs (E_k, A_k) defining the DAEs in descriptor variable form [61, 38, 49], which assures that the DAE hence defined has, in its semi-explicit form, a finite index. In the same vein we may mention switched linear differential systems (SLDS) studied in [40, 42, 41], where state jumps are derived implicitly from so-called *gluing conditions* which may be derived from physical constraints (like the conservation of charge in circuits). However well-posedness issues are still open for SLDS. Linear complementarity systems (LCS) [21, 30, 56, 57] form another class of switching systems close to switching DAEs (with similar state re-initialization mapping) and to the HOSP. In particular the state jump mappings for dissipative LCS coincide with the sweeping process one, for relative degree one systems [3, Proposition 2.65][19][18, p.319]. Results in [27, Chapter 11] on velocity jumps in mechanical systems with C^0 bilateral constraints are worth mentioning. We note finally that a significant feature of the HOSP (and of passive LCS) is their ability to be discretized with suitable, convergent event-capturing schemes [1, Section 5] [2, 29].

5.1 The SISO case

Let us still assume that $m = 1$, and let $z_1(\tau^+) = 0$ on some interval $]t, t+\sigma]$, $\sigma > 0$. On $]t, t+\sigma]$, the function $z_1(\cdot)$ is analytic and we have $z_1(\tau) = z_2(\tau) = ... = z_r(\tau) = 0$. From (5.3) we have $\lambda(\tau) = \chi_r(\tau) \geqslant 0$ on $]t, t+\sigma[$, and $\dot{z}_r(\tau^+) \geqslant 0$ also. It can also be shown that $\dot{z}_r(\tau^+) > 0 \Rightarrow \chi_r(\tau) = 0$. Thus $0 \leqslant \dot{z}_r(\tau^+) \perp \lambda(\tau) = \chi_r(\tau) \geqslant 0$ holds[5]. Using (2.7)

[5] Remind that $\chi_r(\cdot)$ is right-continuous, as well as $\dot{z}_r(\cdot)$, being $z_r(\cdot)$ in $\mathcal{F}_\infty(I; I\!R)$.

this gives rise to the *contact linear complementarity problem (LCP)* with unknown $\lambda(\tau)$:

$$0 \leqslant \lambda(\tau) \perp \dot{z}_r(\tau^+) = w^{(r)}(\tau^+) = CA^rW^{-1}\begin{pmatrix} 0 \\ \xi(\tau) \end{pmatrix} + CA^{r-1}B\lambda(\tau) + \bar{G}\mathcal{W}(\tau) \geqslant 0, \tag{5.5}$$

for all $\tau \in]t, t+\sigma[$. The fact that $CA^{r-1}B > 0$ guarantees that this LCP has a unique solution for any $\xi(t)$ and $\mathcal{W}(t)$. This solution is easily obtained as:

$$\lambda(\tau) = \begin{cases} \textbf{(i)}\ 0 \ \text{ if } \ CA^rW^{-1}\begin{pmatrix} 0 \\ \xi(\tau) \end{pmatrix} + \bar{G}\mathcal{W}(\tau) \geqslant 0 \\ \textbf{(ii)}\ -(CA^{r-1}B)^{-1}\left[CA^rW^{-1}\begin{pmatrix} 0 \\ \xi(\tau) \end{pmatrix} + \bar{G}\mathcal{W}(\tau)\right] \ \text{otherwise.} \end{cases} \tag{5.6}$$

In case (5.6) **(i)** with >0, the trajectory satisfies $z_i(\tau) > 0$ in a right neighborhood of τ (hence by assumption this may happen only at $\tau = t+\sigma$). Otherwise $\dot{z}_r(\tau^+) = 0$, and $z_i(\tau) = 0$ in a right neighborhood of τ. Case (5.6) **(ii)** corresponds to a mode in which the system is equivalent to the DAE (in linear implicit form, or descriptor variable system):

$$E\dot{z}(t) = WAW^{-1}z(t) + WB\lambda(t) + H\mathcal{W}(t), \ \ z_1(t^+) = 0, \tag{5.7}$$

with $E = \begin{pmatrix} 0_{(r-1)\times(r-1)} & 0_{(r-1)\times(n-r+1)} \\ 0_{(n-r+1)\times(r-1)} & I_{n-r+1} \end{pmatrix}$ and $\lambda(t)$ is in (5.6) **(ii)**, so that the right-hand side of (5.7) is linear in z. In a semi-explicit form, the contact DAE may be written as:

$$\begin{cases} \dot{z}(t) = WAW^{-1}z(t) + WB\lambda(t) + H\mathcal{W}(t) \\ h(z) = z_1(t) = 0 \end{cases} \tag{5.8}$$

It can be checked that this DAE has index $r+1$ since $\frac{d^{r+1}}{dt^{r+1}}h(z(t)) = z_1^{(r+1)}(t) = \ddot{z}_r(t)$ contains the term $CA^{r-1}B\dot{\lambda}(t)$. We infer that the HOSP is a DDI which allows to switch between two systems, *i.e.* the ODE in (2.8) with $\lambda = 0$ and $z_1 > 0$, and the DAE in (5.7) with index $r+1$, $\lambda > 0$ and $z_1 = 0$ [6]. The switching times from the DAE to the ODE are defined as the time when the signum of $CA^rW^{-1}\begin{pmatrix} 0 \\ \xi(\tau) \end{pmatrix} + \bar{G}\mathcal{W}(\tau)$ changes (because of the exogenous terms), implying that λ vanishes. The switching times from the ODE to the DAE may occur after a state jump. Due to the possibly non zero coefficients e_i, there may also exist phases of motion with "rebounds" on the surface $z_1 = 0$, before the solution remains on this surface on a non trivial time interval.

5.2 The MIMO case

5.2.1 *The canonical state space representation*

The canonical transformation yielding (2.6) extends to $m \geqslant 2$, $m < n$, for systems with vector relative degree $\bar{r} = (r, r, .., r)^T \in \mathbb{R}^m$, which satisfy $CA^{i-1}B = 0$ for all $1 \leqslant i \leqslant r-1$, and $CA^{r-1}B \in \mathbb{R}^{m\times m}$ is nonsingular. We shall adopt the notations: $\bar{z}_i \stackrel{\Delta}{=} (z_i^1, z_i^2, ..., z_i^r)^T \in \mathbb{R}^r$, $1 \leqslant i \leqslant m$, $\xi \in \mathbb{R}^{n-mr}$, $z^1 \stackrel{\Delta}{=} (w_1, w_2, ..., w_m)^T = (z_1^1, z_2^1, ..., z_m^1)^T \in \mathbb{R}^m$, $z^i \stackrel{\Delta}{=} (z_1^i, z_2^i, ..., z_m^i)^T \in \mathbb{R}^m$, $1 \leqslant i \leqslant r$, the zero dynamics vector

[6] The case where $\lambda = 0$ and $z_1 = 0$ is called a degenerate case in-between the two modes.

$\xi \in I\!R^{n-mr}$, the state vector given by $z = (\bar{z}_1^T, \bar{z}_2^T, ..., \bar{z}_m^T, \xi^T)^T \in I\!R^n$, $B_\xi \in I\!R^{(n-mr)\times m}$, $A_\xi \in I\!R^{(n-mr)\times(n-mr)}$, $WB \in I\!R^{n\times m}$, $CW^{-1} \in I\!R^{m\times n}$. In case $m = 1$ we have denoted $z_i \triangleq z_1^i$ for $1 \leqslant i \leqslant r$. The canonical form then reads:

$$\begin{cases} \dot{z}_i^1(t) = z_i^2(t) \\ \dot{z}_i^2(t) = z_i^3(t) \\ \vdots \\ \dot{z}_i^{r-1}(t) = z_i^r(t) \\ \dot{z}_i^r(t) = (CA^rW^{-1})_{i\bullet} z(t) + (CA^{r-1}B)_{i\bullet}\lambda(t) + (\bar{G}\mathcal{W}(t))_i \\ \dot{\xi}(t) = A_\xi \xi(t) + B_\xi z^1 + G_\xi \mathcal{U}(t), \end{cases} \quad 1 \leqslant i \leqslant m \tag{5.9}$$

where $(CA^rW^{-1})_{i\bullet} = C_iA^rW^{-1} \in I\!R^{1\times n}$, $(CA^{r-1}B)_{i\bullet} = C_iA^{r-1}B \in I\!R^{1\times m}$, $1 \leqslant i \leqslant m$. The system in (5.9) can be rewritten compactly as:

$$\begin{cases} \dot{z}^1(t) = z^2(t) \\ \dot{z}^2(t) = z^3(t) \\ \vdots \\ \dot{z}^{r-1}(t) = z^r(t) \\ \dot{z}^r(t) = CA^rW^{-1}z(t) + CA^{r-1}B\lambda(t) + \bar{G}\mathcal{W}(t) \\ \dot{\xi}(t) = A_\xi\xi(t) + B_\xi z^1 + G_\xi\mathcal{U}(t), \end{cases} \tag{5.10}$$

with $\bar{G} \in I\!R^{m\times(r+1)p}$. The set Φ can be generalized to $\Phi_m = I\!R_+^m$, $T_{\Phi_m}(Z^i) = \times_{k=1}^m T_\Phi^i(Z_k^i)$ [1, Remark 7], and from [53, Proposition 3.1.10]: $N_{T_{\Phi_m}(Z^i)}(z^{i+1}) = \times_{k=1}^m N_{T_\Phi^i(Z_k^i)}(z_k^i)$, with $Z^i \triangleq (z^{1,T}, z^{2,T}, ..., z^{i,T})^T$, $Z_k^i \triangleq (z_k^1, z_k^2, ..., z_k^i)^T$. The tangent and the normal cones in the right-hand side of (3.8) can be calculated, see an example in Appendix F. We denote $d\nu^i = (d\nu_1^i, d\nu_2^i, ..., d\nu_m^i)^T$ and we still impose (3.11) for each $d\nu^i$, $1 \leqslant i \leqslant r$, with $d\nu^i = dz^i - z^{i+1}$ (see (3.10). The state jump rule in (3.13) is unchanged for all $1 \leqslant i \leqslant r-1$, replacing z_i by z^i. For $i = r$ we obtain using (3.12):

$$\{z^r\}(t^+) = -e_r\{z^r\}(t^-) + (1+e_r)\mathrm{proj}_{(CA^{r-1}B)^{-1}}\left[T_{\Phi_m}^{r-1}(\{Z^{r-1}\}(t^-)); \{z^r\}(t^-)\right] \tag{5.11}$$

The generalization of the MDI in (3.10) (3.11) is as follows:

$$\begin{cases} dz^1 - z^2(t)dt \in -\partial\psi_{T_{\Phi_m}^0(\{Z^0\}(t^-))}(\zeta^1(t)) \\ dz^2 - z^3(t)dt \in -\partial\psi_{T_{\Phi_m}^1(\{Z^1\}(t^-))}(\zeta^2(t)) \\ \vdots \\ dz^{r-1} - z^r(t)dt \in -\partial\psi_{T_{\Phi_m}^{r-2}(\{Z^{r-2}\}(t^-))}(\zeta^{r-1}(t)) \\ dz^r - CA^rW^{-1}z(t)dt - \bar{G}\mathcal{W}(t)dt \in -(CA^{r-1}B)\,\partial\psi_{T_{\Phi_m}^{r-1}(\{Z^{r-1}\}(t^-))}(\zeta^r(t)) \\ d\xi = (A_\xi\xi(t) + B_\xi z_1(t) + G_\xi\mathcal{U}(t))dt. \end{cases} \tag{5.12}$$

As indicated in [1, Remarks 15, 17], Proposition 3.2, Theorem 4.2, and Proposition 4.1 continue to hold when $CA^{r-1}B = (CA^{r-1}B)^T > 0$. Provided that $CA^{r-1}B$ is a Stieltjes matrix, Theorems 4.3 and 4.4, as well as Corollaries 4.1 and 4.2, continue to hold also [1, Remark 20 ii)]. Imposing the Stieltjes property allows one to reproduce some parts of the proof, like in part **(b-1)** in the proof of Theorem 4.4 where it secures that $(CA^{r-1}B)^{-1}(\dot{z}_r(t) - d^T\bar{z}(t) - d_\xi^T\xi(t) - \bar{G}\mathcal{W}(t)) = -(CA^{r-1}B)^{-1}(d_\xi^T\xi(t) - \bar{G}\mathcal{W}(t)) \in$

$-\partial\psi_{T_{\Phi_m^{r-1}(0,\ldots,0)}}(0) = \mathbb{R}_+$ when $d_\xi^T\xi(t) - \bar{G}\mathcal{W}(t) \leqslant 0$. Let now $\{z^i\}(t^-) \leqslant 0$, $1 \leqslant i \leqslant r-1$, and $\{z^r\}(t^-) < 0$ (all inequalities are componentwise). Thus (5.11) holds, and $\mathrm{proj}_{(CA^{r-1}B)^{-1}}\left[T_{\Phi_m}^{r-1}(\{Z^{r-1}\}(t^-)); \{z^r\}(t^-)\right] = \{0^m\}$ since $T_{\Phi_m}^{r-1}(\{Z^{r-1}\}(t^-)) = \mathbb{R}_+^m$, so that $\{z^r\}(t^+) = -e_r\{z^r\}(t^-) > 0$. We obtain the inclusion $-(1+e_r)\{z^r\}(t^-) \in -(CA^{r-1}B)\; N_{\mathbb{R}_+^m}(0^m)$, because $\zeta^r(t) = 0$ at an impact time. Therefore we obtain the inclusion $-(1+e_r)(CA^{r-1}B)^{-1}\{z^r\}(t^-) \in \mathbb{R}_+^m$ which makes sense if $CA^{r-1}B$ is a Stieljes matrix.

Another peculiarity of the MIMO case is that $CA^{r-1}B$ is usually non-diagonal, implying couplings between the variables at impacts. Indeed assume that $d\nu_j^r(\{t\}) \neq 0$ at some t, for some $1 \leqslant j \leqslant m$. From $\{z^r\}(t^+) - \{z^r\}(t^-) = CA^{r-1}B\; d\nu^r(\{t\})$ it follows that some variables $\{z_i^r\}$, $i \neq j$, may jump.

