On Local LTI Model Coherence for LPV Interpolation

Qinghua Zhang, Lennart Ljung, Rik Pintelon

On Local LTI Model Coherence for LPV Interpolation

Qinghua Zhang, Lennart Ljung, Rik Pintelon

***Abstract*—In the local approach to linear parameter varying (LPV) system identification, it is widely acknowledged that locally estimated linear state-space models should be made *coherent* before being interpolated, but the accurate meaning of the term "coherent" or "coherence" is rarely defined. The purpose of this paper is to analyze the relevance of two existing definitions and to point out the consequence of this analysis on the practice of LPV system identification.**

***Index Terms*—System identification, LPV model, Coherent local linear models.**

I. INTRODUCTION

Linear parameter varying (LPV) models are widely used in nonlinear control systems [1], [2], [3], [4], [5], [6], [7], [8], [9]. In order to build such models from sensor data, different methods for LPV system identification have been reported [10], [11], [12], [13], [14], [15], [5], [16], [17], [18]. These methods are usually classified into the local approach or the global approach. While each of the two approaches has its advantages and drawbacks, this paper is focused on the local approach. In this approach, *interpolation* is essential to establishing a global LPV model from a collection of locally estimated linear time invariant (LTI) models [14], [19], [20], [21]. As each LTI *state-space* model can be estimated in an arbitrary state basis, independently from each other, the interpolation of a collection of such models is not a trivial problem.

It is widely acknowledged in the LPV system identification literature that it is important to use a *coherent* collection of local models for the purpose of interpolation. However, the accurate meaning of the term "coherent" or "coherence" in this context is rarely defined. To our knowledge, two different definitions have been proposed in [20] and in [22], which will be respectively recalled as Definitions 2 and 3 in Section III. *The purpose of this paper is to analyze the relevance of these two definitions and to point out the consequence of this analysis on the practice of LPV system identification.*

Typically, locally estimated LTI state-space models are transformed into *some canonical form* before being interpolated, by (implicitly) assuming that local models are coherent if they are in the same canonical form. For example, some identification methods are based on the modal form [23], on the controllable canonical form [24], on the balanced form [25], on a zero-pole decomposition-based form [26], [19], and on a normalized observability matrix-based form [20].

It is shown in [22] that, without any global structural assumption, locally estimated LTI models *do not* contain sufficient information to make themselves coherent. This result is in the sense of the local model coherence definition formulated in the same cited paper. It is thus important to investigate the relevance of this definition, because of the important impact of the aforementioned result on the practice of LPV system identification following the local approach. As a matter of fact, this result is in contradiction with the practice of interpolating local models in some canonical form as recalled above. This paper will carefully analyze the relevance of the coherence definition formulated in [22] and its consequence. Here let us make a related remark: the reported works based on different canonical forms naturally raise *the compatibility issue*: given a set of local LTI state-space models, the interpolations based on different canonical forms lead certainly to different LPV models, but do they have (closely) the same input-output (I-O) behavior? By clarifying the definition of local model coherence, this paper will bring some hint to answer this question.

Only *state-space* models are considered in this paper, as local model coherence is not relevant for other models. For shorter expressions, the words "state-space" will be omitted from terms like "local state-space model" and "LTI state-space model".

II. PROBLEM STATEMENT

As the coherence of local models is a concept that essentially concerns the deterministic part of LPV systems, in this paper *the stochastic part will be neglected* in order to use shorter notations.

Let $u(t) \in \mathbb{R}^q$ and $y(t) \in \mathbb{R}^s$ be respectively the input and the output at discrete time instant $t = 0, 1, 2, \ldots$, and $p(t)$ be the scheduling variable evolving within a compact set $\mathbb{S}$ of scheduling values. An LPV system is described by the state-space model

$$x(t+1) = A(p(t))x(t) + B(p(t))u(t) \tag{1a}$$

$$y(t) = C(p(t))x(t) + D(p(t))u(t) \tag{1b}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, and $A(p(t))$, $B(p(t))$, $C(p(t))$, $D(p(t))$ are matrices of appropriate sizes depending on $p(t) \in \mathbb{S}$.

This formulation is sometimes referred to as *static* p-dependent (or static parameter-dependent) LPV systems, as opposed to *dynamic* p-dependent LPV systems, in which the

Q. Zhang is with Inria, IFSTTAR, Université de Rennes, Campus de Beaulieu, 35042 Rennes Cedex, France, phone: +33 2 9984 7463, email: qinghua.zhang@inria.fr. L. Ljung is with Dept. of Electrical Engineering, Linköping University, SE-581 83 Linköping, Sweden, phone: +46 13 281310, email: ljung@isy.liu.se. R. Pintelon is with Dept. ELEC of the Vrije Universiteit Brussel, phone +32 2 6292944, email rik.pintelon@vub.ac.be.

system matrices may depend on the scheduling values at different instants, for example, $A(p(t), p(t-1), \ldots, p(t-l))$. In the local approach, each local LTI model is estimated from data collected around a working point, typically corresponding to a value of $p(t)$. If the working point was characterized by $p(t), p(t-1), \ldots, p(t-l)$ with $p(t)$ evolving with time, the working point would move away, and it would be difficult to collect sensor data around a fixed working point. For this reason, only static p-dependent LPV systems are considered in this paper.

