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Abstract

A new estimator for extreme quantiles is proposed under the log-generalized Weibull-tail model, introduced by (de Valk, C., *Extremes*, pp. 661–686, vol. 19, 2016). This model relies on a new regular variation condition which, in some situations, permits to extrapolate further into the tails than the classical assumption in extreme-value theory. The asymptotic normality of the estimator is established and its finite sample properties are illustrated both on simulated and real datasets.

Keywords: Extreme quantile, Extreme-value theory, Extended regular variation
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1. Introduction

Let X be a random variable with distribution function $F(\cdot) = \mathbb{P}(X \le \cdot)$ and survival function $S := 1 - F$. Starting from a $n-$sample from X, our goal is to estimate extreme quantiles from S of level $1 - \beta_n$ with $\beta_n \to 0$ as $n \to \infty$. Recall that a quantile of level $1 - \beta$ is given by $Q(\beta) := \inf\{y;\ S(y) \le \beta\}$. The rate of convergence of β_n to zero drives the difficulty of the estimation problem. Indeed, if $n\beta_n \to 0$ as $n \to \infty$, then $Q(\beta_n)$ is asymptotically almost surely larger than the sample maxima. In finance or insurance contexts, an extreme quantile is interpreted as the Value-at-Risk associated with an extreme loss, see [10, 19] for links between extreme-value theory and risk theory. In environmental applications, an extreme quantile coincides with the return level associated with an exceptional climatic event (extreme rainfalls [6], extreme wind velocities [16], extreme wave heights [18], river peak flows [17],...).

Dedicated methods have been designed to address the estimation of extreme quantiles, see [9, Chapter 6] or [15, Chapter 4], for an overview. Most of them rely on an extended regular variation assumption on the function Q. Recently, an alternative method has been initiated by Cees de Valk in a series of papers [21, 22], the goal being to estimate "more" extreme quantiles *i.e.* associated with sequences β_n tending to zero at a faster rate than in the previously mentioned approaches [9, 15]. The idea is to put the extended regular variation assumption on the function $V(\cdot) := \ln Q(1/\exp \cdot)$ rather than on $Q(\cdot)$, see Paragraph 1.1 for technical details and Paragraph 1.2

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for examples. Dedicated estimation methods are introduced in [23]. The goal of this work is to contribute to the popularity of this model by proposing alternative estimators, which are more efficient than the initial ones [23] in some situations.

1.1. Tail model

Let X be a random variable with survival function S. For the sake of simplicity, we assume in what follows that $S(1) = 1$ *i.e.* X is almost surely larger than 1. The tail model considered in this work is given by

$$S(x) = \exp(-V^{\leftarrow}(\ln x)),\ x \geq 1, \tag{1}$$

where $V^{\leftarrow}(\cdot) := \inf\{y;\ V(y) \geq \cdot\}$ is the generalized inverse of $V(\cdot) = \ln Q(1/\exp\cdot)$ with Q the quantile function. The function V is supposed to be of extended regular variation with index $\theta \in \mathbb{R}$. More specifically, there exists a positive function a (called the auxiliary function) such that

$$\lim_{x\to\infty} \frac{V(tx) - V(x)}{a(x)} = \int_1^t u^{\theta-1} du =: L_\theta(t),\ \text{for all } t > 0. \tag{2}$$

The class of extended regularly varying functions is denoted by $\mathcal{ERV}(\theta)$. Model (1) is referred to as the "log-generalized Weibull-tail model" [21, 22, 23]. From [15, Corollary 1.1.10], a sufficient condition for (2) is

(A1) V is differentiable with derivative V' satisfying

$$\lim_{x\to\infty} \frac{V'(tx)}{V'(x)} = t^{\theta-1}. \tag{3}$$

Such a function V' is said to be regularly varying with index $\theta - 1$ and this property is denoted by $V' \in \mathcal{RV}(\theta - 1)$. We refer to [5] for a general account on regular variation theory. Moreover, under **(A1)**, a possible choice of auxiliary function in (2) is $a(x) = xV'(x)$.