Remark 5.1 *In the HOSP framework we allow for different restitution coefficients e_i for each variable z_i (in the SISO case), or each vector z^i (in the MIMO case). However we do not allow one coefficient $e_{i,j}$* per *component z_j^i*, i.e. *we take $e_{i,j} = e_i$ for all $1 \leqslant j \leqslant m$, $1 \leqslant i \leqslant r$. This is the same in Mechanics where Moreau's impact rule has one global coefficient, while other models may consider one coefficient for each constraint [18].*

5.2.2 *A MIMO example from Mechanics*

Let us consider the flexible mechanical system in Figure 1a. To simplify the presentation we assume that all five masses are equal to $M > 0$, and all springs stiffnesses are equal to $k > 0$. The masses coordinates are q_i, $1 \leqslant i \leqslant 5$, thus $n = 10$, and the two unilateral constraints defining the normal form "outputs" are $w_1(q,t) = q_5 - u_1(t)$, $w_2(q,t) = q_3 - u_2(t)$, $u(t) = (u_1(t), u_2(t), u_3(t))^T$, thus $m = 2$, $n = 10$, $p = 3$, $r = 2$, $\bar{r} = (2,2)^T$. The dynamics is given by:

$$\begin{cases} M\ddot{q}_1 = k(q_4 - q_1) + k(q_2 - q_1) + u_3(t) \\ M\ddot{q}_2 = k(q_3 - q_2) + k(q_1 - q_2) \\ M\ddot{q}_3 = k(q_2 - q_3) + \lambda_2 \\ M\ddot{q}_4 = k(q_1 - q_4) + k(q_5 - q_4) \\ M\ddot{q}_5 = k(q_4 - q_5) + \lambda_1. \end{cases} \tag{5.13}$$

The normal canonical form is given by:

$$\begin{cases} \dot{z}_1^1 = z_1^2 \\ \dot{z}_1^2 = \frac{k}{M}(\xi_5 - z_1^1) + \frac{\lambda_1}{M} - \ddot{u}_1(t) \\ \dot{z}_2^1 = z_2^2 \\ \dot{z}_2^2 = \frac{k}{M}(\xi_3 - z_2^1) + \frac{\lambda_2}{M} - \ddot{u}_2(t) \\ \dot{\xi} = A_\xi\xi + B_\xi z^1 + G_\xi\mathcal{U}(t), \end{cases} \tag{5.14}$$

where $z_1^1 = w_1 = q_5 - u_1(t)$, $z_2^1 = w_2 = q_3 - u_2(t)$, $z^1 = (z_1^1, z_2^1)^T$, $z^2 = (z_1^2, z_2^2)^T$, $\bar{z}_1 = (z_1^1, z_1^2)^T$, $\bar{z}_2 = (z_2^1, z_2^2)^T$, $CAB = I_2$, $\xi = (q_1\;\dot{q}_1\;q_2\;\dot{q}_2\;q_4\;\dot{q}_4)^T \in \mathbb{R}^6$, $A_\xi =$

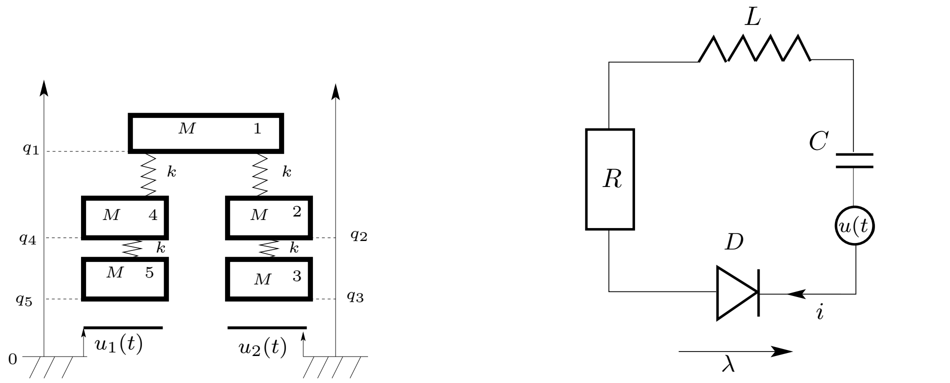
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{2k}{M} & 0 & \frac{k}{M} & 0 & \frac{k}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k}{M} & 0 & -\frac{2k}{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{k}{M} & 0 & 0 & 0 & -\frac{2k}{M} & 0 \end{pmatrix}$, $B_\xi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{k}{M} \\ 0 & 0 \\ \frac{k}{M} & 0 \end{pmatrix}$, $G_\xi \mathcal{U} = (0, \frac{u_3}{M}, 0, 0, 0, 0)^T$, $F = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, E has all entries equal to zero except $E_{2,3} = \frac{1}{M}$, so that $CE = 0_{2\times 3}$. The zero dynamics contains the dynamics of masses 1, 2, 4, connected by linear springs. In the context of unilaterally constrained mechanical systems, our framework is therefore more stringent than the usual second order sweeping process whose expression does not require second order derivatives of the time-dependent part of the constraints. Notice that the zero dynamics intervenes here also in the $\bar{z}$-dynamics (lines $\dot{z}_1^2$ and $\dot{z}_2^2$ in (5.14)). The matrix CAB is a Stieltjes matrix in this case. More generally, let (1.1) represent the dynamics of a (linear) Lagrangian system with mass matrix M, generalized coordinate $q \in I\!R^{\frac{n}{2}}$, subjected to the unilateral constraint $\tilde{C}q \geqslant 0$ (*i.e.*, $C = (\tilde{C}, 0_{m\times n/2})$). Then $CB = 0$, $CAB = \tilde{C}M^{-1}\tilde{C}^T \geqslant 0$. The sign of the off-diagonal terms depend on the kinetic angles between the constraints [18].

Lemma 5.1 *Consider a Lagrangian mechanical system with linear dynamics, generalized coordinates* $q \in I\!R^{\frac{n}{2}}$, *m unilateral constraints* $w = \tilde{C}q \geqslant 0$. *Its decoupling matrix* $\tilde{C}M^{-1}\tilde{C}^T$ *is a Stieltjes matrix if and only if the rows* $\tilde{C}_{i\bullet}$, $1 \leqslant i \leqslant m$, *are independent, and the kinetic angles* $\theta_{ij} = \pi - \arccos\left(\frac{\tilde{C}_{i\bullet}M^{-1}\tilde{C}_{j\bullet}^T}{\sqrt{\tilde{C}_{i\bullet}M^{-1}\tilde{C}_{i\bullet}^T}\sqrt{\tilde{C}_{j\bullet}M^{-1}\tilde{C}_{j\bullet}^T}}\right)$, $i \neq j$, *belong to* $\left[\frac{\pi}{2}, \pi\right]$. *In this case the vector relative degree is* $\bar{r} = (2, ..., 2)^T$.

Proof: Since the constraints are independent (which implies that $m \leqslant \frac{n}{2}$), we obtain $\tilde{C}M^{-1}\tilde{C}^T > 0$. The condition on the kinetic angles imply that $\tilde{C}_{i\bullet}M^{-1}\tilde{C}_{j\bullet}^T \geqslant 0$ for all i and j in $\{1, m\}$, hence the decoupling matrix is Stieltjes. The last statement follows from the fact that $CB = 0$. ■

Therefore the HOSP framework allows to show the well-posedness (existence and uniqueness of regular solutions) of linear mechanical systems subjected to a class of unilateral constraints. Our results depart from those in [25] since we show existence and uniqueness of RCSLBV solutions (and not RCLBV), and with an arbitrary number of constraints. Mechanical systems however have a specific structure that is not used in the HOSP formalism. Let $\bar{M} = \mathrm{diag}(M, M)$, $\bar{C} = (0_{m\times\frac{n}{2}}\ \tilde{C})$, $x_1 = q$, $x_2 = \dot{q}$, from which we infer that in (1.1), $B = \bar{M}^{-1}\bar{C}^T = \begin{pmatrix} 0_{\frac{n}{2}\times m} \\ M^{-1}\tilde{C}^T \end{pmatrix}$. When written at the velocity level, the unilateral constraints $w(t) = \tilde{C}x_1(t) \geqslant 0$ become $\dot{w}(t) = \bar{C}x(t) \geqslant 0$. From the complementarity constraints $0 \leqslant \bar{C}x(t) \perp \lambda(t) \geqslant 0 \Leftrightarrow \lambda(t) \in -N_{I\!R^m_+}(\bar{C}x(t))$, we deduce that the dynamics are a differential inclusion of the form $\dot{x}(t) - Ax(t) - Eu(t) \in -\bar{M}^{-1}\bar{M}B\, N_{I\!R^m_+}(B^T\bar{M}x(t)) = -\bar{M}^{-1}\ N_{\bar{\Phi}}(x(t))$, where $\bar{\Phi} = \{x \in I\!R^n \mid B^T\bar{M}x = \bar{C}x = \tilde{C}x_2 \geqslant 0\} = I\!R^{\frac{n}{2}} \times \Phi$, $\Phi = \{x_2 \in I\!R^{\frac{n}{2}} \mid \tilde{C}x_2 \geqslant 0\}$, and the chain rule for proper, lower semicontinuous convex functions has been used. This shows that mechanical systems with the velocity constraint

$\bar{C}x \geqslant 0$, possess a quite specific set-valued right-hand side due to the relationship between B and $\bar{C}$: $\bar{M}B = \bar{C}^T$, which is a passivity-like input/output constraint. However since the constraint is on the position, Moreau's sweeping process replaces the set $\bar{\Phi}$ by its tangent cone linearization cone. Since we have $N_{\bar{\Phi}}(x) = \begin{pmatrix} \partial_{x_1}\psi_\Phi(x_2) \\ \partial_{x_2}\psi_\Phi(x_2) \end{pmatrix} = \begin{pmatrix} 0^{\frac{n}{2}} \\ N_\Phi(x_2) \end{pmatrix}$, this boils down to set the set-valued right-hand side as $-\bar{M}^{-1}\begin{pmatrix} 0^{\frac{n}{2}} \\ N_{T_\Phi(x_1)}(x_2) \end{pmatrix}$, with $T_\Phi(x_1) = \{v \in I\!R^{\frac{n}{2}} \mid \tilde{C}_{i\bullet}v \geqslant 0, i \in \mathcal{I}(x_1\}$, $\mathcal{I}(x_1) = \{i \in \{1,m\} \mid \tilde{C}_{i\bullet}x_1 = 0\}$ is the index set of active constraints. We thus obtain the second order sweeping process in the original coordinates: $\dot{x}(t) - Ax(t) - Eu(t) \in -\bar{M}^{-1}\begin{pmatrix} 0^{\frac{n}{2}} \\ N_{T_\Phi(x_1)}(x_2) \end{pmatrix}$.
As shown in [19], a quite similar manipulation holds when the triplet (A, B, C) is positive real, allowing to transform (1.1) into a first order sweeping process.



(a) A mechanical system with two unilateral constraints .

(b) Circuit with an ideal diode and a voltage source.

Figure 1. Examples from mechanics and circuits.

5.2.3 *Complementarity switching DAEs*

Due to the constraint imposed on $d\nu_r$, the complementarity in Theorem 5.1 continues to hold in the MIMO case. Thus the contact LCP can be derived. Provided that $CA^{r-1}B$ is a P-matrix (which is guaranteed if it is a Stieljes matrix), the contact LCP in (5.5) which holds on time intervals where $z^1 = 0$ has a unique solution $\lambda(t)$ for any $\xi(t)$ and $\mathcal{W}(t)$, and it has 2^m modes corresponding to $\dot{z}_i^r(\tau^+) > 0$ and $\lambda_i(\tau) = 0$ (detachment from $z_i^1 = 0$), or $\dot{z}_i^r(\tau^+) = 0$ and $\lambda_i(\tau) > 0$ (contact remains active at $z_i^1 = 0$). Let us now assume that $z_i^1(\tau) = 0$ for all $i \in \mathcal{I} \subseteq \{1, m\}$, $z_1^j(\tau) > 0$ for all $j \in \bar{\mathcal{I}} = \{1, m\} \setminus \mathcal{I}$, and all $\tau \in]t, t+\sigma]$, $\sigma > 0$. We denote $\text{card}(\mathcal{I}) = m'$, $\text{card}(\bar{\mathcal{I}}) = \bar{m}$, $\bar{m} + m' = m$[7], $\lambda_{\mathcal{I}} = (\lambda_{i_1}, \lambda_{i_2}, ..., \lambda_{i_{m'}})^T$ with $i_j \in \mathcal{I}$, $z_{\mathcal{I}}^i$ is defined similarly as $z_{\mathcal{I}}^i = (z_{i_1}^i, z_{i_2}^i, ..., z_{i_{m'}}^i)^T$. We have $\lambda_{\mathcal{I}}(\tau) \geqslant 0$, $z_{\mathcal{I}}^1(\tau) = 0$, and $\lambda_{\bar{\mathcal{I}}}(\tau) = 0$, $z_{\bar{\mathcal{I}}}^1(\tau) > 0$, on $]t, t+\sigma]$. Thus we obtain

[7] Both m' and $\bar{m}$ are simple functions of time and state along the HOSP solutions.

from (5.9):

$$\dot{z}^r_{\mathcal{I}}(\tau) = (CA^rW^1)_{\mathcal{I}\bullet} z(\tau) + (CA^{r-1}B)_{\mathcal{I}\mathcal{I}}\lambda_{\mathcal{I}}(\tau) + (\bar{G}\mathcal{W}(\tau))_{\mathcal{I}} \quad (= 0), \tag{5.15}$$

where $(CA^rW^1)_{\mathcal{I}\bullet} \in I\!R^{m'\times n}$ is made of the rows of CA^rW^1 indexed in $\mathcal{I}$, $(CA^{r-1}B)_{\mathcal{I}\mathcal{I}} \in I\!R^{m'\times m'}$ is the principal submatrix of $CA^{r-1}B$ obtained by deleting rows and columns indexed in $\bar{\mathcal{I}}$, from which we infer the contact LCP for active constraints in $\mathcal{I}$ on $]t, t+\sigma[$:

$$0 \leqslant \lambda_{\mathcal{I}}(\tau) \perp (CA^rW^{-1})_{\mathcal{I}\bullet} z(\tau) + (CA^{r-1}B)_{\mathcal{I}\mathcal{I}}\lambda_{\mathcal{I}}(\tau) + (\bar{G}\mathcal{W}(\tau))_{\mathcal{I}} \geqslant 0. \tag{5.16}$$