Based on the fact that the LPV system (1) becomes an LTI system when the scheduling variable $p(t)$ is maintained at a fixed value, the following definition aims at establishing a link between LPV and LTI models.

Consider a set of m LTI models indexed by the integer $i = 1, 2, \ldots, m$,

$$x(t+1) = A_i x(t) + B_i u(t) \tag{2a}$$

$$y(t) = C_i x(t) + D_i u(t) \tag{2b}$$

where the input $u(t) \in \mathbb{R}^q$, the output $y(t) \in \mathbb{R}^s$, the state $x(t) \in \mathbb{R}^n$, and the matrices A_i, B_i, C_i, D_i are of appropriate sizes.

The notation

$$\sigma_i \triangleq (A_i, B_i, C_i, D_i) \tag{3}$$

will be used to denote the *matrices* characterizing the i-th LTI model (2), or the LTI model itself *by abuse of notation*. The set of LTI models will be denoted by

$$\Sigma = \{\sigma_i : i = 1, 2, \ldots, m\}. \tag{4}$$

Definition 1: A set of local LTI models

$$\Sigma^* = \{(A_i^*, B_i^*, C_i^*, D_i^*) : i = 1, 2, \ldots, m\} \tag{5}$$

is called a *multi-snapshot* of the LPV system (1) captured at

$$\mathbb{P} = \{p_1, \ldots, p_m\} \subset \mathbb{S}, \tag{6}$$

if, for all $i = 1, \ldots, m$,

$$A_i^* = A(p_i), \; B_i^* = B(p_i), \; C_i^* = C(p_i), \; D_i^* = D(p_i). \tag{7}$$

□

In the local approach to LPV system identification, it is assumed that, if the multi-snapshot Σ^* involves a sufficient number of local LTI models corresponding to the scheduling values in $\mathbb{P}$ appropriately located in $\mathbb{S}$, the interpolation of the matrices contained in Σ^* leads to a good approximation of the LPV system (1) in the sense that, for any scheduling value $p \in \mathbb{S}$, the interpolation produces a local model

$$\check{\sigma}(p) = (\check{A}(p), \check{B}(p), \check{C}(p), \check{D}(p)) \tag{8}$$

close to the corresponding matrices of the LPV system (1) captured at the same scheduling value, namely $(A(p), B(p), C(p), D(p))$.

In practice, a set $\hat{\Sigma}$ of local LTI models $\hat{\sigma}_i$ are estimated, each from the I-O data collected around a working point corresponding to one of the scheduling values in $\mathbb{P}$. In general, an LTI model is estimated up to an arbitrary similarity transformation, *i.e.*, each estimated LTI model

$$\hat{\sigma}_i = (\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i) \tag{9}$$

is related to the multi-snapshot Σ^* through

$$\hat{A}_i = T_i A_i^* T_i^{-1}, \quad \hat{B}_i = T_i B_i^*, \quad \hat{C}_i = C_i^* T_i^{-1}, \quad \hat{D}_i = D_i^*,$$

or equivalently,

$$\hat{A}_i = T_i A(p_i) T_i^{-1}, \quad \hat{B}_i = T_i B(p_i), \tag{10a}$$

$$\hat{C}_i = C(p_i) T_i^{-1}, \quad \hat{D}_i = D(p_i), \tag{10b}$$

where $T_i \in \mathbb{R}^{n \times n}$ is some invertible matrix for each $i = 1, \ldots, m$, if the estimation error is neglected.

As the true multi-snapshot Σ^* is unknown, the practical interpolation has to be based on the estimated model set $\hat{\Sigma}$. Due to the arbitrary and unknown transformation matrices T_i, the interpolation of $\hat{\Sigma}$ cannot lead to the same LPV model as the interpolation of Σ^*. However, *it is expected that*, after applying an appropriately chosen state transformation to every estimated local model $\hat{\sigma}_i \in \hat{\Sigma}$, so that the transformed local models are "coherent", the interpolation will lead to an LPV model exhibiting an I-O behavior close to that of the LPV system (1).

It follows from the above discussion that the interpolation of a "coherent" set of local LTI models should lead to an LPV model that is equivalent to the LPV system (1) in terms of I-O behavior, up to interpolation errors. In particular, consider the case where the scheduling variable $p(t)$ evolves within the values of the finite set $\mathbb{P}$ only, so that the interpolation of the local models is *trivial*[1]. This restriction would make the interpolation useless in practice, but it simplifies the following theoretical analysis. In this case, a coherent set of local LTI models should satisfy the following requirement.

