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Delaunay Triangulations of Points on Circles*

Vincent Despré † Olivier Devillers† Hugo Parlier‡
Jean-Marc Schlenker‡

Abstract

Delaunay triangulations of a point set in the Euclidean plane are ubiquitous in a number of computational sciences, including computational geometry. Delaunay triangulations are not well defined as soon as 4 or more points are concyclic but since it is not a generic situation, this difficulty is usually handled by using a (symbolic or explicit) perturbation. As an alternative, we propose to define a canonical triangulation for a set of concyclic points by using a max-min angle characterization of Delaunay triangulations. This point of view leads to a well defined and unique triangulation as long as there are no symmetric quadruples of points. This unique triangulation can be computed in quasi-linear time by a very simple algorithm.

1 Introduction

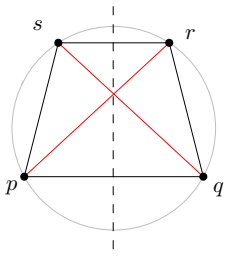
Let P be a set of points in the Euclidean plane. If we assume that P is in general position and in particular do not contain 4 concyclic points, then the Delaunay triangulation $DT(P)$ is the unique triangulation over P such that the (open) circumdisk of each triangle is empty. $DT(P)$ has a number of interesting properties. The one that we focus on is called the *max-min angle* property. For a given triangulation τ, let $A(\tau)$ be the list of all the angles of τ sorted from smallest to largest. $DT(P)$ is the triangulation which maximizes $A(\tau)$ for the lexicographical order [4, 7] . In dimension 2 and for points in general position, this max-min angle property characterizes Delaunay triangulations and highlights one of their most important features: they don't contain skinny triangles with small angles. We call

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such triangulations max-min angle Delaunay triangulations or simply Delaunay triangulations as the two notions are equivalent for points in general position. We call a quadrilateral $pqrs$ symmetric if there is a symmetry that exchange p with q and r with s or, equivalently, if the two diagonals pr and qs have the same length. In such a quadrilateral, the four points are concyclic. In this paper, we show that considering the max-min angle characterization allows to significantly weaken the notion of general position for P. We show the following theorem.



Theorem 1. *If a set of points in the Euclidean plane does not contain any symmetric quadruples then it has a unique max-min angle Delaunay triangulation.*

Notice that the two possible triangulations of a symmetric quadrilateral have exactly the same angles (and same diagonal length), and thus unicity is impossible when symmetric quadrilaterals are allowed. As an immediate corollary to the theorem above we obtain the following:

Corollary 2. *A set of points in the Euclidean plane with distinct pairwise distances admits a unique max-min angle Delaunay triangulation.*

The usual and generic way to address degeneracies is the use of symbolic perturbations [2,5,6,8] that perturb the input point set by an infinitesimal quantity in a way that is guaranteed to remove degeneracies. This approach has been used for Delaunay triangulations and allows to draw diagonals in a set of concyclic points in a consistent but not really meaningful manner [1,3]. Our result allows to use the max-min Delaunay inside such a set of concyclic points yielding a meaningful result to triangulate configurations that are usually considered as degenerate. To this aim we need an efficient algorithm for this special case. Such approach has the advantage on symbolic perturbation to define the triangulation independently of a numbering of the points or of the point coordinates in a particular reference frame. Notice that in symmetric configurations some cases remain degenerate and symbolic perturbation cannot help in such a case (unless the perturbation keeps the points concyclic which seems unpractical).

Theorem 3. *Fix n points ordered along a Euclidean circle without any symmetric quadruples. The unique max-min angle Delaunay triangulation on the n points can be computed using $O(n)$ arithmetic operations.*

This note is structured as follows: We give some structural lemmas about concyclic points in Section 2. We prove Theorem 1 in Section 3. We then describe our algorithm in a simple setting and prove a weaker version of Theorem 3 in Section 4. We want to present the general idea which is nice and very natural

without hiding it in technical details. The details comes in Section 5. Finally, we give an extended algorithm in Section 6 that can be applied to the most general case.