1.2. Properties and examples

Condition **(A1)** generalizes the tail model introduced in [8, 12] where it is assumed that the function V in (2) is asymptotically proportional to L_τ for some $\tau \in [0, 1]$. One can then easily show that such a tail parameter τ coincides with the index θ of extended regular variation in the situation where $\theta \in [0, 1]$. In terms of Maximum Domain of Attraction (MDA), the following result has been established in [2, Proposition 4]:

Lemma 1. *Assume F is twice differentiable.*

(i) If **(A1)** *holds with $\theta < 1$ then $F \in$ MDA(Gumbel).*
(ii) If $F \in$ MDA(Fréchet) then **(A1)** *holds with $\theta = 1$.*
(iii) If **(A1)** *holds with $\theta > 1$ then F does not belong to any MDA.*

It thus appears that model **(A1)** with $\theta \leq 1$ is of particular interest since it is associated with most distributions in MDA(Gumbel) $\cup$ MDA(Fréchet). The situation $\theta > 1$ which does not correspond to any domain of attraction is sometimes referred to as super-heavy tails, see for instance [3]. The following examples are taken from [2, Proposition 3]:

Example 1. *Let $x^* := \sup\{x \geq 1,\ F(x) < 1\}$ be the endpoint of F. Then, under some monotonicity assumptions:*

(i) *If* $V^{\leftarrow}(\ln\cdot) \in \mathcal{RV}(1/\beta)$, $\beta > 0$, *then* **(A1)** *holds with* $\theta = 0$. *In this case,* F *is referred to as a Weibull tail-distribution, see for instance [4, 11, 14]. Such distributions encompass Gaussian, Gamma, Exponential and strict Weibull distributions.*

(ii) $V^{\leftarrow} \in \mathcal{RV}(1/\beta)$, $0 < \beta < 1$ *if and only if* **(A1)** *holds with* $\theta = \beta > 0$. *Here,* F *is called a log-Weibull tail-distribution, see [3, 8, 12], the most popular example being the lognormal distribution.*

(iii) $1 \le x^* < \infty$ *and* $V^{\leftarrow}(\ln x^* + \ln(1-1/\cdot)) \in \mathcal{RV}_{-1/\beta}$, $\beta < 0$ *if and only if* **(A1)** *holds with* $\theta = \beta < 0$. *This case corresponds to distributions with a Weibull tail behavior in the neighborhood of a finite endpoint.*

We also refer to Table 1 for examples of distributions corresponding to the three above families: $\theta = 0$, $\theta > 0$ and $\theta < 0$.

1.3. Outline

The inference aspects associated with model (1) are examined in Section 2: Estimators for extreme quantiles are introduced as well as estimators for the extended regular variation index θ and the auxiliary function a. The asymptotic distributions of these estimators are established in Section 3. Their finite sample performance are investigated in Section 4 on simulated data and compared to the proposals introduced in [23]. Finally, an illustration on real data is presented in Section 5. Proofs are postponed to Section 6.

2. Inference

Let $X_1, \dots, X_n$ be n independent copies of a random variable X distributed as in (1). The associated ordered statistics are denoted by $X_{1,n} \le \dots \le X_{n,n}$ throughout the paper. Starting from this random sample, we focus on the estimation of extreme quantiles *i.e.* $Q(u) := S^{\leftarrow}(u) = \exp(V(\ln(1/u)))$ when $u \to 0$. Two situations for the level u are considered.

Intermediate case. If $u = \alpha_n$ where α_n is an intermediate level satisfying $\alpha_n \to 0$ and $n\alpha_n \to \infty$ as $n \to \infty$, a natural estimator is obtained by replacing Q by its empirical counterpart $\hat{Q}_n$. More precisely, $Q(\alpha_n)$ is estimated by $\hat{Q}_n(\alpha_n) = X_{n-\lfloor n\alpha_n \rfloor,n}$.

Extreme case. If $u = \beta_n$ where β_n is an extreme level such that $n\beta_n \to c \ge 0$ as $n \to \infty$, a simple order statistic cannot be used. Extrapolation beyond the sample should be performed. Starting from an intermediate level $\alpha_n := k_n/n$ where $k_n \to \infty$ and $k_n/n \to 0$, we propose to estimate $Q(\beta_n)$ by

$$\check{Q}_n(\beta_n) := X_{n-k_n,n} \exp\left(\hat{a}_n(\ln(n/k_n)) L_{\hat{\theta}_n}\left(\frac{\ln \beta_n}{\ln(k_n/n)}\right)\right), \tag{4}$$

where $\hat{\theta}_n$ and $\hat{a}_n(\ln(n/k_n))$ are suitable estimators of θ and $a(\ln(n/k_n))$. The rationale behind (4) is based on (2) which basically means that for α close to 0 and for all $t > 0$,

$$\ln Q(t\alpha) \approx \ln Q(\alpha) + a(\ln(1/\alpha)) L_\theta\left(1 + \frac{\ln(t)}{\ln(\alpha)}\right).$$

Estimator (4) is then obtained by taking $\alpha = k_n/n$ and $t = n\beta_n/k_n$ and by replacing the unknown quantities $Q(k_n/n)$, $a(\ln(n/k_n))$ and θ by their corresponding estimators. Since k_n/n is an intermediate level, $Q(k_n/n)$ is estimated by $\hat{Q}_n(k_n/n) = X_{n-k_n,n}$.