Let us recall that if $CA^{r-1}B$ is a P-matrix, so is any of its principal submatrices and so are their inverses. Thus the contact LCPs as in (5.16) always have a unique solution, whatever the number of active constraints. By assumption we have on $]t, t+\sigma[$: $\lambda_{\mathcal{I}}(\tau) = -(CA^{r-1}B)^{-1}_{\mathcal{I}\mathcal{I}}\left[(CA^rW^{-1})_{\mathcal{I}\bullet} z(\tau) + (\bar{G}\mathcal{W}(t))_{\mathcal{I}}\right] \geqslant 0$. Similarly as for the case $m = 1$, the LCP in (5.16) rules the possible detachments from the active constraints at $\tau = t+\sigma$. It is noteworthy that $\lambda_{\bar{\mathcal{I}}}(\tau)$ may influence $z^i_{\bar{\mathcal{I}}}$, indeed we have:

$$\begin{array}{rl} \dot{z}^r_{\bar{\mathcal{I}}}(\tau) & = (CA^rW^{-1})_{\bar{\mathcal{I}}\bullet} z(\tau) + (CA^{r-1}B)_{\bar{\mathcal{I}}\mathcal{I}}\lambda_{\mathcal{I}}(\tau) + (\bar{G}\mathcal{W}(\tau))_{\bar{\mathcal{I}}} \\ & = [(CA^rW^{-1})_{\bar{\mathcal{I}}\bullet} - (CA^{r-1}B)_{\bar{\mathcal{I}}\mathcal{I}} CA^{r-1}B)^{-1}_{\mathcal{I}\mathcal{I}}(CA^rW^{-1})_{\mathcal{I}\bullet}]z(\tau) \\ & \quad +(\bar{G}\mathcal{W}(\tau))_{\bar{\mathcal{I}}} - (CA^{r-1}B)_{\bar{\mathcal{I}}\mathcal{I}}(CA^{r-1}B)^{-1}_{\mathcal{I}\mathcal{I}}(\bar{G}\mathcal{W}(t))_{\mathcal{I}} \end{array} \tag{5.17}$$

where $(CA^{r-1}B)_{\bar{\mathcal{I}}\mathcal{I}} \in I\!R^{(m-m')\times m'}$ is the submatrix of $CA^{r-1}B$ obtained from the rows indexed in $\bar{\mathcal{I}}$ and the columns indexed in $\mathcal{I}$. It represents the couplings between the unilateral constraints, and it makes the DAE vector field depend on the contact LCP solution. The contact DAE in a semi-explicit form is made of the dynamics in (5.10) with the constraint $z^1_{\mathcal{I}} = 0$. It follows that this is a DAE with (vector) index $(r+1, r+1, ..., r+1)^T \in I\!R^m$.

Therefore the HOSP is a DDI which switches between 2^m DAEs with index $r+1$, and with vector fields as in (5.15) (5.17), for varying index sets $\mathcal{I}$ and $\bar{\mathcal{I}}$. We might name it complementarity switching DAEs. It is noteworthy that if the contact LCP implies a switch between a DAE associated with the index set of active constraints $\mathcal{I}_1$ and another DAE with index set of active constraints $\mathcal{I}_2$, then one may have (a) $\mathcal{I}_1 \subset \mathcal{I}_2$, (b) $\mathcal{I}_1 \supset \mathcal{I}_2$ (c) $\mathcal{I}_1 \neq \mathcal{I}_2$ with neither (a) nor (b). Case (a) means that active constraints are added possibly implying jumps in $\bar{z}$ solution of the MDI (see calculations in section 6 on a particular case), case (b) means that some active constraints are deactivated (according to solutions of the contact LCP in (5.16) at time $t+\sigma$), case (c) is the more general situation with a switching event accompanied with state jumps and detachments ruled by the contact LCP. These various transitions are taken into account in the framework of tracking control of complementarity Lagrangian systems in [43].

Remark 5.2 *The complementarity switching DAEs can also be viewed in the original x-dynamics (1.1), with DAEs in semi-explicit form. It corresponds to modes* $w_i(x,u) = 0$ *and* $w_j(x,u) > 0$, $i, j \in \{1, ..., m\}$. *With each mode is associated a multiplier solution of the contact LCP, that modifies the DAE right-hand side. Getting back to the discrepancies with respect to switching DAEs as in [38, 58, 61, 60]: in the HOSP the choice of the DAEs vector fields is dictated in part by the complementarity problem in (5.16).*

5.3 HOSP with time-dependent-switching state feedback

Let us assume that on $[0,\tau_1[$, $\tau_1 \geqslant \delta > 0$, we apply the feedback $u_1(x,t) = K_1 x + v_1(t)$, and on $[\tau_1,\tau_2[$, $\tau_2 \geqslant \tau_1 + \delta$ for some $\delta > 0$, we apply $u_2(x,t) = K_2 x + v_2(t)$, for some matrices K_1, K_2 and exogenous signals $v_1(\cdot)$, $v_2(\cdot)$ satisfying Assumptions 4.1, or 1', or 1". This gives rise to the switched closed-loop dynamics (instead of (1.1)):

$$
\begin{array}{ll}
(a) \left\{ \begin{array}{l} \dot{x}(t) = (A+EK_1)x(t) + B\lambda + Ev_1(t) \\ 0 \leqslant w_1(t) = (C+FK_1)x(t) + Fv_1(t) \\ x(0^-) = x_0, \end{array} \right. & \text{for all } t \in [0,\tau_1[\\
(b) \left\{ \begin{array}{l} \dot{x}(t) = (A+EK_2)x(t) + B\lambda + Ev_2(t) \\ 0 \leqslant w_2(t) = (C+FK_2)x(t) + Fv_2(t) \end{array} \right. & \text{for all } t \in [\tau_1,\tau_2[.
\end{array}
\tag{5.18}
$$

Let us assume that each subsystem $(A+EK_i, B, C+FK_i)$, $i=1,2$, has a vector relative degree $\bar{r}_i = (r_i, r_i, ..., r_i)^T$, so that one can associate with each of them a canonical representation as in (2.6) after the transformation $z = W_i x + T_i \mathcal{U}_i$, $i=1,2$. Using Corollary 4.2, we can prove the well-posedness of the HOSP associated with the quadruple $(A+EK_1, B, C+FK_1, v_1(t))$ on $[0,\tau_1[$ for any x_0. Given any $x(\tau_1^-)$, the well-posedness of the HOSP associated with the quadruple $(A+EK_2, B, C+FK_2, v_2(t))$ on $[\tau_1,\tau_2[$ can be proved also. If τ_1 is equal to a state jump time for subsystem (5.18) (a), then one can apply the state reinitialization mapping (3.13) (5.11) and (2.1) to this subsystem to obtain $z_{s_1}(\tau_1^+)$ and $x(\tau_1^+) = W_1^{-1}(z_{s_1}(\tau_1^+) - T_1\mathcal{U}_1(\tau_1))$, where $z_{s_1} = (\bar{z}_{s_1}^T, \xi_{s_1}^T)^T$ denotes here the state variable of the canonical state representation (2.6) of subsystem (5.18) (a). Then in a second step one can apply the state reinitialization mapping (3.13) (5.11) and (2.1) for the subsystem (5.18) (b), considering $z_{s_2,0} \triangleq W_2 x(\tau_1^+) + T_2\mathcal{U}_2(\tau_1) = W_2 W_1^{-1}(z_{s_1}(\tau_1^+) - T_1\mathcal{U}_1(\tau_1)) + T_2\mathcal{U}_2(\tau_1)$ as an initial condition, compute $z_{s_2}(\tau_1^+)$ and obtain a reinitialized state $x(\tau_1^{++}) = W_2^{-1}(z_{s_2}(\tau_1^+) - T_2\mathcal{U}_2(\tau_1))$ for subsystem (5.18) (b), where the upperscript $++$ indicates that the jump mapping has been applied twice at $t = \tau_1$[8]. If τ_1 is not equal to a state jump time for subsystem (5.18) (a), then one can initialize subsystem (5.18) (b) with $x(\tau_1)$, with which is associated $z_{s_2}(\tau_1^-) \triangleq W_2^{-1}(x(\tau_1) - T_2\mathcal{U}_2(\tau_1))$. Therefore, we see that the HOSP state jump rule furnishes a natural way to switch between the two subsystems in (5.18), which are themselves complementarity switching DAEs. One can then define a sequence of switching times $\{\tau_i\}_{i\geqslant 0}$, $\tau_{i+1} \geqslant \tau_i + \delta$. We thus have proved the following:

Proposition 5.1 *Consider the unilaterally constrained system in (1.1). Let us define the sequence* $\{\tau_i\}_{i\geqslant 0}$, $\tau_{i+1} \geqslant \tau_i + \delta$, $\delta > 0$ *of switching times, and define* $u = K_{i+1}x + v_{i+1}$ *on* $[\tau_i, \tau_{i+1})$. *Assume that each subsystem* $(A+EK_i, B, C+FK_i)$ *has a vector relative degree* $\bar{r}_i = (r_i, ..., r_i)^T$, *and that* $(C+FK_i)(A+EK_i)^{r_i-1}B$ *is a Stieltjes matrix for each* i. *Assume further that the functions* $v_i : \mathbb{R}_+ \to \mathbb{R}^{p_i}$, *satisfy Assumptions 4.1, 1' or 1". Then the switching HOSP system admits a global, unique regular solution in the sense of Definition 4.1.*

[8] The need for two sequential state jumps has been noticed in [62, Lemma 3.1] in the context of switching DAEs, when a change of state space is performed in each mode.

Remark 5.3 *There is a fundamental difference between the exogenous switching times in this section, and the complementarity switching times (*i.e.*, the switching times due to the complementarity conditions): the latter admit accumulation times (where the crossing of the accumulation time is taken care of by the multiplier λ), while the former need some dwell time $\delta > 0$. An open problem is to design state-dependent switching times (other than the complementarity ones) between HOSPs as in (5.18). This however creates serious difficulties, like how to avoid finite accumulations or even an infinity of switching times at one instant (the so-called beating phenomenon in some classes of impulsives ODEs [8]), or continuation after such accumulations, or sliding modes,* etc.

Example 5.1 *Let us provide an academic example from circuits with ideal diodes. Let us consider the simple circuit in Figure 1b. Its dynamics is given by [18,* Example 5.16*]:*

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{R}{L}x_2(t) - \frac{1}{LC}x_1(t) - \frac{1}{L}\lambda(t) + \frac{1}{L}u(t) \\ 0 \leqslant w(t) = -x_2(t) \perp \lambda(t) \geqslant 0, \end{cases} \tag{5.19}$$

where $x_1(t)$ is the charge of the capacitor, $x_2(t) = i(t)$ is the current through the circuit, $u(t)$ is a voltage source. We first propose two dynamic feedback controllers:

$$(\mathbf{a}) \begin{cases} u_1(x,\lambda,t) = \lambda - Lx_3 + v_1(t) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = \lambda(t) \end{cases} \qquad (\mathbf{b}) \begin{cases} u_2(x,\lambda,t) = \lambda + Lx_3 - x_4(t) + v_2(t) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = \lambda(t) \end{cases} \tag{5.20}$$

The corresponding canonical dynamics are given by:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \begin{array}{l} -\frac{R}{L}x_2(t) - \frac{1}{LC}x_1(t) \\ -x_3(t) + \frac{v_1(t)}{L} \end{array} \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = \lambda(t) \\ w(t) = -x_2(t) \end{cases} \Leftrightarrow \begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = z_3(t) \\ \dot{z}_3(t) = -\frac{R}{L}z_3(t) - \frac{1}{LC}z_2(t) + \lambda(t) - \frac{\ddot{v}_1(t)}{L} \\ \dot{\xi}(t) = z_1(t) \\ z_1(t) = -x_2(t) \end{cases} \tag{5.21}$$

for (5.20) **(a)**, *and*

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \begin{array}{l} -\frac{R}{L}x_2(t) - \frac{1}{LC}x_1(t) + x_3(t) \\ -\frac{x_4(t)}{L} + \frac{v_2(t)}{L} \end{array} \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = \lambda(t) \\ w(t) = -x_2(t) \end{cases} \Leftrightarrow \begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = \begin{array}{l} -\frac{R}{L}z_2(t) - \frac{1}{L}z_1(t) - F_\xi\xi(t) \\ -G_\xi z_1(t) + \frac{\lambda(t)}{L} - \frac{\dot{v}_2}{L} \end{array} \\ \dot{\xi}(t) = A_\xi\xi(t) + B_\xi z_1(t) \\ z_1(t) = -x_2(t) \end{cases} \tag{5.22}$$

*for (5.20) (***(b)***, where $\xi(t) \in I\!R^2$, $F_\xi(sI_2 - A_\xi)^{-1}B_\xi + G_\xi = \frac{LCs^2+RCs-1}{Cs(s-L)}$, $s \in \mathbb{C}$, $(A_\xi, B_\xi, F_\xi, G_\xi)$ a minimal representation, and $x_4 = F_\xi\xi + G_\xi z_1$. The proof is outlined in section G. It is possible to apply Proposition 5.1 to a switching system with (5.21) and (5.22).*

6 The higher-order bouncing ball

In this section we consider a very particular form of the “input” term in (2.6) and of the chain of integrators, in order to mimic the mechanical bouncing ball. The focus is put on the existence of trajectories of the HOSP, which possess accumulations of state jumps (the solutions of the HOSP admit infinity of discontinuities in finite time, as functions of local bounded variations do). The results demonstrate that the dynamics of the HOSP may be quite complex when external excitation and non-zero restitution coefficients are considered. Notice that the exhibited complexity could not be shown in the autonomous framework of [1].

Definition 6.1 A left accumulation of state reinitialization times t_k at t, is an accumulation on the left of t. In other words, $\lim_{k\to+\infty} t_k = t$ with $t_k < t$ for all $k \geqslant 0$.

In Mechanics, the bouncing ball example shows that a constant input to a chain of two integrators, and a restitution $e \in (0,1)$ can produce left accumulations of velocity jumps. The position is time-continuous and made of portions of paraboloids, velocities are piecewise linear functions of time, and the force input signal is constant. It is of interest to study whether or not a constant input in (2.6) can also induce left accumulations of state jumps, when $r \geqslant 3$. We denote $T_r(s) = \sum_{i=0}^{r-1} CA^i E s^{r-1-i} + Fs^r$, where $s \in \mathbb{C}$ is the Laplace variable and we denote the Laplace transform as $\mathcal{L}[\cdot]$.