Requirement 1: A candidate set Σ of *coherent* local models regarding the LPV system (1) must satisfy the property that, when the scheduling variable $p(t)$ evolves within the finite set $\mathbb{P}$, the *trivial interpolation* (see footnote 1) of Σ leads to an LPV model exhibiting the same I-O behavior as the LPV system (1). □

It is expected that, by appropriately choosing a transformation matrix $\hat{T}_i$ for every estimated local model $\hat{\sigma}_i$, the similarity transformations

$$\hat{\sigma}_i \xrightarrow{\hat{T}_i} \sigma_i \tag{11}$$

will lead to a set of coherent local models σ_i. As each locally estimated model $\hat{\sigma}_i$ is related to the LPV system (1) through (10), it depends on a *single* scheduling value $p_i \in \mathbb{P}$. Moreover, each chosen transformation matrix $\hat{T}_i$ must be specific to the estimated local model $\hat{\sigma}_i$ and independent of the time t, hence each transformed local model σ_i depends also on a *single* scheduling value $p_i \in \mathbb{P}$. Indeed, if each transformed local

[1] Given a set of m local models, with each local model σ_i corresponding to one of the m scheduling value $p_i \in \mathbb{P}$, when $p(t)$ evolves within $\mathbb{P}$, the interpolation for $p(t) = p_i$ is *trivial*, because the result is simply the given local model σ_i, without any interpolation error.

model σ_i depended on several scheduling values, their interpolation would lead to a *dynamic* p-dependent LPV model. This motivates the following requirement.

Requirement 2: A candidate set of *coherent* local models

$$\{\sigma_i = (A_i, B_i, C_i, D_i) : i = 1, \dots, m\} \tag{12}$$

regarding the LPV system (1) captured at

$$\mathbb{P} = \{p_1, \dots, p_m\} \subset \mathbb{S} \tag{13}$$

must satisfy the property that each local model σ_i corresponds to a single scheduling value p_i, in the sense that there exist invertible transformation matrices $\tilde{T}_i \in \mathbb{R}^{n\times n}$ such that, for all $i = 1, \dots, m$,

$$A_i = \tilde{T}_i A(p_i)\tilde{T}_i^{-1}, \quad B_i = \tilde{T}_i B(p_i), \tag{14a}$$
$$C_i = C(p_i)\tilde{T}_i^{-1}, \quad D_i = D(p_i), \tag{14b}$$

where $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are the p-dependent matrices involved in the definition of the LPV system (1). □

Notice that the transformation matrices $\tilde{T}_i$ appearing in (14a) satisfy $\tilde{T}_i = \hat{T}_i T_i$ with the matrices T_i and $\hat{T}_i$ as involved in (10) and (11) respectively.

Of course, Requirements 1 and 2 represent necessary conditions that a coherent local model set must satisfy, not sufficient conditions. Nevertheless, they will exclude most candidate definitions of coherent local model sets, as shown in the following sections.

III. Two existing definitions

Let us examine two existing definitions of local model coherence in order to gain more insight about what a relevant definition should be.

The following definition was formulated in [22].

Definition 2: A set of LTI models

$$\{\sigma_i = (A_i, B_i, C_i, D_i) : i = 1, \dots, m\} \tag{15}$$

is coherent regarding the LPV system (1) captured at

$$\mathbb{P} = \{p_1, \dots, p_m\} \subset \mathbb{S}, \tag{16}$$

if there exists an invertible transformation matrix $T \in \mathbb{R}^{n\times n}$, *common to all the LTI models* σ_i, such that, for all $i = 1, \dots, m$,

$$A_i = TA(p_i)T^{-1}, \quad B_i = TB(p_i), \tag{17a}$$
$$C_i = C(p_i)T^{-1}, \quad D_i = D(p_i), \tag{17b}$$

where $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are the p-dependent matrices involved in the definition of the LPV system (1). □

It is important to notice that the above transformation matrice T is *common* to to all the LTI models σ_i. Under this definition, when $p(t)$ evolves within the finite set $\mathbb{P}$, the trivial interpolation of a coherent set of LTI models leads to an LPV model that is related to the LPV system (1) through the *constant* transformation matrix T at every time instant t. It is clear that the similarity transformation with the constant matrix T preserves the I-O behavior of the LPV system, hence Definition 2 satisfies Requirement 1. Moreover, each local model σ_i satisfying (17) depends on a single value $p_i \in \mathbb{P}$, hence Requirement 2 is also satisfied.

In general, each locally estimated LTI model $\hat{\sigma}_i$ is related to the LPV system through a different transformation matrix T_i, as expressed in (10), whereas in Definition 2 the transformation matrix T is common to all the local LTI models. A natural question is if this definition can be relaxed to different transformation matrices. In the classical linear time varying (LTV) system theory [27], it is well known that *time varying* linear state transformations preserve the I-O behavior. An LPV system, with its system matrices $A(p(t)), B(p(t)), C(p(t)), D(p(t))$ depending on the time t through the scheduling variable $p(t)$, is also an LTV system. For this reason, it seems not necessary to restrict the definition of local model coherence to a constante transformation matrix T. Indeed, the following definition, formulated in [20], follows this direction.