	$\bar{F}(x)$	θ	ρ	ρ'	ρ''
$\boldsymbol{\theta = 0}$					
Gamma $(a > 0, s > 0)$	$\frac{1}{s^a \Gamma(a)} \int_x^\infty t^{a-1} e^{-t/s} dt$, $x \geq 0$	0	-1	0	-1
Weibull $(k \neq 1, \lambda > 0)$	$e^{-(x/\lambda)^k}$, $x \geq 0$	0	$-\infty$	0	-1
Gaussian $(\mu \in \mathbb{R}, \sigma > 0)$	$\frac{1}{\sigma\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2\right) dt$, $x \in \mathbb{R}$	0	-1	0	-1
$\boldsymbol{\theta > 0}$					
Lognormal $(\mu \in \mathbb{R}, \sigma > 0)$	$\frac{1}{\sigma\sqrt{2\pi}} \int_x^\infty \frac{1}{t} \exp\left(-\frac{(\ln t - \mu)^2}{2\sigma^2}\right) dt$, $x \geq 0$	$1/2$	-1	-1	-1
Burr $(\lambda > 0, c > 0, k > 0)$	$\left(1 + \left(\frac{x}{\lambda}\right)^c\right)^{-k}$, $x \geq 0$	1	$-\infty$	-1	-2
Pareto-like	$1/U^{\leftarrow}(x)$, $U(x) = x\left(1 + 2\ln^2(x)\right)$	1	-1	-1	-1
Super heavy-tail	$e^{-\ln^{1/2}(x)}$, $x \geq 1$	2	$-\infty$	$-\infty$	$-\infty$
$\boldsymbol{\theta < 0}$					
Finite endpoint $(x^* > 0)$	$\exp\left(-\frac{1}{\ln x^* - \ln x}\right)$, $x \in (0, x^*)$	-1	$-\infty$	-1	-2

Table 1: Examples of distributions verifying **(A1)** and **(A2)** with associated values of θ, ρ, ρ' and $\rho'' := \max(\rho, \rho' - 1)$, see (13).

Parameters estimation. Let us now propose new estimators of θ and $a(\ln(n/k_n))$. To this end, for $j \in \{1,2\}$, consider the statistic

$$M_n^{(j)} := \frac{1}{k_n} \sum_{i=0}^{k_n-1} \left(\ln_2(X_{n-i,n}) - \ln_2(X_{n-k_n,n})\right)^j,$$

where $\ln_2 := \ln\ln$, as well as the functions

$$\mu_b(x,\zeta) := \int_0^1 L_\zeta^b\left(1 + \frac{\ln(1/s)}{x}\right) ds \text{ and } \Psi_x(\zeta) := \frac{\mu_1^2(x,\zeta)}{\mu_2(x,\zeta)},$$

defined for $x > 0$, $b \in \mathbb{N} \setminus \{0\}$ and $\zeta < 1$. Let us mention that $\mu_1(x,0) = \mathrm{e}^x E_1(x)$ where $E_1(x) := \int_x^\infty u^{-1}\mathrm{e}^{-u} du$ is the exponential integral, see for instance [1, eq 5.1.1]. Furthermore, it can be shown (see Lemma 5) that Ψ_x is a decreasing function, at least for x large enough, and thus its generalized inverse $\Psi_x^{\leftarrow}$ is well defined for x large enough. The following statistics are then introduced:

$$\hat\theta_{n,+}^{(M)} := \frac{M_n^{(1)}}{\mu_1(\ln(n/k_n),0)}, \tag{5}$$

$$\hat\theta_{n,-}^{(M)} := \Psi_{\ln(n/k_n)}^{\leftarrow}\left(\frac{(M_n^{(1)})^2}{M_n^{(2)}}\right), \tag{6}$$

$$\hat\theta_n^{(M)} := \hat\theta_{n,+}^{(M)} + \hat\theta_{n,-}^{(M)} \text{ and} \tag{7}$$

$$\hat a_n^{(M)}(\ln(n/k_n)) := \frac{\ln X_{n-k_n,n}}{\mu_1(\ln(n/k_n),\hat\theta_{n,-}^{(M)})} M_n^{(1)}. \tag{8}$$