Lemma 6.1 *Consider $r \geqslant 2$ and $m = 1$. Let us consider the HOSP with external inputs and non zero restitution coefficients e_i in (3.5), with all $e_j = 0$ except $e_i > 0$ for some $i \in \{2, ..., r\}$. Assume that $F = 0$, $CA^r = 0$ and $CA^{r-1}B > 0$, and that $\mathcal{L}[u(t)] = \frac{a}{T_r(s)}$, with $a < 0$, a constant and $T_r(s) \neq 0$.* **(i)** *A unique regular solution exists globally for all $a < 0$ and any initial condition.* **(ii)** *Let $\{t_k\}$ be a strictly increasing sequence, with $\lim_{k\to+\infty} t_k < +\infty$. Then a solution $z(\cdot)$ satisfying for all $k \geqslant 0$: $z_1(t_k) = 0$, $\Delta_k = t_{k+1} - t_k > 0$, $z_i(t_k^-) < 0$, $z_1(t) > 0$ on $]t_k, t_{k+1}[$, and $\lim_{k\to+\infty} z_j(t_k^+) = 0$ for all $j \in \{1, ..., r\}$, exists if and only if*

$$e_i \in \left]0, \frac{(i-1)!(r-i+1)!}{r! - (i-1)!(r-i+1)!}\right[. \tag{6.1}$$

Proof: **(i)** The function $u(\cdot)$ is defined as the inverse Laplace transform $u(t) = \mathcal{L}^{-1}\left(\frac{a}{T_r(s)}\right)$. There exists a realization (M, N, P) of the transfer function $\frac{1}{T_r(s)}$ such that $\frac{1}{T_r(s)} = P(sI_{r-1} - M)^{-1}N$, $\dot{v}(t) = Mv(t) + Na$, $u(t) = Pv(t)$. Thus $v(\cdot)$ is analytic and so is $u(\cdot)$. Corollary 4.2 applies.

(ii) Only if : From the data of the Lemma, we get $T_r(\frac{d}{dt})u(t) = a < 0$ so that from (2.6) the system is a chain of integrators with a constant input a. The equality of measures in (3.10) is given by:

$$\begin{cases} dz_1 = z_2(t)dt + d\nu_1 \\ dz_2 = z_3(t)dt + d\nu_2 \\ \vdots \\ dz_{r-1} = z_r(t)dt + d\nu_{r-1} \\ dz_r = CA^{r-1}Bd\nu_r + a\,dt. \end{cases} \tag{6.2}$$

It is also worth noting that $z_i(t_k^-) < 0$ ($\Rightarrow z_i(t_k^+) > 0$) is necessary for the accumulation of "bounces" to exist, since $z_j(t_k^+) = 0$ for all $j \neq i$, due to $e_j = 0$. Integrating on $]t_k, t[$ we obtain

$$z_1(t) = \sum_{j=0}^{r-1} z_{r-j}(t_k^+) \frac{(t-t_k)^{r-1-j}}{(r-1-j)!} + \frac{a}{r!}(t-t_k)^r \tag{6.3}$$

on $[t_k, t_{k+1}[$. Since $z_1(t_k) = 0$ for all k and all $e_j = 0$ except e_i, we get from (6.3)

$$z_i(t_k^+) \frac{\Delta_k^{i-2}}{(i-1)!} + \frac{a}{r!}\Delta_k^{r-1} = 0 \tag{6.4}$$

where we also used that $i \geqslant 2$. It follows that

$$\Delta_k = \left(-\frac{r!}{a} \frac{z_i(t_k^+)}{(i-1)!} \right)^{\frac{1}{r+1-i}} \tag{6.5}$$

where we used that $0 \leqslant i-2 \leqslant r-2 < r-1$ and $\Delta_k > 0$ for all $k < +\infty$. Notice that the right-hand side of (6.5) is positive as $z_i(t_k^+) > 0$. Now we have

$$z_{r-\alpha}(t_{k+1}^-) = z_i(t_k^+) \frac{\Delta_k^{\alpha-r+i}}{(\alpha-r+i)!} + \frac{a}{(\alpha+1)!}\Delta_k^{\alpha+1} \tag{6.6}$$

for $\alpha \in \{0, ..., r-i\}$. Taking $\alpha = r-i$ and recalling from Proposition 3.2 that $z_i(t_{k+1}^+) = -e_i z_i(t_{k+1}^-)$ when $z_i(t_{k+1}^-) < 0$, we obtain from (6.6) and (6.5)

$$z_i(t_{k+1}^+) = e_i \left(-1 + \frac{r!}{(i-1)!(r-i+1)!} \right) z_i(t_k^+) \tag{6.7}$$

from which we deduce that

$$z_i(t_{k+1}^+) = \left(e_i \left(-1 + \frac{r!}{(i-1)!(r-i+1)!} \right) \right)^{k+1} z_i(t_0^+), \quad k \geqslant 0. \tag{6.8}$$

One checks that provided $z_i(t_0^+) > 0$ then $z_i(t_{k+1}^+) > 0$ for all $1 \leqslant k < +\infty$ as $\frac{r!}{(i-1)!(r-i+1)!} > 1$, see Lemma A.1 in the Appendix. The necessary part is proved since (6.7) is a geometric series. It also follows from (6.5) and (6.7) that

$$\Delta_k = -\frac{r!}{a(i-1)!} \bar{e}_i^k z_i(t_0^+) \tag{6.9}$$

with $\bar{e}_i = e_i \left(-1 + \frac{r!}{(i-1)!(r-i+1)!} \right)$. From (6.9) it follows that the total duration of the sequence of infinite impacts is bounded and equal to:

$$\sum_{k=0}^{+\infty} \Delta_k = \frac{-r!\; z_i(t_0^+)}{a(i-1)!} \frac{1}{1-\bar{e}_i}. \tag{6.10}$$

<u>If</u> : Let the system be initialized at $z(0^-)$ and let $z(0^+)$ be the solution of the state jump rule in Proposition 3.2. Then from (6.3) there exists $t^* < +\infty$ such that for $t > t^*$ one has $z_1(t) < 0$ and $z_1(t^*) = 0$. Since the reasoning applies to any initial condition on $z(\cdot)$, one deduces that there exists a contact time for the initial data $z_j(0^-) = 0$ and $z_i(0^-) < 0$ at some bounded $t_0 > 0$. Using (6.8) with $k = 0$ it follows that $z_i(t_1^-) < 0$ and $z_i(t_1^+) > 0$. And so on. Therefore (6.1) is sufficient for an accumulation to exist. ■

The Mechanical bouncing ball model implies $E = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = (1\ 0)$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so that $T_r(s) = \sum_{i=0}^{1} CA^i E s^{1-i} = 1$. In the case of a triple integrator $z_1^{(3)}(t) = u(t) + \lambda(t)$, one gets that either $e_2 = 0$ and $e_3 \in \left]0, \frac{1}{2}\right[$, or $e_3 = 0$ and $e_2 \in \left]0, \frac{1}{2}\right[$. If $i = r$ or $i = 2$ then $e_i \in \left]0, \frac{1}{r-1}\right[$. Since imposing $w(0) \geqslant 0$ implies that $w(\cdot)$ is continuous (as a consequence of (3.13) for $i = 1$, which implies that $\{z_1\}(t^+) = \{z_1\}(t^-)$ at all times $t \geqslant 0$ since $T_\Phi^0 = \Phi = I\!R_+$), one recovers the case of the bouncing ball where $r = i = 2$ so $e_2 \in]0, 1[$. If $i = \alpha + 1$ or $i = r - \alpha + 1$ for some α, then $e_i \in \left]0, \frac{1}{\frac{r!}{\alpha!(r-\alpha)!} - 1}\right[$ in (6.1). If $e_i \geqslant 0$ but it does not satisfy the constraint in (6.1), then the above accumulation may not exist ($e_i = 0$) or the solution may diverge ($e_i \geqslant \frac{(i-1)!(r-i+1)!}{r! - (i-1)!(r-i+1)!}$). Notice finally that after the finite time of state jump accumulation has been reached, the system may enter a phase of persistent contact with $\bar{z}(t) = 0$. Then the multiplier $\lambda(t) = \chi_r(t)$ is a solution of the contact LCP (see section 5) and takes care of maintaining the trajectories on the unilateral constraint boundary.

7 Fixed points, stability, positive invariance

7.1 Existence and uniqueness of fixed points

Consider the MDI in (3.10) (3.11) (or equivalently (4.3) (4.4) (3.11)) in a MIMO setting as (5.10), then fixed points $z^* = (\bar{z}^{*,T}, \xi^{*,T})^T$ have to satisfy the generalized equation:

$$\left\{ \begin{array}{l} 0 = WAW^{-1} z^* + N\chi^*(t) + H\mathcal{W}(t) \\ \bar{\chi}^*(t) \in - \begin{pmatrix} \partial\psi_{\Phi_m}(z^{1,*}) \\ \partial\psi_{T_{\Phi_m}(z^{1,*})}(z^{2,*}) \\ \vdots \\ \partial\psi_{T_{\Phi_m}^{r-1}(z^{1,*},\ldots,z^{r-1,*})}(z^{r,*}) \end{pmatrix} \end{array} \right. \tag{7.1}$$

for all $t \geqslant 0$, where N is in (4.4), $\bar{\chi} = (\chi_1, ..., \chi_r)^T \in I\!R^{rm}$, $\chi = (\bar{\chi}, 0_{n-rm})^T$.

Example 7.1 *Consider the classical bouncing ball dynamics:* $\dot{z}_1(t) = z_2(t)$, $\dot{z}_2(t) = u(t) + \lambda$, $0 \leqslant z_1(t) \perp \chi_2(t) \geqslant 0$, *where* $\lambda = \chi_2(t)$ *outside impact times. We suppose* $z_1(0) \geqslant 0$ *so that* $d\nu_1 = 0$. *Fixed points are solutions of* $\chi_2(t) = -u(t)$, $0 \leqslant z_1^* \perp$

$\chi_2(t) \geqslant 0$. *Suppose that* $u(t) < 0$, *then the unique fixed point is* $z_1^* = z_2^* = 0$ *with* $\chi_2^*(t) = -u(t) > 0$ *(this is the static equilibrium of the ball on the ground). This justifies why* $\bar{\chi}^*$ *may be time-dependent in (7.1) while the set-valued term is independent of time.*

Equilibria occur outside jump times, hence $\bar{z}^* = (z^{1,*,T}, 0_m, ..., 0_m)^T$, $\chi_i^*(t) = 0$, $1 \leqslant i \leqslant r-1$, $\chi_r(t) \geqslant 0$. This allows us to rewrite equivalently (7.1) as:

$$\begin{cases} (a) \ \ 0 \in \partial\psi_{\Phi}(z^{1,*}) \\ (b) \ \ 0 \in \partial\psi_{T_{\Phi_m}^{i-1}(z^{1,*},0,\ldots,0)}(0), \ \ 2 \leqslant i \leqslant r-1 \\ (c) \ \ 0 \in CA^r W^{-1}(z^{1,*}, 0_{r-1}, \xi^{*,T})^T + \bar{G}\mathcal{W}(t) + CA^{r-1}B\chi_r^*(t) \\ (d) \ \ 0 = A_\xi \xi^* + B_\xi z^{1,*} + G_\xi \mathcal{U}(t) \\ (e) \ \ \chi_r^*(t) \in -\partial\psi_{T_{\Phi_m}^{r-1}(z^{1,*},0^m,\ldots,0^m)}(0) \end{cases} \quad (7.2)$$

The inclusions in (7.2) (a) (b) are trivially satisfied since the right-hand sides are cones. Let $m = 1$. We can study first two cases: (i) $z_1^* > 0 \Rightarrow \chi_r(t) = 0$ and fixed points are solutions of $d_1 z_1^* + d_\xi^T \xi^* + \bar{G}\mathcal{W}(t) = 0$ and $A_\xi \xi^* + B_\xi z_1^* + G_\xi \mathcal{U}(t) = 0$. (ii) $z_1^* = 0 \Rightarrow \chi_r(t) \geqslant 0$ and fixed points are solutions of $0 \in d_\xi^T \xi^* + \bar{G}\mathcal{W}(t) + CA^{r-1}B\chi_r^*(t)$, $A_\xi \xi^* + G_\xi \mathcal{U}(t) = 0$, $\chi_r^*(t) \geqslant 0$ (we made use of (2.4) to simplify the equations). Various conditions can be derived from these equations to study the existence and uniqueness of equilibrium points.

Proposition 7.1 *Suppose that* $CA^{r-1}B > 0$, *and* A_ξ *is full rank. Let* $m = 1$, $d_1 - d_\xi^T A_\xi^{-1} B_\xi \neq 0$, *and assume that* $d_\xi^T A_\xi^{-1} G_\xi \mathcal{U}(t) - \bar{G}\mathcal{W}(t)$ *is non constant. Then necessarily* $z_1^* = 0$ *(the equilibrium, if it exists, necessarily occurs on the unilateral constraint boundary). On the boundary the equilibrium exists if and only if* $(CA^{r-1}B)^{-1}(d_\xi^T A_\xi^{-1} G_\xi \mathcal{U}(t) - \bar{G}\mathcal{W}(t)) \geqslant 0$.

Proof: Let $z_1^* > 0$, then $\chi_r^*(t) = 0$ and using (7.2) (c) (d) we obtain $z_1^* = (d_1 - d_\xi^T A_\xi^{-1} B_\xi)^{-1}(d_\xi^T A_\xi^{-1} G_\xi \mathcal{U}(t) - \bar{G}\mathcal{W}(t))$, which is not possible if the term between brackets is not constant. The second result follows from (7.2) (c) (d) (e), using (2.4). One obtains $(CA^{r-1}B)^{-1}(d_\xi^T A_\xi^{-1} G_\xi \mathcal{U}(t) - \bar{G}\mathcal{W}(t)) = \chi_r^*(t)$, while (c) (e) with $z_1^* = 0$ is equivalent to $\chi_r^*(t) \in I\!R_+$. The result follows. ■

Proposition 7.1 allows us to determine when the mixed LCP: $Ax^* + B\lambda^*(t) + Eu(t) = 0$, $0 \leqslant w^* = Cx^* + Fu(t) \perp \lambda^*(t) \geqslant 0$ has a unique solution $(x^*, \lambda^*(t))$ with $w^* = 0$, and $u(t)$ not identically zero (the exogenous term "pushes" the system on $\mathrm{bd}(\Phi_u)$, in a sort of static equilibrium on a moving constraint). In both (2.11) and (5.14) the matrix A_ξ is full rank. When $m \geqslant 2$, the rth row of WAW^{-1} in (2.4) is equal to $(D_1, D_2, ..., D_r, D_\xi)$ with $D_i \in I\!R^{m\times m}$, $1 \leqslant i \leqslant r$, $D_\xi \in I\!R^{m\times(n-rm)}$. Also $\bar{G} \in I\!R^{m\times(r+1)p}$, $G_\xi \in I\!R^{(n-mr)\times rp}$, and $\Phi = (I\!R^+)^m$. Then (7.2) (c) (d) (e) becomes:

$$\begin{cases} D_1 z^{1,*} + D_\xi \xi^* + \bar{G}\mathcal{W}(t) + CA^{r-1}B\chi_r(t) = 0 \\ A_\xi \xi^* + B_\xi z^{1,*} + G_\xi \mathcal{U}(t) = 0 \\ \chi_r(t) \in -\partial\psi_{T_\Phi^{r-1}(z^{1,*},0^m,\ldots,0^m)}(0^m) \end{cases} \quad (7.3)$$

In general one may have equilibria with $z_i^{1,*} = 0$ and $z_j^{1,*} > 0$, $i \neq j$. We do not study further the generalized equation (7.3), however conditions which guarantee that it has a unique solution are given in section 7.2.1, Remark 7.2.