Definition 3: A set of LTI models

$$\{\sigma_i = (A_i, B_i, C_i, D_i) : i = 1, \dots, m\} \tag{18}$$

is coherent regarding the LPV system (1) captured at

$$\mathbb{P} = \{p_1, \dots, p_m\} \subset \mathbb{S}, \tag{19}$$

if there exists a *matrix-valued function* $T : \mathbb{S} \to\in \mathbb{R}^{n\times n}$, such that, for all $i = 1, \dots, m$,

$$A_i = T(p_i)A(p_i)T^{-1}(p_i), \quad B_i = T(p_i)B(p_i), \tag{20a}$$
$$C_i = C(p_i)T^{-1}(p_i), \quad D_i = D(p_i), \tag{20b}$$

where $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ are the matrix functions involved in the definition of the LPV system (1). □

More accurately, the definition originally formulated in [20] states that, for each $i = 1, \dots, m$, A_i is an *evaluation of* $T(p)A(p)T^{-1}(p)$ *for some* $T(p)$ *with* $p = p_i$, and similarly for B_i, C_i, D_i. Though in [20] nothing is said about the properties of the matrix-valued function $T(p)$, some regularity properties seem necessary, otherwise $T_i = T(p_i)$ could be arbitrary matrices, and any set of locally estimated LTI models $\hat{\sigma}_i$ would be coherent in the sense of Definition 3, according to (10). It seems reasonable to assume that $T(p)$ is bounded and differentiable with bounded derivatives, so is its inverse $T^{-1}(p)$.

By restricting $T_i = T(p_i)$ with regularity assumptions, Definition 3 may be a reasonable relaxation of Definition 2, taking into account the fact that *time varying* transformations generally preserve the I-O behavior of an LTV or LPV system, as known from the classical LTV system theory [27].

IV. p-dependent transformations

However, as pointed out in [28], by applying a p-dependent linear transformation $z(t) = T(p(t))x(t)$, the LPV system (1) becomes

$$\begin{aligned} z(t+1) = {} & T(p(t+1))A(p(t))T^{-1}(p(t))z(t) \\ & + T(p(t+1))B(p(t))u(t) \end{aligned} \tag{21a}$$
$$y(t) = C(p(t))T^{-1}(p(t))z(t) + D(p(t))u(t), \tag{21b}$$

hence $A(p(t))$ is transformed as

$$A(p(t)) \longrightarrow \bar{A}(p(t), p(t+1)) \tag{22}$$

with

$$\bar{A}(p(t), p(t+1)) \triangleq T(p(t+1))A(p(t))T^{-1}(p(t)). \tag{23}$$

In general $T(p(t+1)) \neq T(p(t))$, hence in (23) the two matrices surrounding $A(p(t))$ are not the inverse of each other.

Definition 3 is based on a p-dependent transformation matrix $T(p_i)$, yet the same one is present at both sides of $A(p_i)$ in (20), *i.e.*,

$$A_i = T(p_i)A(p_i)T^{-1}(p_i). \tag{24}$$

Apparently, this A_i is in a form not in agreement with the matrix defined in (23). When the scheduling variable $p(t)$ evolves within the finite set $\mathbb{P}$, the trivial interpolation of a coherent set of local LTI models, in the sense of Definition 3, does not correspond to a (time varying) linear transformation of the LPV system (1), hence it may not preserve the I-O behavior of the LPV system (1). In other words, Definition 3 does not satisfy Requirement 1, in general.

Of course, in the case of constant transformation T, like in Definition 2, the two equations (23) and (24) are in agreement. The problem revealed in the above discussion is due to the fact that a p-dependent linear transformation $T(p(t))$ applied to a *static* p-dependent LPV system leads to a *dynamic* p-dependent LPV system (21), in general. It then seems that only constant transformations T can satisfy Requirement 1.

Nevertheless, it has been reported in [28] that, in some particular case, there do exist p-dependent transformations preserving the *static* p-dependence of LPV models. How can this be possible? Let us look at a simple example.

Consider an LPV system with $\mathbb{S} = \mathbb{P} = \{p_1, p_2\}$ and the p-dependent matrices $A(\cdot), B(\cdot), T(\cdot)$ defined as

$$A(p_1) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \; A(p_2) = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}, \tag{25a}$$

$$B(p_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; B(p_2) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \tag{25b}$$

$$T(p_1) = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \; T(p_2) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}. \tag{25c}$$

The matrices $C(\cdot)$ and $D(\cdot)$ do not matter in this example, because in (21b) the scheduling variable appears only as $p(t)$.

It can be readily checked that, in this example,

$$T(p_2)A(p_1)T^{-1}(p_1) = T(p_1)A(p_1)T^{-1}(p_1) = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \tag{26a}$$

$$T(p_1)A(p_2)T^{-1}(p_2) = T(p_2)A(p_2)T^{-1}(p_2) = \begin{bmatrix} 0.5 & 1.5 \\ 0.5 & 1.5 \end{bmatrix} \tag{26b}$$

and

$$T(p_2)B(p_1) = T(p_1)B(p_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{27a}$$

$$T(p_1)B(p_2) = T(p_2)B(p_2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \tag{27b}$$

Therefore, for this example with the p-dependent matrices $A(\cdot), B(\cdot), T(\cdot)$ defined as in (25), the apparently dynamic p-dependent state equation (21a) can also be written as

$$\begin{aligned} z(t+1) = {} & T(p(t))A(p(t))T^{-1}(p(t))z(t) \\ & + T(p(t))B(p(t))u(t), \end{aligned}$$

which becomes a static p-dependent state equation!