We conclude this section by giving the main ideas leading to the estimators (7) and (8) of respectively θ and $a[\ln(n/k_n)]$. The estimator (7) is similar in spirit to the moment estimator introduced in [7]. Its construction is based on the following two results. Letting $\theta_+ := \theta \vee 0$ and $\theta_- := \theta \wedge 0$, for any increasing function $V \in \mathcal{ERV}(\theta)$,

$$\lim_{x\to\infty} \frac{V(x)}{a(x)} \ln\frac{V(tx)}{V(x)} = L_{\theta_-}(t), \tag{9}$$

locally uniformly in $(0,\infty)$, see [15, Lemma B.3.16]. Moreover, one has (see for instance [15, Eq. 3.5.5]),

$$\lim_{x\to\infty}\frac{a(x)}{V(x)} = \theta_+.$$

Plugging $x := \ln(1/\alpha)$ and $t := 1 + \ln(s)/\ln(\alpha)$ in (9) yields the approximation

$$\ln_2 Q(s\alpha) - \ln_2 Q(\alpha) \approx \theta_+ L_0\left(1 + \frac{\ln s}{\ln\alpha}\right), \tag{10}$$

as $\alpha \to 0$ and for all $s \in (0,1)$. Integrating with respect to s on $(0,1)$ leads to

$$\int_0^1 \ln_2 Q(s\alpha) - \ln_2 Q(\alpha) ds \Big/ \int_0^1 L_0\left(1 + \frac{\ln s}{\ln\alpha}\right) ds \approx \theta_+.$$

Considering $\alpha = k_n/n$ where k_n is an intermediate sequence such that $k_n \to \infty$ and $k_n/n \to 0$ and replacing Q by its empirical estimator $\hat Q_n$ lead to the estimator (5) of θ_+. Similarly, remark that (10) leads to the approximation

$$\left(\int_0^1 \ln_2 Q(s\alpha) - \ln_2 Q(\alpha) ds\right)^2 \Big/ \int_0^1 \left(\ln_2 Q(s\alpha) - \ln_2 Q(\alpha)\right)^2 ds \approx \Psi_{\ln(1/\alpha)}(\theta_-),$$

as $\alpha \to 0$. Replacing again in the previous approximation α by k_n/n and Q by its empirical counterpart suggests to estimate θ_- by (6). Finally, estimator (8) is obtained by remarking that, from (10):

$$\frac{\ln Q(\alpha)}{a(\ln(1/\alpha))} \int_0^1 \ln \frac{\ln Q(s\alpha)}{\ln Q(\alpha)} ds \approx \mu_1(\ln(1/\alpha), \theta_-),$$

for α close to 0. Replacing α by k_n/n, Q by $\hat{Q}_n$ and θ_- by $\hat{\theta}_{n,-}^{(M)}$ gives (8).

3. Main results

3.1. Quantile estimation: Intermediate case

Let us first focus on the asymptotic behavior of the quantile estimator in the intermediate case.

Theorem 1. *Under model (1), assume that* **(A1)** *holds. For all intermediate level α_n (i.e. such that $\alpha_n \to 0$ and $n\alpha_n \to \infty$ as $n \to \infty$), one has*

$$\frac{(n\alpha_n)^{1/2} \ln(1/\alpha_n)}{a(\ln(1/\alpha_n))} \ln \left(\frac{\hat{Q}_n(\alpha_n)}{Q(\alpha_n)} \right) \xrightarrow{d} \mathcal{N}(0, 1).$$

First, remark that introducing $k_n = \lfloor n\alpha_n \rfloor$ and choosing $a(t) = tV'(t)$ (see Paragraph 1.1), the above asymptotic normality result can be rewritten as

$$\frac{k_n^{1/2}}{V'(\ln(n/k_n))} \ln \left(\frac{\hat{Q}_n(k_n/n)}{Q(k_n/n)} \right) \xrightarrow{d} \mathcal{N}(0, 1).$$