7.2 Stability and stabilization

The canonical form (5.10) is obtained by considering λ as an "input" and w as an "output", therefore its use for control with $u(\cdot)$ differs from the usual case where the canonical form is obtained with the input $u(\cdot)$. In addition one has to cope with the unilateral constraints and the state jumps.

7.2.1 *Asymptotic stability* via *passification*

Let us consider the MIMO system in (5.10). We can write the associated MDI formalism as

$$dz = WAW^{-1}z(t)dt + \bar{N}d\nu + H\mathcal{W}(t)dt, \tag{7.4}$$

with $d\nu = (d\nu_1^T, ..., d\nu_r^T)^T \in I\!R^{rm}$, $d\nu_i \in I\!R^m$ [9], $\bar{N} = \begin{pmatrix} I_{m(r-1)} & 0_{(r-1)m\times m} \\ 0_{m\times m(r-1)} & CA^{r-1}B \\ 0_{(n-mr)\times m(r-1)} & 0_{(n-mr)\times m} \end{pmatrix} \in I\!R^{n\times mr}$, $CA^{r-1}B \in I\!R^{m\times m}$. We recall that $H\mathcal{W}(t) = \begin{pmatrix} 0^{m(r-1)} \\ \bar{G}\mathcal{W}(t) \\ G_\xi \mathcal{U}(t) \end{pmatrix}$, see (2.7) and (5.10), and that $\mathcal{W}(t)$ and $\mathcal{U}(t)$ involve $u^{(i)}(t)$ for $0 \leqslant i \leqslant r$. In the autonomous case, the stability of the trivial solution $z^* = 0$ is shown in [1, Proposition 10] under a positive real [17, Definition 2.29] condition of the triplet $(WAW^{-1}, \hat{N}, \hat{C})$, where when $m = 1$, $\hat{N} = \begin{pmatrix} \hat{\hat{N}} \\ 0_{(n-r)\times r} \end{pmatrix} \in I\!R^{n\times r}$, $\hat{\hat{N}} = \begin{pmatrix} I_{r-1} & 0^{r-1} \\ 0_{r-1} & CA^{r-1}B \end{pmatrix} \in I\!R^{r\times r}$ is a leading principal submatrix of N in (4.4), $\hat{C} = (I_r, 0_{r\times(n-r)})$, thus defining a system with r inputs and r outputs. Then the matrix $J = \begin{pmatrix} \hat{\hat{N}}^{-1} & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & J_\xi \end{pmatrix}$, $0 < J_\xi = J_\xi^T \in I\!R^{(n-r)\times(n-r)}$, is a solution of the KYP Lemma LMI associated with the triplet $(WAW^{-1}, \hat{N}, \hat{C})$ [17, Lemma 3.1]. It is noteworthy that this is different from the positive realness condition that is alluded to in Remark 4.2, which does not concern the same system. Let us extend this result to the MIMO case by letting $u = Ky + v$ for some $K \in I\!R^{p\times n}$ and $y = Lx$ is a measurable output and v the new input. The triplet (A, B, C) in (1.1) is transformed into the triplet $(\tilde{A}, B, \tilde{C}) \triangleq (A + EKL, B, C + FKL)$, which does not necessarily have the same relative degree as (A, B, C). Letting $v = 0$ and still denoting the vector relative degree of $(\tilde{A}, B, \tilde{C})$ as $\bar{r} = (r, ..., r)^T \in I\!R^m$, we can use the canonical transformation to

[9] Written here with some abuse of notation.

obtain:

$$
\begin{aligned}
&dz = W\tilde{A}W^{-1}z(t)dt + \tilde{N}d\nu, \;\; \tilde{N} = \begin{pmatrix} \tilde{\tilde{N}} \\ 0_{(n-rm)\times rm} \end{pmatrix} \in I\!R^{n\times rm} \\
&\tilde{\tilde{N}} = \begin{pmatrix} I_{m(r-1)} & 0_{m(r-1)\times m} \\ 0_{m\times m(r-1)} & \tilde{C}\tilde{A}^{r-1}B \end{pmatrix} \in I\!R^{rm\times rm}
\end{aligned}
\tag{7.5}
$$

with $\tilde{C}\tilde{A}^{r-1}B \in I\!R^{m\times m}$. The next proposition states the stability of the origin for the dynamical system $\mathbf{HOSP}(z_0, [0, T[)$, and extends the results of section 4.5 in [1] to the MIMO case with feedback. Let $\mathcal{C} \stackrel{\Delta}{=} (I_{mr}, 0_{mr\times(n-rm)})$, $\mathcal{A} \stackrel{\Delta}{=} W\tilde{A}W^{-1}$, then the triplet $(\mathcal{A}, \tilde{N}, \mathcal{C})$ defines a system with rm inputs and rm outputs. The next proposition states conditions under which one can find a common Lyapunov function for the HOSP seen as switching DAEs, with possible state jumps and solutions being higher degree distributions.

Proposition 7.2 *Suppose that the pair $(\mathcal{C}, \mathcal{A})$ is observable, the pair $(\mathcal{A}, \tilde{N})$ is controllable, and the matrix $\mathcal{A}$ is exponentially stable. If the transfer matrix $\mathcal{C}(sI_n - \mathcal{A})^{-1}\tilde{N}$ is strictly positive real, then the trivial solution of the (autonomous) HOSP is Lyapunov stable and globally attractive.*

Proof: From the KYP Lemma for SPR systems [17, Lemma 3.11] there exists $I\!R^{n\times n} \ni \mathcal{P} = \mathcal{P}^T > 0$ such that $\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} < 0$ and $\mathcal{P}\tilde{N} = \mathcal{C}^T$. The second equality implies that $\mathcal{P} = \begin{pmatrix} P & 0_{rm\times(n-rm)} \\ 0_{(n-rm)\times rm} & P_\xi \end{pmatrix}$ with $P\tilde{\tilde{N}} = I_{rm}$. Thus the strict positive realness implies that $\tilde{\tilde{N}} = P^{-1} > 0$, which in turn implies that $\tilde{C}\tilde{A}^{r-1}B > 0$. Let us now show that $\mathcal{P}$ defines a quadratic storage function for the autonomous HOSP (including state jumps), then derive a dissipation inequality. The transfer matrix being SPR, it is clear that on any interval (τ, s) not containing any state jump time, $s > \tau$, the dissipation equality $V(\{z\}(s)) - V(\{z\}(\tau)) = \int_{(\tau,s)} \underbrace{\frac{d\nu}{dt}(t)^T\mathcal{C}\{z\}(t)}_{=\chi_r(t)^T z^1(t)=0}\,dt + \int_{(\tau,s)}\{z\}(t)^T(\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A})\{z\}(t)dt$ holds with the storage function $V(\{z\}) = \frac{1}{2}\{z\}^T\mathcal{P}\{z\}$ [17, Example 4.65]. It is then sufficient to check that the storage function is non-increasing at state-discontinuity times. We can calculate that $V(t^+) - V(t^-) = (\{Z\}^r(t^+) + \{Z\}^r(t^-))^T\tilde{\tilde{N}}^{-1}(\{Z\}^r(t^+) - \{Z\}^r(t^-))$, where $\{Z\}^r = (\{z\}^1, \{z\}^2, ..., \{z\}^r)^T$ (see section 5.2). Using (7.5) we have $\{Z\}^r(t^+) - \{Z\}^r(t^-) = \tilde{\tilde{N}}d\nu(\{t\})$, hence we obtain $V(t^+) - V(t^-) = (\{Z\}^r(t^+) + \{Z\}^r(t^-))^T d\nu(\{t\}) = \sum_{i=1}^r \langle\{z\}^i(t^+) + \{z\}^i(t^-), \eta_i(t)\rangle$, with $\eta_i(t) \in -\partial\psi_{T^{i-1}_{\Phi_m}(\{Z\}^{i-1}(t^-))}(\zeta_i(t))$. We can rewrite $\{z\}^i(t^+) + \{z\}^i(t^-) = \zeta^i(t) - \frac{1-e_i}{1+e_i}\{z\}^i(t^-)$, and since $\{0\}$ belongs to the normal cone computed at any point of its domain of definition, it follows by maximal monotonicity of the normal cone mapping (to a convex set) that $\langle\{z\}^i(t^+) + \{z\}^i(t^-), \eta_i(t)\rangle \leqslant 0$ for all $1 \leqslant i \leqslant r$. Thus Lyapunov stability is shown. Next we have for all $s \geqslant 0$: $V(\{z\}(s)) - V(\{z\}(0)) \leqslant -\lambda_{\max}(-\mathcal{A}^T\mathcal{P} - \mathcal{P}\mathcal{A})\int_{[0,s]}||\{z\}(t)||^2 dt \leqslant -\frac{\lambda_{\max}(-\mathcal{A}^T\mathcal{P}-\mathcal{P}\mathcal{A})}{\lambda_{\min}(\mathcal{P})}\int_{[0,s]}V(\{z\}(t))dt$. Assume now that $V(t) \to V_\infty > 0$ as $t \to +\infty$. Thus there exists $T < +\infty$ and $\delta > 0$ such that $V(t) \geqslant \delta$ for all $t \geqslant T$. We obtain for all $s > T$:

$$\begin{array}{rl} V(\{z\}(s)) & \leqslant V(\{z\}(0)) - \frac{\lambda_{\max}(-\mathcal{A}^T\mathcal{P}-\mathcal{P}\mathcal{A})}{\lambda_{\min}(\mathcal{P})} \left(\int_{[0,T]} V(\{z\}(t))dt - \int_{[T,s]} V(\{z\}(t))dt \right) \\ & \leqslant V(\{z\}(0)) - \frac{\lambda_{\max}(-\mathcal{A}^T\mathcal{P}-\mathcal{P}\mathcal{A})}{\lambda_{\min}(\mathcal{P})} \int_{[0,T]} V(\{z\}(t))dt - \frac{\lambda_{\max}(-\mathcal{A}^T\mathcal{P}-\mathcal{P}\mathcal{A})}{\lambda_{\min}(\mathcal{P})} \delta(s-T) \end{array} \tag{7.6}$$

Letting $s \to +\infty$ in both sides yields a contradiction, showing that $V(t) \to 0$ and consequently $\{z\}(t) \to 0$ as $t \to +\infty$, for any initial condition. ■

Remark 7.1 *Continuity in the initial conditions usually does not hold in the MIMO case, a well-known fact in unilaterally constrained mechanical systems [*18, 9*]. From Lemma 5.1 we can even exhibit cases of well-posed HOSP which have solutions which are discontinuous with respect to the initial data due to the kinetic angle values [*18*]. This precludes in general the application of the Krasovskii-LaSalle invariance principle, except in cases where continuity holds [*36, Theorem 6.3.1*].*

Remark 7.2 *We do not investigate here conditions that guarantee the existence of K such that given (A, B, C) and L, then $(\mathcal{A}, \tilde{N}, \mathcal{C})$ is SPR (which is a novel kind of passification by feedback, where the passification of the operator $d\nu \mapsto \mathcal{C}z$ is done using the input $u(\cdot)$). The KYP Lemma yields a nonlinear matrix inequality in $\mathcal{P}$ and K. It is noteworthy that $(\mathcal{A}, \tilde{N}, \mathcal{C})$ may be SPR while (A, B, C) is not,* e.g. *for all systems with $r \geqslant 2$. We have exhibited a Lyapunov function and shown global stability of the origin, which proves that the origin is the unique fixed point under the above SPR condition. This could be shown directly as proved in the next lemma.*

Lemma 7.1 *Under the SPR constraint, the origin is the unique fixed point of the HOSP.*

Proof: *Let us consider (7.5). Mimicking (7.1), fixed points have to satisfy the generalized equation:*

$$\left\{ \begin{array}{l} 0^n = \mathcal{A}z^* + \tilde{N}\bar{\chi}^* \\ \bar{\chi}^* \in - \begin{pmatrix} \partial\psi_{\Phi_m}(z^{1,*}) \\ \partial\psi_{T_{\Phi_m}(z^{1,*})}(0) \\ \vdots \\ \partial\psi_{T^{r-1}_{\Phi_m}(z^{1,*},0,\ldots,0))}(0) \end{pmatrix} \end{array} \right. \tag{7.7}$$

with $z^* = (z^{1,*}, 0, ..., 0, \xi^{*,T})^T$. By SPRness we have $-\mathcal{A}^T\mathcal{P} - \mathcal{P}\mathcal{A} > 0 \Rightarrow -\mathcal{P}\mathcal{A} > 0$. From $\mathcal{P}\tilde{N} = \mathcal{C}^T$, we obtain $0^n = \mathcal{P}\mathcal{A}z^* + \mathcal{C}^T\bar{\chi}^* \Leftrightarrow -\mathcal{P}\mathcal{A} \begin{pmatrix} z^{1,*} \\ 0^m \\ \vdots \\ 0^m \\ \xi^* \end{pmatrix} = \begin{pmatrix} \bar{\chi}^* \\ 0^{(n-rm)\times rm} \end{pmatrix}$.

