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Abstract

Let k be a fixed integer. We determine the complexity of finding a p-partition $(V_1, \dots, V_p)$ of the vertex set of a given digraph such that the maximum out-degree of each of the digraphs induced by V_i, $(1 \le i \le p)$ is at least k smaller than the maximum out-degree of D. We show that this problem is polynomial-time solvable when $p \ge 2k$ and $\mathcal{NP}$-complete otherwise. The result for $k = 1$ and $p = 2$ answers a question posed in [3]. We also determine, for all fixed non-negative integers k_1, k_2, p, the complexity of deciding whether a given digraph of maximum out-degree p has a 2-partition (V_1, V_2) such that the digraph induced by V_i has maximum out-degree at most k_i for $i \in [2]$. It follows from this characterization that the problem of deciding whether a digraph has a 2-partition (V_1, V_2) such that each vertex $v \in V_i$ has at least as many neighbours in the set V_{3-i} as in V_i, for $i = 1, 2$ is $\mathcal{NP}$-complete. This solves a problem from [6] on majority colourings.

Keywords: 2-partition, maximum out-degree reducing partition, $\mathcal{NP}$-complete, polynomial algorithm.

1 Introduction

Notation and terminology generally follow [2]. However we recall the useful notations and definitions in Section 2. A **p-partition** of a graph or digraph G is a vertex partition $(V_1, V_2, \dots, V_p)$ of its vertex set $V(G)$.

It is a well-known and easy fact that every undirected graph G admits a 2-partition such that the degree of each vertex in its part is at most half of its degree in G and such a partition can be found by a greedy algorithm (or by considering a maximum-cut partition). So we have the following.

Proposition 1.1.

(i) Every graph G has a 2-partition (V_1, V_2) such that $d_{G\langle V_i\rangle}(v) \le d_G(v)/2$ for all $i \in \{1, 2\}$ and all $v \in V_i$.

(ii) Every graph G has a 2-partition (V_1, V_2) with $\Delta(G\langle V_i\rangle) \le \Delta(G)/2$ for $i = 1, 2$.

Thomassen [10] constructed an infinite class of strongly connected digraphs $\mathcal{T} = T_1, T_2, \dots, T_k, \dots$ with the property that for each k, T_k is k-out-regular and has no even directed cycle. As remarked by Alon in [1] this implies that we cannot expect any directed analogues of the statements in Proposition 1.1.

Proposition 1.2. *Let k be a positive integer. For every 2-partition (V_1, V_2) of T_k, some vertex has all its k out-neighbours in the same part as itself, so* $\max\{\Delta^+(D\langle V_1\rangle), \Delta^+(D\langle V_2\rangle)\} = \Delta^+(D)$.

This is due to the simple fact that if a digraph D has a 2-partition (V_1, V_2) such that the bipartite digraph induced by the arcs between the two sets has minimum out-degree at least 1, then this digraph, and hence also D, has an even directed cycle.

Alon [1] also remarked that it is always possible to split $V(D)$ into three sets such that each of the induced subdigraphs has smaller maximum out-degree than D (see Theorem 6.5). In Proposition 5.1, we generalize this to all values of k. We show that for every positive integer k, there is a $(2k+1)$-partition of $V(D)$ such that the out-degree of every vertex x in its part is at most $d^+_D(x) - k$ or 0 if $d^+_D(x) < k$.

The digraphs in $\mathcal{T}$ show that one cannot always obtain a 2-partition of a digraph such that in each subdigraph induced by the parts, the out-degree of every vertex or the maximum out-degree is smaller than in the original graph. So it is natural to ask whether the existence of such a partition can be decided in polynomial time.

A **k-all-out-degree-reducing p-partition** of a digraph D is a p-partition $(V_1, \dots, V_p)$ of V such that $d^+_{D\langle V_i\rangle}(v) \leq \max\{0, d^+_D(v) - k\}$ for all $1 \leq i \leq p$ and all $v \in V_i$. A **k-max-out-degree-reducing p-partition** of a digraph D is a p-partition $(V_1, \dots, V_p)$ of V such that $\Delta^+(D\langle V_i\rangle) \leq \max\{0, \Delta^+(D) - k\}$ for $i \in [p]$. Observe that a k-all-out-degree-reducing p-partition is also a k-max-out-degree-reducing p-partition. However the converse is not necessarily true. So for fixed integers k and p, we are interested in the problems of deciding whether a given digraph admits one of the above defined partitions.

Problem 1.3 (k-ALL-OUT-DEGREE-REDUCING p-PARTITION).
Input: a digraph D;
Question: Does D have a p-partition (V_1, V_2) with $d^+_{D\langle V_i\rangle}(v) \leq \max\{0, d^+_D(v) - k\}$ for $i \in [p]$?

Problem 1.4 (k-MAX-OUT-DEGREE-REDUCING p-PARTITION).
Input: a digraph D;
Question: Does D have a p-partition (V_1, V_2) with $\Delta^+(D\langle V_i\rangle) \leq \max\{0, \Delta^+(D) - k\}$ for $i \in [p]$?

We first consider the case of 2-partitions. The complexity of 1-MAX-OUT-DEGREE-REDUCING 2-PARTITION was posed in the paper [3] in which the complexity of a large number of other 2-partition problems is established. We also consider a closely related kind of 2-partitions: A **$(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-partition** of a digraph is a 2-partition (V_1, V_2) such that $\Delta^+(D\langle V_i\rangle) \leq k_i$ for $i \in \{1, 2\}$. Note that if a digraph is r-out-regular, then a $(\Delta^+ \leq r-k, \Delta^+ \leq r-k)$-partition is also a k-max-out-degree-reducing 2-partition and a k-all-out-degree-reducing 2-partition. We thus consider the following problem.

Problem 1.5 ($(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-PARTITION).
Input: a digraph D;
Question: Does D have a 2-partition (V_1, V_2) with $\Delta^+(D\langle V_i\rangle) \leq k_i$ for $i \in \{1, 2\}$?

When $k_1 = k_2 = 0$ the problem is the same as just asking whether D is bipartite which is clearly polynomial-time solvable. If D is a symmetric digraph, then there is a one-to-one correspondence between the set of $(\Delta^+ \leq k, \Delta^+ \leq k)$-partitions of D and the so-called k-improper 2-colourings of $UG(D)$, the underlying (undirected) graph of D. A 2-colouring is **k-improper** if no vertex has more than k neighbours with the same colour as itself. Cowen et al. [4] proved that for any $k \geq 1$, deciding whether a graph has a k-improper 2-colouring is $\mathcal{NP}$-complete. Consequently, $(\Delta^+ \leq k, \Delta^+ \leq k)$-PARTITION is $\mathcal{NP}$-complete for all $k \geq 1$.

On the other hand, Proposition 1.1 (ii) can be translated as follows to symmetric digraphs.