If, moreover,

$$k_n^{1/2}/V'(\ln(n/k_n)) \to \infty \text{ as } n \to \infty, \tag{11}$$

then

$$\frac{k_n^{1/2}}{(n/k_n)U'(n/k_n)} \left(\hat{Q}_n(k_n/n) - Q(k_n/n) \right) \xrightarrow{d} \mathcal{N}(0, 1),$$

where $U(\cdot) = Q(1/\cdot)$ is the tail quantile function. This result coincides with [15, Theorem 2.2.1] established under a von Mises' condition for the maximum domain of attraction of an extreme-value distribution. Clearly, (11) holds when $\theta < 1$ since, in this case, $V'(t) \to 0$ as $t \to \infty$. Moreover, if $F \in$ MDA(Fréchet) then $\theta = 1$ from Lemma 1(ii) and $U \in \mathcal{RV}(\gamma)$ for some $\gamma > 0$. It thus follows that $V'(\ln t) = tU'(t)/U(t) \to \gamma$ as $t \to \infty$ and (11) is verified. The case $\theta > 1$ is not relevant here, since, in this case, F does not belong to any domain of attraction, see Lemma 1(iii).
Second, under additional conditions,

$$\frac{\hat{a}_n^{(M)}(\ln(1/\alpha_n))}{a(\ln(1/\alpha_n))} \xrightarrow{\mathbb{P}} 1,$$

see Theorem 4 below, and thus

$$\frac{(n\alpha_n)^{1/2} \ln(1/\alpha_n)}{\hat{a}_n^{(M)}(\ln(1/\alpha_n))} \ln \left(\frac{\hat{Q}_n(\alpha_n)}{Q(\alpha_n)} \right) \xrightarrow{d} \mathcal{N}(0, 1)$$

which provides a way for constructing asymptotic confidence intervals for intermediate quantiles $Q(\alpha_n)$ based on $\hat{Q}_n(\alpha_n)$. Letting u_ζ the $(1+\zeta)/2$th quantile from a standard Gaussian distribution,

$$\left[\hat{Q}_n(\alpha_n) \exp\left(-\frac{\hat{a}_n^{(M)}(\ln(1/\alpha_n))}{(n\alpha_n)^{1/2} \ln(1/\alpha_n)} u_\zeta \right) ; \hat{Q}_n(\alpha_n) \exp\left(\frac{\hat{a}_n^{(M)}(\ln(1/\alpha_n))}{(n\alpha_n)^{1/2} \ln(1/\alpha_n)} u_\zeta \right) \right]$$

is an asymptotic confidence interval for $Q(\alpha_n)$ of confidence level ζ.

3.2. Quantile estimation: Extreme case

Our next goal is to establish the asymptotic normality of $\check{Q}_n(\beta_n)$ for an extreme level β_n satisfying $n\beta_n \to c \geq 0$. A second-order condition is needed on $V \in \mathcal{ERV}(\theta)$ to control the rate of convergence in (2):

(A2) There exist a function $\tilde{A}$ with $\tilde{A}(x) \to 0$ as $x \to \infty$ and $\rho < 0$ such that for all $t > 0$,

$$\lim_{x\to\infty} \frac{1}{\tilde{A}(x)} \left(\frac{V(tx) - V(x)}{a(x)} - L_\theta(t) \right) = H_{\theta,\rho}(t) := \int_1^t u^{\theta-1} L_\rho(u) du.$$

locally uniformly for $t > 0$.

Note that **(A2)** also provides the rate of convergence in (9). Indeed, from [15, Lemma B.3.16], condition **(A2)** with $\theta \neq \rho$ entails that there exists a function A with $A(x) \to 0$ as $x \to \infty$ such that

$$\lim_{x\to\infty} \frac{1}{A(x)} \left(\frac{V(x)}{a(x)} \ln \frac{V(tx)}{V(x)} - L_{\theta_-}(t) \right) = H_{\theta_-,\rho'}(t). \tag{12}$$

The function $|A|$ is regularly varying with index $\rho' \leq 0$ where, according to [15, Lemma B.3.16],

$$\rho' = \begin{cases} \rho & \text{if } \theta < \rho, \\ \theta & \text{if } \rho < \theta \leq 0, \\ -\theta & \text{if } (0 < \theta < -\rho \text{ and } l \neq 0), \\ \rho & \text{if } (0 < \theta < -\rho \text{ and } l = 0) \text{ or } (\theta \geq -\rho), \end{cases} \tag{13}$$

with, for $\theta > 0$,

$$l := \lim_{x\to\infty} \left(V(x) - \frac{a(x)}{\theta} \right).$$

Let us also introduce the positive function B defined by $B(x) := \max(|\tilde{A}(x)|, |A(x)|/x)$. It is easily checked that B is regularly varying with index $\rho'' := \max(\rho, \rho' - 1)$. We are now in position to establish the asymptotic distribution of $\check{Q}_n(\beta_n)$ for general estimators of θ and $a(\ln(n/k_n))$ satisfying the condition:

(A3) There exist a sequence $\sigma_n \to 0$ and a random vector (B, Θ, Λ) such that

$$\sigma_n^{-1} \left\{ \frac{\ln X_{n-k_n,n} - \ln Q(k_n/n)}{a(\ln(n/k_n)) H_{\theta,0}(d_n)}, \hat{\theta}_n - \theta, \frac{L_\theta(d_n)}{H_{\theta,0}(d_n)} \left(\frac{\hat{a}_n(\ln(n/k_n))}{a(\ln(n/k_n))} - 1 \right) \right\} \xrightarrow{d} (\Omega, \Theta, \Lambda),$$

where $d_n := \ln(1/\beta_n)/\ln(n/k_n)$.

Theorem 2. *Under model (1), assume conditions* **(A2)**, **(A3)** *hold. Let (k_n) and (β_n) be two sequences such that $n\beta_n \to c \geq 0$, $k_n \to \infty$, $k_n/n \to 0$, $d_n \to d \in [1, \infty]$, $\sigma_n \ln(d_n) \to 0$ and $\sigma_n^{-1} \tilde{A}(\ln(n/k_n)) \to 0$ as $n \to \infty$. Then,*

$$\frac{\sigma_n^{-1}}{a(\ln(n/k_n)) H_{\theta,0}(d_n)} \ln \left(\frac{\check{Q}_n(\beta_n)}{Q(\beta_n)} \right) \xrightarrow{d} \Omega + \Theta + \Lambda.$$

Under the conditions of Theorem 2, three situations can arise for the extreme quantile level β_n. The first one is when $d_n \to 1$ which corresponds to the least extreme case. This condition is achieved for instance when $n\beta_n \to c > 0$ and $\ln(k_n)/\ln(n) \to 0$ as $n \to \infty$. In this situation, a Taylor expansion yields

$$H_{\theta,0}(d_n) \overset{d_n \to 1}{\sim} (d_n - 1)^2/2 \to 0. \tag{14}$$

The second case corresponds to the situation where $d_n \to d \in (1, \infty)$. Here,

$$H_{\theta,0}(d_n) \overset{d_n \to d}{\longrightarrow} H_{\theta,0}(d) > 0. \tag{15}$$

Note that for these two situations, $\sigma_n \ln(d_n) \to 0$ is a consequence of the assumption $\sigma_n \to 0$ as $n \to \infty$. Finally, the most extreme case occurs when $d_n \to \infty$ leading to

$$H_{\theta,0}(d_n) \overset{d_n \to \infty}{\sim} \begin{cases} d_n^\theta \ln(d_n)/\theta & \text{if } \theta > 0, \\ \ln^2(d_n)/2 & \text{if } \theta = 0, \\ 1/\theta^2 & \text{if } \theta < 0. \end{cases} \tag{16}$$

As expected, the rate of convergence in Theorem 2 is getting worse when the quantile level β_n is getting more extreme. Let us also highlight that, when $\theta < 0$, the rates of convergence in situations $d_n \to d > 1$ and $d_n \to \infty$ are of the same order.
To conclude this section, let us give the following consistency result.

Proposition 1. *Under the conditions of Theorem 2,*

$$\frac{\hat{a}_n(\ln(n/k_n))}{a(\ln(n/k_n))} \xrightarrow{\mathbb{P}} 1 \text{ and } \frac{H_{\hat{\theta}_n,0}(d_n)}{H_{\theta,0}(d_n)} \xrightarrow{\mathbb{P}} 1. \tag{17}$$

and therefore

$$\frac{\sigma_n^{-1}}{\hat{a}_n(\ln(n/k_n))H_{\hat{\theta}_n,0}(d_n)} \ln\left(\frac{\check{Q}_n(\beta_n)}{Q(\beta_n)}\right) \xrightarrow{d} \Omega + \Theta + \Lambda.$$

Proposition 1 can be used to construct asymptotic confidence intervals for extreme quantiles $Q(\beta_n)$ based on $\check{Q}_n(\beta_n)$, see (19) below.

3.3. Parameters estimation

First, the asymptotic distribution of the estimator of θ proposed in (7) is provided.

