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Design of interval observers and controls for PDEs using finite-element approximations

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Abstract

Synthesis of interval state estimators is investigated for the systems described by a class of parabolic Partial Differential Equations (PDEs). First, a finite-element approximation of a PDE is constructed and the design of an interval observer for the derived ordinary differential equation is given. Second, the interval inclusion of the state function of the PDE is calculated using the error estimates of the finite-element approximation. Finally, the obtained interval estimates are used to design a dynamic output stabilizing control. The results are illustrated by numerical experiments with an academic example and the Black-Scholes model of financial market.

Key words: Interval observers, PDE, Finite-element approximation

1 Introduction

Model complexity is a key issue for development of control and observation algorithms. Sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics, as well as the models of other physical phenomena, can be formalized similarly in terms of PDEs, whose distributed nature introduces an additional level of intricacy. That is why control and estimation of PDEs is a very popular direction of research nowadays [3, 37, 7, 36, 29, 34, 2, 21, 16, 27, 25]. In this class of models, where the system state is a function of the space at each instant of time, the problem of its explicit measurement is natural, since only pointwise and discrete space measurements are realizable by a sensor [19, 40]. Frequently, in order to design a state estimator, the finite-dimensional approximation approach is used [1, 8, 39, 15], then the observation problem is addressed with the well-known tools available for finite-dimensional systems, while the convergence assessment has to be performed with respect to the solutions of the original distributed system.

After complexity, another difficulty for synthesis of an estimator or controller consists in the model uncertainty (unknown parameters or/and external disturbances). Presence of uncertainty implies that the design of a conventional estimator, converging to the ideal value of the state, is difficult to achieve. In this case a set-membership or interval estimation becomes more attainable: an observer can be constructed such that using the input-output information it evaluates the set of admissible values (interval) for the state at each instant of time. The interval width is proportional to the size of the model uncertainty (it has to be minimized by tuning the observer parameters). There are several approaches to design the interval/set-membership estimators [18, 24, 30]. This work is devoted to the interval observers [30, 28, 33, 32, 10], which form a subclass of set-membership estimators and whose design is based on the monotone systems theory [13, 35, 20]. The idea of the interval observer design has been proposed rather recently in [14], but it has already received numerous extensions for various classes of dynamical models. In the present paper an extension of this approach for the estimation of systems described by PDEs is discussed.

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An interval observer for systems described by PDEs using the finite-dimensional approximation approach has been proposed in [23], in the present work the proofs of those results are given, with the additional design of an output stabilizing control and an application to a model of financial market. Using the discretization error estimates from [41], the enveloping interval for solutions of the PDE is evaluated. An interesting feature of the proposed approach is that being applied to a nonlinear PDE, assuming that all nonlinearities are bounded and treated as perturbations, then the proposed interval observer is linear and can be easily implemented providing bounds on solutions of the originally nonlinear PDE (under the hypothesis that these solutions exist). The proposed control strategy disposes a similar advantage, since it is designed for a finite-dimensional model, but guaranteeing boundedness of trajectories for an uncertain distributed dynamics.

The outline of this paper is as follows. After preliminaries in Section 2, and an introduction of the distributed system properties in Section 3, the interval observer design is given in Section 4. The design of an output control algorithm based on interval estimates is considered in Section 5. The results of numerical experiments are presented in Section 6.

2 Preliminaries

The real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$. Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$. The symbols I_n, $E_{n\times m}$ and E_p denote the identity matrix with dimension $n \times n$, the matrix with all elements equal 1 with dimensions $n \times m$ and $p \times 1$, respectively.

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. The relation $P \prec 0$ ($P \succ 0$) means that the matrix $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ is negative (positive) definite. Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and $|A| = A^+ + A^-$.

Lemma 1 [10] *Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$. If $A \in \mathbb{R}^{m\times n}$ is a constant matrix, then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \quad (1)$$

2.1 Nonnegative continuous-time linear systems

A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues have negative real parts, and it is called Metzler if all its elements outside the main diagonal are nonnegative. Any solution of the linear system

$$\dot{x} = Ax + B\omega(t),\ \omega : \mathbb{R}_+ \to \mathbb{R}^q_+, \quad (2)$$
$$y = Cx + D\omega(t),$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and a Metzler matrix $A \in \mathbb{R}^{n\times n}$, is elementwise nonnegative for all $t \geq 0$ provided that $x(0) \geq 0$ and $B \in \mathbb{R}^{n\times q}_+$ [13, 35, 20]. The output solution $y(t)$ is nonnegative if $C \in \mathbb{R}^{p\times n}_+$ and $D \in \mathbb{R}^{p\times q}_+$. Such a dynamical system is called cooperative (monotone) or nonnegative if only initial conditions in $\mathbb{R}^n_+$ are considered [13, 35, 20].

For a Metzler matrix $A \in \mathbb{R}^{n\times n}$ its stability can be checked verifying a Linear Programming (LP) problem $A^{\mathrm{T}}\lambda < 0$ for some $\lambda \in \mathbb{R}^n_+ \setminus \{0\}$, or the Lyapunov matrix equation $A^{\mathrm{T}}P + PA \prec 0$ for a diagonal matrix $P \in \mathbb{R}^{n\times n}$, $P > 0$ (in the general case the matrix P should not be diagonal). The L_1 and L_∞ gains for nonnegative systems (2) have been studied in [4, 9], for this kind of systems these gains are interrelated. The conventional results and definitions on the L_2/L_∞ stability for linear systems can be found in [22].

3 Distributed systems

In this section basic facts on finite-dimensional approximations of a PDE and some auxiliary results are given.

3.1 Preliminaries

If X is a normed space with norm $||\cdot||_X$, $\Omega \subset \mathbb{R}^n$ is an open set for some $n \geq 1$ and $\phi : \Omega \to X$, define

$$||\phi||^2_{L^2(\Omega,X)} = \int_\Omega ||\phi(s)||^2_X ds,\ ||\phi||_{L^\infty(\Omega,X)} = ess\sup_{s\in\Omega} ||\phi(s)||_X.$$

By $L^\infty(\Omega, X)$ and $L^2(\Omega, X)$ denote the set of functions $\Omega \to X$ with the properties $||\cdot||_{L^\infty(\Omega,X)} < +\infty$ and $||\cdot||_{L^2(\Omega,X)} < +\infty$, respectively. Denote $I = [0, 1]$, let $C^k(I, \mathbb{R})$ be the set of functions having continuous derivatives through the order $k \geq 0$ on I. For any $q > 0$ and an open interval $I' \subset I$ define $W^{q,\infty}(I', \mathbb{R})$ as a subset of functions $y \in C^{q-1}(I', \mathbb{R})$ with an absolutely continuous $y^{(q-1)}$ and with $y^{(q)}$ essentially bounded on I', $||y||_{W^{q,\infty}} = \sum_{i=0}^q ||y^{(i)}||_{L^\infty(I',\mathbb{R})}$. Denote by $H^q(I, \mathbb{R})$ with $q \geq 0$ the Sobolev space of functions with derivatives through order q in $L^2(I, \mathbb{R})$, and for $q < 0$ the corresponding dual spaces, while by $H^q_0(I, \mathbb{R})$ a closure of C^∞ functions having compact support in I with respect to the norm in $H^q(I, \mathbb{R})$.

For two functions $z_1, z_2 : I \to \mathbb{R}$ their relation $z_1 \leq z_2$ has to be understood as $z_1(x) \leq z_2(x)$ for all $x \in I$, the inner product is defined in a standard way:

$$\langle z_1, z_2\rangle = \int_0^1 z_1(x)z_2(x)dx \quad z_1, z_2 \in L^2(I, \mathbb{R}).$$

3.2 Approximation

Following [41], consider the following PDE with homogeneous Dirichlet boundary conditions:

$$\begin{aligned}\rho(x)\frac{\partial z(x,t)}{\partial t} &= L[x, z(x,t)] + r(x,t) \quad \forall (x,t) \in I \times (0,T),\\ z(x,0) &= z_0(x) \quad \forall x \in I,\\ 0 = z(0,t) &= z(1,t) \quad \forall t \in (0,T),\end{aligned} \tag{3}$$

where $I = [0,1]$ and $T > 0$,

$$L(x,z) = \frac{\partial}{\partial x}\left(a(x)\frac{\partial z}{\partial x}\right) - b(x)\frac{\partial z}{\partial x} - q(x)z,$$

$r \in L^\infty(I \times [0,T], \mathbb{R})$, $a, b, q, \rho \in L^\infty(I, \mathbb{R})$ and there exist $a_0, a_1, \rho_0, \rho_1 \in \mathbb{R}_+$ such that

$$0 < a_0 \le a(x) \le a_1, \; 0 < \rho_0 \le \rho(x) \le \rho_1 \quad \forall x \in I,$$

and $a', b' \in L^2(I, \mathbb{R})$, where $a' = \partial a(x)/\partial x$.