The inclusion in (7.7) implies that $-(\mathcal{P}\mathcal{A})_{1,1}z^{1,*} - (\mathcal{P}\mathcal{A})_{1,n-rm}\xi^* \in -\partial\psi_{\Phi_m}(z^{1,*}) \subset \mathbb{R}^m$ and $-(\mathcal{P}\mathcal{A})_{n-rm,1}z^{1,*} - (\mathcal{P}\mathcal{A})_{n-rm,n-rm}\xi^* = 0^{n-rm}$, with $-(\mathcal{P}\mathcal{A})_{1,1} > 0$ and $-(\mathcal{P}\mathcal{A})_{n-rm,n-rm} > 0$ since $-\mathcal{P}\mathcal{A} > 0$. Let us now denote the ith unit vector of $\mathbb{R}^m$

as $\mathbf{e}_i$*. We have* $-\partial\psi_{\Phi_m}(z^{1,*}) = \{v \in I\!R^m | v = \sum_{i\in\mathcal{I}} \alpha_i \mathbf{e}_i, \alpha_i \geqslant 0\}$*,* $\mathcal{I} = \{i \in \{1,m\} | z_i^{1,*} = 0\}$*. Assume that* $z_j^{1,*} > 0$ *for some indices* $j \in \mathcal{J} \triangleq \{1,m\} \setminus \mathcal{I} = \{j_1, ..., j_q\}$*,* $q \leqslant m$*. Then* $\mathbf{e}_j^T(-(\mathcal{PA})_{1,1} z^{1,*} - (\mathcal{PA})_{1,n-rm}\xi^*) = \sum_{i\in\mathcal{I}} \alpha_i \mathbf{e}_j^T \mathbf{e}_i = 0$*. Let us define* $\mathcal{J} = \{j_1, ..., j_q\}$*,* $E_{\mathcal{J}} = (\mathbf{e}_{j_1}, ..., \mathbf{e}_{j_q})$*, and* $z_{\mathcal{J}}^{1,*} = (z_{j_1}^{1,*}, ..., z_{j_q}^{1,*})^T$*. We get* $-E_{\mathcal{J}}^T(\mathcal{PA})_{1,1} E_{\mathcal{J}} z_{\mathcal{J}}^{1,*} - E_{\mathcal{J}}^T(\mathcal{PA})_{1,n-rm}\xi^* = 0^m$*, with* $-E_{\mathcal{J}}^T(\mathcal{PA})_{1,1}E_{\mathcal{J}} > 0$ *since* $q \leqslant m$*. From the second equality we get* $\xi^* = -(\mathcal{PA})_{n-rm,n-rm}^{-1}(\mathcal{PA})_{n-rm,1} E_{\mathcal{J}} z_{\mathcal{J}}^{1,*}$*, so that we obtain the equality* $-E_{\mathcal{J}}^T\{(\mathcal{PA})_{1,1} - (\mathcal{PA})_{1,n-rm}(\mathcal{PA})_{n-rm,n-rm}^{-1}(\mathcal{PA})_{n-rm,1}\} E_{\mathcal{J}} z_{\mathcal{J}}^{1,*} = 0^q$*. The term between bracket is the Schur complement of the matrix* $(\mathcal{PA})_{n-rm,n-rm}$ *in the matrix* $\begin{pmatrix} (\mathcal{PA})_{1,1} & (\mathcal{PA})_{1,n-rm} \\ (\mathcal{PA})_{n-rm,1} & (\mathcal{PA})_{n-rm,n-rm} \end{pmatrix} \in I\!R^{(n-(r-1)m)\times(n-(r-1)m)}$*. This last matrix is a principal submatrix of* $\mathcal{PA}$ *and is therefore negative definite [*13, Proposition 8.2.7*], consequently it has full rank* $n-(r-1)m$*. Now applying [*13, Fact 6.4.20*] we infer that rank*$[(\mathcal{PA})_{1,1} - (\mathcal{PA})_{1,n-rm}(\mathcal{PA})_{n-rm,n-rm}^{-1}(\mathcal{PA})_{n-rm,1}] = m$*, and we deduce that rank*$[E_{\mathcal{J}}^T\{(\mathcal{PA})_{1,1} - (\mathcal{PA})_{1,n-rm}(\mathcal{PA})_{n-rm,n-rm}^{-1}(\mathcal{PA})_{n-rm,1}\}E_{\mathcal{J}}] = m$*, and thus* $z_{\mathcal{J}}^{1,*} = 0^q$*. This is a contradiction and we infer that* $z^{1,*} = 0^m$*, so that* $\xi^* = 0$*: the only equilibrium is the origin.* ■

7.2.2 *Finite-time stabilization on* $\{x \in I\!R^n \mid w(x,t) = Cx + Fu(t) = 0\}$

Under the proposed feedback for Proposition 7.2 and Lemma 7.1 to apply, the origin in the z-dynamics is also the origin in the x-dynamics, and it belongs to the constraint boundary. We note however that $\bar{\chi}^* = 0^{rm}$, which may be unwanted in some applications. As shown in section 6 the constraint boundary may be attained in finite-time after an accumulation of state re-initialization times (a Zeno behaviour similar to what is observed in the bouncing-ball mechanical system [37]), and with a positive multiplier $\chi_r(t)$ to "maintain" the state on the constraint boundary (in the case of Lemma 6.1, one has $\chi_r^* = -a$, see also Example 7.1). In this setting the input $u(\cdot)$ contact LCP can be designed to force a particular mode (*i.e.*, a set $\mathcal{I}$) of the contact LCP in (5.16) through the term $\bar{G}\mathcal{W}(\cdot)$. Due to space limitations we do not study more deeply this approach, which is however an important field of investigation for all systems subjected to unilateral constraints (see [18, 43] for the case of tracking control of Lagrangian systems, [20, 50] for finite-time stabilization on constraints).

7.3 Positive invariance

Nagumo's theorem states a sufficient condition for the positive invariance of sets $\mathcal{S}$ [15], for ODEs $\dot{x}(t) = f(x(t),t)$ with unique solutions, as $f(x,t) \in T_{\mathcal{S}}(x)$ for all $x \in \mathcal{S}$, where $T_{\mathcal{S}}(x)$ is the tangent cone (or in a more general setting the contingent cone [6]) to $\mathcal{S}$ at x. When the system is autonomous the condition is also necessary [34, 5]. Its geometric interpretation is clear: on the boundary of $\mathcal{S}$ the vector field points inwards $\mathcal{S}$. It extends to systems with non-unique solutions, to differential inclusions [6, 14], impulsive differential inclusions [7], ODEs with distributions [34], with the so-called viability property. For the second order sweeping process applied to Lagrangian systems, Moreau's viability Lemma holds [18, Lemma 5.1] [47, Proposition 2.4], which states that if the initial position is admissible, then it is sufficient that the velocity belongs to the tangent cone linearization cone for almost all times, to guarantee that positions are admissible for all times. The HOSP is a differential inclusion which guarantees that $z^1(t^+) \in \Phi_m$ for any initial condition $\{z\}(0^-)$ (or, in the x-dynamics (1.1), the positive invariance of the set $\Phi_u = \{x \in \mathbb{R}^n \mid Cx + Fu \geqslant 0\}$). Moreau's viability Lemma is extended as follows.

Lemma 7.2 *Assume that z is a solution of HOSP($z_0; I$), that $z^1(0) \in \Phi_m$, and that $\{z^2\}(t) \in T_{\Phi_m}(z^1(t))$ for Lebesgue-almost all $t \in I$. Then $z^1(t) \in \Phi_m$ for all $t \in I$.*

Proof: Suppose that there exists $t_1 > 0$ such that $z^1(t_1) \notin \Phi_m$, that is there exists a nonempty set $\mathcal{J} = \{j_1, j_2, ..., j_q\}$ such that $z^1_{j_i}(t_1) < 0$ for all $j_i \in \mathcal{J}$. By absolute continuity of $z^1(\cdot)$ we have $z^1_{j_i}(t_1) - z^1_{j_i}(0) = \int_{[0,t_1]}\{z^2_{j_i}\}(s)ds < 0$ since by assumption $z^1_{j_i}(0) \geqslant 0$. From the definition of the tangent cone and the fact that $T_{\Phi_m}(Z^i) = \times_{k=1}^m T^i_\Phi(Z^i_k)$, it follows that almost everywhere $\{z^2_{j_i}\}(t) \in T_\Phi(z^1_{j_i}(t)) = \mathbb{R}_+$. Hence a contradiction and we infer that $\mathcal{J} = \emptyset$. ■

Let us now prove the following result, which we may name the HOSP Viability Lemma of order 3, while Lemma 7.2 may be named the Viability Lemma of order 2.

Lemma 7.3 *Let z be a solution of HOSP($z_0; I$), with $z^1(\cdot)$, $\{z^2\}(\cdot)$ and $\{z^3\}(\cdot)$ absolutely continuous on $I = [0, T[$, $T > 0$. Assume that $z^1(0) \in \Phi_m$, $\{z^2\}(0) \in T_{\Phi_m}(z^1(0))$, and $\{z^3\}(t) \in T^2_{\Phi_m}(z^1(t), \{z_2\}(t))$ for all $t \in I$. Then $z^1(t) \in \Phi_m$ for all $t \in I$ and $\{z_2\}(t) \in T_{\Phi_m}(z^1(t))$ for all $t \in I$.*

Proof: First of all we remark that since $T_{\Phi_m}(Z^i) = \times_{k=1}^m T^i_\Phi(Z^i_k)$ we can do the proof for the case $m = 1$ in order to simplify the presentation. We can therefore adopt the notation for the SISO case in the following. Let us consider two cases: **a)** $z_1(0) = 0$ with subcases **a1)** $\{z_2\}(0) = 0$, and **a2)** $\{z_2\}(0) > 0$, and **b)** $z_1(0) > 0$. Let us analyse case **b)**. If $z_1(t) > 0$ for all $t \in I$ then $T_\Phi(z_1(t)) = \mathbb{R}$ for all $t \in I$ and $\{z_2\}(t) \in T_\Phi(z_1(t))$ for all $t \in I$. If there exists $t_1 > 0$ such that $z_1(t_1) = 0$, then we can go to case **a)**, changing $t = t_1$ to $t = 0$. Let us therefore analyse case **a)**. In case **a2)** we have $z_1(t) - z_1(0) = \int_0^t \{z_2\}(s)ds$ for all $t \in I$, by the absolute continuity. Also $\{z_2\}(t) > 0$ in a right neighborhood of $t = 0$, so that $z_1(t) > 0$ in this neighborhood, and we are back to case **b)**. In case **a1)**, we have $z_1(0) = \{z_2\}(0) = 0$ so $\{z_3\}(0) \geqslant 0$. We consider two subcases **a11)** $\{z_3\}(0) = 0$, and **a12)** $\{z_3\}(0) > 0$. Let us start with **a12)**. In a right neighborhood of $t = 0$ we have by continuity $\{z_3\}(t) > 0$, consequently $\{z_2\}(t) > 0$ and $z_1(t) > 0$ in this neighborhood, since $\{z_2\}(t) - \{z_2\}(0) = \int_0^t \{z_3\}(s)ds$ and $z_1(t) - z_1(0) = \int_0^t \{z_2\}(s)ds$ for all $t \in I$. So we are back to case **b)**. We can now split case **a11)** into three subcases: **a111)** $\{z_4\}(0) = 0$, **a112)** $\{z_4\}(0) > 0$, **a113)** $\{z_4\}(0) < 0$. In case **a113)** we have $\{z_3\}(t) - \{z_3\}(0) = \int_0^t \{z_4\}(s)ds$ so that $\{z_3\}(t) = \int_0^t \{z_4\}(s)ds < 0$ in a right neighborhood of $t = 0$. However by assumption $\{z_3\}(t) \in T^2_\Phi(z_1(t), \{z_2\}(t))$ for all $t \in I$, so in particular $\{z_3\}(0) \in T^2_\Phi(0, 0) = \mathbb{R}_+$ so that $\{z_3\}(0) \geqslant 0$ in a right neighborhood of $t = 0$: this is a contradiction and we infer that case **a113)** is impossible. Thus only **a111)** and **a112)** are possible. In case **a112)** we have $z_1(0) = \{z_2\}(0) = \{z_3\}(0) = 0$ and

$\{z_4\}(0) > 0$. Again by integrating in a right neighborhood of $t = 0$ we get $\{z_3\}(t) > 0$, $\{z_2\}(t) > 0$ and $\{z_1\}(t) > 0$ in this neighborhood and we are back to case **b)**. Case **a111)** is $z_1(0) = \{z_2\}(0) = \{z_3\}(0) = \{z_4\}(0) = 0$. As we saw in case **a113)** we cannot have $\{z_4\}(t) < 0$ in a right neighborhood of $t = 0$. Thus we can only get $\{z_4(t)\} \geqslant 0$, *i.e*, either $\{z_4\}(t) = 0$ almost everywhere or not in a right neighborhood of $t = 0$. Integrating again on $[0, t]$ for some small enough $t > 0$ we infer that in this neighborhood $z_1(t) \in \Phi$ with either $z_1(t_1) = 0$ or $z_1(t_1) > 0$ for some $t_1 > 0$, and in both cases using similar argulents as above we deduce that $\{z_2(t)\}(t) \in T_\Phi(z_1(t))$ for all $t \in [0, t_1]$. Since the solution is supposed to exist on I the result is proved. ■

Viability Lemmas of higher order could be proved in a similar way.

8 Conclusions

In this article we have shown that the autonomous high-order sweeping process introduced in [1], can be extended when an exogenous term (a control, or a disturbance $u(\cdot)$) acts in the smooth dynamics as well as in the inequality function, and is well-posed provided that the exogenous term satisfies some analycity conditions. A detailed well-posedness analysis is presented. The link with complementarity systems and switching DAEs is studied. Stability and stabilization by state feedback issues are analysed under positive real constraints. The so-called high-order bouncing ball illustrates how complex such differential inclusions (with distribution solutions) may be, with Zeno behaviours for transitions between constrained and unconstrained modes. It is noteworthy that the time-discretization of the HOSP which is detailed in [1], also applies to the non-autonomous case studied above.

Topics of future investigations are numerous: relax the uniqueness of solutions which is not a crucial property for stability; relax the Stieltjes property of the matrix $CA^{r-1}B$ in the MIMO case; analyse the MIMO case with vector relative degree $\bar{r} = (r_1, r_2, ..., r_m)^T$, with $r_i \neq r_j$, $i \neq j$, which would allow to switch between DAEs with different indices; find conditions such that switching feedback controllers allow the HOSP to coincide with given switching DAEs and/or Linear Complementarity Systems (said another way: try to recast larger classes of switching DAEs or LCSs into an HOSP framework); use switching strategies in the HOSP with time-dependent switching state feedback controllers, to study new stabilization strategies, with possibly non monotonic Lyapunov functions; study cases of globally well-posed nonlinear HOSP (involving products of distributions) relying on nonstandard analysis [32, 10, 11]; analyse the case of mixed unilateral and bilateral (equality) constraints; analyse switching between systems (HOSP) with varying state dimension (switching dynamic feedback controllers).

Appendix A Auxiliary lemma

Lemma A.1 *Let* $r \geqslant 2$ *and* $i \in \{2, ..., r\}$. *Then* $\frac{r!}{(i-1)!(r-i+1)!} > 1$. *Consequently one has* $\frac{(i-1)!(r-i+1)!}{r!-(i-1)!(r-i+1)!} > 0$.

