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# Out-degree reducing partitions of digraphs

J. Bang-Jensen<sup>1</sup>, S. Bessy<sup>2</sup>, F. Havet<sup>3</sup> and A. Yeo<sup>1,4</sup>

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<sup>1</sup> Department of Mathematics and Computer Science, University of Southern Denmark, Odense DK-5230, Denmark, Email: {jbj,yeo}@imada.sdu.dk. Financial support: Danish research council, grant number 1323-00178B and Labex UCN@Sophia

<sup>2</sup> LIRMM, Université de Montpellier, Montpellier, France. Email: stephane.bessy@lirmm.fr. Financial support: OSMO project, Occitanie regional council.

<sup>3</sup> Université Côte d’Azur, CNRS, I3S and INRIA, Sophia Antipolis, France. Email: frederic.havet@cnrs.fr.

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<sup>4</sup> Department of Mathematics, University of Johannesburg, Auckland Park, 2006 South Africa.

## Abstract

Let  $k$  be a fixed integer. We determine the complexity of finding a  $p$ -partition  $(V_1, \dots, V_p)$  of the vertex set of a given digraph such that the maximum out-degree of each of the digraphs induced by  $V_i$ , ( $1 \leq i \leq p$ ) is at least  $k$  smaller than the maximum out-degree of  $D$ . We show that this problem is polynomial-time solvable when  $p \geq 2k$  and  $\mathcal{NP}$ -complete otherwise. The result for  $k = 1$  and  $p = 2$  answers a question posed in [3]. We also determine, for all fixed non-negative integers  $k_1, k_2, p$ , the complexity of deciding whether a given digraph of maximum out-degree  $p$  has a 2-partition  $(V_1, V_2)$  such that the digraph induced by  $V_i$  has maximum out-degree at most  $k_i$  for  $i \in [2]$ . It follows from this characterization that the problem of deciding whether a digraph has a 2-partition  $(V_1, V_2)$  such that each vertex  $v \in V_i$  has at least as many neighbours in the set  $V_{3-i}$  as in  $V_i$ , for  $i = 1, 2$  is  $\mathcal{NP}$ -complete. This solves a problem from [6] on majority colourings.

**Keywords:** 2-partition, maximum out-degree reducing partition,  $\mathcal{NP}$ -complete, polynomial algorithm.

## 1 Introduction

Notation and terminology generally follow [2]. However we recall the useful notations and definitions in Section 2. A  $p$ -**partition** of a graph or digraph  $G$  is a vertex partition  $(V_1, V_2, \dots, V_p)$  of its vertex set  $V(G)$ .

It is a well-known and easy fact that every undirected graph  $G$  admits a 2-partition such that the degree of each vertex in its part is at most half of its degree in  $G$  and such a partition can be found by a greedy algorithm (or by considering a maximum-cut partition). So we have the following.

### Proposition 1.1.

- (i) Every graph  $G$  has a 2-partition  $(V_1, V_2)$  such that  $d_{G[V_i]}(v) \leq d_G(v)/2$  for all  $i \in \{1, 2\}$  and all  $v \in V_i$ .
- (ii) Every graph  $G$  has a 2-partition  $(V_1, V_2)$  with  $\Delta(G[V_i]) \leq \Delta(G)/2$  for  $i = 1, 2$ .

Thomassen [10] constructed an infinite class of strongly connected digraphs  $\mathcal{T} = T_1, T_2, \dots, T_k, \dots$  with the property that for each  $k$ ,  $T_k$  is  $k$ -out-regular and has no even directed cycle. As remarked by Alon in [1] this implies that we cannot expect any directed analogues of the statements in Proposition 1.1.

**Proposition 1.2.** *Let  $k$  be a positive integer. For every 2-partition  $(V_1, V_2)$  of  $T_k$ , some vertex has all its  $k$  out-neighbours in the same part as itself, so  $\max\{\Delta^+(D[V_1]), \Delta^+(D[V_2])\} = \Delta^+(D)$ .*

This is due to the simple fact that if a digraph  $D$  has a 2-partition  $(V_1, V_2)$  such that the bipartite digraph induced by the arcs between the two sets has minimum out-degree at least 1, then this digraph, and hence also  $D$ , has an even directed cycle.

Alon [1] also remarked that it is always possible to split  $V(D)$  into three sets such that each of the induced subdigraphs has smaller maximum out-degree than  $D$  (see Theorem 6.5). In Proposition 5.1, we generalize this to all values of  $k$ . We show that for every positive integer  $k$ , there is a  $(2k + 1)$ -partition of  $V(D)$  such that the out-degree of every vertex  $x$  in its part is at most  $d_D^+(x) - k$  or 0 if  $d_D^+(x) < k$ .

The digraphs in  $\mathcal{T}$  show that one cannot always obtain a 2-partition of a digraph such that in each subdigraph induced by the parts, the out-degree of every vertex or the maximum out-degree is smaller than in the original graph. So it is natural to ask whether the existence of such a partition can be decided in polynomial time.

A  **$k$ -all-out-degree-reducing  $p$ -partition** of a digraph  $D$  is a  $p$ -partition  $(V_1, \dots, V_p)$  of  $V$  such that  $d_{D\langle V_i \rangle}^+(v) \leq \max\{0, d_D^+(v) - k\}$  for all  $1 \leq i \leq p$  and all  $v \in V_i$ . A  **$k$ -max-out-degree-reducing  $p$ -partition** of a digraph  $D$  is a  $p$ -partition  $(V_1, \dots, V_p)$  of  $V$  such that  $\Delta^+(D\langle V_i \rangle) \leq \max\{0, \Delta^+(D) - k\}$  for  $i \in [p]$ . Observe that a  $k$ -all-out-degree-reducing  $p$ -partition is also a  $k$ -max-out-degree-reducing  $p$ -partition. However the converse is not necessarily true. So for fixed integers  $k$  and  $p$ , we are interested in the problems of deciding whether a given digraph admits one of the above defined partitions.

**Problem 1.3** ( $k$ -ALL-OUT-DEGREE-REDUCING  $p$ -PARTITION).

Input: a digraph  $D$ ;

Question: Does  $D$  have a  $p$ -partition  $(V_1, V_2)$  with  $d_{D\langle V_i \rangle}^+(v) \leq \max\{0, d_D^+(v) - k\}$  for  $i \in [p]$ ?

**Problem 1.4** ( $k$ -MAX-OUT-DEGREE-REDUCING  $p$ -PARTITION).

Input: a digraph  $D$ ;

Question: Does  $D$  have a  $p$ -partition  $(V_1, V_2)$  with  $\Delta^+(D\langle V_i \rangle) \leq \max\{0, \Delta^+(D) - k\}$  for  $i \in [p]$ ?

We first consider the case of 2-partitions. The complexity of 1-MAX-OUT-DEGREE-REDUCING 2-PARTITION was posed in the paper [3] in which the complexity of a large number of other 2-partition problems is established. We also consider a closely related kind of 2-partitions: A  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -**partition** of a digraph is a 2-partition  $(V_1, V_2)$  such that  $\Delta^+(D\langle V_i \rangle) \leq k_i$  for  $i \in \{1, 2\}$ . Note that if a digraph is  $r$ -out-regular, then a  $(\Delta^+ \leq r - k, \Delta^+ \leq r - k)$ -partition is also a  $k$ -max-out-degree-reducing 2-partition and a  $k$ -all-out-degree-reducing 2-partition. We thus consider the following problem.