Let $\Delta = \{x_j\}_{j=0}^{N'}$ for some $N' > 0$, where $0 = x_0 < x_1 < \cdots < x_{N'} = 1$, and $I_j = (x_{j-1}, x_j)$, $h_j = x_j - x_{j-1}$, $h = \max_{1\le j\le N'} h_j$. Let $P_s(I')$ be the set of polynomials of the degree less than $s+1$, $s > 0$ on an interval $I' \subseteq I$, then adopt the notation:

$$\begin{aligned}M^{s,\Delta} = \{&v \in C^0(I,\mathbb{R}) : v(x) = v_j(x) \; \forall x \in I_j,\\ &v_j \in P_s(I_j) \; \forall 1 \le j \le N'\}\end{aligned}$$

and $M = M_0^{s,\Delta} = \{v \in M^{s,\Delta} : v(0) = v(1) = 0\}$.

Introduce a bilinear form:

$$\mathcal{L}(y,v) = -\langle ay', v'\rangle - \langle by', v\rangle - \langle qy, v\rangle \quad y, v \in H^1(I,\mathbb{R}),$$

and define

$$\lambda \ge \frac{1}{2a_0}(\operatorname{ess\,sup}_{x\in I} b^2(x) - \operatorname{ess\,inf}_{x\in I} q(x)).$$

The continuous-time Galerkin approximation $Z(\cdot,t) \in M$ to the solution $z(x,t)$ of the parabolic system (3) is defined by

$$\begin{aligned}\left\langle \rho\frac{\partial Z}{\partial t}, \Phi\right\rangle &= \mathcal{L}(Z,\Phi) + \langle r, \Phi\rangle \quad \forall \Phi \in M, \; \forall t \in (0,T);\\ \mathcal{L}(Z - z_0, \Phi) - \lambda\langle Z - z_0, \Phi\rangle &= 0 \quad \forall \Phi \in M, \; t = 0.\end{aligned} \tag{4}$$

Assumption 1 *There exist $s > 0$, $l_1 > 0$ and $l_2 > 0$ such that the solution z of* (3) *belongs to* $L^\infty([0,T), W^{s+1,\infty}(I,\mathbb{R}))$ *and* $\partial z/\partial t \in L^2([0,T), H^{s+1}(I,\mathbb{R}))$,

$$||z||_{L^\infty([0,T),W^{s+1,\infty}(I,\mathbb{R}))} \le l_1, \; ||\partial z/\partial t||_{L^2([0,T),H^{s+1}(I,\mathbb{R}))} \le l_2.$$

Proposition 2 [41] *Let Assumption 1 be satisfied, then there is $\varrho > 0$ such that*

$$||Z - z||_{L^\infty(I\times(0,T),\mathbb{R})} \le \varrho h^{s+1}(l_1 + l_2),$$

where z and Z are solutions of (3) *and* (4), *respectively.*

Remark 3 *Since the operator $A = \frac{\partial}{\partial x}\left(a(x)\frac{\partial}{\partial x}\right) : L^2(I,\mathbb{R}) \to D(A) \subset L^2(I,\mathbb{R})$ with $D(A) = H_0^1(I,\mathbb{R}) \cap H^2(I,\mathbb{R})$ is closed, then for $z_0 \in D(A)$ and $\frac{\partial r}{\partial t} \in C((0,T), L^2(I,\mathbb{R})$ using, for example, [31, Ch. 1, Corollary 2.5] we derive $z \in C^0([0,T), D(A))$. Taking into account that $Z \in C^0([0,T), D(A))$ we conclude that in the latter case the obtained estimate on $Z - z$ holds for all $t \in [0,T)$ and $x \in I$.*

Remark 4 *The constants l_1 and l_2 depend on the original solution z and may be evaluated* a priori *from the domain of application, while ϱ needs a numeric experimentation to be estimated. Thus, in order to be applied, the result of this proposition can also be interpreted as the existence for any $\bar{\varrho} > 0$ a sufficiently small discretization step $h > 0$ such that $\varrho h^{s+1}(l_1 + l_2) \le \bar{\varrho}$.*

In order to calculate Z, let $\Phi_j \in M$, $1 \le j \le N$ with $N \ge N'$ be a basis in M, then following the Galerkin method [38] the solution $Z(x,t)$ of (4) can be presented as

$$Z(x,t) = \sum_{j=1}^N \xi_j(t)\Phi_j(x),$$

where $\xi = [\xi_1 \ldots \xi_N]^T \in \mathbb{R}^N$ is the vector of coefficients satisfying the ODEs for all $1 \le j \le N$:

$$\begin{aligned}&\left\langle \rho\sum_{i=1}^N \dot{\xi}_i\Phi_i, \Phi_j\right\rangle = \mathcal{L}(\sum_{i=1}^N \xi_i\Phi_i, \Phi_j) + \langle r, \Phi_j\rangle \quad \forall t \in (0,T);\\ &\mathcal{L}(\sum_{i=1}^N \xi_i(0)\Phi_i - z_0, \Phi_j) - \lambda\left\langle \sum_{i=1}^N \xi_i(0)\Phi_i - z_0, \Phi_j\right\rangle = 0,\end{aligned}$$

which finally can be presented in the form (*a.a.* means "for almost all"):

$$\Upsilon\dot{\xi}(t) = \Lambda\xi(t) + \bar{r}(t) \quad a.a. \; t \in (0,T); \; \Psi\xi(0) = \varpi,$$

where for all $1 \le i,j \le N$

$$\begin{aligned}&\Upsilon_{j,i} = \langle \rho\Phi_i, \Phi_j\rangle, \; \Lambda_{j,i} = \mathcal{L}(\Phi_i, \Phi_j), \; \bar{r}_j = \langle r, \Phi_j\rangle,\\ &\Psi_{j,i} = \mathcal{L}(\Phi_i, \Phi_j) - \lambda\langle \Phi_i, \Phi_j\rangle, \; \varpi_j = \mathcal{L}(z_0, \Phi_j) - \lambda\langle z_0, \Phi_j\rangle.\end{aligned}$$

Under the introduced restrictions on (3) and by construction of the basis functions Φ_j, we assume that the matrices Υ and Ψ are nonsingular, therefore

$$\dot{\xi}(t) = A\xi(t) + G\bar{r}(t) \quad a.a. \; t \in (0,T), \; \xi(0) = \xi_0, \tag{5}$$

where $A=\Upsilon^{-1}\Lambda\in\mathbb{R}^{N\times N}$, $G=\Upsilon^{-1}$, $\xi_0=\Psi^{-1}\varpi\in\mathbb{R}^N$ and $\bar{r}\in L^\infty([0,T),\mathbb{R}^N)$. Then for any $\xi_0\in\mathbb{R}^N$ the corresponding solution $\xi\in C^0([0,T),\mathbb{R}^N)$ to Cauchy problem (5) can be easily calculated.

3.3 Interval estimates

For $\phi\in\mathbb{R}$ define two operators $\cdot^+$ and $\cdot^-$ as follows:

$$\phi^+=\max\{0,\phi\},\ \phi^-=\phi^+-\phi.$$

Lemma 5 *Let $s,\underline{s},\overline{s}:I\to\mathbb{R}$ admit the relations $\underline{s}\le s\le\overline{s}$, then for any $\phi:I\to\mathbb{R}$ we have*

$$\left\langle\underline{s},\phi^+\right\rangle-\left\langle\overline{s},\phi^-\right\rangle\le\langle s,\phi\rangle\le\left\langle\overline{s},\phi^+\right\rangle-\left\langle\underline{s},\phi^-\right\rangle.$$

PROOF. By definition,

$$\langle s,\phi\rangle=\left\langle s,\phi^+-\phi^-\right\rangle=\left\langle s,\phi^+\right\rangle-\left\langle s,\phi^-\right\rangle$$

and the functions ϕ^+,ϕ^- take only positive values, then $\langle\underline{s},\phi^+\rangle\le\langle s,\phi^+\rangle\le\langle\overline{s},\phi^+\rangle$, $\langle\underline{s},\phi^-\rangle\le\langle s,\phi^-\rangle\le\langle\overline{s},\phi^-\rangle$ and the result follows by substitution.