Proposition 1.6. *Every symmetric digraph with maximum out-degree K has a $(\Delta^+ \leq \lfloor K/2 \rfloor, \Delta^+ \leq \lfloor K/2 \rfloor)$-partition.*

As we saw in Proposition 1.2, this result does not extend to general digraphs. Hence it is natural to ask about the complexity of $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-PARTITION when restricted to digraphs with small maximal out-degree.

In the first part of the paper, we prove that 1-ALL-OUT-DEGREE-REDUCING 2-PARTITION and 1-MAX-OUT-DEGREE-REDUCING 2-PARTITION can be solved in polynomial time. This answers the question posed in [3] affirmatively. Then we derive a complete characterization of the complexity of Problem 1.5 in terms of the values of k_1, k_2 and use it to prove that k-ALL-OUT-DEGREE-REDUCING 2-PARTITION and k-MAX-OUT-DEGREE-REDUCING 2-PARTITION are $\mathcal{NP}$-complete for all values of k higher than 1. As a consequence of these results, we solve an open problem from [6] on majority colourings.

Next, in Section 5, we consider p-partitions for $p \geq 3$. We show that every digraph admits a k-all-out-degree-reducing $(2k+1)$-partition. This implies that k-ALL-OUT-DEGREE-REDUCING p-PARTITION and k-MAX-OUT-DEGREE-REDUCING p-PARTITION are polynomial-time solvable for $p \geq 2k+1$ as the answer is always 'Yes'. We also characterize the digraphs having a k-all-out-degree-reducing $2k$-partition, which implies that k-ALL-OUT-DEGREE-REDUCING $2k$-PARTITION and k-MAX-OUT-DEGREE-REDUCING $2k$-PARTITION are polynomial-time solvable. Finally, we show that, for any $k > 1$ and $3 \leq p \leq 2k-1$, the problems k-ALL-OUT-DEGREE-REDUCING p-PARTITION and k-MAX-OUT-DEGREE-REDUCING k-PARTITION are $\mathcal{NP}$-complete.

We conclude with some remarks and related open problems.

2 Notation and definitions

We use the shorthand notation $[k]$ for the set $\{1, 2, \ldots, k\}$. Let $D = (V, A)$ be a digraph with vertex set V and arc set A.

Given an arc $uv \in A$, we say that u **dominates** v and v is **dominated** by u. If uv or vu (or both) are arcs of D, then u and v are **adjacent**. If neither uv or vu exist in D, then u and v are **non-adjacent**. The **underlying graph** of a digraph D, denoted by $UG(D)$, is obtained from D by suppressing the orientation of each arc and deleting multiple copies of the same edge (coming from directed 2-cycles). A digraph D is **connected** if $UG(D)$ is a connected graph, and the **connected components** of D are those of $UG(D)$.

A (u, v)**-path** is a directed path from u to v. A digraph is **strongly connected** (or **strong**) if it contains a (u, v)-path for every ordered pair of distinct vertices u, v. A digraph D is k**-strong** if for every set S of less than k vertices the digraph $D - S$ is strong. A **strong component** of a digraph D is a maximal subdigraph of D which is strong. A strong component is **trivial**, if it has order 1. An **initial** (resp. **terminal**) strong component of D is a strong component X with no arcs entering (resp. leaving) X in D.

The **subdigraph induced** by a set of vertices X in a digraph D, denoted by $D\langle X \rangle$, is the digraph with vertex set X and which contains those arcs from D that have both end-vertices in X. When X is a subset of the vertices of D, we denote by $D - X$ the subdigraph $D\langle V \setminus X \rangle$. If D' is a subdigraph of D, for convenience we abbreviate $D - V(D')$ to $D - D'$.

The **in-degree** (resp. **out-degree**) of v, denoted by $d^-_D(v)$ (resp. $d^+_D(v)$), is the number of arcs from $V \setminus \{v\}$ to v (resp. v to $V \setminus \{v\}$). A digraph is k**-out-regular** if all its vertices have out-degree k and it is k**-regular** if every vertex has both in-degree and out-degree k. A **sink** is a vertex with out-degree 0 and a **source** is a vertex with in-degree 0. The **degree** of v, denoted by $d_D(v)$, is given by $d_D(v) = d^+_D(v) + d^-_D(v)$. Finally the **maximum out-degree** and **maximum in-degree** of D are respectively denoted by $\Delta^+(D)$ and $\Delta^-(D)$.

An **out-tree** rooted at the vertex s, also called an s**-out-tree**, is a connected digraph T^+_s such that $d^-_{T^+_s}(s) = 0$ and $d^-_{T^+_s}(v) = 1$ for every vertex v different from s. Equivalently, for every $v \in V(T^+_s) \setminus \{s\}$ there is a unique (s, v)-path in T^+_s.

An **oriented graph** is a digraph with no directed 2-cycle. A **tournament** is a digraph is an oriented graph in which any two vertices are adjacent; in other words, for every two distinct vertices u and v, either uv or vu is an arc but not both.

A k**-colouring** of a graph G is a function $f : V(G) \to [k]$. A colouring f is **proper** if $f(u) \neq f(v)$ for every edge $uv \in E(G)$. A graph is k**-colourable** if it admits a proper k-colouring. It is k**-degenerate** if each of its subgraphs has a vertex of degree at most k. It is well-known that a k-degenerate graph is $(k+1)$-colourable.

In our $\mathcal{NP}$-completeness proofs we use reductions from the well-known 3-SAT problem and from MONOTONE NOT-ALL-EQUAL-3-SAT. The later is the variant where the boolean formula $\mathcal{F}$ to be satisfied consists of clauses all of whose literals are non-negated variables and we seek a truth assignment such that each clause will get both a true and a false literal. This problem is also $\mathcal{NP}$-complete [8].

3 1-out-degree reducing partitions of digraphs

In this section we prove that 1-ALL-OUT-DEGREE-REDUCING 2-PARTITION and 1-MAX-OUT-DEGREE-REDUCING 2-PARTITION are solvable in polynomial time for $k = 1$.

Part (i) of the theorem below follows from a result of Seymour [9] (see also [6]) but we include the short proof for completeness (and we use the same idea to prove (ii)). We shall use the following result, due to Robertson, Seymour, and Thomas.

Theorem 3.1 (Robertson, Seymour, and Thomas [7]). *Deciding whether a given digraph has an even directed cycle is polynomial-time solvable.*

Theorem 3.2. *Let D be a digraph.*

(i) D admits a 1-all-out-degree-reducing 2-partition if and only if every non-trivial terminal strong component contains an even directed cycle.

(ii) D admits a 1-max-out-degree-reducing 2-partition if and only if every terminal strong component contains an even directed cycle or a vertex with out-degree less than $\Delta^+(D)$.