**Problem 1.5**  $((\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -PARTITION).

Input: a digraph  $D$ ;

Question: Does  $D$  have a 2-partition  $(V_1, V_2)$  with  $\Delta^+(D\langle V_i \rangle) \leq k_i$  for  $i \in \{1, 2\}$ ?

When  $k_1 = k_2 = 0$  the problem is the same as just asking whether  $D$  is bipartite which is clearly polynomial-time solvable. If  $D$  is a symmetric digraph, then there is a one-to-one correspondence between the set of  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -partitions of  $D$  and the so-called  $k$ -improper 2-colourings of  $UG(D)$ , the underlying (undirected) graph of  $D$ . A 2-colouring is  **$k$ -improper** if no vertex has more than  $k$  neighbours with the same colour as itself. Cowen et al. [4] proved that for any  $k \geq 1$ , deciding whether a graph has a  $k$ -improper 2-colouring is  $\mathcal{NP}$ -complete. Consequently,  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -PARTITION is  $\mathcal{NP}$ -complete for all  $k \geq 1$ .

On the other hand, Proposition 1.1 (ii) can be translated as follows to symmetric digraphs.

**Proposition 1.6.** *Every symmetric digraph with maximum out-degree  $K$  has a  $(\Delta^+ \leq \lfloor K/2 \rfloor, \Delta^+ \leq \lfloor K/2 \rfloor)$ -partition.*

As we saw in Proposition 1.2, this result does not extend to general digraphs. Hence it is natural to ask about the complexity of  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -PARTITION when restricted to digraphs with small maximal out-degree.

In the first part of the paper, we prove that 1-ALL-OUT-DEGREE-REDUCING 2-PARTITION and 1-MAX-OUT-DEGREE-REDUCING 2-PARTITION can be solved in polynomial time. This answers the

question posed in [3] affirmatively. Then we derive a complete characterization of the complexity of Problem 1.5 in terms of the values of  $k_1, k_2$  and use it to prove that  $k$ -ALL-OUT-DEGREE-REDUCING 2-PARTITION and  $k$ -MAX-OUT-DEGREE-REDUCING 2-PARTITION are  $\mathcal{NP}$ -complete for all values of  $k$  higher than 1. As a consequence of these results, we solve an open problem from [6] on majority colourings.

Next, in Section 5, we consider  $p$ -partitions for  $p \geq 3$ . We show that every digraph admits a  $k$ -all-out-degree-reducing  $(2k+1)$ -partition. This implies that  $k$ -ALL-OUT-DEGREE-REDUCING  $p$ -PARTITION and  $k$ -MAX-OUT-DEGREE-REDUCING  $p$ -PARTITION are polynomial-time solvable for  $p \geq 2k+1$  as the answer is always ‘Yes’. We also characterize the digraphs having a  $k$ -all-out-degree-reducing  $2k$ -partition, which implies that  $k$ -ALL-OUT-DEGREE-REDUCING  $2k$ -PARTITION and  $k$ -MAX-OUT-DEGREE-REDUCING  $2k$ -PARTITION are polynomial-time solvable. Finally, we show that, for any  $k > 1$  and  $3 \leq p \leq 2k-1$ , the problems  $k$ -ALL-OUT-DEGREE-REDUCING  $p$ -PARTITION and  $k$ -MAX-OUT-DEGREE-REDUCING  $k$ -PARTITION are  $\mathcal{NP}$ -complete.

We conclude with some remarks and related open problems.

## 2 Notation and definitions

We use the shorthand notation  $[k]$  for the set  $\{1, 2, \dots, k\}$ . Let  $D = (V, A)$  be a digraph with vertex set  $V$  and arc set  $A$ .

Given an arc  $uv \in A$ , we say that  $u$  **dominates**  $v$  and  $v$  is **dominated** by  $u$ . If  $uv$  or  $vu$  (or both) are arcs of  $D$ , then  $u$  and  $v$  are **adjacent**. If neither  $uv$  or  $vu$  exist in  $D$ , then  $u$  and  $v$  are **non-adjacent**. The **underlying graph** of a digraph  $D$ , denoted by  $UG(D)$ , is obtained from  $D$  by suppressing the orientation of each arc and deleting multiple copies of the same edge (coming from directed 2-cycles). A digraph  $D$  is **connected** if  $UG(D)$  is a connected graph, and the **connected components** of  $D$  are those of  $UG(D)$ .

A  $(u, v)$ -**path** is a directed path from  $u$  to  $v$ . A digraph is **strongly connected** (or **strong**) if it contains a  $(u, v)$ -path for every ordered pair of distinct vertices  $u, v$ . A digraph  $D$  is  $k$ -**strong** if for every set  $S$  of less than  $k$  vertices the digraph  $D - S$  is strong. A **strong component** of a digraph  $D$  is a maximal subdigraph of  $D$  which is strong. A strong component is **trivial**, if it has order 1. An **initial** (resp. **terminal**) strong component of  $D$  is a strong component  $X$  with no arcs entering (resp. leaving)  $X$  in  $D$ .

The **subdigraph induced** by a set of vertices  $X$  in a digraph  $D$ , denoted by  $D\langle X \rangle$ , is the digraph with vertex set  $X$  and which contains those arcs from  $D$  that have both end-vertices in  $X$ . When  $X$  is a subset of the vertices of  $D$ , we denote by  $D - X$  the subdigraph  $D(V \setminus X)$ . If  $D'$  is a subdigraph of  $D$ , for convenience we abbreviate  $D - V(D')$  to  $D - D'$ .

The **in-degree** (resp. **out-degree**) of  $v$ , denoted by  $d_D^-(v)$  (resp.  $d_D^+(v)$ ), is the number of arcs from  $V \setminus \{v\}$  to  $v$  (resp.  $v$  to  $V \setminus \{v\}$ ). A digraph is  $k$ -**out-regular** if all its vertices have out-degree  $k$  and it is  $k$ -**regular** if every vertex has both in-degree and out-degree  $k$ . A **sink** is a vertex with out-degree 0 and a **source** is a vertex with in-degree 0. The **degree** of  $v$ , denoted by  $d_D(v)$ , is given by  $d_D(v) = d_D^+(v) + d_D^-(v)$ . Finally the **maximum out-degree** and **maximum in-degree** of  $D$  are respectively denoted by  $\Delta^+(D)$  and  $\Delta^-(D)$ .

An **out-tree** rooted at the vertex  $s$ , also called an  $s$ -**out-tree**, is a connected digraph  $T_s^+$  such that  $d_{T_s^+}^-(s) = 0$  and  $d_{T_s^+}^-(v) = 1$  for every vertex  $v$  different from  $s$ . Equivalently, for every  $v \in V(T_s^+) \setminus \{s\}$  there is a unique  $(s, v)$ -path in  $T_s^+$ .

An **oriented graph** is a digraph with no directed 2-cycle. A **tournament** is a digraph is an oriented graph in which any two vertices are adjacent; in other words, for every two distinct vertices  $u$  and  $v$ , either  $uv$  or  $vu$  is an arc but not both.

A  $k$ -**colouring** of a graph  $G$  is a function  $f : V(G) \rightarrow [k]$ . A colouring  $f$  is **proper** if  $f(u) \neq f(v)$  for every edge  $uv \in E(G)$ . A graph is  $k$ -**colourable** if it admits a proper  $k$ -colouring. It is  $k$ -**degenerate** if each of its subgraphs has a vertex of degree at most  $k$ . It is well-known that a  $k$ -degenerate graph is  $(k+1)$ -colourable.