Proof: A simple calculation shows that when $i = 2$ and $i = r$ then $\frac{r!}{(i-1)!(r-i+1)!} = r$.

Let us consider first $r = 2\alpha + 1$, $\alpha \in \mathbb{N}$, $\alpha \geqslant 1$, and $i \leqslant \frac{r}{2} + 1 \Leftrightarrow i \leqslant \alpha + 1$ since i and α are integers. Then $r - i + 1 \geqslant \alpha + 1$. Thus $r - i + 1 \geqslant i$. Let us examine the term $(i-1)!(r-i+1)!$. Let us increment i to $i+1$. We get $i!(r-i)!$. Doing so the term $(i-1)!$ is multiplied by i while the term $(r-i+1)!$ is divided by $r-i+1$. Since $r-i+1 \geqslant i$ one finds that $i!(r-i)! \leqslant (i-1)!(r-i+1)!$. By induction it follows that the maximum value is attained at $i = 2$. The problem is symmetric in the sense that $(i-1)!(r-i+1)!$ has the same value for $i = k$ and $i = r - k + 2$. By the symmetry one may now consider $r \geqslant i \geqslant \frac{r}{2} + 1$ and conclude similarly. The reasoning can be applied for $i = 2, 3, ..., k$ with $k < \frac{r}{2} + 1$, which shows that the maximum is indeed attained for $i = 2$ and $i = r$. In the even case with $r = 2\alpha$, $\alpha \in \mathbb{N}$, $\alpha \geqslant 1$, one gets $i < \frac{r}{2} + 1 \Leftrightarrow i \leqslant \alpha$ so that $r - i + 1 > \alpha$ and $r - i + 1 > i$. A reasoning similar to the odd case applies and by symmetry one considers integers $r \geqslant i \geqslant \alpha + 2$. For $i = \alpha + 1$ one gets $\frac{r!}{(i-1)!(r-i+1)!} = \frac{r!}{\alpha!\alpha!} > 1$ for all $r \geqslant 2$. Therefore $-1 + \frac{r!}{(i-1)!(r-i+1)!} > 0$, and the last statement follows.

■

Appendix B Calculations for the proof of i) of Theorem 4.2

Let us consider (4.6). From Lemma E.1 we obtain

$$\|\{z\}(t)\|^2 \leqslant \|z_0\|^2 + \alpha t + (2\Lambda + 1) \int_0^t (\|z_0\|^2 + \alpha s) e^{(2\Lambda+1)(t-s)} ds. \tag{B 1}$$

Integrating by parts one gets

$$\int_0^t s e^{-(2\Lambda+1)s} ds = t e^{-(2\Lambda+1)t} - \frac{1}{(2\Lambda+1)^2} \left(e^{-(2\Lambda+1)t} - 1 \right).$$

One also computes that

$$(2\Lambda + 1) \|z_0\|^2 \int_0^t e^{(2\Lambda+1)(t-s)} ds = \|z_0\|^2 \left(e^{(2\Lambda+1)t} - 1 \right).$$

From these two expressions one can obtain the upperbound in i) of Theorem 4.2.

Appendix C Some mathematical definitions

The next notions may be found in [1, 48, 39] Let I denote a non-degenerate real interval (not empty nor reduced to a singleton).

• By $z \in BV(I; \mathbb{R}^n)$ it is meant that z is a $\mathbb{R}^n$-valued function of Bounded Variation if there exists a constant $C > 0$ such that for all finite sequences $t_0 < t_1 < .... < t_N$ (N arbitrary) of points of I, we have

$$\sum_{i=1}^{N} \|z(t_i) - z(t_{i-1})\| \leqslant C.$$

Let J be a subinterval of I. The real number

$$\mathrm{var}(z, J) \triangleq \sup \sum_{i=1}^{N} \|z(t_i) - z(t_{i-1})\|,$$

where the supremum is taken with respect to all the finite sequences $t_0 < t_1 < ... < t_N$ (N arbitrary) of points of J, is called the variation of z in J.

Any BV function has a countable set of discontinuity points and is almost everywhere differentiable. A BV function defined on $[a, b] \subset I$ possesses left-limits in $]a, b]$ and right-limits in $[a, b[$. Moreover, the functions $t \mapsto z(t^+) \triangleq \lim_{s\to t, s>t} z(s)$ and $t \mapsto z(t^-) \triangleq \lim_{s\to t, s<t} z(s)$ are both BV functions.

• We denote by $LBV(I; \mathbb{R}^n)$ the space of functions of Locally Bounded Variation, *i.e.* of bounded variation on every compact subinterval of I.

• We denote by $RCLBV(I; \mathbb{R}^n)$ the space of Right-Continuous functions of Locally Bounded Variation. It is known that if $z \in RCLBV(I; \mathbb{R}^n)$ and $[a, b]$ denotes a compact subinterval of I, then z can be represented in the form:

$$z(t) = \mathcal{J}_z(t) + [z](t) + \zeta_z(t), \text{forall } t \in [a, b],$$

where $\mathcal{J}_z$ is a jump function, $[z]$ is an absolutely continuous function and ζ_z is a singular function. Here $\mathcal{J}_z$ is a jump function in the sense that $\mathcal{J}_z$ is right-continuous and given any $\varepsilon > 0$, there exist finitely many points of discontinuity $t_1, ..., t_N$ of $\mathcal{J}_z$ such that $\sum_{i=1}^{N} \|\mathcal{J}_z(t_i) - \mathcal{J}_z(t_i^-)\| + \varepsilon >$ $\mathrm{var}(\mathcal{J}_z, [a, b])$, $[z]$ is an absolutely continuous function in the sense that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^{N} \|[z](\beta_i) - [z](\alpha_i)\| < \varepsilon$, for any collection of disjoint subintervals $]\alpha_i, \beta_i] \subset [a, b] (1 \leqslant i \leqslant N)$ such that $\sum_{i=1}^{N} (\beta_i - \alpha_i) < \delta$, and ζ_z is a singular function in the sense that ζ_z is a continuous and bounded variation function on $[a, b]$ such that $\dot{\zeta}_z = 0$ almost everywhere on $[a, b]$.

• By $z \in RCSLBV(I; \mathbb{R}^n)$ it is meant that z is a Right-Continuous function of Special Locally Bounded Variation, i.e. z is of bounded variation and can be written as the sum of a jump function and an absolutely continuous function on every compact subinterval of I. So, if $z \in RCSLBV(I; \mathbb{R}^n)$ then

$$z = [z] + \mathcal{J}_z \tag{C 1}$$

where $[z]$ is a locally absolutely continuous function called the absolutely continuous component of z and $\mathcal{J}_z$ is uniquely defined up to a constant by

$$\mathcal{J}_z(t) = \sum_{t \geqslant t_n} z(t_n^+) - z(t_n^-) = \sum_{t \geqslant t_n} z(t_n) - z(t_n^-) \tag{C 2}$$

where $t_1, t_2, ..., t_n, ...$ denote the countably many points of discontinuity of z in I. Notice that the notion of solutions that is used in [58, 60] for switched DAEs (with exogenous switching times) has the same structure as (C 1) with a "smooth" and a jump parts (without finite accumulations of jump instants).

Differential (or Stieltjes) measure. Let $z \in LBV(I; \mathbb{R}^n)$ be given. We denote by dz the Stieltjes or differential measure generated by z. For $a \leqslant b$, $a, b \in I$ one has $dz([a, b]) =$

$z(b^+) - z(a^-)$, $dz([a,b[) = z(b^-) - z(a^-)$, $dz(]a,b]) = z(b^+) - z(a^+)$, $dz(]a,b[) = z(b^-) - z(a^+)$. In particular, we have $dz(\{a\}) = z(a^+) - z(a^-)$.

Appendix D Proof of (4.11)

One has $F(t) = \int_0^t e^{-A_\xi s} G_\xi \mathcal{U}(s) ds$, so that from the analycity property

$$e^{A_\xi t} F(t) = \sum_{k=0}^{\infty} \left(e^{A_\xi \cdot} F(\cdot)\right)^{(k)} (0) \frac{t^k}{k!} = \sum_{k=1}^{\infty} (e^{A_\xi \cdot} F(\cdot))^{(k)} (0) \frac{t^k}{k!}. \tag{D 1}$$

From the Leibniz formula and since $F(0) = 0$ we deduce that

$$e^{A_\xi t} F(t) = \sum_{k=1}^{\infty} \left(\sum_{i=0}^{k-1} C_k^i A_\xi^i F^{(k-i)}(0) \right) \frac{t^k}{k!}. \tag{D 2}$$

Let us now calculate $F^{(k)}(0)$ for $k \geqslant 1$. From the Leibniz formula

$$F^{(k+1)}(0) = \sum_{i=0}^{k} C_k^i (-1)^i A_\xi^i G_\xi \mathcal{U}^{(k-i)}(0), \tag{D 3}$$

for all $k \geqslant 0$. Consequently for $0 \leqslant i \leqslant k-1$:

$$F^{(k-i)}(0) = \sum_{j=0}^{k-i-1} C_{k-i-1}^j (-1)^j A_\xi^j G_\xi \mathcal{U}^{(k-i-1-j)}(0), \tag{D 4}$$

and we obtain:

$$e^{A_\xi t} F(t) = \sum_{k=1}^{\infty} \left(\sum_{i=0}^{k-1} C_k^i A_\xi^i \sum_{j=0}^{k-i-1} C_{k-i-1}^j (-1)^j A_\xi^j G_\xi \mathcal{U}^{(k-i-1-j)}(0) \right) \frac{t^k}{k!}, \tag{D 5}$$

as well as (4.11) using (D 1), (D 2) and (D 4).

Appendix E Gronwall's Lemma

Lemma E.1 *[33, Lemma 4] Let $x(\cdot)$, $f(\cdot)$, $g(\cdot)$ and $y(\cdot)$ be real valued functions defined on an interval $[a,b]$ and either continuous or of bounded variation. Let $g(\cdot)$ and $y(\cdot)$ be nonnegative and let η be a nondecreasing continuous functional defined on $[a,b]$. If for all t in $[a,b]$*

$$x(t) \leqslant f(t) + g(t) \int_a^t y(s) x(s) d\eta \tag{E 1}$$

then

$$x(t) \leqslant f(t) + g(t) \int_a^t f(s) y(s) \exp\left(\int_s^t g(r) y(r) dr \right) d\eta(s) \tag{E 2}$$

■

Appendix F Normal and tangent cones (MIMO systems)

Let us explain how the normal and tangent cones may be calculated in the MIMO case $m = 2$, that is $\Phi_2 = \mathbb{R}^2_+$. For the numerical integration of the HOSP, such calculations are not necessary, since this is taken care of by solving linear complementarity problems (see [1, Section 5] for details on the time-discretization of the HOSP). We have $z^1 = (w_1, w_2)^T = (z^1_1, z^1_2)^T$, $z^2 = (z^2_1, z^2_2)^T$, $z^3 = (z^3_1, z^3_2)^T$. The tangent cones are given by:

$$T_{\Phi_2}(z^1) = \begin{cases} \mathbb{R}^2 & \text{if} \quad z^1_1 > 0, z^1_2 > 0 \\ \mathbb{R}_+ \times \mathbb{R} & \text{if} \quad z^1_1 = 0, z^1_2 > 0 \\ \mathbb{R} \times \mathbb{R}_+ & \text{if} \quad z^1_1 > 0, z^1_2 = 0 \\ \mathbb{R}_+ \times \mathbb{R}_+ & \text{if} \quad z^1_1 = 0, z^1_2 = 0 \end{cases} = \times_{i=1,2} T_\Phi(z^1_i) \tag{F 1}$$

$$T^2_{\Phi_2}(z^1, z^2) = T^2_{\Phi_2}(Z^2) = T_{T_{\Phi_2}(z^1)}(z^2) =$$

$$\begin{cases} T_{\mathbb{R}^2}(z^2) = \mathbb{R}^2 & \text{if} \quad z^1_1 > 0, z^1_2 > 0 \\ T_{\mathbb{R}_+ \times \mathbb{R}}(z^2) = \begin{cases} \mathbb{R}^2 & \text{if } z^2_1 > 0 \\ \mathbb{R}_+ \times \mathbb{R} & \text{if } z^2_1 = 0 \end{cases} & \text{if} \quad z^1_1 = 0, z^1_2 > 0 \\ T_{\mathbb{R} \times \mathbb{R}_+}(z^2) = \begin{cases} \mathbb{R}^2 & \text{if } z^2_2 > 0 \\ \mathbb{R} \times \mathbb{R}_+ & \text{if } z^2_2 = 0 \end{cases} & \text{if} \quad z^1_1 > 0, z^1_2 = 0 \\ T_{\mathbb{R}_+ \times \mathbb{R}_+}(z^2) = \begin{cases} \mathbb{R}^2 & \text{if } z^2_1 > 0, z^2_2 > 0 \\ \mathbb{R}_+ \times \mathbb{R} & \text{if } z^2_1 = 0, z^2_2 > 0 \\ \mathbb{R} \times \mathbb{R}_+ & \text{if } z^2_1 > 0, z^2_2 = 0 \\ \mathbb{R}_+ \times \mathbb{R}_+ & \text{if } z^2_1 = 0, z^2_2 = 0 \end{cases} & \text{if} \quad z^1_1 = 0, z^1_2 = 0. \end{cases} \tag{F 2}$$

The normal cones are calculated accordingly, for instance:

$$\partial\psi_{T_{T_{\Phi_2}(z^1)}(z^2)}(z^3) = \begin{cases} N_{\mathbb{R}^2}(z^3) = \{0\} & \begin{array}{ll} \text{if} & z^1_1 > 0, z^1_2 > 0 \\ \text{or} & z^2_2 > 0, z^1_1 > 0, z^1_2 = 0 \\ \text{or} & z^2_1 > 0, z^2_2 > 0, z^1_1 = 0, z^1_2 = 0 \end{array} \\ N_{\mathbb{R}_+ \times \mathbb{R}}(z^3) & \begin{array}{ll} \text{if} & z^2_1 = 0, z^1_1 = 0, z^1_2 > 0 \\ \text{or} & z^2_1 = 0, z^2_2 > 0, z^1_1 = 0, z^1_2 = 0 \end{array} \end{cases} \tag{F 3}$$

with $N_{\mathbb{R}_+ \times \mathbb{R}}(z^3) = \begin{cases} \mathbb{R}_- \times \{0\} & \text{if} \quad z^3_1 = 0 \\ \{0^2\} & \text{if} \quad z^3_1 > 0 \end{cases}$.