This example illustrates the existence of p-dependent transformations preserving *static* p-dependence of LPV models, suggesting that Definition 3 may satisfy Requirement 1 in such circumstances. However, equalities (26) and (27) are due to some degeneracy in the system model. It remains to analyze if such degenerate models are significant in practice.

V. Excluding degenerate models

In this section different conditions will be formulated in order to exclude degenerate models like the example built in (25).

A. Discretized continuous time systems

In most applications, the true systems are in continuous time, yet discrete time models are often used for efficient numerical computations related to sampled sensor data. In this case, when the scheduling variable is maintained at a constant value, say $p(t) = p_i$, the discrete time system (1) corresponds to a discretized LTI model. By assuming a zero order hold at the (controlled) input, the matrix $A(p(t))$ in (1) corresponds to

$$A(p_i) = e^{A_c(p_i)\tau} \tag{28}$$

for some matrix $A_c(p_i)$ characterizing the continuous time system and for some sampling period τ. As such an exponential matrix cannot be singular, in this case singular matrices like the examples of $A(p_1)$ and $A(p_2)$ in (25a) are excluded.

Theorem 1: Let $\Sigma = \{\sigma_1, \ldots, \sigma_m\}$ be a candidate set of coherent local models regarding the LPV system (1), with each local model σ_i corresponding to one scheduling value $p_i \in \mathbb{P}$. Assume that the multi-snapshot of the LPV system (1), captured at $\mathbb{P}$ as defined in Definition 1, is such that at least one of the matrices $A_i^* = A(p_i)$ is non singular. Then Requirements 1 and 2 are satisfied by Σ, *if and only if* Σ is a set of coherent local models in the sense of Definition 2. □

As discussed above, for discretized continuous time systems, the matrices $A(p_i) = A_i^*$ are non singular, thus Theorem 1 applies in this case.

Proof of Theorem 1.

It is straightforward to check the sufficiency, *i.e.*, a coherent set of local models in the sense of Definition 2 satisfy Requirements 1 and 2. Let us prove the necessity in the following.

Let

$$\Sigma = \{\sigma_i = (A_i, B_i, C_i, D_i) : i = 1, \ldots, m\} \tag{29}$$

be a set of local LTI models satisfying Requirements 1 and 2, and assume that, in the corresponding multi-snapshot, at least one of the matrices $A(p_i) = A_i^*$ is non singular. It will be shown that Σ is a coherent set of local models in the sense of Definition 2.

When $p(t)$ evolves within the finite set $\mathbb{P}$, according to Requirement 1, the trivial interpolation of Σ leads to an LPV model exhibiting the same I-O behavior as the LPV system (1).

As pointed out in [28], to preserve the I-O behavior of the LPV system (1), the interpolated LPV model must be related to (1) through a (generally time varying) invertible transformation matrix $T(p(t))$ such that, at each instant t, the interpolated matrix $\breve{A}(p(t))$ satisfies

$$\breve{A}(p(t)) = T(p(t+1))A(p(t))T^{-1}(p(t)) \tag{30}$$

where $A(\cdot)$ is the p-dependent matrix involved in the LPV system (1).

Still consider $p(t)$ evolving within the finite set $\mathbb{P}$, and assume that $p(t) = p_i \in \mathbb{P}$ and $p(t+1) = p_j \in \mathbb{P}$. Then the trivial interpolation yields $\breve{A}(p(t)) = A_i$, with the same A_i as the one involved in (29). Hence equation (30) becomes

$$A_i = T(p_j)A(p_i)T^{-1}(p_i). \tag{31}$$

This result shows that A_i depends on both p_i and p_j, which is *apparently* in contradiction with Requirement 2, which states that each local model σ_i depends on a single scheduling value $p_i \in \mathbb{P}$, so does A_i as part of σ_i as expressed in (29). The only way to avoid this contradiction is that

$$T(p_j)A(p_i)T^{-1}(p_i) = T(p_i)A(p_i)T^{-1}(p_i) \tag{32}$$

despite the fact that $p_j \neq p_i$, like the equalities (26) in the illustrative example, so that

$$A_i = T(p_i)A(p_i)T^{-1}(p_i) \tag{33}$$

depends on the single scheduling value p_i.

Unlike in the illustrative example where $A(p_i)$ was singular, in the statement of Theorem 1, at least one of the matrices $A(p_i)$, *say* $A(p_1)$, *is assumed non singular*.