2 Properties of concyclic points

For an integer $n \geq 4$, let C be the unit circle in the Euclidean plane and $P = (p_0, \cdots, p_{n-1})$ be a set of points of C listed in (cyclic) counterclockwise order around C. We call *chords* the segments of the form $[p_i p_{i+1}]$ and *diagonals* the other possible segments. We also call *ears* the diagonals of the form $[p_i p_{i+2}]$. Note that if $n = 4$, all segments are ears or chords, but otherwise there are diagonals that are not ears.

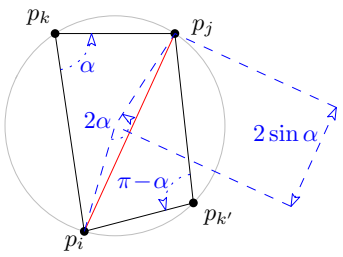
We are looking for a triangulation τ of a polygon $P = p_0 \cdots p_{n-1}$ which we think of as a decomposition of the polygon into triangles, or alternatively as a collection of edges that cut the polygon into triangles. By an Euler characteristic argument, τ contains $n-3$ diagonals and τ cuts P into $n-2$ triangles. The resulting triangles have a total of $3n-6$ angles, all strictly between 0 and π. We denote by $A(\tau) = (a_0, \cdots, a_{3n-7})$ the set of these angles listed in increasing order: $a_0 \leq a_1 \leq \cdots \leq a_{3n-7}$. For different triangulations τ, we order the resulting $A(\tau)$ according to lexicographical order providing the resulting triangulations with an order. A triangulation τ_0 is said to be *angle maximal* if $A(\tau_0) \geq A(\tau)$ for all triangulations τ of P.

We choose τ to be one of the triangulations having the list of angles $A(\tau)$ which is maximal for the lexicographical order. Let $D(\tau) = (d_0, \cdots, d_{n-3})$ be the list of the diagonals of τ such that $\ell(d_0) \leq \ell(d_1) \leq \cdots \leq \ell(d_{n-3})$. For simplicity, in the sequel we will not distinguish between a diagonal and its length (thus d_i means either the diagonal d_i or its length $\ell(d_i)$). As before, for different triangulations τ, we order the resulting $D(\tau)$ according to lexicographical order. A triangulation τ_0 is said to be *length maximal* if $D(\tau_0) \geq D(\tau)$ for all triangulations τ of P.

Lemma 4. *A triangulation of a set of concyclic points P is angle maximal if and only if it is length maximal.*

Proof. Let τ be a triangulation of P. Each chord $[p_i p_{i+1}]$ is incident to a triangle of the form $p_i p_{i+1} p_j$. All the vertices p_j for $j \neq i, i+1$ lie on the same circular arc of C between p_{i+1} and p_i.

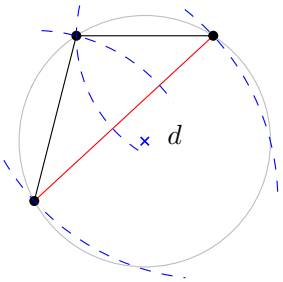
Given a diagonal or a chord d the inscribed angle theorem shows that the angle in the triangle incident to d (on a given side of d) at the vertex opposite to d depends only on d and not on the position of this vertex on C. Namely, on the side of d that contains the origin this angle is $\arcsin \frac{d}{2}$ and $\pi - \arcsin \frac{d}{2}$ when the origin is on the other side. Since $\arcsin \frac{d}{2}$ is an increasing function of d when $d \in [0, 2]$ the angles in the triangulation are:



$$\begin{aligned}\arcsin \tfrac{d_0}{2} \leq \arcsin \tfrac{d_1}{2} \leq \quad \ldots \quad \leq \arcsin \tfrac{d_{n-3}}{2} \\ \leq \pi - \arcsin \tfrac{d_{n-3}}{2} \leq \quad \ldots \quad \leq \pi - \arcsin \tfrac{d_1}{2} \leq \pi - \arcsin \tfrac{d_0}{2}\end{aligned}$$

and for a chord $p_i p_{i+1}$ $\arcsin \frac{\|p_i p_{i+1}\|}{2}$ or $\pi - \arcsin \frac{\|p_i p_{i+1}\|}{2}$ depending on the side of the origin with respect to $p_i p_{i+1}$. Since the angles associated to chords do not depend on a particular triangulation, they are not relevant when comparing the angles of two triangulations and the above correspondence between angles associated to diagonals and lengths of these diagonals allows to conclude. □

We will now show that ears are extremal for the lexicographical order of $DT(P)$.

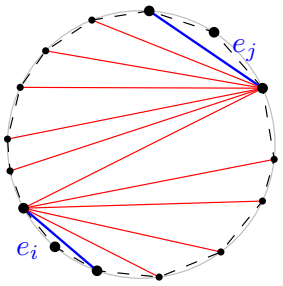


Lemma 5. *For any triangulation of a set of concyclic points, the smallest diagonal, d_0, is always an ear.*

Proof. Let d be a diagonal of the triangulation, then the two other edges of the triangle incident to d on the side of d that does not contain the origin are shorter than d. Thus if d is the shortest diagonal, these two edges are chords and d is an ear. □

We denote by e_i the ear $[p_{i-1}p_{i+1}]$.

Lemma 6. *Let e_i and e_j be two non-crossing ears. There exists a triangulation containing both e_i and e_j.*



Proof. The triangulation with diagonals

$$e_i, [p_{i-1}p_{i+2}], [p_{i-1}p_{i+3}], \ldots, [p_{i-1}p_{j-2}], [p_{i-1}p_{j-1}]$$

$$e_j, [p_{j-1}p_{j+2}], [p_{j-1}p_{j+3}], \ldots, [p_{j-1}p_{i-2}], [p_{j-1}p_{i-1}]$$

is such a triangulation. □

We can deduce strong structural properties about $DT(P)$ from the previous lemma.

Proposition 7. *If P is a set of concyclic points, then the dual of $DT(P)$ is a path.*

Proof. We first prove that $DT(P)$ has at most two ears. Let us consider a triangulation T_0 that contains three ears $\{e_i, e_j, e_k\}$ such that $|e_i| \geq |e_j| \geq |e_k|$. Let T be the triangulation given by Lemma 6 for the ears e_i and e_j. In T all the diagonals different from e_i and e_j have length strictly bigger than e_j. Then the list of all the diagonals sorted by length of T has the form $(e_j, d, \cdots)$ or $(e_j, e_i, d, \cdots)$ with $|d| > |e_j|$. However, for T_0 we have $(e_k, e_j, \cdots)$ or $(e_k, e_j, e_i, \cdots)$ that is strictly smaller than the list of T_0 for the lexicographic order. It means that T_0 cannot be $DT(P)$ and thus that $DT(P)$ has at most two ears. So, the dual of $DT(P)$ is a tree with exactly two leaves and and as such is a path. □

Let E be the set of all pairs of disjoint ears of P. An element of E is said to be maximal if its shortest ear has maximal length among all elements of E. Lemma 6 implies that, for any maximal element $\{e_i, e_j\}$ of E, there exists a triangulation which has as set of ears exactly e_i and e_j. This enables us to show the following.

Lemma 8. *$DT(P)$ has a maximal element of E as a subset.*

Proof. Assume that P does not contain four points in symmetric position. Let (e_i, e_j) be a maximal couple of E with e_i smaller than e_j.