Lemma 6 *Let there exist $\underline{\xi},\overline{\xi}\in C^0([0,T),\mathbb{R}^N)$ such that for the solution ξ of (5) we have*

$$\underline{\xi}(t)\le\xi(t)\le\overline{\xi}(t)\quad\forall t\in[0,T),$$

then for the solution Z of (4),

$$\underline{Z}(x,t)\le Z(x,t)\le\overline{Z}(x,t)\quad\forall(x,t)\in I\times[0,T)\quad(6)$$

and $\underline{Z},\overline{Z}\in C^0(I\times[0,T),\mathbb{R})$, where

$$\begin{aligned}\underline{Z}(x,t)&=\sum_{j=1}^{N}(\underline{\xi}_j(t)\Phi_j^+(x)-\overline{\xi}_j(t)\Phi_j^-(x)),\\ \overline{Z}(x,t)&=\sum_{j=1}^{N}(\overline{\xi}_j(t)\Phi_j^+(x)-\underline{\xi}_j(t)\Phi_j^-(x)).\end{aligned}\quad(7)$$

PROOF. The result follows from the definitions of $\Phi_j^+(x),\Phi_j^-(x)$ and Lemma 1:

$$\begin{aligned}Z(x,t)&=\sum_{j=1}^{N}\xi_j(t)\Phi_j(x)=\sum_{j=1}^{N}\xi_j(t)[\Phi_j^+(x)-\Phi_j^-(x)]\\&\le\sum_{j=1}^{N}\overline{\xi}_j(t)\Phi_j^+(x)-\underline{\xi}_j(t)\Phi_j^-(x)=\overline{Z}(x,t),\end{aligned}$$

similarly for $\underline{Z}(x,t)$. The needed continuity of $\underline{Z},\overline{Z}$ is deduced from similar properties of Φ_j and $\underline{\xi},\overline{\xi}$ since by construction $\xi\in C^0([0,T),\mathbb{R}^N)$.

The result Lemma 6 connects the interval estimates obtained for a real vector ξ and the approximated solution Z, and can be extended to z as follows:

Lemma 7 *Let Assumption 1 be satisfied and there exist $\underline{Z},\overline{Z}\in L^\infty(I\times[0,T),\mathbb{R})$ such that (6) be true for the solution Z of (4), then there is $\varrho>0$ such that for the solution z of (3),*

$$\underline{z}(x,t)\le z(x,t)\le\overline{z}(x,t)\quad(8)$$

for all $x\in I$ and almost all $t\in[0,T)$, where $\underline{z},\overline{z}\in L^\infty(I\times[0,T),\mathbb{R})$ given by

$$\begin{aligned}\underline{z}(x,t)&=\underline{Z}(x,t)-\varrho h^{s+1}(l_1+l_2),\\ \overline{z}(x,t)&=\overline{Z}(x,t)+\varrho h^{s+1}(l_1+l_2).\end{aligned}\quad(9)$$

PROOF. The result can be justified by applying the estimates on the error $Z-z$ given in Proposition 2:

$$\begin{aligned}z(t,x)&\le Z(t,x)+\varrho h^{s+1}(l_1+l_2)\\&\le\overline{Z}(x,t)+\varrho h^{s+1}(l_1+l_2)=\overline{z}(x,t)\end{aligned}$$

for almost all $(x,t)\in I\times[0,T)$, and similarly for $\underline{z}(x,t)$.

Therefore, according to lemmas 6 and 7, in order to calculate interval estimates for (3) it is enough to design an interval observer for (5).

4 Interval observer design

Assume that the state $z(x,t)$ is available for measurements in certain points $x_i^m\in I$ for $1\le i\le p$:

$$y_i(t)=z(x_i^m,t)+\nu_i(t),\quad(10)$$

where $y(t),\nu(t)\in\mathbb{R}^p$, $\nu\in L^\infty(\mathbb{R}_+,\mathbb{R}^p)$ is the measurement noise. Under Assumption 1 from Proposition 2, for a finite-element approximation we can assign

$$y_i(t)=Z(x_i^m,t)+\nu_i(t)+e_i(t),$$

where $||e||_{L^\infty([0,T),\mathbb{R}^p)}\le\varrho h^{s+1}(l_1+l_2)$ for some $\varrho>0$, $e=[e_1\dots e_p]^T$. Next,

$$y_i(t)=\sum_{j=1}^{N}\xi_j(t)\Phi_j(x_i^m)+\nu_i(t)+e_i(t)$$

and

$$y(t)=C\xi(t)+v(t),\quad(11)$$

with $v(t) = \nu(t) + e(t) \in \mathbb{R}^p$ being the new measurement noise, and $C \in \mathbb{R}^{p\times N}$ is the appropriate matrix:

$$C = \begin{bmatrix} \Phi_1(x_1^m) & \dots & \Phi_N(x_1^m) \\ \vdots & \ddots & \vdots \\ \Phi_1(x_p^m) & \dots & \Phi_N(x_p^m) \end{bmatrix}.$$

We will also assume that in (3),

$$r(x,t) = \sum_{k=1}^{m} r_{1k}(x)u_k(t) + r_0(x,t),$$

where $u(t) \in \mathbb{R}^m$ is a control (known input), $r_{1k} \in L^\infty(I,\mathbb{R})$ and $r_0 \in L^\infty(I\times[0,T),\mathbb{R})$. Then in (5):

$$G\bar{r}(t) = G\begin{bmatrix} \langle\sum_{k=1}^m r_{1k}(x)u_k(t) + r_0(x,t), \Phi_1\rangle \\ \vdots \\ \langle\sum_{k=1}^m r_{1k}(x)u_k(t) + r_0(x,t), \Phi_N\rangle \end{bmatrix}$$
$$= Bu(t) + Gd(t),$$

where

$$B = G\begin{bmatrix} \langle r_{11},\Phi_1\rangle & \dots & \langle r_{1m},\Phi_1\rangle \\ \vdots & \ddots & \vdots \\ \langle r_{11},\Phi_N\rangle & \dots & \langle r_{1m},\Phi_N\rangle \end{bmatrix} \in \mathbb{R}^{N\times m},\ d(t) = \begin{bmatrix} \langle r_0,\Phi_1\rangle \\ \vdots \\ \langle r_0,\Phi_N\rangle \end{bmatrix} \in \mathbb{R}^N$$

is an external unknown disturbance.

The idea of the work consists in design of an interval observer for the approximation (5), (11) with the aim to calculate an interval estimate for the state of (3), (10) taking into account the approximation error evaluated in Proposition 2 and the results of lemmas 6 and 7. For this purpose we need the following hypothesis.

Assumption 2 *Let $\underline{z}_0 \le z_0 \le \overline{z}_0$ for some known $\underline{z}_0, \overline{z}_0 \in L^\infty(I,\mathbb{R})$, two functions $\underline{r}_0, \overline{r}_0 \in L^\infty(I\times[0,T),\mathbb{R})$ and a constant $\nu_0 > 0$ be given such that*

$$\underline{r}_0(x,t) \le r_0(x,t) \le \overline{r}_0(x,t),\ |\nu(t)| \le \nu_0 \text{ a.a. } (x,t) \in I\times(0,T).$$

Assumption 3 *There are a matrix $L \in \mathbb{R}^{N\times p}$ and a Metzler matrix $D \in \mathbb{R}^{N\times N}$ s.t. the matrices $A - LC$ and D have the same eigenvalues and the pairs $(A - LC, \chi_1)$, (D, χ_2) are observable for some $\chi_1 \in \mathbb{R}^{1\times N}$, $\chi_2 \in \mathbb{R}^{1\times N}$.*