In our  $\mathcal{NP}$ -completeness proofs we use reductions from the well-known 3-SAT problem and from MONOTONE NOT-ALL-EQUAL-3-SAT. The later is the variant where the boolean formula  $\mathcal{F}$  to be satisfied consists of clauses all of whose literals are non-negated variables and we seek a truth assign-

ment such that each clause will get both a true and a false literal. This problem is also  $\mathcal{NP}$ -complete [8].

### 3 1-out-degree reducing partitions of digraphs

In this section we prove that 1-ALL-OUT-DEGREE-REDUCING 2-PARTITION and 1-MAX-OUT-DEGREE-REDUCING 2-PARTITION are solvable in polynomial time for  $k = 1$ .

Part (i) of the theorem below follows from a result of Seymour [9] (see also [6]) but we include the short proof for completeness (and we use the same idea to prove (ii)). We shall use the following result, due to Robertson, Seymour, and Thomas.

**Theorem 3.1** (Robertson, Seymour, and Thomas [7]). *Deciding whether a given digraph has an even directed cycle is polynomial-time solvable.*

**Theorem 3.2.** *Let  $D$  be a digraph.*

- (i)  *$D$  admits a 1-all-out-degree-reducing 2-partition if and only if every non-trivial terminal strong component contains an even directed cycle.*
- (ii)  *$D$  admits a 1-max-out-degree-reducing 2-partition if and only if every terminal strong component contains an even directed cycle or a vertex with out-degree less than  $\Delta^+(D)$ .*

*In both cases above, the desired 2-partition can be constructed in polynomial time when it exists.*

**Proof.** Let  $X_1, \dots, X_r$  be the terminal strong components of  $D$  ordered in such a way that  $X_1, \dots, X_q$  are non-trivial and  $X_{q+1}, \dots, X_r$  are trivial. Set  $S = \bigcup_{i=q+1}^r V(X_i)$ . Observe that  $S$  is the set of sinks of  $D$ .

(i) Suppose first that  $D$  admits a 1-all-out-degree-reducing 2-partition, then that partition restricted to  $X_i$ ,  $1 \leq i \leq q$ , would induce a bipartite spanning subdigraph of  $X_i$  with an even directed cycle.

Assume now that  $X_i$  contains an even directed cycle  $C_i$  for all  $i \in [q]$ . First properly 2-colour all the cycles  $C_1, C_2, \dots, C_q$  and colour the vertices of  $S$  with colour 1. If there exists an uncoloured vertex, then there must also exist an uncoloured vertex with an arc to a coloured one (as we have coloured at least one vertex in every terminal strong component). Give this vertex the opposite colour of its coloured out-neighbour. Repeating this procedure until all vertices have been coloured gives us a 2-colouring where every vertex not in  $S$  has an out-neighbour of different colour to itself. From this 2-colouring, we obtain the desired partition.

(ii) The necessity is seen as above. Now assume that every terminal component  $X_i$ ,  $i \in [r]$ , contains either an even directed cycle or a vertex of out-degree less than  $\Delta^+(D)$ . Pick an even directed cycle  $C_i$  for each terminal component with such a cycle and a vertex  $z_j$  with  $d^+(z_j) < \Delta^+(D)$  for the other terminal components (this includes the trivial ones). Let  $Z$  be the union of the vertices  $z_j$ . Now 2-colour all the even directed cycles and colour the vertices of  $Z$  with colour 1. As above we can extend this colouring into a 2-colouring of  $D$  where every vertex not in  $Z$  has an out-neighbour of different colour to itself. This 2-colouring correspond to the desired partition.

The complexity claim follows from Theorem 3.1 and the fact that our proof is constructive.  $\square$

We will show in Theorem 4.8 that  $k$ -ALL-OUT-DEGREE-REDUCING 2-PARTITION and  $k$ -MAX-OUT-DEGREE-REDUCING 2-PARTITION are  $\mathcal{NP}$ -complete for  $k > 1$ .

### 4 2-partitions with restricted maximum out-degrees

In this section we consider Problem 1.5 and determine its complexity for all possible values of the parameters  $k_1, k_2$ . By symmetry, we may assume that we always have  $k_1 \leq k_2$ . Recall that when  $k_1 = k_2 = 0$  the problem is the same as just asking whether  $D$  is bipartite which is polynomial-time solvable, so we may assume below that  $k_2 > 0$ .

The following gadget, depicted in Figure 1, turns out to be very useful in our constructions. An  $(x, y)$ - $(i, p)$ -**connector** is the digraph with vertex set  $\{x, y, s\} \cup T \cup U \cup U'$  with  $|T| = i$  and  $|U| = |U'| = p$  with all arcs from  $x$  to  $T$ , all arcs from  $T$  to  $U$ , all arcs between  $U$  and  $U'$ , except one arc  $u'u$  for some  $u \in U$  and  $u' \in U'$ , all arcs from  $s$  to  $U' \setminus \{u'\}$  arcs  $u's$  and  $sy$ . Observe that in an  $(x, y)$ - $(i, p)$ -connector, all vertices have out-degree  $p$  except  $x$  and  $y$  which have out-degree  $i$  and  $0$  respectively.

The next two lemmas illustrate the usefulness of connectors.

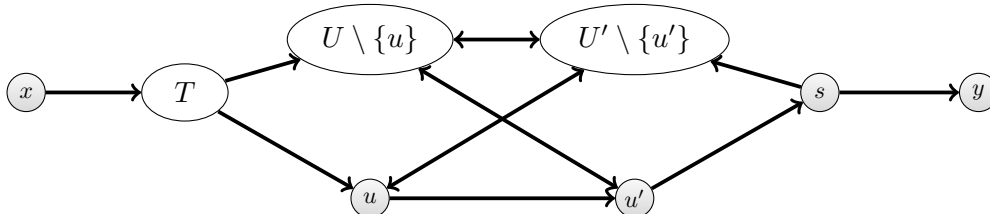


Figure 1: An  $(x, y)$ - $(i, p)$ -**connector**, where  $|T| = i$  and  $|U \setminus \{u\}| = p - 1$  and  $|U' \setminus \{u'\}| = p - 1$ .

**Lemma 4.1.** *Let  $k_1, k_2, i$  be three positive integers, with  $1 \leq k_1 \leq k_2$ , let  $D$  be a digraph and let  $x, y$  be two vertices in  $D$ . Let  $D'$  be the digraph obtained from  $D$  by adding an  $(x, y)$ - $(i, p)$ -connector.  $D'$  has a  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -partition if and only if  $D$  has one.*

**Proof.** Clearly, if  $D'$  has a  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -partition, then its restriction to  $V(D)$  is also a  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -partition.