Lemma 6 yields a triangulation whose smallest edge has length $|e_i|$. Lemma 5 implies that $DT(P)$ has an ear e_k as smallest diagonal. Let e_l be another ear of $DT(P)$. On the one hand, comparing the two triangulations, Lemma 4 gives that $|e_k| \geq |e_i|$ since $DT(P)$ is length maximal. On the other hand, comparing the two pair of ears $\{e_i, e_j\}$ and $\{e_k, e_l\}$, maximality of $\{e_i, e_j\}$ in E yields $|e_i| \geq |e_k|$. Thus $|e_k| = |e_i|$ and the two ears of $DT(P)$ form also a maximal pair of E. □

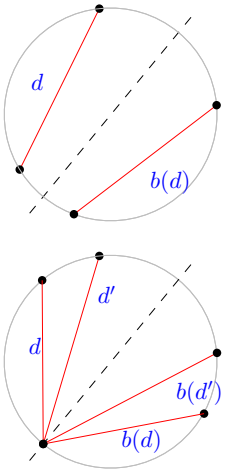
This lemma is the key of the construction of $DT(P)$. Amongst all ears, the longest one is not a good candidate because it can enforce shorter ear afterwards while the second (non intersecting) longest ear is always part of a good triangulation.

3 Uniqueness

We prove uniqueness in this section.

Proof of Theorem 1. Let P be a minimal set of concyclic points admitting two

distinct Delaunay triangulations without any symmetric quadruple of points. Let us assume that there exist two disjoint Delaunay triangulations of P, τ_1 and τ_2. By minimality of P they cannot share a diagonal. By Lemma 4, there is a length preserving bijective map b between the diagonals of τ_1 and τ_2. Let d be a diagonal of τ_1. Note that d and $b(d)$ must share a single point, otherwise their endpoints form a symmetric quadruple. In addition, if d' is another diagonal of τ_1, then d, $b(d)$, d' and $b(d')$ cannot all share the same point, again because this would create a symmetric quadruple. This implies that each pair is associated to a different point. However, the points inside ears of τ_1 or τ_2 cannot be one of these shared points. And since τ_1 and τ_2 have at least 2 ears each, those edges have to be different since the triangulations do not share any diagonals. This implies that at most $n-4$ points are the end points of these pairs. This contradicts the fact that there are $n-3$ pairs. Hence it is impossible to have two disjoint Delaunay triangulations of P and this completes the proof. □



4 A simplified algorithm

Lemma 8 suggests an algorithm. We need to find an ear that belongs to all the maximal pairs of E. We first describe a simplified version of the algorithm that works in the case where P does not admits two diagonals of the same length. This is a stronger condition not having any quadruple of points in symmetric position because it also forbids two diagonals of the same length in the event they share a vertex. The algorithm works as follows.

Consider a set of points P. We first compute the three longest ears of P. If the two longest ears of P are disjoint then we add the second longest to the output triangulation τ_o. Otherwise we add the third longest edge to τ_o. Let i be the index of the ear e_i that we just added to τ_o. Now, we remove p_i from P, and proceed inductively until we reach a pentagon where a brute force calculation can easily be done.

To prove that this algorithm has the correct output and to compute running time, we will need some notation. We denote by P_k the set of k points obtained after $n-k$ steps of the algorithm where $n \geq k \geq 5$. We relabel the remaining points from 0 to $k-1$. We denote by e_i^k the ear in position i in P_k. We denote by $(se_0^k, \cdots, se_{n-k}^k)$ the sorted list of the ears of P_k such that $se_0^k > \cdots > se_{n-k}^k$.

Proposition 9. *The triangulation obtained is the unique Delaunay triangulation:* τ_o=$DT(P)$.