Thus, by Assumption 2 three intervals $[\underline{z}_0, \overline{z}_0]$, $[\underline{r}_0(x,t), \overline{r}_0(x,t)]$ and $[-\nu_0, \nu_0]$ determine for all $(x,t) \in I\times[0,T)$ in (3), (10) uncertainty of values of z_0, $r_0(x,t)$ and $\nu(t)$, respectively. Using Lemma 5 we obtain:

$$\underline{d}(t) \le d(t) \le \overline{d}(t) \quad \forall t \in [0,T),$$
$$\underline{d}(t) = \begin{bmatrix} \langle \underline{r}_0, \Phi_1^+\rangle - \langle \overline{r}_0, \Phi_1^-\rangle \\ \vdots \\ \langle \underline{r}_0, \Phi_N^+\rangle - \langle \overline{r}_0, \Phi_N^-\rangle \end{bmatrix},\ \overline{d}(t) = \begin{bmatrix} \langle \overline{r}_0, \Phi_1^+\rangle - \langle \underline{r}_0, \Phi_1^-\rangle \\ \vdots \\ \langle \overline{r}_0, \Phi_N^+\rangle - \langle \underline{r}_0, \Phi_N^-\rangle \end{bmatrix}$$

and under Assumption 1

$$-V \le v(t) \le V = \nu_0 + \varrho h^{s+1}(l_1 + l_2).$$

Finally,

$$\underline{\xi}_0 \le \xi(0) \le \overline{\xi}_0,$$

where

$$\underline{\xi}_0 = (\Psi^{-1})^+\underline{\varpi} - (\Psi^{-1})^-\overline{\varpi},\ \overline{\xi}_0 = (\Psi^{-1})^+\overline{\varpi} - (\Psi^{-1})^-\underline{\varpi}$$

and $\underline{\varpi}_j \le \varpi_j \le \overline{\varpi}_j$ for all $1 \le j \le N$ with

$$\begin{aligned}\overline{\varpi}_j = {} & \lambda[\langle \overline{z}_0, \Phi_j^-\rangle - \langle \underline{z}_0, \Phi_j^+\rangle] - \langle a\underline{z}_0', \Phi_j'^+\rangle + \langle a\overline{z}_0', \Phi_j'^-\rangle \\ & - \langle b^+\underline{z}_0' - b^-\overline{z}_0', \Phi_j^+\rangle + \langle b^+\overline{z}_0' - b^-\underline{z}_0', \Phi_j^-\rangle \\ & - \langle q^+\underline{z}_0 - q^-\overline{z}_0, \Phi_j^+\rangle + \langle q^+\overline{z}_0 - q^-\underline{z}_0, \Phi_j^-\rangle, \\ \underline{\varpi}_j = {} & \lambda[\langle \underline{z}_0, \Phi_j^-\rangle - \langle \overline{z}_0, \Phi_j^+\rangle] - \langle a\overline{z}_0', \Phi_j'^+\rangle + \langle a\underline{z}_0', \Phi_j'^-\rangle \\ & - \langle b^+\overline{z}_0' - b^-\underline{z}_0', \Phi_j^+\rangle + \langle b^+\underline{z}_0' - b^-\overline{z}_0', \Phi_j^-\rangle \\ & - \langle q^+\overline{z}_0 - q^-\underline{z}_0, \Phi_j^+\rangle + \langle q^+\underline{z}_0 - q^-\overline{z}_0, \Phi_j^-\rangle,\end{aligned}$$

According to Assumption 3 (which is always satisfied if the pair (A,C) is observable, for example) and [32] there is a nonsingular matrix $S \in \mathbb{R}^{N\times N}$ such that $D = S(A - LC)S^{-1}$. Now, applying the results of [14, 6] two bounded estimates $\underline{\xi}, \overline{\xi} \in C^0([0,T),\mathbb{R}^N)$ can be calculated, based on the available information on these intervals and $y(t)$, such that

$$\underline{\xi}(t) \le \xi(t) \le \overline{\xi}(t) \quad \forall t \in [0,T). \tag{12}$$

For this purpose, following [14, 6], rewrite (5):

$$\dot{\xi}(t) = (A - LC)\xi(t) + Bu(t) + Ly(t) - Lv(t) + Gd(t).$$

In the new coordinates $\zeta = S\xi$, (5) takes the form:

$$\dot{\zeta}(t) = D\zeta(t) + SBu(t) + SLy(t) + \delta(t), \tag{13}$$

$\delta(t) = S[Gd(t) - Lv(t)]$. And using Lemma 1 we obtain

$$\underline{\delta}(t) \le \delta(t) \le \overline{\delta}(t),$$

where $\underline{\delta}(t) = (SG)^+\underline{d}(t) - (SG)^-\overline{d}(t) - |SL|E_pV$ and $\overline{\delta}(t) = (SG)^+\overline{d}(t) - (SG)^-\underline{d}(t) + |SL|E_pV$. Next, for the system (13) an interval observer can be proposed:

$$\begin{aligned}\dot{\underline{\zeta}}(t) &= D\underline{\zeta}(t) + SBu(t) + SLy(t) + \underline{\delta}(t),\\ \dot{\overline{\zeta}}(t) &= D\overline{\zeta}(t) + SBu(t) + SLy(t) + \overline{\delta}(t),\\ \underline{\zeta}(0) &= S^{+}\underline{\xi}_0 - S^{-}\overline{\xi}_0,\ \overline{\zeta}(0) = S^{+}\overline{\xi}_0 - S^{-}\underline{\xi}_0,\\ \underline{\xi}(t) &= (S^{-1})^{+}\underline{\zeta}(t) - (S^{-1})^{-}\overline{\zeta}(t),\\ \overline{\xi}(t) &= (S^{-1})^{+}\overline{\zeta}(t) - (S^{-1})^{-}\underline{\zeta}(t),\end{aligned} \tag{14}$$

where the relations (1) are used to calculate the initial conditions for $\underline{\zeta}, \overline{\zeta}$ and the estimates $\underline{\xi}, \overline{\xi}$.

Proposition 8 *Let assumptions 1, 2 and 3 be satisfied. Then for* (5), (11) *with the interval observer* (14) *the relations* (12) *are fulfilled and* $\underline{\xi}, \overline{\xi} \in C^0([0,T),\mathbb{R}^N)$*. In addition,* $\underline{\xi}, \overline{\xi} \in L^\infty([0,T),\mathbb{R}^N)$ *if* $A - LC$ *is Hurwitz.*

PROOF. By Assumption 1 $z \in L^\infty([0,T), W^{s+1,\infty}(I,\mathbb{R}))$, then $\xi, \zeta, \underline{\xi}, \overline{\xi} \in C^0([0,T),\mathbb{R}^N)$ and $\xi, \zeta \in L^\infty([0,T),\mathbb{R}^N)$ by construction. Define two estimation errors

$$\underline{e}(t) = \zeta(t) - \underline{\zeta}(t),\ \overline{e}(t) = \overline{\zeta}(t) - \zeta(t),$$

which yield the differential equations:

$$\dot{\underline{e}}(t) = D\underline{e}(t) + \delta(t) - \underline{\delta}(t),\ \dot{\overline{e}}(t) = D\overline{e}(t) + \overline{\delta}(t) - \delta(t).$$

By Assumption 2 and the previous calculations,

$$\delta(t) - \underline{\delta}(t) \geq 0,\ \overline{\delta}(t) - \delta(t) \geq 0\ \forall t \in [0,T).$$

If D is a Metzler matrix, since all inputs of $\underline{e}(t)$, $\overline{e}(t)$ are positive and $\underline{e}(0) \geq 0$, $\overline{e}(t) \geq 0$, then $\underline{e}(t) \geq 0$, $\overline{e}(t) \geq 0$ for all $t \geq 0$ [13, 35]. The property (12) follows from these relations. If $A - LC$ is Hurwitz, D possesses the same property, since all inputs $\delta(t) - \underline{\delta}(t)$, $\overline{\delta}(t) - \delta(t)$ are bounded, then $\underline{e}, \overline{e} \in L^\infty([0,T),\mathbb{R}^N)$ and the boundedness of $\underline{\xi}, \overline{\xi}$ is followed by the boundedness of ξ.