In both cases above, the desired 2-partition can be constructed in polynomial time when it exists.

Proof. Let $X_1, \ldots, X_r$ be the terminal strong components of D ordered in such a way that $X_1, \ldots, X_q$ are non-trivial and $X_{q+1}, \ldots, X_r$ are trivial. Set $S = \bigcup_{i=q+1}^{r} V(X_i)$. Observe that S is the set of sinks of D.

(i) Suppose first that D admits a 1-all-out-degree-reducing 2-partition, then that partition restricted to X_i, $1 \leq i \leq q$, would induce a bipartite spanning subdigraph of X_i with an even directed cycle.

Assume now that X_i contains an even directed cycle C_i for all $i \in [q]$. First properly 2-colour all the cycles $C_1, C_2, \ldots, C_q$ and colour the vertices of S with colour 1. If there exists an uncoloured vertex, then there must also exist an uncoloured vertex with an arc to a coloured one (as we have coloured at least one vertex in every terminal strong component). Give this vertex the opposite colour of its coloured out-neighbour. Repeating this procedure until all vertices have been coloured gives us a 2-colouring where every vertex not in S has an out-neighbour of different colour to itself. From this 2-colouring, we obtain the desired partition.

(ii) The necessity is seen as above. Now assume that every terminal component X_i, $i \in [r]$, contains either an even directed cycle or a vertex of out-degree less than $\Delta^+(D)$. Pick an even directed cycle C_i for each terminal component with such a cycle and a vertex z_j with $d^+(z_j) < \Delta^+(D)$ for the other terminal components (this includes the trivial ones). Let Z be the union of the vertices z_j. Now 2-colour all the even directed cycles and colour the vertices of Z with colour 1. As above we can extend this colouring into a 2-colouring of D where every vertex not in Z has an out-neighbour of different colour to itself. This 2-colouring correspond to the desired partition.

The complexity claim follows from Theorem 3.1 and the fact that our proof is constructive. □

We will show in Theorem 4.8 that k-ALL-OUT-DEGREE-REDUCING 2-PARTITION and k-MAX-OUT-DEGREE-REDUCING 2-PARTITION are $\mathcal{NP}$-complete for $k > 1$.

4 2-partitions with restricted maximum out-degrees

In this section we consider Problem 1.5 and determine its complexity for all possible values of the parameters k_1, k_2. By symmetry, we may assume that we always have $k_1 \leq k_2$. Recall that when $k_1 = k_2 = 0$ the problem is the same as just asking whether D is bipartite which is polynomial-time solvable, so we may assume below that $k_2 > 0$.

The following gadget, depicted in Figure 1, turns out to be very useful in our constructions. An (x, y)**-**(i, p)**-connector** is the digraph with vertex set $\{x, y, s\} \cup T \cup U \cup U'$ with $|T| = i$ and $|U| = |U'| = p$ with all arcs from x to T, all arcs from T to U, all arcs between U and U', except one arc $u'u$ for some $u \in U$ and $u' \in U'$, all arcs from s to $U' \setminus \{u'\}$ arcs $u's$ and sy. Observe that in an (x, y)-(i, p)-connector, all vertices have out-degree p except x and y which have out-degree i and 0 respectively.

The next two lemmas illustrate the usefulness of connectors.

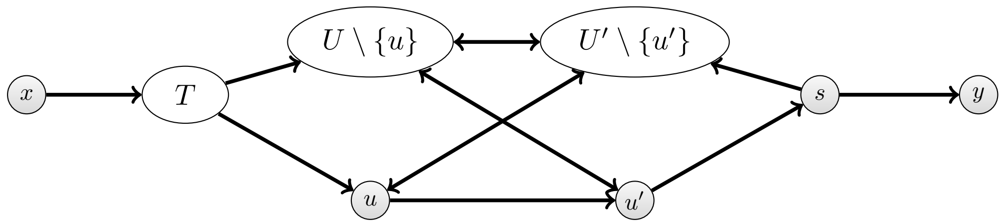

Figure 1: An (x, y)**-**(i, p)**-connector**, where $|T| = i$ and $|U \setminus \{u\}| = p - 1$ and $|U \setminus \{u'\}| = p - 1$.

Lemma 4.1. *Let k_1, k_2, i be three positive integers, with $1 \le k_1 \le k_2$, let D be a digraph and let x, y be two vertices in D. Let D' be the digraph obtained from D by adding an (x, y)-(i, p)-connector. D' has a $(\Delta^+ \le k_1, \Delta^+ \le k_2)$-partition if and only if D has one.*

Proof. Clearly, if D' has a $(\Delta^+ \le k_1, \Delta^+ \le k_2)$-partition, then its restriction to $V(D)$ is also a $(\Delta^+ \le k_1, \Delta^+ \le k_2)$-partition.

Assume now that D has a $(\Delta^+ \le k_1, \Delta^+ \le k_2)$-partition (V_1, V_2). By symmetry, we may assume that $x \in V_1$. Now one easily checks that $(V_1 \cup U \cup \{s\}, V_2 \cup T \cup U')$ is a $(\Delta^+ \le k_1, \Delta^+ \le k_2)$-partition of D'. Indeed, even if $y \in V_1$ we have $d^+_{V_1}(s) \le 1 \le k_1$. $\square$

Lemma 4.2. *Let k_1, k_2, p be non-negative integers with $0 \le k_1 \le k_2$. If $(\Delta^+ \le k_1, \Delta^+ \le k_2)$-*Partition *is $\mathcal{NP}$-complete for digraphs with maximum out-degree p, then $(\Delta^+ \le k_1, \Delta^+ \le k_2)$-*Partition *is also $\mathcal{NP}$-complete for digraphs for strong $(p+1)$-out-regular digraphs.*

Proof. First assume that $1 \le k_1 \le k_2$. Then we can use Lemma 4.1 quite directly. Consider a digraph D with maximum out-degree p, and let $\{v_1, \dots, v_n\}$ be its vertex set. For $j \in [n]$, let $i_j = p+1-d^+_D(v_j)$. Informally, i_j is the number of out-neighbours we must add to v_j so that it gets out-degree $p + 1$. Observe that for every j we have $i_j \ge 1$, because $\Delta^+(D) \le p$. Let D' be the digraph obtained by adding a (v_j, v_{j+1})-$(i_j, p + 1)$-connector for every $j \in [n]$ (with $v_{n+1} = v_1$). It is simple matter to check that D' is $(p + 1)$-out-regular and strong because every i_j is at least 1. Moreover, Lemma 4.1 implies that D' has a $(\Delta^+ \le k_1, \Delta^+ \le k_2)$-partition if and only if D has one.