Assume now that  $D$  has a  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -partition  $(V_1, V_2)$ . By symmetry, we may assume that  $x \in V_1$ . Now one easily checks that  $(V_1 \cup U \cup \{s\}, V_2 \cup T \cup U')$  is a  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -partition of  $D'$ . Indeed, even if  $y \in V_1$  we have  $d_{V_1}^+(s) \leq 1 \leq k_1$ .  $\square$

**Lemma 4.2.** *Let  $k_1, k_2, p$  be non-negative integers with  $0 \leq k_1 \leq k_2$ . If  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -PARTITION is  $\mathcal{NP}$ -complete for digraphs with maximum out-degree  $p$ , then  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -PARTITION is also  $\mathcal{NP}$ -complete for digraphs for strong  $(p + 1)$ -out-regular digraphs.*

**Proof.** First assume that  $1 \leq k_1 \leq k_2$ . Then we can use Lemma 4.1 quite directly. Consider a digraph  $D$  with maximum out-degree  $p$ , and let  $\{v_1, \dots, v_n\}$  be its vertex set. For  $j \in [n]$ , let  $i_j = p + 1 - d_D^+(v_j)$ . Informally,  $i_j$  is the number of out-neighbours we must add to  $v_j$  so that it gets out-degree  $p + 1$ . Observe that for every  $j$  we have  $i_j \geq 1$ , because  $\Delta^+(D) \leq p$ . Let  $D'$  be the digraph obtained by adding a  $(v_j, v_{j+1})$ - $(i_j, p + 1)$ -connector for every  $j \in [n]$  (with  $v_{n+1} = v_1$ ). It is simple matter to check that  $D'$  is  $(p + 1)$ -out-regular and strong because every  $i_j$  is at least 1. Moreover, Lemma 4.1 implies that  $D'$  has a  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -partition if and only if  $D$  has one.

Now assume that we have  $0 = k_1 < k_2$ . In this case we will need to put connectors between adjacent vertices to insure that Lemma 4.1 holds. Indeed if a digraph  $D$  has a  $(\Delta^+ = 0, \Delta^+ \leq k_2)$ -partition and  $xy$  is an arc of  $D$ , then the digraph obtained from  $D$  by adding an  $(x, y)$ - $(p + 1 - d_D^+(x), p)$ -connector to  $D$  admits also a  $(\Delta^+ = 0, \Delta^+ \leq k_2)$ -partition. The proof of this statement is similar to the one of Lemma 4.1 using the fact that as  $xy$  is an arc of  $D$  then we cannot have  $x \in V_1$  and  $y \in V_1$  in any  $(\Delta^+ = 0, \Delta^+ \leq k_2)$ -partition  $(V_1, V_2)$  of  $D$ .

Now let  $D$  be a digraph with maximum out-degree  $p$ . It is easy to check that the digraph obtained by adding a new vertex to  $D$  with two out-neighbours in  $D$  has a  $(\Delta^+ = 0, \Delta^+ \leq k_2)$ -partition if and only if  $D$  has one. So let  $s$  a new vertex and let  $T$  be a binary  $s$ -out-tree with  $|V(D)|$  leaves (i.e. every vertex of  $T$  has out-degree 2 except the leaves which have out-degree 0). We construct  $D'$  by adding a copy of  $T$  to  $D$  and identifying the vertices of  $D$  with the leaves of  $T$ . Note that  $V(D') = V(D)$ . By repeating the previous remark, we obtain that  $D'$  admits a  $(\Delta^+ = 0, \Delta^+ \leq k_2)$ -partition if and only if  $D$  has one. To conclude we build  $D''$  by adding a  $(v, u)$ - $(p + 1 - d_D^+(v), p + 1)$ -connector to  $D'$  for every arc  $uv$  of the copy of  $T$  and a  $(s, w)$ - $(p - 1, p + 1)$ -connector for an out-neighbour  $w$  of  $s$ . Using the modified version of Lemma 4.1 for  $(\Delta^+ = 0, \Delta^+ \leq k_2)$ -partitions, we conclude that  $D$  has such

a partition if, and only if,  $D''$  has one. Moreover, by construction, it is clear that  $D''$  is strong and  $(p + 1)$ -out-regular.  $\square$

Obviously every digraph of maximum out-degree  $k \leq \max\{k_1, k_2\}$  has a  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -partition. As we now show, just increasing the maximum out-degree one above this value results in a shift in complexity from trivial to  $\mathcal{NP}$ -complete, even if we also require that the digraph is strongly connected and out-regular.

**Theorem 4.3.** *For every choice of non-negative integers  $k_1, k_2$  with  $\max\{1, k_1\} < k_2$ , the  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -PARTITION problem is  $\mathcal{NP}$ -complete for strong  $(k_2 + 1)$ -out-regular digraphs.*

**Proof.** Let us call a 2-colouring  $c : V \rightarrow \{1, 2\}$  **good** if the 2-partition induced by  $c$  is a  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -partition. We start by describing a reduction from 3-SAT to  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -PARTITION in graphs of maximum out-degree  $k_2 + 1$  and then show how to modify the proof to work for strong and  $(k_2 + 1)$ -out-regular digraphs using Lemma 4.1.

We first make some observations about gadgets that force certain vertices to have colour 1 or 2 in any good 2-colouring. Let  $X$  be the digraph that we obtain from a copy of the Thomassen digraph  $T_{k_2-1}$  (it exists because  $k_2 > 1$ ) by adding one new vertex  $v$  and all possible arcs from  $V(T_{k_2-1})$  to  $v$ . It follows from Proposition 1.2 that in any good 2-colouring  $c$  of a digraph containing an induced copy of  $X$  the vertex  $v$  must have  $c(v) = 2$ . Let  $Z$  be the digraph obtained by taking  $k_2 + 1$  copies  $X_i$ ,  $i \in [k_2 + 1]$  of  $X$ , where  $v_i$  denotes the copy of  $v$  in  $X_i$ ,  $i \in [k_2 + 1]$  and a new vertex  $w$  and adding the arcs of  $\{v_1 v_{1+i} \mid i \in [k_2]\} \cup \{v_1 w\}$ . By the remark above, for every good 2-colouring of a digraph containing an induced copy of  $Z$ , we have  $c(w) = 1$ .

When we say below that a certain vertex  $u$  has colour 1 or colour 2 we mean that we use a private copy of either  $Z$  with  $u = w$  or  $X$  with  $u = v$  to enforce that in all good 2-colourings of  $D$  the vertex  $u$  will have the desired colour. Now let  $W$  be a digraph containing  $k_1 + k_2 + 2$  vertices  $v, \bar{v}, a_1, \dots, a_{k_1}, b_1, \dots, b_{k_2}$  and the arcs of  $\{v\bar{v}, \bar{v}v\} \cup \{a_1 v, a_1 \bar{v}, b_1 v, b_1 \bar{v}\} \cup \{a_1 a_{j+1} \mid j \in [k_1 - 1]\} \cup \{b_1 b_{j+1} \mid j \in [k_2 - 1]\}$ . By adding suitable copies of  $X, Z$  we can ensure that for every good colouring of the digraph we construct below we have  $c(a_h) = 1$  for  $h \in [k_1]$  and  $c(b_h) = 2$  for  $h \in [k_2]$ . This implies that in every good colouring we have  $c(v) = r$  and  $c(\bar{v}) = 3 - r$  for some  $r \in \{1, 2\}$ .

Now we are ready to construct a digraph  $D = D(\mathcal{F})$  from a given instance  $\mathcal{F}$  of 3-SAT. Let  $\mathcal{F}$  have variables  $x_1, x_2, \dots, x_n$  and clauses  $C_1, C_2, \dots, C_m$ : represent each variable  $x_i$  by a copy  $W_i$  of  $W$  where the vertices  $v_i, \bar{v}_i$  correspond to  $v$  and  $\bar{v}$  in  $W$  and play the role of  $x_i, \bar{x}_i$ , respectively. For each clause  $C_j$ , we add a new vertex  $c_j$  of colour 2,  $k_2 - 2$  arcs from  $c_j$  to private (to  $c_k$ ) vertices of colour 2 and three arcs from  $c_j$  to the three vertices that correspond to its literals. So, if  $C_j = (x_1 \vee \bar{x}_8 \vee x_9)$  then we add the arcs  $c_j v_1, c_j \bar{v}_8$  and  $c_j v_9$ . This completes the construction of  $D$ . Clearly  $D$  can be constructed in polynomial time given  $\mathcal{F}$ . The fact that  $c_j$  must have colour 2 and already has  $k_2 - 2$  out-neighbours of colour 2 implies that at least one of the vertices corresponding to the literals of  $C_j$  must have colour 1 in any good colouring. Now it is easy to see that if we associate colour 1 with *true*, then  $D$  has a good colouring if and only if  $\mathcal{F}$  is satisfiable. This proves that  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -PARTITION is  $\mathcal{NP}$ -complete for digraphs of maximum out-degree  $k_2 + 1$  as it is easy to check that  $\Delta^+(D) \leq k_2 + 1$ .