As $p(t)$ is allowed to evolve arbitrarily within the finite set $\mathbb{P}$, $p(t) = p_i$ and $p(t+1) = p_j$ can be any pair of scheduling values within $\mathbb{P}$, hence equality (32) holds for all $p_i \in \mathbb{P}$ and $p_j \in \mathbb{P}$. In particular, for $i = 1$,

$$T(p_j)A(p_1)T^{-1}(p_1) = T(p_1)A(p_1)T^{-1}(p_1). \tag{34}$$

The matrix $A(p_1)$ is assumed non singular, and $T^{-1}(p_1)$ is by definition non singular, hence equation (34) implies

$$T(p_j) = T(p_1), \tag{35}$$

which holds for all $p_j \in \mathbb{P}$. In other words, the matrices $T(p_j)$, for all $p_j \in \mathbb{P}$, are all equal to a common matrix, say T. Hence (33) becomes, for all $p_i \in \mathbb{P}$,

$$A_i = TA(p_i)T^{-1}, \tag{36}$$

and similarly,

$$B_i = TB(p_i), \quad C_i = C(p_i)T^{-1}, \quad D_i = D(p_i). \tag{37}$$

Therefore, Σ as expressed in (29) is a coherent set of local models in the sense of Definition 2.

The necessity is thus proved, and then Theorem 1 is established. □

B. Matrix rank condition

Theorem 1 mainly concerns discretized continuous time systems. For systems intrinsically in discrete time, it may happen that the matrix $A(p)$ is singular for all $p \in \mathbb{P}$. The following result addresses this case.

Theorem 2: Let $\Sigma = \{\sigma_1, \ldots, \sigma_m\}$ be a candidate set of coherent local models regarding the LPV system (1), with each local model σ_i corresponding to one scheduling value $p_i \in \mathbb{P}$. Assume that the multi-snapshot of the LPV system (1), captured at $\mathbb{P}$ as defined in Definition 1, is such that the matrix sum

$$\sum_{i=1}^{m} \left[A_i^*(A_i^*)^T + B_i^*(B_i^*)^T\right] \tag{38}$$

is positive definite. Then Requirements 1 and 2 are satisfied by Σ, *if and only if* Σ is a set of coherent local models in the sense of Definition 2. □

This result differs from Theorem 1 in that the full rank (positive definiteness) of the matrix sum (38) replaces the full rank of at least one of A_i^*.

Each term $A_i^*(A_i^*)^T$ or $B_i^*(B_i^*)^T$ in (38) can contribute to increase the rank of the matrix sum, and never to decrease the rank, as each term is positive semi-definite. This sum is similar to an empirical covariance matrix estimator, which is typically of full rank. Hence Theorem 2 is based on a condition much weaker than the one in Theorem 1, since the matrix sum (38) is almost always of full rank in practice, even if each term in it is rank deficient.

Proof of Theorem 2.

It is straightforward to check the sufficiency, *i.e.*, a coherent set of local models in the sense of Definition 2 satisfy Requirements 1 and 2. Let us prove the necessity in the following.

The beginning of the proof of Theorem 1 can be copied exactly here, till equation (32), before using the non singularity of one of the matrices $A(p_i)$, which was assumed in Theorem 1 only.

In (32), the invertible matrix $T^{-1}(p_i)$ can be removed from both sides, therefore,

$$T(p_j)A(p_i) = T(p_i)A(p_i), \tag{39}$$

which holds for all $p_j \in \mathbb{P}$ and all $p_i \in \mathbb{P}$, and in particular,

$$T(p_1)A(p_i) = T(p_i)A(p_i). \tag{40}$$

Subtract each side of equation (40) from the corresponding side of equation (39), then

$$[T(p_j) - T(p_1)]A(p_i) = 0. \tag{41}$$

In this last equation, p_i can be any value within $\mathbb{P}$. In addition, $A_i^* = A(p_i)$ as in Definition 1. Therefore,

$$[T(p_j) - T(p_1)] \sum_{i=1}^{m} A_i^*(A_i^*)^T = 0. \tag{42}$$

This reasoning is then similarly repeated for $T(p(t+1))B(p(t))$ in (21a), yielding

$$[T(p_j) - T(p_1)] \sum_{i=1}^{m} B_i^*(B_i^*)^T = 0. \tag{43}$$

These results are then combined into

$$[T(p_j) - T(p_1)] \sum_{i=1}^{m} \left[A_i^* (A_i^*)^T + B_i^* (B_i^*)^T \right] = 0. \quad (44)$$

The matrix sum in (38) is assumed positive definite in the statement of Theorem 2, hence

$$T(p_j) - T(p_1) = 0, \quad (45)$$

which holds for all $p_j \in \mathbb{P}$. In other words, all the matrices $T(p_j)$ are equal to a common matrix, say T, for $p_j \in \mathbb{P}$.

Therefore, Σ is a coherent set of local models in the sense of Definition 2.

The necessity is thus proved, and then Theorem 2 is established. □

C. Minimal state LPV models

In an *affine LPV system*, the matrices $A(p), B(p), C(p), D(p)$ are affine functions of p.