Now assume that we have $0 = k_1 < k_2$. In this case we will need to put connectors between adjacent vertices to insure that Lemma 4.1 holds. Indeed if a digraph D has a $(\Delta^+ = 0, \Delta^+ \le k_2)$-partition and xy is an arc of D, then the digraph obtained from D by adding an (x, y)-$(p + 1 - d^+_D(x), p)$-connector to D admits also a $(\Delta^+ = 0, \Delta^+ \le k_2)$-partition. The proof of this statement is similar to the one of Lemma 4.1 using the fact that as xy is an arc of D then we cannot have $x \in V_1$ and $y \in V_1$ in any $(\Delta^+ = 0, \Delta^+ \le k_2)$-partition (V_1, V_2) of D.

Now let D be a digraph with maximum out-degree p. It is easy to check that the digraph obtained by adding a new vertex to D with two out-neighbours in D has a $(\Delta^+ = 0, \Delta^+ \le k_2)$-partition if and only if D has one. So let s a new vertex and let T be a binary s-out-tree with $|V(D)|$ leaves (i.e. every vertex of T has out-degree 2 except the leaves which have out-degree 0). We construct D' by adding a copy of T to D and identifying the vertices of D with the leaves of T. Note that $V(D') = V(T)$. By repeating the previous remark, we obtain that D' admits a $(\Delta^+ = 0, \Delta^+ \le k_2)$-partition if and only if D has one. To conclude we build D'' by adding a (v, u)-$(p + 1 - d^+_{D'}(v), p + 1)$-connector to D' for every arc uv of the copy of T and a (s, w)-$(p - 1, p + 1)$-connector for an out-neighbour w of s. Using the modified version of Lemma 4.1 for $(\Delta^+ = 0, \Delta^+ \le k_2)$-partitions, we conclude that D has such a partition if, and only if, D'' has one. Moreover, by construction, it is clear that D'' is strong and $(p+1)$-out-regular. $\square$

Obviously every digraph of maximum out-degree $k \leq \max\{k_1, k_2\}$ has a $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-partition. As we now show, just increasing the maximum out-degree one above this value results in a shift in complexity from trivial to $\mathcal{NP}$-complete, even if we also require that the digraph is strongly connected and out-regular.

Theorem 4.3. *For every choice of non-negative integers k_1, k_2 with $\max\{1, k_1\} < k_2$, the $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-*Partition *problem is $\mathcal{NP}$-complete for strong (k_2+1)-out-regular digraphs.*

Proof. Let us call a 2-colouring $c : V \to \{1, 2\}$ **good** if the 2-partition induced by c is a $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-partition. We start by describing a reduction from 3-SAT to $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-Partition in graphs of maximum out-degree $k_2 + 1$ and then show how to modify the proof to work for strong and (k_2+1)-out-regular digraphs using Lemma 4.1.

We first make some observations about gadgets that force certain vertices to have colour 1 or 2 in any good 2-colouring. Let X be the digraph that we obtain from a copy of the Thomassen digraph T_{k_2-1} (it exists because $k_2 > 1$) by adding one new vertex v and all possible arcs from $V(T_{k_2-1})$ to v. It follows from Proposition 1.2 that in any good 2-colouring c of a digraph containing an induced copy of X the vertex v must have $c(v) = 2$. Let Z be the digraph obtained by taking k_2+1 copies X_i, $i \in [k_2+1]$ of X, where v_i denotes the copy of v in X_i, $i \in [k_2+1]$ and a new vertex w and adding the arcs of $\{v_1v_{1+i} \mid i \in [k_2]\} \cup \{v_1w\}$. By the remark above, for every good 2-colouring of a digraph containing an induced copy of Z, we have $c(w) = 1$.

When we say below that a certain vertex u has colour 1 or colour 2 we mean that we use a private copy of either Z with $u = w$ or X with $u = v$ to enforce that in all good 2-colourings of D the vertex u will have the desired colour. Now let W be a digraph containing $k_1 + k_2 + 2$ vertices $v, \bar{v}, a_1, \ldots, a_{k_1}, b_1 \ldots, b_{k_2}$ and the arcs of $\{v\bar{v}, \bar{v}v\} \cup \{a_1v, a_1\bar{v}, b_1v, b_1\bar{v}\} \cup \{a_1a_{j+1} \mid j \in [k_1-1]\} \cup \{b_1b_{j+1} \mid j \in [k_2-1]\}$. By adding suitable copies of X, Z we can ensure that for every good colouring of the digraph we construct below we have $c(a_h) = 1$ for $h \in [k_1]$ and $c(b_h) = 2$ for $h \in [k_2]$. This implies that in every good colouring we have $c(v) = r$ and $c(\bar{v}) = 3 - r$ for some $r \in \{1, 2\}$.

Now we are ready to construct a digraph $D = D(\mathcal{F})$ from a given instance $\mathcal{F}$ of 3-SAT. Let $\mathcal{F}$ have variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$: represent each variable x_i by a copy W_i of W where the vertices $v_i, \bar{v}_i$ correspond to v and $\bar{v}$ in W and play the role of $x_i, \bar{x}_i$, respectively. For each clause C_j, we add a new vertex c_j of colour 2, $k_2 - 2$ arcs from c_j to private (to c_k) vertices of colour 2 and three arcs from c_j to the three vertices that correspond to its literals. So, if $C_j = (x_1 \vee \bar{x}_8 \vee x_9)$ then we add the arcs $c_jv_1, c_j\bar{v}_8$ and c_jv_9. This completes the construction of D. Clearly D can be constructed in polynomial time given $\mathcal{F}$. The fact that c_j must have colour 2 and already has $k_2 - 2$ out-neighbours of colour 2 implies that at least one of the vertices corresponding to the literals of C_j must have colour 1 in any good colouring. Now it is easy to see that if we associate colour 1 with *true*, then D has a good colouring if and only if $\mathcal{F}$ is satisfiable. This proves that $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-Partition is $\mathcal{NP}$-complete for digraphs of maximum out-degree $k_2 + 1$ as it is easy to check that $\Delta^+(D) \leq k_2 + 1$.

To obtain the result on strong (k_2+1)-out-regular digraphs, we first show how to obtain a strong superdigraph D' of D with the desired colouring property. First observe that in D no arc enters a copy of X unless this is inside a copy of Z and for every copy of Z one copy of X has no arcs entering it. By adding a new vertex s, sufficiently (but still polynomial in the size of $\mathcal{F}$) many new vertices and the arcs of an out-tree of maximum out-degree k_2 rooted at s, we can obtain that s is the root of an out-tree T_s^+ whose only intersection with $V(D)$ is in its leaves where T_s^+ has exactly one leaf in each copy of X.