Appendix G Calculation of the canonical form (5.22)

Starting from the x-dynamics in (5.22), one calculates $w(s) = \frac{Cs^2 - CLs}{s^2(LCs^2+RCs-1)}\lambda(s) - \frac{Cs^3}{s^2(LCs^2+RCs-1)}v_2(s)$. Starting from the z_2-dynamics one finds that $w(s) = z_1(s) = \frac{Cs^2}{s^2(LCs^2+RCs-1)}\lambda(s) - \frac{Cs^3}{s^2(LCs^2+RCs-1)}v_2(s) - \frac{LCs}{s^2(LCs^2+RCs-1)}sx_4(s)$. Let $sx_4(s) = H(s)z_1(s) + G(s)v_2(s)$ for some tranfer functions $H(s)$ and $G(s)$ to be calculated. We obtain $z_1(s) = \left(1 + \frac{LCH(s)}{LCs^2+RCs-1}\right)^{-1} \frac{C}{LCs^2+RCs-1}\lambda(s) - \left(1 + \frac{LCH(s)}{LCs^2+RCs-1}\right)^{-1} \frac{Cs+LCG(s)}{LCs^2+RCs-1}v_2(s)$. Equalling both expressions we obtain $H(s) = \frac{1}{LC}\left(\frac{s(LCs^2+RCs-1)}{s-L} - \frac{(LCs^2+RCs-1)(s-L)}{s-L}\right) = \frac{LCs^2+RCs-1}{C(s-L)}$. Similar calculations yield $\left(1+\frac{LCH(s)}{LCs^2+RCs-1}\right)^{-1}\frac{Cs+LCG(s)}{LCs^2+RCs-1} = \frac{C(s-L)s+C(s-L)LG(s)}{s(LCs^2+RCs-1)}$ while we want this expression to equal $\frac{Cs}{LCs^2+RCs-1}$. This yields $G(s)=0$. Hence $x_4(s) = \frac{LCs^2+RCs-1}{C(s-L)s}z_1(s)$. Since $\dot{z}_2(t) = -\frac{R}{L}z_2(t) - \frac{1}{LC}z_1(t) - x_4(t) - \frac{\dot{v}_2(t)}{L} + \frac{\lambda(t)}{L}$, the result follows.

References

[1] V. Acary, B. Brogliato, D. Goeleven, 2008 "Higher order Moreau's sweeping process: Mathematical formulation and numerical simulation", Mathematical Programming A, vol.113, pp.133-217.

[2] V. Acary, B. Brogliato, 2008 *Numerical Methods for Nonsmooth Dynamical Systems. Applications in Mechanics and Electronics*, Springer Verlag, LNACM 35, Heidelberg.

[3] V. Acary, O. Bonnefon, B. Brogliato, 2011 *Nonsmooth Modeling and Simulation for Switched Circuits*, Springer Verlag, LNEE 69.

[4] S. Adly, T. Haddad, L. Thubault, 2014 "Convex sweeping process in the framework of measure differential inclusions and evolutional variational inequalities", Mathematical Programming B, vol.148, pp.5-47.

[5] J.P. Aubin, A. Cellina, 1984 *Differential Inclusions. Set-Valued Maps and Viability Theory*, Springer Verlag, Grundlehren der mathematischen Wissenschften 264.

[6] J.P. Aubin, 1990 "A survey of viability theory", SIAM Journal of Control and Optimization, vol.28, no 4, pp.749-788, July.

[7] J.P. Aubin, J. Lygeros, M. Quincampoix, S. Sastry, N. Seube, 2002 "Impulse differential inclusions: A viability approach to hybrid systems", IEEE Transactions on Automatic Control, vol.47, no 1, pp.2-20.

[8] D.D. Bainov, P.S. Simeonov, 1989 *Systems with Impulse Effects. Stability, Theory and Applications*, Ellis Horwood Limited, Chichester, England.

[9] P. Ballard, 2000 "The dynamics of discrete mechanical systems with perfect unilateral constraints", Archive for Rational Mechanics and Analysis, vol.154, no 3, pp.199-274.

[10] A. Benveniste, T. Bourke, B. Caillaud, M. Pouzet, 2012 "Non-standard semantics of hybrid systems modelers", Journal of Computer and System Sciences, vol.78, no 3, pp.877-910.

[11] A. Benveniste, B. Caillaud, H. Elmqvist, K. Ghorbal, M. Otter, M. Pouzet, 2017 "Structural analysis of multi-mode DAE systems", INRIA Research Report 8933, July 2016, `https://hal.inria.fr/hal-01343967/file/RR8933V2.pdf`.

[12] A. Berman, R.J. Plemmons, 1979 *Nonnegative Matrices in the Mathematical Sciences*, Computer Science and Applied Mathematics, Academic press.

[13] D.S. Bernstein, 2005 *Matrix Mathematics. Theory, Facts, and Formulas with Application to Linear Systems Theory*, Princeton University Press.

[14] P. Bettiol, A. Bressan, R. Vinter, 2010 "On trajectories satisfying a state constraint: $W^{1,1}$ estimates and counterexamples", SIAM J. Control and Optimization, vol.48, no 7, pp.4664-4679.

[15] F. Blanchini, 1999 "Set invariance in control", Automatica, vol.35, pp.1747-1767.

[16] B. Brogliato, 2009 "Some results on the optimal control with unilateral state constraints", Nonlinear Analysis: Theory, Methods and Applications, vol.70, 15 May, pp.3626-3657.

[17] B. Brogliato, R. Lozano, B. Maschke, O. Egeland, 2007 *Dissipative Systems Analysis and Control*, Springer Verlag, Communications and Control Eng., London.

[18] B. Brogliato, 2016 *Nonsmooth Mechanics. Models, Dynamics and Control*, Springer Verlag London, 3rd Edition.

[19] B. Brogliato, L. Thibault, 2010 "Existence and uniqueness of solutions for non-autonomous

complementarity dynamical systems", Journal of Convex Analysis, vol.17, no 3-4, pp.961-990.

[20] A. Cabot, L. Paoli, 2007 "Asymptotics for some vibro-impact problem with linear dissipation term", Journal de Mathématiques Pures et Appliquées, vol.87, no 3, pp.291-323.

[21] M.K. Camlibel, W.P.M.H. Heemels, J.M. Schumacher, 2002 "On linear passive complementarity systems", European Journal of Control, vol.8, no 3, pp. 220-237.

[22] G. Colombo, R. Henrion, N.D. Hoang, B.S. Mordukhovich, 2016 "Optimal control of the sweeping process over polyhedral controlled sets", Journal of Differential Equations, vol.260, pp.3397-3447.

[23] G. Colombo, R. Henrion, N.D. Hoang, B.S. Mordukhovich, 2015 "Discrete approximations of a controlled sweeping process", Set-Valued and Variational Analysis, vol.23, pp.69-86.

[24] R.W. Cottle, J.S. Pang, R.E. Stone, 1992 *The Linear Complementarity Problem*, Computer Science and Scientific Computing, Academic Press.

[25] R. Dzonou, M.D.P. Monteiro Marques, L. Paoli, 2009 "A convergence result for a vibro-impact problem with a general inertia operator", Nonlinear Dynamics, vol.58, no 1-2, pp.361-384.

[26] J.F. Edmond, L. Thibault, 2006 "BV solutions of nonconvex sweeping process differential inclusion with perturbation", Journal of Differential Equations, vol.226, no 1, pp.135-179.

[27] C. Glocker, 2001 *Set-Valued Force Laws. Dynamics of Non-Smooth Systems*, Springer-Verlag Berlin Heidelberg, Lecture Notes in Applied Mechanics, vol.1.

[28] S. Greenhalg, V. Acary, B. Brogliato, 2013, "On preserving dissipativity properties of linear complementarity systems with the θ-method", Numerische Mathematik, vol.125, pp.601-637.

[29] L. Han, A. Tiwari, M.K. Camlibel, J.-S. Pang, 2009 "Convergence of time-stepping schemes for passive and extended linear complementarity systems", SIAM Journal on Numerical Analysis, vol.47, no 5, pp.3768-3796.

[30] W.P.M.H. Heemels, J.M. Schumacher, S. Weiland, 2000 "Linear complementarity systems", SIAM Journal on Applied Mathematics, vol. 60, no 4, pp. 1234-1269.

[31] W.P.M.H. Heemels, M.K. Camlibel, J.M. Schumacher, B. Brogliato, 2011 "Observer-based control of linear complementarity systems", Int. Journal of Robust and Nonlinear Control, vol.21, no 10, pp.1193-1218.

[32] N.S.P. Hyun, E.I. Verriest, 2017 "Causal modeling in impulsive systems: A new rigorous non-standard analysis approach", Nonlinear Analysis: Hybrid Systems, `http://dx.doi.org/10.1016/j.nahs.2017.01.007`.

[33] G.S. Jones, 1964 "Fundamental inequalities for discrete and discontinuous functional equations", J. Soc. Indust. Appl. Math., vol.12, no 1, pp.43-57, March.

[34] D. Kinzebulatov, 2007 "Systems with distributions and viability theorem", Journal of Mathematical Analysis and Applications, vol.331, pp.1046-1067.

[35] M. Kunze, M.D.P. Monteiro Marques, 2000 "An introduction to Moreau's sweeping process", in *Impacts in Mechanical Systems. Analysis and Modelling*, Springer Verlag Lecture Notes in Physics 551, B. Brogliato (Ed.), pp.1-60.

[36] R.I. Leine, N. van de Wouw, 2008 *Stability and Convergence of Mechanical Systems with Unilateral Constraints*, Springer Verlag, Lecture Notes in Applied and Computational Mechanics, vol.36.

[37] R.I. Leine, T.F. Heimsch, 2012 "Global uniform symptotic attractive stablity of the non-autonomous bouncing ball system", Physica D, vol.241, pp.2029-2041.

[38] D. Liberzon, S. Trenn, 2012 "Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability", Automatica, vol.48, no 5, pp.954-963.

[39] M.D.P. Monteiro Marques, 1993 *Differential Inclusions in Nonsmooth Mechanical Problems. Shocks and Dry Friction*, Birkhauser, Progress in Nonlinear Differential Equations and Their Applications.

[40] J.C. Mayo-Maldonado, P. Rapisarda, 2014 "Modelling of switching dynamics in electrical

systems", 21st Int. Symposium on Mathematical Theory of Networks and Systems, July 7-11, Groningen, The Netherlands, pp.985-992.

[41] J.C. Mayo-Maldonado, P. Rapisarda, 2016 "On positive-realness and Lyapunov functions for switched linear differential systems", IEEE Transactions on Automatic Control, vol.61, no 8, pp.2239-2244.

[42] J.C. Mayo-Maldonado, P. Rapisarda, P. Rocha, 2014 "Stability of switched linear differential systems", IEEE Transactions on Automatic Control, vol.59, no 8, pp.2038-2051.

[43] C.I. Morarescu, B. Brogliato, 2010 "Trajectory tracking control of multiconstraint complementarity Lagrangian systems", IEEE Transactions on Automatic Control, vol.55, no 6, pp.1300-1313, June 2010.

[44] J.J. Moreau, 1971 "Rafle par un convexe variable I", Séminaire Analyse Convexe Montpellier, Exposé 15.

[45] J.J. Moreau, 1972 "Rafle par un convexe variable II", Séminaire Analyse Convexe Montpellier, Exposé 3.

[46] J.J. Moreau, 1977 "Evolution problem associated with a moving convex set in a Hilbert space", Journal of Differential Equations, vol.26, pp.347-374.

[47] J.J. Moreau, 1988, "Unilateral contact and dry friction in finite freedom dynamics", in *Nonsmooth Mechanics and Applications*, J.J. Moreau, P.D. Panagiotopoulos (Eds.), CISM Courses and Lectures no 302, Int. Center for Mechanical Sciences, Springer Verlag, pp.1-82.

[48] J.J. Moreau, 1988 "Bounded variation in time", in *Topics in Nonsmooth Dynamics*, J.J. Moreau, P.D. Panagiotopoulos, G. Strang (Eds.), pp.1-74, Birkhauser, Basel.

[49] P. Nanez, R.G. Sanfelice, N. Quijano, 2017 "On an invariance principle for differential-algebraic equations with jumps and its application to switched differential-algebraic equations", Mathematics of Control, Signals and Systems, 29:5, doi:10.1007/s00498-016-0185-2.

[50] Y. Or, A.D. Ames, 2011 "Stability and completion of Zeno equilibria in Lagrangian hybrid systems", IEEE Transactions on Automatic Control, vol.56, no 6, pp.1322-1336.

[51] D. Percivale, 1985 "Uniqueness in the elastic bounce problem", Journal of Differential Equations, vol.56, no 2, pp.206-215.

[52] D. Percivale, 1991 "Uniqueness in the elastic bounce problem II", Journal of Differential Equations, vol.90, pp.304-315.

[53] R.T. Rockafellar, R.J.B. Wets, 1998 *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften, vol.317, Springer Verlag Berlin.

[54] P. Sannuti, 1983 "Direct singular perturbation analysis of high gain and cheap control problem", Automatica, vol.19, no 1, pp.41-51.

[55] S. Sastry, 1999 *Nonlinear Systems, Analysis, Stability, and Control*, Interdisciplinary Applied Mathematics, Springer-Verlag New York.

[56] A. van der Schaft, J.M. Schumacher, 1996 "The complementary-slackness class of hybrid systems", Mathematics of Control, Signals and Systems, vol.9, no 3, pp.266-301.

[57] J. Shen, J.S. Pang, 2006 "Linear complementarity systems: Zeno states", SIAM J. on Control and Optimization, vol.44, no 3, pp.1040-1066.

[58] A. Tanwani, S. Trenn, 2017 "Determinability and state estimation for switched differential-algebraic equations", Automatica, vol.76, pp.17-31.

[59] L. Thibault, 2003 "Sweeping process with regular and nonregular sets", Journal of Differential Equations, vol.193, no 1, pp.1-26.

[60] S. Trenn, 2009, *Distributional Differential Algebraic Equations*, (Ph.D. thesis), Germany: Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau.

[61] S. Trenn, 2012 "Switched differential algebraic equations", in *Dynamics and Control of Switched Electronic Systems. Advanced Perspectives for Modeling, Simulation and Control of Power Converters*, Springer London, F. Vasca and L. Iannelli (Eds.), pp.189-216.

[62] L. Zhou, D.W.C. Ho, G. Zhai, 2013 "Stability analysis of switched linear singular systems", Automatica, vol.49, pp.1481-1487.