Remark 9 *In order to regulate the estimation accuracy it is worth to strengthen the conditions of stability for* $\underline{\xi}, \overline{\xi}$ *(Hurwitz property of the matrix* $A - LC$*) to a requirement that the* L_∞ *gain of the transfer* $\begin{bmatrix}\delta - \underline{\delta}\\ \overline{\delta} - \delta\end{bmatrix} \to \begin{bmatrix}\overline{e}\\ \underline{e}\end{bmatrix}$ *is less than* γ *for some* $\gamma > 0$*. To this end, coupling this restriction with the conditions of Assumption 3 the following nonlinear matrix inequalities can be obtained:*

$$\begin{bmatrix} W^T + W + I_N & P \\ P & \gamma^2 I_N \end{bmatrix} \preceq 0, \tag{15}$$

$$W + Z \geq 0,\ P > 0,\ Z > 0, \tag{16}$$

$$SA - FC = P^{-1}WS, \tag{17}$$

which have to be solved with respect to diagonal matrices $P \in \mathbb{R}^{N\times N}$ *and* $Z \in \mathbb{R}^{N\times N}$*, nonsingular matrices* $S \in \mathbb{R}^{N\times N}$ *and* $W \in \mathbb{R}^{N\times N}$*, some* $F \in \mathbb{R}^{N\times p}$ *and* $\gamma > 0$*. Then* $D = P^{-1}W$ *and* $L = S^{-1}F$*. It is easy to see that this system can be easily solved iteratively: first, a solution* $P^{-1}W$ *of the LMIs* (15), (16) *can be found for given* $N > 0$ *with optimally tuned* $\gamma > 0$*, second, the existence of a solution* S *and* F *of the LMI* (17) *can be checked. If such a solution does not exist, then another iteration can be performed for some other values of* N*.*

Theorem 10 *Let assumptions 1, 2 and 3 be satisfied and the matrix* $A - LC$ *be Hurwitz. Then for* (3), (10) *with the interval observer* (7), (9), (14) *the relations* (8) *are fulfilled and* $\underline{z}, \overline{z} \in L^\infty(I \times [0,T),\mathbb{R})$*.*

PROOF. Since all conditions of Proposition 8 are satisfied, then the property (12) for $\xi(t)$ is true. Next, all restrictions of Lemma 6 are verified and the interval estimate (6) for $Z(t)$ is justified. Finally, the needed interval estimates for $z(t)$ can be obtained by applying Lemma 7.

Remark 11 *The designed interval observer can also be applied to a nonlinear PDE. If in Assumption 2,*

$$\underline{r}_0(x,t) \leq r_0(x,t,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial t}) \leq \overline{r}_0(x,t)$$

for some known $\underline{r}_0, \overline{r}_0 \in L^\infty(I \times [0,T),\mathbb{R})$ *for all* $x \in I$*,* $t \in [0,T)$ *and the corresponding solutions* $z(x,t)$ *(provided that they exist for such a nonlinear PDE and the Galerkin method can be applied), then the interval observer* (7), (9), (14) *preserves its form and the result of Theorem 10 stays correct. In such a case the proposed interval observer can be used for a fast and reliable calculation of envelops for solutions of nonlinear PDEs.*

5 Control design

In this section the interval observer (14) is used to design a control law ensuring stabilization of the finite-dimensional approximation (5), (11) in the spirit of [11], which implies also (under additional mild restrictions) the stabilization of (3).

In Theorem 10 the gain L together with the transformation matrix S have been used to guarantee the properties of positivity and stability for the dynamics of estimation errors $\underline{e}(t)$, $\overline{e}(t)$. The positivity property has been obtained uniformly in $u(t)$. Thus, the control design can be applied in order to ensure boundedness of the observer estimates $\underline{z}(x,t)$, $\overline{z}(x,t)$, that in its turn (since $\underline{z}(x,t) \leq z(x,t) \leq \overline{z}(x,t)$ for almost all $(x,t) \in I\times[0,T)$, see (8)) will provide boundedness of $z(x,t)$. An advantage of this approach is that the system (3) is uncertain, distributed and the state of that system cannot be measured (it is infinite-dimensional), while the observer (14) together with (7), (9) is a completely known linear system with the accessible state $\underline{\zeta}(t), \overline{\zeta}(t)$ [11]. An obstacle is that the dimension of the state of (14) is $2N$, while the dimension of the control is m, similarly to (5).

In our work, the control is chosen as a conventional state linear feedback:

$$u(t) = \underline{K}\underline{\zeta}(t) + \overline{K}\overline{\zeta}(t) \quad (18)$$

where $\underline{K}, \overline{K} \in \mathbb{R}^{m\times N}$ are two feedback matrix gains to be designed. To this end, let us consider the combined system, which consists of (13), (14) and (18):

$$\dot{\eta}(t) = (\widetilde{A} + \widetilde{B}K\Gamma)\eta(t) + \widetilde{\Delta}(t), \quad (19)$$

where $\eta = [\zeta^{\mathrm{T}}\ \underline{\zeta}^{\mathrm{T}}\ \overline{\zeta}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{3N}$ is the combined state and

$$\widetilde{A} = \begin{bmatrix} SAS^{-1} & 0 & 0 \\ SLCS^{-1} & D & 0 \\ SLCS^{-1} & 0 & D \end{bmatrix},\ \widetilde{B} = \begin{bmatrix} SB \\ SB \\ SB \end{bmatrix},\ K = [0\ \underline{K}\ \overline{K}],$$

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_N & 0 \\ 0 & 0 & I_N \end{bmatrix},\ \widetilde{\Delta}(t) = \begin{bmatrix} SGd(t) \\ \underline{\delta}(t) + SLv(t) \\ \overline{\delta}(t) + SLv(t) \end{bmatrix}.$$

Proposition 12 *Let assumptions 2 and 3 be satisfied. Then for* (5), (11) *with the interval observer* (14) *and the control law* (18) *the relations* (12) *are satisfied. In addition,* $\xi, \underline{\xi}, \overline{\xi} \in L^\infty([0,T), \mathbb{R}^N)$ *if there exists a matrix* $\widetilde{X} \in \mathbb{R}^{3N\times 3N}$ *in the form*

$$\widetilde{X} = \begin{bmatrix} X_P & 0\ 0 \\ 0 & \\ 0 & X_Q \end{bmatrix},\ X_P = X_P^T \in \mathbb{R}^{N\times N},\ X_Q = X_Q^T \in \mathbb{R}^{2N\times 2N} \quad (20)$$

and $\widetilde{Y} \in \mathbb{R}^{m\times 2N}$ *that satisfy the matrix inequalities*

$$\widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{A}^T + \widetilde{B}\begin{bmatrix} 0 & \widetilde{Y} \end{bmatrix} + \begin{bmatrix} 0 & \widetilde{Y} \end{bmatrix}^T \widetilde{B}^T \prec 0,\ X_P \succ 0,\ X_Q \succ 0, \quad (21)$$

then $[\underline{K}\ \overline{K}] = \widetilde{Y}X_Q^{-1}$ *and* (5), (11), (14), (18) *is stable.*

PROOF. The relations (12) can be substantiated repeating the same arguments as previously in Proposition 8 (they are independent in control).

Substitution of the control (18) into the equations of the interval observer (14) together with the actual system (13) will give us the equations of the combined system (19), in which $\widetilde{\Delta}(t)$ is bounded since the signals $d(t)$, $\underline{\delta}(t)$, $\overline{\delta}(t)$ and $\nu(t)$ are bounded by Assumption 2 and previous calculations made in Section 4. Calculating derivative of the Lyapunov function $V(\eta) = \eta^{\mathrm{T}}\widetilde{P}\eta$ we obtain

$$\begin{aligned}\dot{V} &= \eta^{\mathrm{T}}[(\widetilde{A} + \widetilde{B}K\Gamma)^{\mathrm{T}}\widetilde{P} + \widetilde{P}(\widetilde{A} + \widetilde{B}K\Gamma)]\eta + 2\eta^{\mathrm{T}}\widetilde{P}\widetilde{\Delta} \\ &\le \eta^{\mathrm{T}}[(\widetilde{A} + \widetilde{B}K\Gamma)^{\mathrm{T}}\widetilde{P} + \widetilde{P}(\widetilde{A} + \widetilde{B}K\Gamma) + \chi I_{3N}]\eta \\ &\quad + \chi^{-1}\widetilde{\Delta}^{\mathrm{T}}\widetilde{P}^2\widetilde{\Delta}\end{aligned}$$

for some $\chi > 0$. Therefore, to prove the proposition we need to ensure stability of the matrix $\widetilde{A} + \widetilde{B}K\Gamma$ by verifying the Lyapunov equation for the matrix $\widetilde{P} = \widetilde{X}^{-1}$:

$$(\widetilde{A} + \widetilde{B}K\Gamma)^{\mathrm{T}}\widetilde{P} + \widetilde{P}(\widetilde{A} + \widetilde{B}K\Gamma) \prec 0,$$

which can be rewritten as

$$\widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{A}^{\mathrm{T}} + \widetilde{B}K\Gamma\widetilde{X} + \widetilde{X}\Gamma^{\mathrm{T}}K^{\mathrm{T}}\widetilde{B}^{\mathrm{T}} \prec 0,$$

and after the transformation

$$\widetilde{B}K\Gamma\widetilde{X} = \widetilde{B}K\begin{bmatrix} 0 & 0\ 0 \\ 0 & \\ 0 & X_Q \end{bmatrix} = \widetilde{B}\begin{bmatrix} 0 & \widetilde{Y} \end{bmatrix}$$

it is equivalent to the stated LMI (21) for the variable $\widetilde{Y} = [\underline{K}\ \overline{K}]X_Q$.