Proof. We first observe that if there is an ear e_i belonging to $DT(P_k)$, then the restriction of $DT(P_k)$ to $P_k - p_i$ is $DT(P_k - p_i)$. We want to show that the chosen ear e_i^k at step k belongs to $DT(P_k)$. By Lemma 8, one of the maximal pair of $E(P_k)$ is in $DT(P_k)$. If se_0^k and se_1^k are non-crossing then (se_0^k, se_1^k) is the unique maximal pair of $E(P_k)$. If se_0^k and se_1^k cross then (se_0^k, se_2^k) and (se_1^k, se_2^k) are the two possible maximal pairs of $E(P_k)$ and at least one is non crossing. In the latter case, se_2^k is the only ear that surely belongs to $DT(P_k)$. This proves that in all cases, the simplified algorithm choose an ear that belongs to $DT(P_k)$. □

Incidentally, this proposition proves the uniqueness of $DT(P)$ when P does not admits two diagonals of the same length using a constructive proof.

Proposition 10. *The simplified algorithm runs using $O(n)$ arithmetic operations.*

Proof. We first compute the three longest ears of the input polygon on P_n. Since there are n ears, it can be computed using $O(n)$ operations. In other words, finding the first ear requires $O(n)$ operations. We want to show that the choices of subsequent ears only require a constant number of operations at each step. We need to update the list of the three longest ears.

Let p_i be the point of P_{k+1} removed at step $k+1$. Note that e_i cannot contain the origin of the circle since such an ear cannot appear as the smallest ear of a pair. The ears of P_k are almost the same as the ears of P_{k+1}. Actually, three ears disappear : $p_{i-2}p_i$, $p_{i-1}p_{i+1}$, and p_ip_{i+2} and two ears appear: $p_{i-2}p_{i+1}$ and $p_{i-1}p_{i+2}$. Since these two ears are longer than the chosen ear $p_{i-1}p_{i+1}$ and at least one of the three longest ear of P_{k+1} remains an ear of P_k we can guarantee that the three longest ear of P_k must be chosen in $p_{i-2}p_{i+1}$, $p_{i-1}p_{i+2}$ and the ears remaining amongst the three longest ear of P_{k+1}. Thus selecting these three ears is done in constant time. □

5 General Case

If we allow ears of equal length but no symmetric quadrilaterals, then two ears of equal length necessarily share a point (see Section 3) and thus there at most two of them (at most three if $n \leq 6$). Instead of the three longest ears, we have to use *all* the ears of the three longest possible lengths reachable by ears (l_0, l_1, l_2). We want to apply Lemma 8 to find an ear that belongs to $DT(P)$. We thus study the possible configurations for the maximal pairs of E. The goal is to find an ear that belongs to all maximal pairs or alternatively find a way to rank those pairs. Let us start with some easy cases.

Case 1: $|se_0| = |se_1| = l_0$**.** Here (se_0, se_1) is the unique maximal pair of ears. This

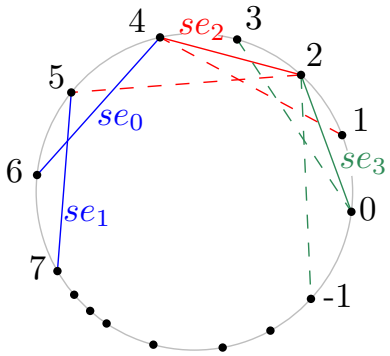
implies that d_0 and d_1 are in $DT(P)$.

Case 2: $|se_0| = l_0$**,** $|se_1| = |se_2| = l_1$**.** Hence (se_0, se_1), (se_0, se_2) and (se_1, se_2) are the only possible maximal pairs. If se_0 crosses both se_1 and se_2, then (se_1, se_2) is the unique maximal pair. If se_0 crosses only se_1 (resp. se_2) then (se_0, se_2) (resp. (se_0, se_1)) is the unique maximal pair and there is a canonical choice. In the remaining case, if we choose to include (se_0, se_1) we can find a triangulation where $|d_0| = l_1$ and $|d_1| > l_1$ using Lemma 6. This implies that (se_1, se_2) cannot be included in $DT(P)$ since it is strictly worse than any such triangulation. Thus $DT(P)$ contains se_0.