According to [29] (see Theorem 2 therein), if two minimal state affine LPV models are I-O equivalent, then they are related by a constant transformation matrix T.

It is also pointed out in [29] that an LPV system with its scheduling variable evolving within a finite set of scheduling values is equivalent to an affine LPV system. See Section III of [29] where such systems are called *linear switched systems*.

Now let us come back to the definitions of local model coherence. Requirements 1 and 2 concern only the case where the scheduling variable $p(t)$ is restricted to the finite set $\mathbb{P}$. In this restricted case, the LPV system (1) is an affine LPV system. Based on the results of [29] recalled above, at least when $p(t)$ evolves within the finite set $\mathbb{P}$, there does not exist any p-dependent transformation of a minimal state LPV model preserving its I-O behavior. Therefore, for minimal state LPV systems, it is not possible to relax Definition 2 to p-dependent transformations $T(p)$ without breaking Requirements 1 and 2. This result is summarized as follows.

Theorem 3: Let $\Sigma = \{\sigma_1, \ldots, \sigma_m\}$ be a candidate set of coherent local models regarding the LPV system (1), with each local model σ_i corresponding to one scheduling value $p_i \in \mathbb{P}$. Assume that, when $p(t)$ evolves within the finite set $\mathbb{P}$, the trivial interpolation of Σ leads to a minimal state LPV model. Then Requirements 1 and 2 are satisfied by Σ, *if and only if* Σ is a set of coherent local models in the sense of Definition 2. □

An affine LPV model is a minimal state model if and only if it is reachable and observable [29]. Good system identification methods usually lead to minimal state models. The example formulated in (25) is clearly not a minimal state model.

VI. Tolerance to lack of coherence

It was shown in the last section that, for most practically significant LPV systems, Definition 2 is the only relevant definition of coherent local models. It is then important to recall that, as reported in [22], without any global structural assumption, locally estimated LTI models *do not* contain sufficient information to make themselves coherent, in the sense of Definition 2.

On the other hand, quite a few LPV system identification methods following the local approach have been reported, in which locally estimated LTI models are typically transformed into some canonical form before being interpolated. For example, some methods are based on the modal form [23], on the controllable canonical form [24], on the balanced form [25], on a zero-pole decomposition-based form [26], [19], and on a normalized observability matrix-based form [20]. It is thus (implicitly) assumed that a collection of local models are "coherent" if they are in the same canonical form. This assumption is obviously in contradiction with the aforementioned result reported in [22]. Nevertheless, satisfactory results have been reported in the above cited examples. Are they so lucky that, in each of these reported works, the chosen particular canonical form coincides with the almost only relevant definition of coherent local models? Do they all work with degenerate models, so that Definition 2 is not the only relevant one? A more plausible explanation is as follows.

It is recently reported in [30] that, if two LPV systems have equivalent I-O behavior when the scheduling variable $p(t)$ is maintained at any scheduling value $p \in \mathbb{S}$, they generally do not exhibit the same I-O behavior when $p(t)$ evolves with time. However, the difference in their I-O behaviors is bounded, with an error bound depending on the evolution speed of $p(t)$ and on how far the two LPV systems are "incoherent" from each other. Moreover, the difference between their I-O behaviors can be made arbitrarily small by restricting the evolution speed of $p(t)$. *It is possible* that, in the reported examples of LPV system identification following the local approach, the scheduling variable evolves slowly enough so that the results of local model interpolation are satisfactory, despite the fact that these local models, in any of the chosen canonical forms, may not be coherent in the sense of Definition 2.

VII. Conclusion

It has been shown in this paper that, for most systems encountered in practice, Definition 2 is the only relevant definition of local model coherence in the local approach to LPV system identification. It is previously reported that, under this definition, locally estimated state-space LTI models without making global structural assumptions do not contain sufficient information to make themselves coherent. Nevertheless, the interpolation of local models can tolerate the lack of coherence, if the scheduling variable varies slowly.

References

[1] F. Wu, "A generalized LPV system analysis and control synthesis framework," *International Journal of Control*, vol. 74, no. 7, pp. 745–759, 2001.

[2] B. Lu and F. Wu, "Switching LPV control designs using multiple parameter-dependent Lyapunov functions," *Automatica*, vol. 40, no. 11, pp. 1973–1980, 2004.

[3] B. Lu, F. Wu, and S. Kim, "Switching LPV control of an F-16 aircraft via controller state reset," *IEEE Transactions on Control Systems Technology*, vol. 14, no. 2, pp. 267–277, 2006.

[4] J. Mohammadpour and C. W. Scherer, eds., *Control of Linear Parameter Varying Systems with Applications*. New York: Springer, 2012.

[5] P. L. dos Santos, T. P. A. Perdicoulis, C. Novara, J. A. Ramos, and D. E. Rivera, eds., *Linear Parameter-Varying System Identification: New developments and trends*. New Jersey: World Scientific Publishing Company, 2012.

[6] O. Sename, P. Gaspar, and J. Bokor, eds., *Robust Control and Linear Parameter Varying Approaches*. Berlin: Springer, 2013.