To obtain the result on strong  $(k_2 + 1)$ -out-regular digraphs, we first show how to obtain a strong superdigraph  $D'$  of  $D$  with the desired colouring property. First observe that in  $D$  no arc enters a copy of  $X$  unless this is inside a copy of  $Z$  and for every copy of  $Z$  one copy of  $X$  has no arcs entering it. By adding a new vertex  $s$ , sufficiently (but still polynomial in the size of  $\mathcal{F}$ ) many new vertices and the arcs of an out-tree of maximum out-degree  $k_2$  rooted at  $s$ , we can obtain that  $s$  is the root of an out-tree  $T_s^+$  whose only intersection with  $V(D)$  is in its leaves where  $T_s^+$  has exactly one leaf in each copy of  $X$ .

Note that every vertex corresponding to a literal has out-degree 1 and that every vertex which does not correspond to a literal has a directed path to at least one vertex that corresponds to a literal (here we use that  $T_{k_2-1}$  is strongly connected). Thus if we add the arcs of the directed cycle  $C = s v_1 v_2 \dots v_n s$ , we obtain the desired strong digraph  $D'$  with  $\Delta^+(D') = k_2 + 1$ . Clearly  $D$  is a subdigraph of  $D'$  so every good 2-colouring of  $D'$  induces a good 2 colouring of  $D$ . Conversely, if  $c$  is a good 2-colouring of  $V(D)$ , then it is still a good 2-colouring of  $D \cup A(C)$  because  $k_2 \geq 2$  and we can extend  $c$  to the non-leaf vertices of  $T_s^+$  (colouring them by 2) because they have out-degree at most  $k_2$ .

It remains to prove that we can also achieve a  $(k_2 + 1)$ -out-regular digraph  $D''$  which is strong and has a good 2-colouring if and only if  $\mathcal{F}$  is satisfiable. To show this we just have to observe that, by Lemma 4.1, for every vertex  $w$  with out-degree  $k < k_2 + 1$  we can add a private  $(w, w) - (k_2 + 1 - k, k_2 + 1)$ -connector.  $\square$

Note that we used the fact that  $k_2 > 1$  at several places in the proof above. One of these was the use of  $T_{k_2-1}$ . Hence there still remains the complexity of  $(\Delta^+ \leq 0, \Delta^+ \leq 1)$ -PARTITION. This was solved by Fraenkel.

**Theorem 4.4** (Fraenkel [5]).  $(\Delta^+ \leq 0, \Delta^+ \leq 1)$ -PARTITION is  $\mathcal{NP}$ -complete on the class of digraphs with in- and out-degree at most 2.

In order to strengthen this and to unify our results we need the following result which can be obtained by modifying the proof in [5]. We give a proof for completeness.

**Theorem 4.5.** For all  $p \geq 2$ ,  $(\Delta^+ \leq 0, \Delta^+ \leq 1)$ -PARTITION is  $\mathcal{NP}$ -complete on the class of strong  $p$ -out-regular digraphs.

**Proof.** By Lemma 4.2, it suffices to prove the statement for  $p = 2$ . A **kernel** in a digraph  $D$  is an independent set  $K$  of vertices such that every vertex in  $V(G) \setminus K$  has an out-neighbour in  $K$ . Note that  $(V_1, V_2)$  is a  $(\Delta^+ \leq 0, \Delta^+ \leq 1)$ -partition of a 2-out-regular digraph  $D$  if and only if  $V_1$  is a kernel of  $D$ . We first recall a (slightly simpler version of) the proof from [5] that deciding whether a digraph has a kernel is  $\mathcal{NP}$ -complete for digraphs of maximum out-degree 2 and then modify that reduction to show that it is  $\mathcal{NP}$ -complete for strong 2-out-regular digraphs.

Let  $W$  denote the digraph defined by

$$V(W) = \{z_1, \dots, z_9\} \quad \text{and} \quad A(W) = \{z_1 z_2, z_2 z_3, z_3 z_1, z_3 z_4, z_4 z_5, z_5 z_6, z_5 z_7, z_6 z_8, z_7 z_9\}.$$

Now let  $\mathcal{F}$  be an instance of 3-SAT with variable  $x_1, \dots, x_n$  and clauses  $C_1, \dots, C_m$ . Free to duplicate one clause, we may assume that  $m$  is odd. Form the digraph  $G = G(\mathcal{F})$  by taking one copy  $W_j$  of  $W$  for each clause  $C_j$ ,  $j \in [m]$  (denoting the vertices of  $W_j$  by  $z_{j,q}$ ,  $q \in [9]$ ) and adding  $2n$  new vertices  $v_1, \bar{v}_1, \dots, v_n, \bar{v}_n$ , where  $v_i, \bar{v}_i$  correspond to the literals  $x_i, \bar{x}_i$  as well as the arcs  $v_i \bar{v}_i, \bar{v}_i v_i$  for  $i \in [n]$ . Finally, we add three arcs from each  $W_j$  to the vertices that correspond to its literals so that the vertex  $z_{j,8}$  is joined to the vertex corresponding to the first literal and the vertex  $z_{j,9}$  is joined to the two vertices corresponding to the second and third literal of  $W_j$ . Thus if  $W_j = (x_4 \vee x_5 \vee \bar{x}_8)$ , then we add the arcs  $z_{j,8} v_4, z_{j,9} v_5, z_{j,9} \bar{v}_8$ . This completes the construction of  $G$ . Note that if  $K$  is a kernel of  $G$ , then for every  $j \in [m]$  we have either  $\{z_{j,2}, z_{j,4}, z_{j,6}\} \subset K$  or  $\{z_{j,2}, z_{j,4}, z_{j,7}\} \subset K$  (or both) and this implies that  $|K \cap \{z_{j,8}, z_{j,9}\}| \leq 1$ . From this it follows that at least one of the vertices corresponding to the literals of  $C_j$  will belong to  $K$ . For each  $i \in [n]$  we have precisely one of  $v_i, \bar{v}_i$  in  $K$  as these vertices are adjacent. Now it is easy to see that  $G$  has a kernel if and only if  $\mathcal{F}$  is satisfiable. This shows that deciding whether a digraph has a kernel and hence  $(\Delta^+ \leq 0, \Delta^+ \leq 1)$ -PARTITION is  $\mathcal{NP}$ -complete for digraphs of maximum out-degree 2.

Let us now prove that it is  $\mathcal{NP}$ -complete for strong 2-out-regular digraphs. Note that in  $G$  every vertex has out-degree at least 1. Let  $H$  be the digraph on six vertices  $a, b, c, d, e, f$  and the arcs  $de, ef, fd, da, eb, fc, ae, bd, bf, cd, ce$ . Let  $G'$  be the digraph obtained from the disjoint union of  $G$  and  $H$  and a directed path  $a_1 a_2 \dots a_m$  by identifying  $a$  and  $a_1$  and adding the arc  $a_m z_{m,3}$ , the arcs  $a_j z_{j,1}$  for  $j \in [m]$  and the arcs  $ud$  for every vertex  $u$  having out-degree 1 in  $G$ . Clearly, the digraph  $G'$  is strong and 2-out-regular.