Note that every vertex corresponding to a literal has out-degree 1 and that every vertex which does not correspond to a literal has a directed path to at least one vertex that corresponds to a literal (here we use that T_{k_2-1} is strongly connected). Thus if we add the arcs of the directed cycle $C = sv_1v_2 \ldots v_ns$, we obtain the desired strong digraph D' with $\Delta^+(D') = k_2 + 1$. Clearly D is a subdigraph of D' so every good 2-colouring of D' induces a good 2 colouring of D. Conversely, if c is a good 2-colouring of $V(D)$, then it is still a good 2-colouring of $D \cup A(C)$ because $k_2 \geq 2$ and we can extend c to the non-leaf vertices of T_s^+ (colouring them by 2) because they have out-degree at most k_2.

It remains to prove that we can also achieve a (k_2+1)-out-regular digraph D'' which is strong and has a good 2-colouring if and only if $\mathcal{F}$ is satisfiable. To show this we just have to observe that, by Lemma 4.1, for every vertex w with out-degree $k < k_2+1$ we can add a private $(w,w)-(k_2+1-k, k_2+1)$-connector. $\square$

Note that we used the fact that $k_2 > 1$ at several places in the proof above. One of these was the use of T_{k_2-1}. Hence there still remains the complexity of $(\Delta^+ \leq 0, \Delta^+ \leq 1)$-Partition. This was solved by Fraenkel.

Theorem 4.4 (Fraenkel [5])**.** $(\Delta^+ \leq 0, \Delta^+ \leq 1)$-Partition *is $\mathcal{NP}$-complete on the class of digraphs with in- and out-degree at most* 2*.*

In order to strengthen this and to unify our results we need the following result which can be obtained by modifying the proof in [5]. We give a proof for completeness.

Theorem 4.5. *For all $p \geq 2$,* $(\Delta^+ \leq 0, \Delta^+ \leq 1)$-Partition *is $\mathcal{NP}$-complete on the class of strong p-out-regular digraphs.*

Proof. By Lemma 4.2, it suffices to prove the statement for $p = 2$. A **kernel** in a digraph D is an independent set K of vertices such that every vertex in $V(G) \setminus K$ has an out-neighbour in K. Note that (V_1, V_2) is a $(\Delta^+ \leq 0, \Delta^+ \leq 1)$-partition of a 2-out-regular digraph D if and only if V_1 is a kernel of D. We first recall a (slightly simpler version of) the proof from [5] that deciding whether a digraph has a kernel is $\mathcal{NP}$-complete for digraphs of maximum out-degree 2 and then modify that reduction to show that it is $\mathcal{NP}$-complete for strong 2-out-regular digraphs.

Let W denote the digraph defined by

$$V(W) = \{z_1, \dots, z_9\} \quad \text{and} \quad A(W) = \{z_1z_2, z_2z_3, z_3z_1, z_3z_4, z_4z_5, z_5z_6, z_5z_7, z_6z_8, z_7z_9\}.$$

Now let $\mathcal{F}$ be an instance of 3-SAT with variable $x_1, \dots, x_n$ and clauses $C_1, \dots, C_m$. Free to duplicate one clause, we may assume that m is odd. Form the digraph $G = G(\mathcal{F})$ by taking one copy W_j of W for each clause C_j, $j \in [m]$ (denoting the vertices of W_j by $z_{j,q}$, $q \in [9]$) and adding $2n$ new vertices $v_1, \bar{v}_1, \dots, v_n, \bar{v}_n$, where $v_i, \bar{v}_i$ correspond to the literals $x_i, \bar{x}_i$ as well as the arcs $v_i\bar{v}_i, \bar{v}_iv_i$ for $i \in [n]$. Finally, we add three arcs from each W_j to the vertices that correspond to its literals so that the vertex $z_{j,8}$ is joined to the vertex corresponding to the first literal and the vertex $z_{j,9}$ is joined to the two vertices corresponding to the second and third literal of W_j. Thus if $W_j = (x_4 \vee x_5 \vee \bar{x}_8)$, then we add the arcs $z_{j,8}v_4, z_{j,9}v_5, z_{j,9}\bar{v}_8$. This completes the construction of G. Note that if K is a kernel of G, then for every $j \in [m]$ we have either $\{z_{j,2}, z_{j,4}, z_{j,6}\} \subset K$ or $\{z_{j,2}, z_{j,4}, z_{j,7}\} \subset K$ (or both) and this implies that $|K \cap \{z_{j,8}, z_{j,9}\}| \leq 1$. From this it follows that at least one of the vertices corresponding to the literals of C_j will belong to K. For each $i \in [n]$ we have precisely one of $v_i, \bar{v}_i$ in K as these vertices are adjacent. Now it is easy to see that G has a kernel if and only if $\mathcal{F}$ is satisfiable. This shows that deciding whether a digraph has a kernel and hence $(\Delta^+ \leq 0, \Delta^+ \leq 1)$-Partition is $\mathcal{NP}$-complete for digraphs of maximum out-degree 2.

Let us now prove that it is $\mathcal{NP}$-complete for strong 2-out-regular digraphs. Note that in G every vertex has out-degree at least 1. Let H be the digraph on six vertices a, b, c, d, e, f and the arcs $de, ef, fd, da, eb, fc, ae, bd, bf, cd, ce$. Let G' be the digraph obtained from the disjoint union of G and H and a directed path $a_1a_2 \dots a_m$ by identifying a and a_1 and adding the arc $a_mz_{m,3}$, the arcs $a_jz_{j,1}$ for $j \in [m]$ and the arcs ud for every vertex u having out-degree 1 in G. Clearly, the digraph G' is strong and 2-out-regular.

Finally let us now prove that G' has a kernel if and only if G has one. This will immediately imply the result. If G has a kernel K, then one can easily check that $K \cup \{b, c\} \cup \{a_j \mid j \text{ odd}\}$ is a kernel of G' (recall that m is odd and that K contains none of $z_{j,1}$, $z_{j,3}$). Assume now that G' has a kernel K'. We have $d \notin K'$, for otherwise b and f are not in K' (because K' is an independent set) and so e has no out-heighbour in K', a contradiction. Now all arcs leaving G in G' have head d, so every vertex of G has an out-neighbour in $K' \cap V(G)$. Hence $K' \cap V(G)$ is a kernel of G. $\square$

Theorem 4.6. *Let k, p be two positive integers k such that $p \geq k+2$. Then* $(\Delta^+ \leq k, \Delta^+ \leq k)$-Partition *is polynomial-time solvable for digraphs of maximum out-degree $k+1$ and $\mathcal{NP}$-complete on the class of strong p-out-regular digraphs.*

Proof. The first part of the claim follows from Theorem 3.2. Below we show how make a reduction from Monotone Not-All-Equal 3-SAT to the $(\Delta^+ \le k, \Delta^+ \le k)$-partition problem in strong $(k+2)$-out-regular digraphs. Combining this with Lemma 4.2 proves the theorem, as $k > 0$.