The last case occurs when $|se_0| = l_0$, $|se_1| = l_1$ and $|se_2| = |se_3| = l_2$. It will require looking at the next possible steps to decide between maximal pairs.

Lemma 11. *Let $n > 8$. When $|se_0| = l_0$, $|se_1| = l_1$ and $|se_2| = |se_3| = l_2$, we can always find a diagonal that belongs to $DT(P)$ by considering a constant number of diagonals.*

Proof. If se_0 and se_1 are non-crossing then we just choose se_1. We now assume that se_0 and se_1 cross. We choose the indices of the points such that $se_3 = e_1$ and $se_2 = e_3$. Since $n > 8$, it is not possible that se_0, se_1, $[p_1p_4]$, $[p_2p_5]$, $[p_{-1}p_2]$ and $[p_0p_3]$ contain three diagonals of the same length. We want to go a step further to choose the best choice between se_2 and se_3. The one leading to the best second diagonal leads to a better d_1 with the same d_0 and so must be in $DT(P)$.



The next choice of se_2 has to be made between $[p_1p_4]$, $[p_2p_5]$, se_0 and se_1 because all of the other ears are shorter than se_2. For se_3, we have $[p_{-1}p_2]$, $[p_0p_3]$, se_0 and se_1. Furthermore $[p_1p_4]$, $[p_2p_5]$, $[p_{-1}p_2]$ and $[p_0p_3]$ must have 4 different lengths since we forbid symmetric quadruples. If there is still an ambiguity, this means that the second possible diagonal must have the same length in the two cases. In addition, if the next diagonal chosen in both situations is se_0, then se_0 is in $DT(P)$. The same holds for se_1. It implies that the only ambiguous case left is when one of se_0 and se_1 has the same length as one of $[p_1p_4]$, $[p_2p_5]$, $[p_{-1}p_2]$ or $[p_0p_3]$. We call $event_i$ the remaining possibilities where we choose se_i as a first ear.

Case 1: $se_0 = e_4$ **and** $se_1 = e_5$ **or** $se_0 = e_5$ **and** $se_1 = e_4$**.** In $event_2$ the next choice is between $[p_1p_4]$, $[p_2p_5]$ and e_5 since e_4 is no longer reachable. Since $[p_2p_5]$ crosses the two others, the unique maximal pair is $([p_1p_4], e_5)$. If $[p_1p_4]$ is the

smallest then $event_3$ cannot reach the same length because $[p_1p_4]$ crosses e_4 and is smaller than e_5. If e_5 is the smallest, then $event_3$ has no other solution but to include e_5 as well (none of its eligible ears have a common point with e_5) and thus e_5 is in $DT(P)$. If they have the same length then e_5 is also in $DT(P)$.

Case 2: $se_0 = e_5$ and $se_1 = e_6$. In $event_3$, se_0 and se_1 are disjoint from $[p_{-1}p_2]$ and $[p_0p_3]$. This implies that $[p_{-1}p_2]$ and $[p_0p_3]$ have different lengths than se_0 and se_1. So, the only possible ambiguity is if the second choice is se_0 or se_1. Since a pair with se_1 is beaten by a pair with se_0, se_0 needs to be chosen. In $event_2$, we can have $|se_0| = |p_1p_4|$. However, in this case $(se_0, [p_1p_4])$ is the unique maximal pair and both ears are selected. So, in this case, se_0 is in $DT(P)$.

Case 3: $se_0 = e_6$. For the same reason as in the previous case, we can assume that $event_3$ selected se_0. In $event_2$, we can have now $|se_0| = |p_2p_5|$. If $(se_0, [p_2p_5])$ is the only maximal pair we are done. So we need to have $|p_1p_4| > |p_2p_5|$. In this case, $(se_0, [p_1p_4])$ is the only maximal pair and se_0 is in $DT(P)$ in both events.