Remark 13 *In order to regulate the estimation accuracy it is worth to strengthen the conditions of stability for* η *to a requirement that the* H_∞ *gain of the transfer* $\widetilde{\Delta} \to z$ *is less than* γ *for some* $\gamma > 0$*, where* $z = H\eta$ *is an auxiliary performance output (for example,* $z = \overline{\zeta} - \underline{\zeta}$ *characterizes the interval estimation accuracy). To this end, consider again the Lyapunov function* $V(\eta) = \eta^T\widetilde{P}\eta$ *whose derivative can be rewritten as follows:*

$$\dot{V} = \begin{pmatrix} \eta \\ \widetilde{\Delta} \end{pmatrix}\begin{pmatrix} (\widetilde{A} + \widetilde{B}K\Gamma)^T\widetilde{P} + \widetilde{P}(\widetilde{A} + \widetilde{B}K\Gamma) + H^TH & \widetilde{P} \\ \widetilde{P} & -\gamma^2 I_{3N} \end{pmatrix}\begin{pmatrix} \eta \\ \widetilde{\Delta} \end{pmatrix} \\ -z^Tz + \gamma^2\widetilde{\Delta}^T\widetilde{\Delta}.$$

As in the proof of Proposition 12 denote $\widetilde{P}^{-1} = \widetilde{X}$*, then the following linear matrix equality can be obtained:*

$$\begin{pmatrix} \widetilde{X} & 0 \\ 0 & I_{3N} \end{pmatrix}\begin{pmatrix} (\widetilde{A} + \widetilde{B}K\Gamma)^T\widetilde{P} + \widetilde{P}(\widetilde{A} + \widetilde{B}K\Gamma) + H^TH & \widetilde{P} \\ \widetilde{P} & -\gamma^2 I_{3N} \end{pmatrix}\begin{pmatrix} \widetilde{X} & 0 \\ 0 & I_{3N} \end{pmatrix} \\ = \begin{pmatrix} \widetilde{A}\widetilde{X} + \widetilde{X}\widetilde{A}^T + \widetilde{B}\begin{bmatrix} 0 & \widetilde{Y} \end{bmatrix} + \begin{bmatrix} 0 & \widetilde{Y} \end{bmatrix}^T\widetilde{B}^T + \widetilde{X}H^TH\widetilde{X} & I_{3N} \\ I_{3N} & -\gamma^2 I_{3N} \end{pmatrix}$$

where

$$\begin{bmatrix} 0 & \widetilde{Y} \end{bmatrix} = K\Gamma\widetilde{X}$$

and $\widetilde{Y} \in \mathbb{R}^{m\times 2N}$ is a new matrix variable. Finally, using Schur complement we derive an LMI:

$$\begin{bmatrix} \tilde{A}\tilde{X}+\tilde{X}\tilde{A}^T+\tilde{B}\begin{bmatrix}0 & \tilde{Y}\end{bmatrix}+\begin{bmatrix}0 & \tilde{Y}\end{bmatrix}^T\tilde{B}^T & I_{3N} & \tilde{X}H \\ I_{3N} & -\gamma^2 I_{3N} & 0_{3N} \\ H\tilde{X} & 0_{3N} & -I_{3N} \end{bmatrix} \prec 0, \quad (22)$$
$$X_P \succ 0,\ X_Q \succ 0$$

which has to be solved with respect to the matrices X_P, X_Q and $\widetilde{Y}$, then $[\underline{K}\ \overline{K}] = \widetilde{Y}X_Q^{-1}$ as in Proposition 12.

Remark 14 *The required gains $\underline{K}$ and $\overline{K}$ exist if the matrix pair (A, B) is controllable (stabilizable). Indeed, impose a restriction that $\underline{K} = \overline{K} = 0.5KS^{-1}$, where $K \in \mathbb{R}^{m\times N}$ now is a new controller gain to find, and consider two auxiliary variables*

$$e(t) = \zeta(t) - \frac{\underline{\zeta}(t) + \overline{\zeta}(t)}{2},\ w(t) = \overline{\zeta}(t) - \underline{\zeta}(t),$$

which correspond to a regulation error with respect to the middle value of the estimated interval and the interval width, and whose dynamics is as follows:

$$\dot{e}(t) = De(t) + \delta(t) - \frac{\underline{\delta}(t) + \overline{\delta}(t)}{2},\ \dot{w}(t) = Dw(t) + \overline{\delta}(t) - \underline{\delta}(t).$$

Obviously, to study the stability property of (5), (11), (14), (18), instead of analysis of the vector η its linear transformation $\eta^* = [\zeta^T\ e^T\ w^T]^T \in \mathbb{R}^{3N}$ can be considered, then $u(t) = KS^{-1}(\zeta(t) - e(t))$ and

$$\dot{\eta}^*(t) = A^*\eta^*(t) + \Delta^*(t),$$

where

$$A^* = \begin{bmatrix} S(A+BK)S^{-1} & -SBKS^{-1} & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{bmatrix},\ \Delta^*(t) = \begin{bmatrix} SGd(t) \\ \delta(t) - \frac{\underline{\delta}(t)+\overline{\delta}(t)}{2} \\ \overline{\delta}(t) - \underline{\delta}(t) \end{bmatrix}.$$

*As before, the vector $\Delta^*(t)$ is bounded, and the system stability follows the same property of the matrix A^*, which has an upper-triangular structure and if the matrix D is stable (*i.e. *the pair (A, C) is observable), then as in the conventional case the separation principle holds and the required conclusion can be justified for a stable matrix $A+BK$. Note that the use in the control of both bounds, $\underline{\zeta}$ and $\overline{\zeta}$, allows to compensate the dependence of the dynamics of ζ on the interval width w, contrarily the case when only one bound is used, as in [26].*

Now, in order to prove boundedness of the state of (3) with application of the control (18), let us replace Assumption 1 with the following one:

Assumption 4 *Let the coefficients of (3), $a, b, q, p \in L^\infty(I, \mathbb{R})$, be smooth on I, $z_0(x) \in H_0^1(I, \mathbb{R})$, $r_0 \in L^2([0, T), L^2(I, \mathbb{R}))$, $r_1 \in L^2(I, \mathbb{R})$ and for any $u \in C^1([0, T), \mathbb{R})$ there exists a weak solution of (3) $z \in L^2([0, T), H_0^1(I, \mathbb{R}))$ with $\dot{z}_t \in L^2([0, T), H^{-1}(I, \mathbb{R}))$.*

Using this regularity hypothesis it is possible to substantiate stabilization by the control (18) of the distributed-parameter system (3):

Theorem 15 *Let assumptions 2, 3 and 4 be satisfied. Then for the system (3), (10) with the interval observer (7), (14) and the control (18), the relations (8) are satisfied, and $z, \underline{Z}, \overline{Z} \in L^\infty(I \times [0, T), \mathbb{R})$.*

PROOF. According to [12] with the restrictions on the coefficients, input and initial conditions of the system (3) introduced in Assumption 4, it holds that

$$z \in L^2([0, T), H^2(I, \mathbb{R})) \cap L^\infty([0, T), H_0^1(I, \mathbb{R})),$$
$$\dot{z}_t \in L^2([0, T), L^2(I, \mathbb{R})),$$

and we have the estimate

$$\underset{0\le t\le T}{\text{esssup}}||z(\cdot, t)||_{H_0^1(I,\mathbb{R})} + ||z(\cdot, t)||_{L^2([0,T),H^2(I,\mathbb{R}))}$$
$$+||\dot{z}_t(\cdot, t)||_{L^2([0,T),L^2(I,\mathbb{R}))}$$
$$\le c\left(||r(\cdot, t)||_{L^2([0,T),L^2(I,\mathbb{R}))} + ||z_0||_{H_0^1(I,\mathbb{R})}\right),$$

where the constant c is depending only on I, T and the coefficient functions a, b, q, p. Note that the above inequality does not imply boundedness of z, and it only states its existence on the interval of time $[0, T)$ (a kind of forward completeness in time). Therefore, Assumption 1 is valid for some $l_1 > 0$ and $l_2 > 0$, then we can apply the result of Proposition 2 to estimate the error of the approximation. Finally, since all conditions are satisfied, the results of Proposition 12 and Lemma 7 are true, then the conclusion on boundedness of $z, \underline{Z}, \overline{Z}$ follows.