Finally let us now prove that  $G'$  has a kernel if and only if  $G$  has one. This will immediately imply the result. If  $G$  has a kernel  $K$ , then one can easily check that  $K \cup \{b, c\} \cup \{a_j \mid j \text{ odd}\}$  is a kernel of  $G'$  (recall that  $m$  is odd and that  $K$  contains none of  $z_{j,1}, z_{j,3}$ ). Assume now that  $G'$  has a kernel  $K'$ . We have  $d \notin K'$ , for otherwise  $b$  and  $f$  are not in  $K'$  (because  $K'$  is an independent set) and so  $e$  has no out-neighbour in  $K'$ , a contradiction. Now all arcs leaving  $G$  in  $G'$  have head  $d$ , so every vertex of  $G$  has an out-neighbour in  $K' \cap V(G)$ . Hence  $K' \cap V(G)$  is a kernel of  $G$ .  $\square$

**Theorem 4.6.** Let  $k, p$  be two positive integers  $k$  such that  $p \geq k + 2$ . Then  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -PARTITION is polynomial-time solvable for digraphs of maximum out-degree  $k + 1$  and  $\mathcal{NP}$ -complete on the class of strong  $p$ -out-regular digraphs.



**Proof.** The first part of the claim follows from Theorem 3.2. Below we show how make a reduction from MONOTONE NOT-ALL-EQUAL 3-SAT to the  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -partition problem in strong  $(k+2)$ -out-regular digraphs. Combining this with Lemma 4.2 proves the theorem, as  $k > 0$ .

The reduction makes use of the following **forcing gadget**, namely the digraph  $F$  whose vertex set is the union of  $X = \{x, x'\}$ ,  $Y = V(T_k)$  (Recall that  $T_k$  is Thomassen's digraph defined in the introduction.) and whose arc set is the union of the arcs of  $T_k$  and all possible arcs from  $Y$  to  $X$ . The **head** of a forcing gadget is the set  $X$ .

**Claim 4.6.1.**

- (i) In a forcing gadget, all vertices have out-degree  $k+2$ , except those of the head which have out-degree 0.
- (ii) In any  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -partition of a digraph which contains a copy of the forcing gadget as an induced subdigraph, the two vertices of the head are in the same part.

*Subproof.* (i) follows from the definition of the forcing gadget as  $T_k$  is  $k$ -out-regular.

(ii) follows from the fact that  $F(Y) = T_k$  has no  $(\Delta^+ \leq k-1, \Delta^+ \leq k-1)$ -partition, implying that in any 2-partition  $(V_1, V_2)$  of  $F$  some vertex of  $Y$  already has its  $k$  out-neighbours in  $Y$  in the same set  $V_i$  as itself and hence both  $x$  and  $x'$  must belong to  $V_{3-i}$ .  $\diamond$

Let  $\mathcal{F}$  be an instance of MONOTONE NOT-ALL-EQUAL  $(k+2)$ -SAT on  $n$  variables  $x_1, \dots, x_n$  and  $m$  clauses  $C_1, \dots, C_m$ . For every  $i \in [n]$ , let  $j_1(i) < j_2(i) \dots < j_{m(i)}(i)$  be the indices of those clauses in which variable  $x_i$  occurs and let  $J(i) = \{j_1(i), \dots, j_{m(i)}(i)\}$ . For each  $j \in [m]$  and  $q \in [k+2]$ , let  $a_{q,j}$  be the unique integer such that if  $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$ , then  $x_{i_q}$  occurs exactly  $a_{q,j} - 1$  times among the clauses  $C_1, \dots, C_{j-1}$ .

Let  $D_{\mathcal{F}}$  be the digraph constructed as follows. For all  $i \in [n]$ , we create a variable gadget  $VG_i$  as follows. We first create the vertices  $\{x_i^j \mid j \in J(i)\}$ . Then for all  $1 \leq p < m(i)$ , we add a forcing gadget with head  $\{x_i^{j_p(i)}, x_i^{j_{p+1}(i)}\}$ . Let  $Y_i^p$  be the set corresponding to  $Y$  in this forcing gadget. This will force all the vertices of  $\{x_i^j \mid j \in J(i)\}$  to be in the same part for any  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -partition.

Then for every clause  $C_j = x_{i_1} \vee x_{i_2} \vee \dots \vee x_{i_{k+2}}$ , we add a vertex  $t_j$ , all the arcs from the set  $\{x_{i_1}^{a_{1,j}}, x_{i_2}^{a_{2,j}}, \dots, x_{i_{k+2}}^{a_{k+2,j}}\}$  to  $t_j$  and the arcs of the complete digraph on  $\{x_{i_1}^{a_{1,j}}, x_{i_2}^{a_{2,j}}, \dots, x_{i_{k+2}}^{a_{k+2,j}}\}$ .

Let  $D'_{\mathcal{F}}$  be the digraph obtained from  $D_{\mathcal{F}}$  as follows. Add a set of  $3m - n$  new vertices  $U = \{u_1, \dots, u_{3m-n}\}$  and let  $f$  be a bijection between  $U$  and  $\{Y_i^p \mid i \in [n], 1 \leq p \leq m(i) - 1\}$ . For each  $j \in [3m - n]$ , we add a  $(u_j, v_j)$ - $(1, k+2)$ -connector with  $v_j$  being an arbitrary vertex in  $f(u_j)$ , and a  $(u_j, u_{j+1})$ - $(k+1, k+2)$ -connector (with  $u_{3m-n+1} = u_1$ ). Finally, for each  $j \in [m]$ , add a  $(t_j, u_1)$ - $(k+2, k+2)$ -connector. We can easily check that  $D'_{\mathcal{F}}$  is strong and  $(k+2)$ -out-regular.

Let us now prove that  $D'_{\mathcal{F}}$  has a  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -partition if and only if  $\mathcal{F}$  admits a **NAE-assignment**, that is a truth assignment such that each clause contains a true literal and a false literal. By Lemma 4.1, as  $k > 0$ , it is equivalent to prove that  $D_{\mathcal{F}}$  has a  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -partition if and only if  $\mathcal{F}$  admits a NAE-assignment.

First suppose that  $\phi$  is a NAE-assignment. Define the following 2-colouring of  $V(D_{\mathcal{F}})$ : for each  $i \in [n]$  colour all vertices of  $\{x_i^j \mid j \in J(i)\}$  by colour 1 and those of  $\bigcup_{p=1}^{m(i)-1} Y_i^p$  by 2 if  $\phi(x_i) = \text{true}$  and otherwise colour all vertices of  $\{x_i^j \mid j \in J(i)\}$  by 2 and those of  $\bigcup_{p=1}^{m(i)-1} Y_i^p$  by 1. Now each  $t_j$ ,  $j \in [m]$ , has at least one in-neighbour of colour  $i$  for  $i \in [2]$ . If it has precisely one of colour  $i$ , we colour it by colour  $i$  and otherwise we colour it arbitrarily. Now it is easy to see that letting  $V_i$  be the set of vertices of colour  $i$ ,  $i = 1, 2$ , we obtain the desired 2-partition of  $D_{\mathcal{F}}$ .