Case 4: $se_0 = e_j$ with $j > 6$. se_1 cannot appear in any maximal pair since se_0 is always better so the only equality holds when both events use se_0 and in this case, we include se_0 in $DT(P)$.

By symmetry between se_2 and se_3, we have no more cases and this concludes the proof. □

The last case is for $n \leq 8$.

6 Extended Algorithm

Algorithm for triangulations without any symmetric quadruples. We want to extend the simplified algorithm of Section 4. First, we see from the last section that we need to consider the four longest ears instead of the three longest. The step that changes is the selection of the ear that we put in $DT(P)$ at each step. We describe it in detail. Firstly, if $n \leq 8$, when we have more than one possible ear to put in $DT(P)$, we just try all possibilities until one becomes strictly better than all the others. We assume that $n > 8$. Let (se_0, se_1, se_2, se_3) be the sorted list of the four longest ears of the current polygon. By considering the cases of the last section, we obtain the following procedure.

- If $|se_0| > |se_1| > |se_2| > |se_3|$ just apply the same rules as in the simplified version.
- Else if $|se_0| = |se_1|$, we put any one (or both) in $DT(P)$.

- Else if $|se_0| > |se_1| = |se_2|$ then we put se_0 in $DT(P)$.
- Else if $|se_0| > |se_1| > |se_2| = |se_3|$, apply:
  - If se_0 and se_1 are non-crossing, put se_1 in $DT(P)$.
  - Else look at the length of the next diagonal coming after the choice of se_2 or se_3.
    * If one is strictly longer, put the corresponding se_i in $DT(P)$.
    * If they are equal, then the length of the second edge must be the length of se_0 and se_1, so put the corresponding ear in $DT(P)$.

This algorithm works by Lemma 11 and runs in $O(n)$ time by Lemma 11 and 10.

Multiple output algorithm. We want to have an answer even in degenerate cases. We first describe an algorithm that outputs all the optimal triangulations. We proceed as follows: if we have many admissible ears, then we create as many triangulations as the number of admissible ears. We carry on constructing a tree of possibly optimal triangulations and we check at each layer which one is the best. If there is more than one possibility, we keep them all. We first consider the critical case of the regular polygons.

Lemma 12. *If P corresponds to a regular polygon then it admits $n \cdot 2^{n-5}$ triangulations as $DT(P)$.*

Proof. Let P be a set of n points forming a regular polygon contained in a circle of radius 1. We first prove that all the triangulations that are dual to a path have the same set of lengths and thus are optimal by Proposition 7. Let T be a triangulation whose dual is a path. A diagonal d have length $f(k) = 2\sin(\frac{k\pi}{n})$ where k is the number of chords on the smallest side of d. f is an increasing function of k since $k \in [0, \lfloor \frac{n}{2} \rfloor]$. An ear has length $f(1)$ and then the length of the next diagonal following the dual of T have length $f(2)$ and so on until we reach $f(\lfloor \frac{n}{2} \rfloor)$ (there is one or two of this length depending on the parity of n) and then the diagonals length decrease to $f(1)$. So, the sorted set of diagonal length of T is $(f(1), f(1), f(2), f(2), \cdots, f(\lfloor \frac{n}{2} \rfloor)(, f(\lfloor \frac{n}{2} \rfloor)))$ and is independent of T.

It remains to count the number of different triangulations dual to a path. Let us choose a random ear e. Then we have to choose the next diagonal after e and so on until we reach another ear, at each step, we can choose between two distinct edges. It means that we have 2^{n-4} triangulations containing e, since a triangulation has $n-3$ edges. There are n distinct possible choices for e but we construct all the triangulations exactly twice (one for each of its ears). We obtain $n \cdot 2^{n-4}/2 = n \cdot 2^{n-5}$ distinct triangulations. □

The complexity of the algorithm is directly linked to the number of possible $DT(P)$ and this number is controlled by the number of symmetric quadruples of P. More precisely:

Proposition 13. *A set of n concyclic points with less than k distinct symmetric quadruples admits h different Delaunay triangulations with $h = O(2^k)$. These triangulations can be enumerated in $O(nh)$ time.*

Proof. Let us assume that a decision made by our algorithm involves k_0 ears of the same length. Then, we have k_0 different ears of the same size such that all the pairs are crossing or disjoint (but not sharing a point) and so there are necessarily at least $k_0(k_0 - 1)$ symmetric quadruples. Now, at each step of the algorithm, we have a number of equivalent sets of diagonals and we want to find all the possibles extensions. Since all the sets have to be equivalent, we must always add ears of the same length to all the current constructions so $k_0(k_0 - 1)$ symmetric quadruples may be used to multiply the number of possible configurations by k_0. Until we reach $n \cdot 2^{n-5}$ possibilities the bigger number that we can obtain is 2^k by using each quadruples separately. It proves that $h = O(2^k)$.

To actually construct the triangulations, we only need a constant number of operations on each node of the tree of configurations. A tree as a linear number of nodes with respect to its number of leaves so the complexity of the algorithm is $O(nh)$. □

Single output algorithm. For computational applications, it is important to have an algorithm that build always the same triangulation on a given set of points P. Perturbations are not useful because if you have two triangulations with the sorted length of diagonals $(a, b, \cdots) < (a', c, \cdots)$ with $a = a'$ then a perturbation may perturb the lengths a and a' so that $a' > a$, yielding an incorrect result. If the frame of coordinates is fixed, we may use it to pick a unique triangulation amongst the optimal ones. Let $p_0 \in P$ be the smallest point for the lexicographic order and label the other points p_i in counterclockwise direction starting at p_0. Now we can choose to set an order on the tree constructed by the previous algorithm as follows: if we cannot decide between a set of ears $(e_i, e_j, \cdots)$, we order the list by label and we put the smallest e_i as the leftmost descendant of the previous node. The algorithm outputs the leftmost solution in $O(n \cdot 2^k)$ time where k is the number of symmetric quadruples of P. Note that it is not possible to only keep the leftmost descendant at each step since it can lead to a non optimal situation.

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References

[1] Pierre Alliez, Olivier Devillers, and Jack Snoeyink. Removing degeneracies by perturbing the problem or the world. *Reliable Computing*, 6:61–79, 2000. URL: http://hal.inria.fr/inria-00338566/, doi:10.1023/A:1009942427413.

[2] Olivier Devillers, Menelaos Karavelas, and Monique Teillaud. Qualitative symbolic perturbation: Two applications of a new geometry-based perturbation framework. *Journal of Computational Geometry*, 8(1):282–315, 2017. URL: https://hal.inria.fr/hal-01586511, doi:10.20382/jocg.v8i1a11.

[3] Olivier Devillers and Monique Teillaud. Perturbations for Delaunay and weighted Delaunay 3D triangulations. *Computational Geometry: Theory and Applications*, 44:160–168, 2011. URL: http://hal.archives-ouvertes.fr/inria-00560388/, doi:10.1016/j.comgeo.2010.09.010.

[4] H. Edelsbrunner. *Algorithms in Combinatorial Geometry*, volume 10 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Heidelberg, West Germany, 1987.

[5] I. Emiris and J. Canny. A general approach to removing degeneracies. *SIAM J. Comput.*, 24:650–664, 1995. doi:10.1137/S0097539792235918.

[6] R. Seidel. The nature and meaning of perturbations in geometric computing. *Discrete Comput. Geom.*, 19:1–17, 1998. doi:10.1007/PL00009330.

[7] Robin Sibson. Locally equiangular triangulations. *The computer journal*, 21(3):243–245, 1978. doi:10.1093/comjnl/21.3.243.

[8] C. K. Yap. Symbolic treatment of geometric degeneracies. *J. Symbolic Comput.*, 10:349–370, 1990. URL: http://www.sciencedirect.com/science/article/pii/S0747717108800697.