6 Example

6.1 Academic example with control

Consider an unstable academic example of (3) with

$$\rho(x) = 0.7\sin(0.67x),\ a(x) = 1.5 + 1.5\cos(0.2x^{0.25}),$$
$$b(x) = -2 + \sin(2\sqrt{x}),$$
$$q(x) = -0.8 - x^2\cos(3x),\ r_1(x) = x^3 + 2.5,$$
$$r_0(x, t) = r_{01}(x)r_{02}(t),\ r_{01}(x) = 0.1\cos(3\pi x),\ |r_{02}(t)| \le 1,$$

and $T = 10$, then $\lambda = 1$ is an admissible choice and r_{02} is an uncertain part of the input r_0 (for simulation $r_{02}(t) = \cos(5t)$), then

$$\underline{r}_0(x, t) = -|r_{01}(x)|,\ \overline{r}_0(x, t) = |r_{01}(x)|.$$

The uncertainty of initial conditions is given by

$$\underline{z}_0(x) = z_0(x) - 1,\ \overline{z}_0(x) = z_0(x) + 1,$$

where $z_0(x) = \sin(\pi x)$ is the function used as the initial condition for simulation. Take $\Delta = \{0, h, 2h, \dots, 1-h, 1\}$ with $h = 1/N'$, and a pyramidal basis

$$\Phi_i(x) = \begin{cases} 0 & x \le x_{i-1}, \\ \frac{x-x_{i-1}}{x_i - x_{i-1}} & x_{i-1} < x \le x_i, \\ \frac{x_{i+1}-x}{x_{i+1}-x_i} & x_i < x \le x_{i+1}, \\ 0 & x \ge x_{i+1} \end{cases} \tag{23}$$

for $i = 0, \dots, N = N'$ (it is assumed $x_{-1} = -h$ and $x_{N+1} = 1 + h$). For simulation we took $N = 10$, then the approximated dynamics (5), (11) is an observable system, and assume that $\varrho h^{s+1}(l_1 + l_2) \le \bar{\varrho} = 0.1$. Let $p = 3$ with $x_1^m = 0.2$, $x_2^m = 0.5$, $x_3^m = 0.8$, and

$$\nu(t) = 0.1[\sin(20t)\ \sin(15t)\ \cos(25t)]^{\mathrm{T}},$$

then $\nu_0 = 0.14$. For calculation of the scalar product in space or for simulation of the approximated PDE in time, the explicit Euler method has been used with the step 0.01. The matrix L is selected to ensure distinct eigenvalues of the matrix $A - LC$ in the interval $[-10.22, -1.4]$, then S^{-1} is composed by eigenvectors of the matrix $A - LC$ and the matrix D is chosen diagonal.

To calculate the control matrix $[\underline{K}\ \overline{K}]$ the LMIs (22) has been used with YALMIP optimization toolbox in Matlab, and it is found with $\gamma = 1.1505$ that

$$\underline{K} = \overline{K} = [-0.034\ 0.126\ 0.122\ 0.076\ 0.185\ -0.018\ -0.508\ -0.022].$$

The results of the interval estimation and control are shown in the Fig. 1 for different instances of time, where red lines corresponds to $Z(x,t)$, while green and blue ones represent $\underline{z}(x,t)$ and $\overline{z}(x,t)$, respectively (20 and 40 points are used for plotting in space and in time).

6.2 Black–Scholes model

The Black–Scholes PDE governs the price evolution of an option under the so-called Black–Scholes model (a mathematical model of a financial market containing de-

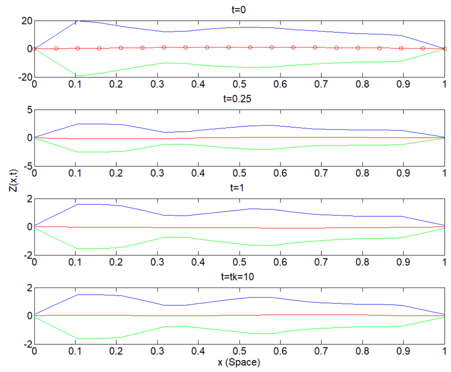

Figure 1. The results of the interval estimation of the academic example for different instants of time: $t = 0,\ 0.25,\ 1,\ 10$ for $N = 10$

rivative investment instruments [1]):

$$\frac{\partial \mathcal{V}(S,t)}{\partial t} = -\frac{1}{2}\sigma^2 S^2 \frac{\partial \mathcal{V}^2(S,t)}{\partial S^2} - (r(t) - q(t))S\frac{\partial \mathcal{V}(S,t)}{\partial S}$$
$$+ r(t)\mathcal{V}(S,t) + g(S) \quad \forall (S,t) \in I \times (0,T),$$
$$\mathcal{V}(S,0) = \mathcal{V}_0 \quad \forall S \in I,$$
$$0 = \mathcal{V}(0,t) = \mathcal{V}(1,t) \quad \forall t \in (0,T),$$

where $\mathcal{V}(S,t)$ is the price of the option, S is the stock price belonging a given interval of admissible prices I; r is the risk-free interest rate, q is the dividend rate of the underlying asset, σ is the volatility of the stock; and $g(S)$ is an inhomogeneous term [5, 17]. Obviously, this equation can be presented in the form of (3) with the following parameters ($x = S$ for a normalized price):

$$a(x) = -0.16x^2(2 + \sin(x))\cos(x) - 0.16x(2 + \sin(x))^2,$$
$$\rho(x) = 1,\ b(x) = -0.06x,\ q(x) = 0.06,\ r_1(x) = 0.8x^2 - 1,$$
$$r_0(x,t) = r_{01}(x) r_{02}(t),\ r_{01}(x) = 0.2\sin^2(3\pi x),\ |r_{02}(t)| \le 1,$$

with $T = 5$, then $\lambda = 1$ is an admissible choice and r_{02} is an uncertain part of the input r_0, the uncertainty of initial conditions (for simulation $z_0(x) = \max(x - 25e^{-0.06x}, 0)$ is given by the interval

$$\underline{z}_0(x) = \max(z_0(x) - 0.1, 0),\ \overline{z}_0(x) = z_0(x) + 0.1.$$

The decomposition basis (23) is taken as in the previous example, points for measurements with $p = 3$ are $x_1^m = 0.2$, $x_2^m = 0.5$, $x_3^m = 0.8$, and

$$\nu(t) = 0.2[\cos(2t)\ \sin(1.8t)\ \cos(3t)]^{\mathrm{T}},$$

then $\nu_0 = 0.217$. For calculation of the scalar product in space or for simulation of the approximated PDE in time,

[1] https://en.wikipedia.org/wiki/Black–Scholes_equation
https://www.theguardian.com/science/2012/feb/12/black-scholes-equation-credit-crunch

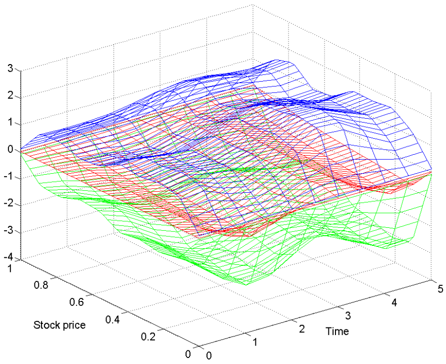

Figure 2. The results of the interval estimation for the Black-Scholes model

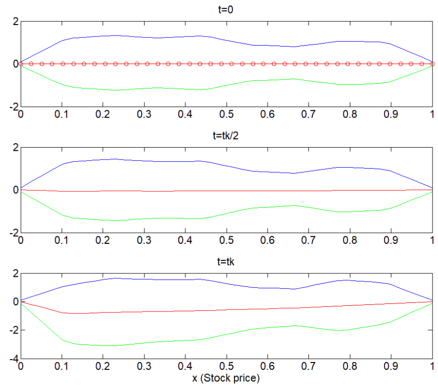

Figure 3. The results of the interval estimation of the Black Scholes model at instant of time time $t = 0,\ \frac{T}{2},\ T$

the implicit Euler method is used with the step 0.01. The matrix L is selected to ensure distinct eigenvalues of the matrix $A - LC$ in the interval $[-8.63, -0.72]$, then S^{-1} is composed by eigenvectors of $A - LC$ and the matrix D is chosen diagonal. The results of the interval estimation are shown in Fig. 2, where the red surface corresponds to $Z(x,t)$, while green and blue ones represent $\underline{z}(x,t)$ and $\overline{z}(x,t)$, respectively (20 and 40 points are used for plotting in space and in time). In the Fig. 3 the interval estimates are shown for different instants of time.