Assume now that  $(V_1, V_2)$  is a good 2-partition of  $D_{\mathcal{F}}$ . The forcing gadgets ensure that in every  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -partition  $(V_1, V_2)$  of  $V(D_{\mathcal{F}})$  all vertices of  $\{x_i^j \mid j \in J(i)\}$  belong to the same set in the partition for all  $i \in [n]$ . Furthermore, because of the complete subdigraphs on the vertices  $\{x_{i_1}^{a_{1,j}}, x_{i_2}^{a_{2,j}}, \dots, x_{i_{k+2}}^{a_{k+2,j}}\}$ ,  $j \in [m]$ , at least one of these vertices is in  $V_1$  and at least one of them is in  $V_2$ . Thus if we assign  $x_i$  the value *true* if  $\{x_i^j \mid j \in J(i)\} \subset V_1$  and *false* otherwise, each clause will have at least one true and at least one false literal.  $\square$

Combining our results above we obtain the following complete classification in terms of  $k_1, k_2$ .

**Theorem 4.7.** *Let  $k_1, k_2$  be non-negative integers. The  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -PARTITION problem is*

- *polynomial-time solvable for all digraphs when  $k_1 = k_2 = 0$ ;*
- *polynomial-time solvable for digraphs of maximum degree  $p \leq \max\{k_1, k_2\}$ ;*
- *$\mathcal{NP}$ -complete for strong  $p$ -out-regular digraphs for all  $p \geq \max\{k_1, k_2\} + 1$  when  $k_1 \neq k_2$ ;*
- *polynomial-time solvable for  $(k_2 + 1)$ -out-regular digraphs and  $\mathcal{NP}$ -complete for strong  $p$ -out-regular digraphs for all  $p \geq \max\{k_1, k_2\} + 2$  when  $k_1 = k_2$ .*

Theorems 4.6 and 3.2 immediately yield the following.

**Theorem 4.8.**  *$k$ -ALL-OUT-DEGREE REDUCING 2-PARTITION and  $k$ -MAX-OUT-DEGREE-REDUCING 2-PARTITION are polynomial-time solvable for  $k = 1$  and  $\mathcal{NP}$ -complete for all integers  $k \geq 2$  even when the input is a strong out-regular digraph.*

## 5 Out-degree reducing $p$ -partitions for $p \geq 3$ .

All our complexity results so far dealt with 2-partition problems. In this section we deal with  $p$ -partitions for  $p \geq 3$ .

The next proposition implies that  $k$ -ALL-OUT-DEGREE-REDUCING  $p$ -PARTITION and  $k$ -MAX-OUT-DEGREE-REDUCING  $p$ -PARTITION are polynomial-time solvable when  $p \geq 2k + 1$ , because the answer is trivially ‘yes’.

**Proposition 5.1.** *Every digraph has a  $k$ -all-out-degree-reducing  $(2k + 1)$ -partition and this is best possible.*

**Proof.** Let  $D$  be a digraph. For each vertex  $v$  pick  $\min\{k, d^+(v)\}$  arcs with tail in  $v$ . Let  $H$  be the subdigraph of  $D$  induced by these arcs. Then  $H$  has a vertex of degree at most  $2k$  and this holds for every subdigraph of  $H$ , so  $UG(H)$  is  $2k$ -degenerate and hence it is  $2k + 1$ -colourable. Let  $(V_1, V_2, \dots, V_{2k+1})$  be a  $(2k + 1)$ -partition of  $D$  induced by a  $(2k + 1)$ -colouring of  $UG(H)$ . It is easy to check that this is a  $k$ -all-out-degree-reducing  $(2k + 1)$ -partition since every arc of  $H$  goes between two different sets in the partition.

The  $k$ -out-regular tournaments show that  $2k + 1$  is best possible for each  $k \geq 1$ . □

The next result implies that  $k$ -ALL-OUT-DEGREE-REDUCING  $p$ -PARTITION and  $k$ -MAX-OUT-DEGREE-REDUCING  $p$ -PARTITION are also polynomial-time solvable when  $p = 2k$ .

**Theorem 5.2.** *Let  $k \geq 2$ . A digraph  $D$  admits a  $k$ -all-out-degree-reducing  $2k$ -partition if and only if no terminal strong component of  $D$  is a  $k$ -regular tournament.*

**Proof.** First assume that some terminal component,  $Q$ , of  $D$  is a  $k$ -regular tournament. This implies that every vertex in  $Q$  has out-degree  $k$  in  $D$  and for any  $2k$ -partition of  $D$  there will be two vertices from  $Q$  in the same part, as  $|V(Q)| = 2k + 1$ . Therefore some vertex will have out-degree at least 1 in its part and therefore not have reduced its out-degree by  $k$ . This proves one direction. We now prove the opposite direction.

Let  $D$  be any digraph of order  $n$  and size  $m$  with no terminal component isomorphic to a  $k$ -regular tournament. We will now show that  $D$  has a  $k$ -all-out-degree-reducing  $2k$ -partition by induction on  $n + m$ . Clearly this holds when  $n + m \leq 3$  so assume that it also holds for all digraphs,  $D'$ , with  $|V(D')| + |E(D')| < n + m$ . We may assume that  $D$  is connected as otherwise we are done by using induction on each connected component. Let  $G$  be the underlying graph of  $D$ . We consider the following three cases which exhaust all possibilities.

**Case 1. There exists a vertex  $x \in V(D)$  with  $d^+(x) > k$ .** If  $N^+(x)$  is independent then let  $v \in N^+(x)$  be arbitrary, and otherwise let  $u, v \in N^+(x)$  be chosen such that  $uv \in A(D)$ . Let  $D' = D \setminus xv$  (i.e. delete the arc  $xv$  from  $D$ ). Let  $Q'$  be any terminal component in  $D'$ . If  $x \notin V(Q')$ , then  $Q'$  is also a terminal component of  $D$  and therefore not a  $k$ -regular tournament. So suppose  $x \in V(Q')$ . Recall that either  $N^+(x)$  is independent or  $xuv$  is a path in  $D$  which implies that

$v \in V(Q')$ . Both cases imply that  $Q'$  is not a tournament. Therefore, by induction, there is a  $k$ -all-out-degree-reducing  $2k$ -partition of  $D'$  and therefore also of  $D$  (using the same partition). This completes Case 1.

**Case 2.  $\Delta^+(D) \leq k$  and  $G$  is not  $2k$ -regular.** Let  $w$  be a vertex having degree at most  $2k - 1$  in  $G$ . Let  $D' = D - w$ . Assume that some terminal component,  $Q'$ , in  $D'$  is a  $k$ -regular tournament. As  $\Delta^+(D) \leq k$ , this implies that  $Q'$  is also a terminal component in  $D$ , a contradiction. Therefore no terminal component in  $D'$  is a  $k$ -regular tournament and by induction there is a  $k$ -all-out-degree-reducing  $2k$ -partition of  $D'$ . Now add  $w$  to a different part to all of its at most  $2k - 1$  neighbours in  $G$ . This gives a  $k$ -all-out-degree-reducing  $2k$ -partition of  $D$ .