[7] C. Hoffmann and H. Werner, "A survey of linear parameter-varying control applications validated by experiments or high-fidelity simulations," *IEEE Transactions on Control Systems Technology (*, vol. 23, no. 2, pp. 416–433, 2015.

[8] S. Maalej, A. Kruszewski, and L. Belkoura, "Robust control for continuous LPV system with restricted-model-based control," *Circuits, Systems, and Signal Processing*, 2017.

[9] L. Kovács, "Linear parameter varying (LPV) based robust control of type-I diabetes driven for real patient data," *Knowledge-Based Systems*, vol. 122, pp. 199–213, 2017.

[10] B. Bamieh and L. Giarré, "Identification of linear parameter varying models," *International Journal of Robust and Nonlinear Control*, vol. 12, pp. 841–853, 2002.

[11] V. Verdult and M. Verhaegen, "Kernel methods for subspace identification of multivariable LPV and bilinear systems," *Automatica*, vol. 41, no. 9, pp. 1557–1565, 2005.

[12] J.-W. Van Wingerden and M. Verhaegen, "Subspace identification of bilinear and LPV systems for open- and closed-loop data," *Automatica*, vol. 45, no. 2, pp. 372–381, 2009.

[13] R. Tóth, *Modeling and Identification of Linear Parameter-Varying Systems*. Springer, 2010.

[14] G. Mercère, H. Palsson, and T. Poinot, "Continuous-time linear parameter-varying identification of a cross: A local approach," *IEEE Trans. on Control System Technology*, vol. 19, no. 1, pp. 64–76, 2011.

[15] P. Lopes dos Santos, T.-P. Azevedo-Perdicoúlis, J. Ramos, J. Martins de Carvalho, G. Jank, and J. Milhinhos, "An LPV modeling and identification approach to leakage detection in high pressure natural gas transportation networks," *IEEE Trans. on Control System Technology*, vol. 19, pp. 77–92, Jan. 2011.

[16] R. Tóth, V. Laurain, M. Gilson, and H. Garnier, "Instrumental variable scheme for closed-loop LPV model identification," *Automatica*, vol. 48, no. 9, pp. 2314–2320, 2012.

[17] Y. Zhao, B. Huang, H. Su, and J. Chu, "Prediction error method for identification of LPV models," *Journal of Process Control*, vol. 22, no. 1, pp. 180–193, 2012.

[18] D. Piga, P. B. Cox, R. Toth, and V. Laurain, "LPV system identification under noise corrupted scheduling and output signal observations," *Automatica*, vol. 53, pp. 329–338, 2015.

[19] J. De Caigny, J. F. Camino, and J. Swevers, "Interpolation-based modeling of MIMO LPV systems," *IEEE Trans. on Control System Technology*, vol. 19, no. 1, pp. 46–63, 2011.

[20] J. De Caigny, R. Pintelon, J. Camino, and J. Swevers, "Interpolated modeling of LPV systems," *IEEE Trans. on Control System Technology*, vol. 22, no. 6, pp. 2232 – 2246, 2014.

[21] D. Vizer, G. Mercère, E. Laroche, and O. Prot, "Linear fractional LPV model identification from local experiments using an h-infinity-based glocal approach," in *Control-oriented modelling and identification : theory and practice* (M. Lovera, ed.), pp. 189–213, The Institution of Engineering and Technology, 2015.

[22] Q. Zhang and L. Ljung, "From structurally independent local LTI models to LPV model," *Automatica*, vol. 84, pp. 232–235, 2017.

[23] J. H. Yung, *Gain scheduling for geometrically nonlinear flexible space structures*. PhD thesis, MIT, Cambridge, 2002.

[24] M. Steinbuch, R. van de Molengraft, and A. van der Voort, "Experimental modelling and LPV control of a motion system," in *American Control Conference*, vol. 2, (Denver), pp. 1374–1379, 2003.

[25] M. Lovera and G. Mercère, "Identification for gain-scheduling: a balanced subspace approach," in *American Control Conference*, (New York City), pp. 858–863, 2007.

[26] J. De Caigny, F. Camino, Juan, and J. Swevers, "Interpolating model identification for SISO linear parameter-varying systems," *Mechanical Systems and Signal Processing*, vol. 23, no. 8, pp. 2395–2417, 2009.

[27] T. Kailath, *Linear Systems*. Prentice-Hall, 1980.

[28] B. Kulcsar and R. Tóth, "On the similarity state transformation for linear parameter-varying systems," in *18th IFAC World Congress*, (Milano), 2011.

[29] M. Petreczky and G. Mercère, "Affine LPV systems: realization theory, input-output equations and relationship with linear switched systems," in *51st IEEE Conference on Decision and Control (CDC)*, (Hawaii, USA), pp. 4511–4516, 2012.

[30] Z. Alkhoury and M. P. G. Mercère, "Comparing global input-output behavior of frozen-equivalent LPV state-space models," in *20th IFAC World Congress*, (Toulouse, France), pp. 10176–10181, 2017.