7 Conclusion

Taking a parabolic PDE with Dirichlet boundary conditions, a method of design of interval observers is proposed, which is based on a finite-element approximation. The errors of discretization given in [41] are taken into account by the interval estimates. The proposed interval observer is used for control of an uncertain PDE system. The efficiency of the proposed interval observer and control is demonstrated through numerical experiments with the academic and Black–Scholes models.

For future research, the passage to finite-element approximation can be avoided developing conditions of positivity of solutions of a PDE. Next, more complex uncertainty of PDE equation can also be incorporated in the design procedure. Besides, this approach can be extended as well to PDEs with Neumann, Robin, or mixed boundary conditions.

References

[1] J. Alvarez and G. Stephanopoulos. An estimator for a class of non-linear distributed systems. *International Journal of Control*, 5(36):787–802, 1982.

[2] N. Barje, M. Achhab, and V. Wertz. Observer for linear distributed-parameter systems with application to isothermal plug-flow reactor. *Intelligent Control and Automation*, 4(4):379–384, 2013.

[3] Kristian Bredies, Christian Clason, Karl Kunisch, and Gregory von Winckel. *Control and optimization with PDE constraints*, volume 164 of *International Series of Numerical Mathematics*. Birkhäuser, Basel, 2013.

[4] C. Briat. Robust stability analysis of uncertain linear positive systems via integral linear constraints: l_1- and l_∞-gain characterizations. In *Proc. 50th IEEE CDC and ECC*, pages 6337–6342, Orlando, 2011.

[5] J.S. Butler and B. Schachter. Unbiased estimation of the Black/Scholes formula. *Journal of Financial Economics*, 15(3):341–357, 1986.

[6] S. Chebotarev, D. Efimov, T. Raïssi, and A. Zolghadri. Interval observers for continuous-time LPV systems with l_1/l_2 performance. *Automatica*, 58(8):82–89, 2015.

[7] M. A. Demetriou. Natural second-order observers for second-order distributed parameter systems. *Systems & Control Letters*, 51(3-4):225–234, 2004.

[8] D. Dochain. State observers for tubular reactors with unknown kinetics. *Journal of Process Control*, 10:259–268, 2000.

[9] Y. Ebihara, D. Peaucelle, and D. Arzelier. L1 gain analysis of linear positive systems and its application. In *Proc. 50th IEEE CDC and ECC*, pages 4029–4035, Orlando, 2011.

[10] D. Efimov, L.M. Fridman, T. Raïssi, A. Zolghadri, and R. Seydou. Interval estimation for LPV systems applying high order sliding mode techniques. *Automatica*, 48:2365–2371, 2012.

[11] D. Efimov, T. Raïssi, and A. Zolghadri. Control of nonlinear and lpv systems: interval observer-based framework. *IEEE Trans. Automatic Control*, 58(3):773–782, 2013.

[12] Lawrence C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence (R.I.), 1998.

[13] L. Farina and S. Rinaldi. *Positive Linear Systems: Theory and Applications*. Wiley, New York, 2000.

[14] J.L. Gouzé, A. Rapaport, and M.Z. Hadj-Sadok. Interval observers for uncertain biological systems. *Ecological Modelling*, 133:46–56, 2000.

[15] G. Hagen and Mezic I. Spillover stabilization in finite-dimensional control and observer design for dissipative evolution equations. *SIAM Journal on Control and Optimization*, 2(42):746–768, 2003.

[16] A. Hasan, O. M. Aamo, and M. Krstic. Boundary observer design for hyperbolic PDE-ODE cascade systems. *Automatica*, 68:75–86, 2016.

[17] O. Hyong-Chol, Jo. Jong-Jun, and Kim Ji-Sok. General properties of solutions to inhomogeneous Black–Scholes equations with discontinuous maturity payoffs. *Journal of Differential Equations*, 260:3151–3172, 2016.

[18] L. Jaulin. Nonlinear bounded-error state estimation of continuous time systems. *Automatica*, 38(2):1079–1082, 2002.

[19] S.B Jrgensen, L. Goldschmidt, and K. Clement. A sensor location procedure for chemical processes. *Comp. Chem. Engng.*, 8:195–204, 1984.

[20] T. Kaczorek. *Positive 1D and 2D Systems*. London, U.K.: Springer-Verlag, communications and control engineering edition, 2002.

[21] N. N. Kamran and S. V. Drakunov. Observer design for distributed parameter systems. In *Proceedings of the Conference on Control and its Applications*, 2015.

[22] Hassan K. Khalil. *Nonlinear Systems*. Prentice Hall PTR, 3rd edition, 2002.

[23] T. Kharkovskaya, D. Efimov, A. Polyakov, and J.-P. Richard. Interval observers for pdes: approximation approach. In *Proc. 10th IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, Monterey, 2016.

[24] M. Kieffer and E. Walter. Guaranteed nonlinear state estimator for cooperative systems. *Numerical Algorithms*, 37:187–198, 2004.

[25] M. Krstic. Compensating actuator and sensor dynamics governed by diffusion pdes. *Systems & Control Letters*, 58:372–377, 2009.

[26] F. Mazenc, T. N. Dinh, and S. I. Niculescu. Robust interval observers and stabilization design for discrete-time systems with input and output. *Automatica*, 49:3490–3497, 2013.

[27] T. Meurer. On the extended Luenberger-type observer for semilinear distributed-parameter systems. *IEEE Transaction Automatic Control*, 58:1732–1743, 2013.

[28] M. Moisan, O. Bernard, and J.L. Gouzé. Near optimal interval observers bundle for uncertain bioreactors. *Automatica*, 45(1):291–295, 2009.

[29] T. D. Nguyen. Second-order observers for second order distributed parameter systems in r2. *Systems & Control Letters*, 57(10):787–795, 2008.

[30] B. Olivier and J.L. Gouzé. Closed loop observers bundle for uncertain biotechnological models. *Journal of Process Control*, 14(7):765–774, 2004.

[31] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.

[32] T. Raïssi, D. Efimov, and A. Zolghadri. Interval state estimation for a class of nonlinear systems. *IEEE Trans. Automatic Control*, 57(1):260–265, 2012.

[33] T. Raïssi, G. Videau, and A. Zolghadri. Interval observers design for consistency checks of nonlinear continuous-time systems. *Automatica*, 46(3):518–527, 2010.

[34] D.L. Russell. *Encyclopedia of Life Support Systems (EOLSS): Control Systems Robotics And Automation*, chapter Distributed parameter systems: An overview. London:EOLSS Publishers, 2003.

[35] H.L. Smith. *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, volume 41 of *Surveys and Monographs*. AMS, Providence, 1995.

[36] A. Smyshlyaev and M. Krstic. Backstepping observers for a class of parabolic pdes. *Systems & Control Letters*, 54(7):613–625, 2005.

[37] Andrey Smyshlyaev and Miroslav Krstic. *Adaptive Control of Parabolic PDEs*. Princeton University Press, 2010.

[38] Vidar Thomée. *Galerkin Finite Element Methods for Parabolic Problems*. Springer, Berlin, 2006.

[39] A. Vande Wouver and M. Zeitz. *Encyclopedia of Life Support Systems (EOLSS)*, chapter State estimation in distributed parameter systems. Eolss Publishers, 2002.

[40] A. Vande Wouwer, N. Point, S. Porteman, and M. Remy. An approach to the selection of optimal sensor locations in distributed parameter systems. *Journal of Process Control*, 10(4):291–300, 2000.

[41] M.F. Wheeler. l^∞ estimates of optimal orders for Galerkin methods for one-dimensional second order parabolic and hyperbolic equations. *SIAM J. Numer. Anal.*, 10(5):908–913, 1973.