**Case 3.  $\Delta^+(D) \leq k$  and  $G$  is  $2k$ -regular.** Note that in that case  $D$  is an oriented graph and  $D$  is  $k$ -regular. Now  $G$  is not a complete graph for otherwise  $D$  would be  $k$ -regular tournament. Moreover, as  $k \geq 2$ , the graph  $G$  is not an odd cycle. Therefore, by Brook's Theorem,  $G$  admits a proper  $2k$ -colouring. This  $2k$ -colouring gives us the desired  $k$ -all-out-degree-reducing  $2k$ -partition of  $D$ .  $\square$

**Theorem 5.3.** *If  $k > 1$  and  $3 \leq p \leq 2k - 1$ , then  $k$ -ALL-OUT-DEGREE-REDUCING  $p$ -PARTITION and  $k$ -MAX-OUT-DEGREE-REDUCING  $p$ -PARTITION are  $\mathcal{NP}$ -complete.*

**Proof.** We give a reduction from  $p$ -COLOURABILITY which consists in deciding whether a given digraph is  $p$ -colourable. This problem is well-known to be  $\mathcal{NP}$ -complete for all  $p \geq 3$ .

We first need to define a gadget  $D_2(x, y)$  as follows. Let  $T$  be a regular or almost regular tournament of order  $p - 1$  and let  $V_1 = \{v \mid d_T^+(v) = k - 1\}$ . Note that  $V_1$  is empty if  $p \leq 2k - 2$  and  $|V_1| = k - 1 = |V(T)|/2$  if  $p = 2k - 1$ .

Let  $D_2(x, y)$  be the digraph obtained from a copy of  $T$  by adding two vertices  $x, y$  and all arcs from  $V(T) \setminus V_1$  to  $\{x, y\}$ , all arcs from  $V_1$  to  $x$  and all arcs from  $y$  to  $V_1$ . Note that  $d^+(x) = 0$  and  $d^+(y) = |V_1|$ .

Note that in both cases above  $x$  and  $y$  are the only non-adjacent vertices in  $D_2(x, y)$  and  $\Delta^+(D_2(x, y)) \leq k$ .

We now define the gadget  $D_n(x_1, x_2, \dots, x_n)$  for  $n \geq 3$  as the union of  $D_2(x_1, x_2), D_2(x_2, x_3), \dots, D_2(x_{n-1}, x_n)$ , where the copies of  $T$  are disjoint. Note that  $d^+(x_1) = 0$  and  $d^+(x_i) \leq k - 1$  for all  $i = 2, 3, \dots, n$  (in fact  $d^+(x_i) = 0$  if  $p < 2k - 1$  and  $d^+(x_i) = k - 1$  otherwise).

We will now reduce an instance of  $p$ -COLOURABILITY to an instance of  $k$ -MAX-OUT-DEGREE-REDUCING  $p$ -PARTITION. Let  $G$  be a graph with vertex set  $v_1, \dots, v_n$ . We will now construct a digraph  $D$  as follows. For each vertex  $v_i \in V(G)$  we let  $D^i$  be a copy of  $D_n(x_1^i, x_2^i, \dots, x_n^i)$ . For each edge  $v_i v_j$  of  $G$  with  $i < j$  add an arc from  $x_j^i$  to  $x_j^j$ . Observe that the set of arcs added by this operation are disjoint, so the resulting digraph  $D$  has out-degree at most  $k$ . Consequently, every  $k$ -max-out-degree-reducing  $p$ -partition and every  $k$ -max-out-degree-reducing  $p$ -partition of  $D$  is equivalent to a proper  $p$ -colouring of the underlying graph  $UG(D)$  of  $D$ .

Hence to prove the theorem, it is enough to show that  $UG(D)$  has a proper  $p$ -colouring if and only if  $G$  does. But this follows directly from the following claim.

**Claim 5.3.1.** *In any  $p$ -colouring of  $UG(D_n(x_1, x_2, \dots, x_n))$ , all the vertices in  $\{x_1, x_2, \dots, x_n\}$  must be coloured the same. Furthermore, there exists a  $p$ -colouring of  $UG(D_n(x_1, x_2, \dots, x_n))$ .*

*Proof of Claim 5.3.1.* We show Claim 5.3.1 is true when  $n = 2$  and then note that this implies that Claim 5.3.1 is true for all  $n$ . Let  $n = 2$ . As  $x_1$  and  $x_2$  are the only non-adjacent vertices in  $D_2(x_1, x_2)$  and  $|V(D_2(x_1, x_2))| = p + 1$  we note that  $x_1$  and  $x_2$  must have the same colour in a proper  $p$ -colouring of  $UG(D_2(x_1, x_2))$ . Conversely if  $x_1$  and  $x_2$  have the same colour all other vertices of  $D_2(x_1, x_2)$  can be given a distinct colour in order to obtain a proper  $p$ -colouring of the underlying graph. This proves Claim 5.3.1 when  $n = 2$ .

When  $n \geq 3$  we note by the above that  $x_1$  and  $x_2$  must be in the same partite set. Analogously  $x_2$  and  $x_3$  must be in the same partite set. Continuing this process we obtain the desired result for  $n \geq 3$ . This completes the proof of Claim 5.3.1.  $\diamond$   $\square$

## 6 Remarks and open questions

A **majority  $k$ -colouring** of a digraph  $D = (V, A)$  is a  $k$ -colouring of the vertices of  $V$  so that each vertex  $v$  has at most  $\frac{d^+(v)}{2}$  out-neighbours with the same colour as itself. It is shown in [6] that every digraph has a majority 4-colouring and the authors conjecture that, in fact, every digraph has a majority 3-colouring. They also asked about the complexity of deciding whether a digraph has a majority 2-colouring. Since a 3-out-regular digraph has a majority 2-colouring if and only if it has a  $(\Delta^+ \leq 1, \Delta^+ \leq 1)$ -partition the following is an immediate consequence of Theorem 4.6.

**Theorem 6.1.** *Deciding whether a digraph has a majority 2-colouring is  $\mathcal{NP}$ -complete even when the input is 3-out-regular and strongly connected.*

In all our  $\mathcal{NP}$ -completeness proofs above on out-regular digraphs, these are far from being also in-regular. Thus it is natural to ask about the complexity in the case of regular digraphs.

**Problem 6.2.** What is the complexity of the  $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$ -partition problem for  $(\max\{k_1, k_2\} + 1)$ -regular digraphs when  $k_1 < k_2$ ?

**Problem 6.3.** What is the complexity of the  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -partition problem for  $(k + 2)$ -regular digraphs?

Theorem 3.2 implies that Problem 6.3 becomes polynomial-time solvable if we replace  $(k + 2)$ -regular by  $(k + 1)$ -regular and that when  $k \geq 2$  a  $(\Delta^+ \leq k, \Delta^+ \leq k)$ -partition always exists in every  $(k + 1)$ -regular digraph as, by a result of Thomassen [11], these all have an even directed cycle (see also [2, Theorem 8.3.7]).

Finally, we can also ask about 2-partitions where the maximum out-degree is reduced in one part whereas it is the maximum in-degree that must be reduced in the other part.

**Problem 6.4.** What is the complexity of the  $(\Delta^+ \leq k_1, \Delta^- \leq k_2)$ -partition problem?

In this paper, we studied partitions such that the out-degree in (the digraph induced by) each part is  $k$  smaller than the out-degree in the whole digraph for some value  $k$  which is fixed and the same for each part. It would be interesting to study the analogous problem where  $k$  depends on the part. In this vein Alon proved the following result.

**Theorem 6.5** ([1]). *Let  $D$  be a digraph of maximum out-degree  $\Delta^+$  and let  $d_1, d_2, \dots, d_p$  non-negative integers satisfying  $d_1 + d_2 + \dots + d_p + (p - 1) \geq 2\Delta^+$ . Then  $D$  has a  $p$ -partition  $(V_1, V_2, \dots, V_p)$  such that  $\Delta^+(D(V_i)) \leq d_i$ .*

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