The reduction makes use of the following **forcing gadget**, namely the digraph F whose vertex set is the union of $X = \{x, x'\}$, $Y = V(T_k)$ (Recall that T_k is Thomassen's digraph defined in the introduction.) and whose arc set is the union of the arcs of T_k and all possible arcs from Y to X. The **head** of a forcing gadget is the set X.

Claim 4.6.1.

(i) In a forcing gadget, all vertices have out-degree $k+2$, except those of the head which have out-degree 0.

(ii) In any $(\Delta^+ \le k, \Delta^+ \le k)$-partition of a digraph which contains a copy of the forcing gadget as an induced subdigraph, the two vertices of the head are in the same part.

Subproof. (i) follows from the definition of the forcing gadget as T_k is k-out-regular.

(ii) follows from the fact that $F\langle Y\rangle = T_k$ has no $(\Delta^+ \le k-1, \Delta^+ \le k-1)$-partition, implying that in any 2-partition (V_1, V_2) of F some vertex of Y already has its k out-neighbours in Y in the same set V_i as itself and hence both x and x' must belong to V_{3-i}. $\Diamond$

Let $\mathcal{F}$ be an instance of Monotone Not-All-Equal $(k+2)$-SAT on n variables $x_1, \dots, x_n$ and m clauses $C_1, \dots, C_m$. For every $i \in [n]$, let $j_1(i) < j_2(i) \cdots < j_{m(i)}(i)$ be the indices of those clauses in which variable x_i occurs and let $J(i) = \{j_1(i), \dots, j_{m(i)}(i)\}$. For each $j \in [m]$ and $q \in [k+2]$, let $a_{q,j}$ be the unique integer such that if $C_j = x_{i_1} \vee x_{i_2} \vee x_{i_3}$, then x_{i_q} occurs exactly $a_{q,j} - 1$ times among the clauses $C_1, \dots, C_{j-1}$.

Let $D_{\mathcal{F}}$ be the digraph constructed as follows. For all $i \in [n]$, we create a variable gadget VG_i as follows. We first create the vertices $\{x_i^j \mid j \in J(i)\}$. Then for all $1 \le p < m(i)$, we add a forcing gadget with head $\{x_i^{j_p(i)}, x_i^{j_{p+1}(i)}\}$. Let Y_i^p be the set corresponding to Y in this forcing gadget. This will force all the vertices of $\{x_i^j \mid j \in J(i)\}$ to be in the same part for any $(\Delta^+ \le k, \Delta^+ \le k)$-partition.

Then for every clause $C_j = x_{i_1} \vee x_{i_2} \vee \ldots \vee x_{i_{k+2}}$, we add a vertex t_j, all the arcs from the set $\{x_{i_1}^{a_{1,j}}, x_{i_2}^{a_{2,j}}, \dots, x_{i_{k+2}}^{a_{k+2,j}}\}$ to t_j and the arcs of the complete digraph on $\{x_{i_1}^{a_{1,j}}, x_{i_2}^{a_{2,j}}, \dots, x_{i_{k+2}}^{a_{k+2,j}}\}$.

Let $D'_{\mathcal{F}}$ be the digraph obtained from $D_{\mathcal{F}}$ as follows. Add a set of $3m-n$ new vertices $U = \{u_1, \dots, u_{3m-n}\}$ and let f be a bijection between U and $\{Y_i^p \mid i \in [n], 1 \le p \le m(i)-1\}$. For each $j \in [3m-n]$, we add a (u_j, v_j)-$(1, k+2)$-connector with v_j being an arbitrary vertex in $f(u_j)$, and a (u_j, u_{j+1})-$(k+1, k+2)$-connector (with $u_{3m-n+1} = u_1$). Finally, for each $j \in [m]$, add a (t_j, u_1)-$(k+2, k+2)$-connector. We can easily check that $D'_{\mathcal{F}}$ is strong and $(k+2)$-out-regular.

Let us now prove that $D'_{\mathcal{F}}$ has a $(\Delta^+ \le k, \Delta^+ \le k)$-partition if and only if $\mathcal{F}$ admits a **NAE-assignment**, that is a truth assignment such that each clause contains a true literal and a false literal. By Lemma 4.1, as $k > 0$, it is equivalent to prove that $D_{\mathcal{F}}$ has a $(\Delta^+ \le k, \Delta^+ \le k)$-partition if and only if $\mathcal{F}$ admits a NAE-assignment.

First suppose that ϕ is a NAE-assignment. Define the following 2-colouring of $V(D_{\mathcal{F}})$: for each $i \in [n]$ colour all vertices of $\{x_i^j \mid j \in J(i)\}$ by colour 1 and those of $\bigcup_{p=1}^{m(i)-1} Y_i^p$ by 2 if $\phi(x_i) = true$ and otherwise colour all vertices of $\{x_i^j \mid j \in J(i)\}$ by 2 and those of $\bigcup_{p=1}^{m(i)-1} Y_i^p$ by 1. Now each t_j, $j \in [m]$, has at least one in-neighbour of colour i for $i \in [2]$. If it has precisely one of colour i, we colour it by colour i and otherwise we colour it arbitrarily. Now it is easy to see that letting V_i be the set of vertices of colour i, $i = 1, 2$, we obtain the desired 2-partition of $D_{\mathcal{F}}$.

Assume now that (V_1, V_2) is a good 2-partition of $D_{\mathcal{F}}$. The forcing gadgets ensure that in every $(\Delta^+ \le k, \Delta^+ \le k)$-partition (V_1, V_2) of $V(D_{\mathcal{F}})$ all vertices of $\{x_i^j \mid j \in J(i)\}$ belong to the same set in the partition for all $i \in [n]$. Furthermore, because of the complete subdigraphs on the vertices $\{x_{i_1}^{a_{1,j}}, x_{i_2}^{a_{2,j}}, \dots, x_{i_{k+2}}^{a_{k+2,j}}\}$, $j \in [m]$, at least one of these vertices is in V_1 and at least one of them is in V_2. Thus if we assign x_i the value $true$ if $\{x_i^j \mid j \in J(i)\} \subset V_1$ and $false$ otherwise, each clause will have at least one true and at least one false literal. $\square$

Combining our results above we obtain the following complete classification in terms of k_1, k_2.

Theorem 4.7. *Let k_1, k_2 be non-negative integers. The* $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-PARTITION *problem is*

- *polynomial-time solvable for all digraphs when* $k_1 = k_2 = 0$*;*
- *polynomial-time solvable for digraphs of maximum degree* $p \leq \max\{k_1, k_2\}$*;*
- $\mathcal{NP}$*-complete for strong p-out-regular digraphs for all* $p \geq \max\{k_1, k_2\} + 1$ *when* $k_1 \neq k_2$*;*
- *polynomial-time solvable for* $(k_2 + 1)$*-out-regular digraphs and* $\mathcal{NP}$*-complete for strong p-out-regular digraphs for all* $p \geq \max\{k_1, k_2\} + 2$ *when* $k_1 = k_2$*.*

Theorems 4.6 and 3.2 immediately yield the following.

Theorem 4.8. k-ALL-OUT-DEGREE REDUCING 2-PARTITION *and* k-MAX-OUT-DEGREE-REDUCING 2-PARTITION *are polynomial-time solvable for* $k = 1$ *and* $\mathcal{NP}$*-complete for all integers* $k \geq 2$ *even when the input is a strong out-regular digraph.*

5 Out-degree reducing p-partitions for $p \geq 3$.

All our complexity results so far dealt with 2-partition problems. In this section we deal with p-partitions for $p \geq 3$.

The next proposition implies that k-ALL-OUT-DEGREE-REDUCING p-PARTITION and k-MAX-OUT-DEGREE-REDUCING p-PARTITION are polynomial-time solvable when $p \geq 2k + 1$, because the answer is trivially 'yes'.

Proposition 5.1. *Every digraph has a k-all-out-degree-reducing* $(2k + 1)$*-partition and this is best possible.*

Proof. Let D be a digraph. For each vertex v pick $\min\{k, d^+(v)\}$ arcs with tail in v. Let H be the subdigraph of D induced by these arcs. Then H has a vertex of degree at most $2k$ and this holds for every subdigraph of H, so $UG(H)$ is $2k$-degenerate and hence it is $2k + 1$-colourable. Let $(V_1, V_2, \dots, V_{2k+1})$ be a $(2k + 1)$-partition of D induced by a $(2k + 1)$-colouring of $UG(H)$. It is easy to check that this is a k-all-out-degree-reducing $(2k + 1)$-partition since every arc of H goes between two different sets in the partition.

The k-out-regular tournaments show that $2k + 1$ is best possible for each $k \geq 1$. □

The next result implies that k-ALL-OUT-DEGREE-REDUCING p-PARTITION and k-MAX-OUT-DEGREE-REDUCING p-PARTITION are also polynomial-time solvable when $p = 2k$.

Theorem 5.2. *Let* $k \geq 2$*. A digraph D admits a k-all-out-degree-reducing* $2k$*-partition if and only if no terminal strong component of D is a k-regular tournament.*

Proof. First assume that some terminal component, Q, of D is a k-regular tournament. This implies that every vertex in Q has out-degree k in D and for any $2k$-partition of D there will be two vertices from Q in the same part, as $|V(Q)| = 2k + 1$. Therefore some vertex will have out-degree at least 1 in its part and therefore not have reduced its out-degree by k. This proves one direction. We now prove the opposite direction.

Let D be any digraph of order n and size m with no terminal component isomorphic to a k-regular tournament. We will now show that D has a k-all-out-degree-reducing $2k$-partition by induction on $n + m$. Clearly this holds when $n + m \leq 3$ so assume that it also holds for all digraphs, D', with $|V(D')| + |E(D')| < n + m$. We may assume that D is connected as otherwise we are done by using induction on each connected component. Let G be the underlying graph of D. We consider the following three cases which exhaust all possibilities.

Case 1. There exists a vertex $x \in V(D)$ with $d^+(x) > k$. If $N^+(x)$ is independent then let $v \in N^+(x)$ be arbitrary, and otherwise let $u, v \in N^+(x)$ be chosen such that $uv \in A(D)$. Let $D' = D \setminus xv$ (i.e. delete the arc xv from D). Let Q' be any terminal component in D'. If $x \notin V(Q')$, then Q' is also a terminal component of D and therefore not a k-regular tournament. So suppose $x \in V(Q')$. Recall that either $N^+(x)$ is independent or xuv is a path in D which implies that

$v \in V(Q')$. Both cases imply that Q' is not a tournament. Therefore, by induction, there is a k-all-out-degree-reducing $2k$-partition of D' and therefore also of D (using the same partition). This completes Case 1.

Case 2. $\Delta^+(D) \leq k$ and G is not $2k$-regular. Let w be a vertex having degree at most $2k-1$ in G. Let $D' = D - w$. Assume that some terminal component, Q', in D' is a k-regular tournament. As $\Delta^+(D) \leq k$, this implies that Q' is also a terminal component in D, a contradiction. Therefore no terminal component in D' is a k-regular tournament and by induction there is a k-all-out-degree-reducing $2k$-partition of D'. Now add w to a different part to all of its at most $2k-1$ neighbours in G. This gives a k-all-out-degree-reducing $2k$-partition of D.

Case 3. $\Delta^+(D) \leq k$ and G is $2k$-regular. Note that in that case D is an oriented graph and D is k-regular. Now G is not a complete graph for otherwise D would be k-regular tournament. Moreover, as $k \geq 2$, the graph G is not an odd cycle. Therefore, by Brook's Theorem, G admits a proper $2k$-colouring. This $2k$-colouring gives us the desired k-all-out-degree-reducing $2k$-partition of D. □

Theorem 5.3. *If $k > 1$ and $3 \leq p \leq 2k-1$, then k-ALL-OUT-DEGREE-REDUCING p-PARTITION and k-MAX-OUT-DEGREE-REDUCING p-PARTITION are $\mathcal{NP}$-complete.*

Proof. We give a reduction from p-COLOURABILITY which consists in deciding whether a given digraph is p-colourable. This problem is well-known to be $\mathcal{NP}$-complete for all $p \geq 3$.

We first need to define a gadget $D_2(x, y)$ as follows. Let T be a regular or almost regular tournament of order $p-1$ and let $V_1 = \{v \mid d^+_T(v) = k-1\}$. Note that V_1 is empty if $p \leq 2k-2$ and $|V_1| = k-1 = |V(T)|/2$ if $p = 2k-1$.

Let $D_2(x, y)$ be the digraph obtained from a copy of T by adding two vertices x, y and all arcs from $V(T) \setminus V_1$ to $\{x, y\}$, all arcs from V_1 to x and all arcs from y to V_1. Note that $d^+(x) = 0$ and $d^+(y) = |V_1|$.

Note that in both cases above x and y are the only non-adjacent vertices in $D_2(x, y)$ and $\Delta^+(D_2(x, y)) \leq k$.

We now define the gadget $D_n(x_1, x_2, \ldots, x_n)$ for $n \geq 3$ as the union of $D_2(x_1, x_2)$, $D_2(x_2, x_3)$, ..., $D_2(x_{n-1}, x_n)$, where the copies of T are disjoint. Note that $d^+(x_1) = 0$ and $d^+(x_i) \leq k-1$ for all $i = 2, 3, \ldots, n$ (in fact $d^+(x_i) = 0$ if $p < 2k-1$ and $d^+(x_i) = k-1$ otherwise).

We will now reduce an instance of p-COLOURABILITY to an instance of k-MAX-OUT-DEGREE-REDUCING p-PARTITION. Let G be a graph with vertex set $v_1, \ldots, v_n$. We will now construct a digraph D as follows. For each vertex $v_i \in V(G)$ we let D^i be a copy of $D_n(x^i_1, x^i_2, \ldots, x^i_n)$. For each edge v_iv_j of G with $i < j$ add an arc from x^i_j to x^i_j. Observe that the set of arcs added by this operation are disjoint, so the resulting digraph D has out-degree at most k. Consequently, every k-max-out-degree-reducing p-partition and every k-max-out-degree-reducing p-partition of D is equivalent to a proper p-colouring of the underlying graph $UG(D)$ of D.

Hence to prove the theorem, it is enough to show that $UG(D)$ has a proper p-colouring if and only if G does. But this follows directly from the following claim.

Claim 5.3.1. *In any p-colouring of $UG(D_n(x_1, x_2, \ldots, x_n))$, all the vertices in $\{x_1, x_2, \ldots, x_n\}$ must be coloured the same. Furthermore, there exists a p-colouring of $UG(D_n(x_1, x_2, \ldots, x_n))$.*

Proof of Claim 5.3.1. We show Claim 5.3.1 is true when $n = 2$ and then note that this implies that Claim 5.3.1 is true for all n. Let $n = 2$. As x_1 and x_2 are the only non-adjacent vertices in $D_2(x_1, x_2)$ and $|V(D_2(x_1, x_2))| = p+1$ we note that x_1 and x_2 must have the same colour in a proper p-colouring of $UG(D_2(x_1, x_2))$. Conversely if x_1 and x_2 have the same colour all other vertices of $D_2(x_1, x_2)$ can be given a distinct colour in order to obtain a proper p-colouring of the underlying graph. This proves Claim 5.3.1 when $n = 2$.

When $n \geq 3$ we note by the above that x_1 and x_2 must be in the same partite set. Analogously x_2 and x_3 must be in the same partite set. Continuing this process we obtain the desired result for $n \geq 3$. This completes the proof of Claim 5.3.1. ◊ □

6 Remarks and open questions

A **majority k-colouring** of a digraph $D = (V, A)$ is a k-colouring of the vertices of V so that each vertex v has at most $\frac{d^+(v)}{2}$ out-neighbours with the same colour as itself. It is shown in [6] that every digraph has a majority 4-colouring and the authors conjecture that, in fact, every digraph has a majority 3-colouring. They also asked about the complexity of deciding whether a digraph has a majority 2-colouring. Since a 3-out-regular digraph has a majority 2-colouring if and only if it has a $(\Delta^+ \leq 1, \Delta^+ \leq 1)$-partition the following is an immediate consequence of Theorem 4.6.

Theorem 6.1. *Deciding whether a digraph has a majority 2-colouring is $\mathcal{NP}$-complete even when the input is 3-out-regular and strongly connected.*

In all our $\mathcal{NP}$-completeness proofs above on out-regular digraphs, these are far from being also in-regular. Thus it is natural to ask about the complexity in the case of regular digraphs.

Problem 6.2. What is the complexity of the $(\Delta^+ \leq k_1, \Delta^+ \leq k_2)$-partition problem for $(\max\{k_1, k_2\} + 1)$-regular digraphs when $k_1 < k_2$?

Problem 6.3. What is the complexity of the $(\Delta^+ \leq k, \Delta^+ \leq k)$-partition problem for $(k+2)$-regular digraphs?

Theorem 3.2 implies that Problem 6.3 becomes polynomial-time solvable if we replace $(k+2)$-regular by $(k+1)$-regular and that when $k \geq 2$ a $(\Delta^+ \leq k, \Delta^+ \leq k)$-partition always exists in every $(k+1)$-regular digraph as, by a result of Thomassen [11], these all have an even directed cycle (see also [2, Theorem 8.3.7]).

Finally, we can also ask about 2-partitions where the maximum out-degree is reduced in one part whereas it is the maximum in-degree that must be reduced in the other part.

Problem 6.4. What is the complexity of the $(\Delta^+ \leq k_1, \Delta^- \leq k_2)$-partition problem?

In this paper, we studied partitions such that the out-degree in (the digraph induced by) each part is k smaller than the out-degree in the whole digraph for some value k which is fixed and the same for each part. It would be interesting to study the analogous problem where k depends on the part. In this vein Alon proved the following result.

Theorem 6.5 ([1])**.** *Let D be a digraph of maximum out-degree Δ^+ and let $d_1, d_2, \ldots, d_p$ non-negative integers satisfying $d_1 + d_2 + \ldots + d_p + (p-1) \geq 2\Delta^+$. Then D has a p-partition $(V_1, V_2, \ldots, V_p)$ such that $\Delta^+(D\langle V_i \rangle) \leq d_i$.*

References

[1] N. Alon. Splitting digraphs. *Combin. Probab. Comput.*, 15:933–937, 2006.

[2] J. Bang-Jensen and G. Gutin. *Digraphs: Theory, Algorithms and Applications, 2nd Edition.* Springer-Verlag, London, 2009.

[3] J. Bang-Jensen and F. Havet. Finding good 2-partitions of digraphs I. Hereditary properties. *Theoretical Computer Science*, 636:85–94, 2016.

[4] Robert Cowen. Some connections between set theory and computer science. In *In: Computational Logic and Proof Theory, Third Kurt Gödel Colloquium, KGC'93, Brno, Czech Republic, August 24-27, 1993, Proceedings, Editors G. Gottlob and A. Leitsch and D. Mundici*, volume 713 of *Lecture Notes in Computer Science*, pages 14–22. Springer, 1993.

[5] A. S. Fraenkel. Planar kernel and grundy with d≤3, d_{out}≤2, d_{in}≤2 are np-complete. *Discrete Applied Mathematics*, 3(4):257–262, 1981.

[6] S. Kreutzer, S.-I Oum, P. Seymour, D. van der Zypen, and D.R. Wood. Majority colourings of digraphs. *Electronic J. Combinatorics*, 24:Paper #P2,25, 2017.

[7] N. Robertson, P.D. Seymour, and R. Thomas. Permanents, Pfaffian orientations, and even directed circuits. *Ann. Math.*, 150:929–975, 1999.

[8] T.J. Schaefer. The complexity of satisfiability problems. In *Proceedings of the 10th Annual ACM Symposium on Theory of Computing (STOC 10)*, pages 216–226, New York, 1978. ACM.

[9] P.D. Seymour. On the two-colouring of hypergraphs. *Quarterly J. Math.*, 25:303–311, 1974.

[10] C. Thomassen. Even cycles in directed graphs. *Eur. J. Combin.*, 6(1):85–89, 1985.

[11] C. Thomassen. The even cycle problem for directed graphs. *J. Amer. Math. Soc.*, 5(2):217–229, 1992.