Theorem 3. *Under model (1), assume that condition* **(A2)** *holds with $\theta \neq \rho$. Let (k_n) be a sequence such that $k_n/\ln^2(n) \to \infty$, $k_n/n \to 0$ and $k_n A^2(\ln(n/k_n))/\ln^2(n/k_n) \to 0$ as $n \to \infty$. Then,*

$$\frac{k_n^{1/2}}{\ln(n/k_n)} \left(\hat{\theta}_n^{(M)} - \theta\right) \xrightarrow{d} \mathcal{N}(0, 1).$$

It is shown in the proof of Theorem 3 that the negative part $\hat{\theta}_{n,-}^{(M)}$ of the estimator converges slower than the positive part $\hat{\theta}_{n,+}^{(M)}$, see (25) and (27). As a consequence, $\hat{\theta}_n^{(M)}$ inherits its asymptotic normality from $\hat{\theta}_{n,-}^{(M)}$. This phenomenon can be explained by the fact that $\hat{\theta}_{n,-}^{(M)}$ is obtained through the inversion of the function $\Psi_{\ln(n/k_n)}$. The rate of convergence of $\hat{\theta}_{n,-}^{(M)}$ thus depends on the first derivative of $\Psi_{\ln(n/k_n)}$ which converges to 0 as $n \to \infty$, see Lemma 5(ii). Note also that from [13, Lemma 1], condition $k_n A^2(\ln(n/k_n))/\ln^2(n/k_n) \to 0$ implies $\ln(k_n)/\ln(n) \to 0$ as $n \to \infty$ and thus $\ln(n/k_n) \sim \ln(n)$. Second, the asymptotic distribution of the estimator of $a(\ln(n/k_n))$ proposed in (8) is established in the following theorem.

Theorem 4. *Under model (1), assume that condition* **(A2)** *holds with $\theta \neq \rho$. Let (k_n) be a sequence such that $k_n/\ln^2(n) \to \infty$, $k_n/n \to 0$ and $k_n B^2(\ln(n/k_n)) \to 0$ as $n \to \infty$. Then,*

$$k_n^{1/2} \left(\frac{\hat{a}_n^{(M)}(\ln(n/k_n))}{a(\ln(n/k_n))} - 1\right) \xrightarrow{d} \mathcal{N}(0, 2).$$

Note that, if $\rho > -1$ and $k_n \tilde{A}^2(\ln(n)) \to 0$, then $k_n/\ln^2(n) \to 0$. Hence, Theorem 4 does not apply when $\rho \in (-1,0)$. Let us stress that this limitation also appears in [8, Theorem 1]. As a consequence of Theorems 1 – 4, the asymptotic normality of the extreme quantile estimator $\check{Q}_n^{(M)}(\beta_n)$ is obtained by considering $\hat{\theta}_n = \hat{\theta}_n^{(M)}$ and $\hat{a}_n(\ln(n/k_n)) = \hat{a}_n^{(M)}(\ln(n/k_n))$ in (4).

Corollary 1. *Under model (1), assume that* **(A2)** *holds with $\theta \neq \rho$. Let (k_n) and (β_n) be two sequences such that $n\beta_n \to c \geq 0$, $k_n/n \to 0$, $k_n B^2(\ln(n/k_n)) \to 0$, $d_n \to d \in [1,\infty]$ and $(\ln(n)\max(1,\ln(d_n)))^2/k_n \to 0$ as $n \to \infty$. Then,*

$$\frac{k_n^{1/2}/\ln(n/k_n)}{a(\ln(n/k_n))H_{\theta,0}(d_n)} \ln\left(\frac{\check{Q}_n^{(M)}(\beta_n)}{Q(\beta_n)}\right) \xrightarrow{d} \mathcal{N}(0,1).$$

The proof consists in showing that the estimators $\hat{\theta}_n^{(M)}$ and $\hat{a}_n^{(M)}(\ln(n/k_n))$ satisfy condition **(A3)** with $(\Omega,\Theta,\Lambda) = (0,\Theta,0)$ where Θ is a standard Gaussian random variable. Hence, in this situation, only the estimator of θ contributes to the asymptotic distribution of $\check{Q}_n^{(M)}(\beta_n)$. It appears that $\ln(n/k_n)a(\ln(n/k_n))H_{\theta,0}(d_n)/k_n^{1/2} \to 0$ is a sufficient condition to ensure that $\check{Q}_n^{(M)}(\beta_n)$ is a relatively consistent estimator of $Q(\beta_n)$, *i.e.* such that $\check{Q}_n^{(M)}(\beta_n)/Q(\beta_n) \xrightarrow{\mathbb{P}} 1$. Recalling that $\ln(n/k_n) \sim \ln n$ as $n \to \infty$, that $B \in \mathcal{RV}(\rho'')$ and $a \in \mathcal{RV}(\theta)$, we end up with a set of three conditions on the sequences (k_n) and (β_n): $k_n B^2(\ln n) \to 0$, $(\ln(n)\max(1,\ln(d_n)))^2/k_n \to 0$ and $(\ln(n)a(\ln n)H_{\theta,0}(d_n))^2/k_n \to 0$ as $n \to \infty$. Let us illustrate how these conditions may limit the extrapolation range β_n depending on the index θ of extended regular variation in three situations:

- Let $\beta_n = c/n$, $c \in (0,1)$. Here $d_n \to 1$ as $n \to \infty$, this is the least extreme case considered in Subsection 3.2, and, in view of (14), $H_{\theta,0}(d_n) \sim (\ln(k_n)/\ln(n))^2/2$. Two constraints arise on the distribution parameters: $\rho \leq -1$ and $\theta \leq 2 - \rho''$. The first one, $\rho \leq -1$, was already imposed by Theorem 4. The second one is fulfilled as soon as $\theta \leq 2$ including MDA(Fréchet), see Lemma 1(ii), finite endpoint, Weibull-tail, log-Weibull tail distributions defined in Example 1 and some super-heavy tail distributions. As an example, all distributions of Table 1 satisfy the above constraints.

- Let $\beta_n = n^{-\tau}$, $\tau > 1$. Here, $d_n = \tau$, this is the second extreme case considered in Subsection 3.2, and, as a particular case of (15), $H_{\theta,0}(d_n)$ is constant. The constraints are: $\rho \leq -1$ and $\theta \leq -1 - \rho''$. The condition on θ is fulfilled by finite endpoint distributions of Example 1(iii), Weibull-tail distributions (Example 1(i)) and some log-Weibull tail distributions (Example 1(ii)). In MDA(Fréchet), $\theta = 1$ and thus the condition on the second order parameters is strengthened: $\rho \leq -2$ and $\rho' \leq -1$. As an example, in Table 1, Lognormal and Pareto-like distributions do not satisfy the above constraints.

- Let $\beta_n = \exp(-cn)$, $c > 0$. Here $d_n \to \infty$ as $n \to \infty$, this is the most extreme case considered in Subsection 3.2. In view of (16), three subcases have to be considered. If $\theta < 0$ then $H_{\theta,0}(d_n)$ is asymptotically constant and the conditions are $\rho \leq -2$ and $\rho' \leq -1$. If $\theta = 0$ then necessarily $\rho' = 0$ in view of (13), $H_{\theta,0}(d_n) \sim (\ln n)^2/2$ and it is not possible to find sequences satisfying the constraints. If $\theta > 0$ then $H_{\theta,0}(d_n) \sim (c^\theta/\theta)n^\theta(\ln n)^{1-\theta}$ and it is not possible either to find sequences satisfying the constraints.

It thus appears that only the first two cases $\beta_n = c/n$, $c \in (0,1)$ and $\beta_n = n^{-\tau}$, $\tau > 1$ are of practical interest. The third situation $\beta_n = \exp(-cn)$, $c > 0$ can be addressed only when $\theta < 0$, *i.e.*

for finite endpoint distributions. For such distributions, the estimation of very extreme quantiles boils down to estimating the endpoint. In the first two cases, a possible choice of the intermediate sequence when $\rho'' < -1$ is $k_n = (\ln n)^{-2\rho''-\varepsilon}$ where $\varepsilon > 0$ is arbitrarily small. Moreover, in the second case where $\beta_n = n^{-\tau}$, $\tau > 1$, it is possible to compare the asymptotic standard deviation of $\ln \check{Q}_n^{(M)}(\beta_n)$, denoted by σ_n, to the one associated with the estimator introduced in [23], denoted by σ_n'. Our Corollary 1 and [23, Corollary 2] yield:

$$\begin{aligned} \sigma_n &\sim H_{\theta,0}(\tau) k_n^{-1/2} (\ln n) a(\ln n) \text{ and} \\ \sigma_n' &\sim (L_\theta^2(\tau) + H_{\theta,0}^2(\tau))^{1/2} k_n^{-1/2} (\ln n) a(\ln n). \end{aligned}$$

Consequently, the asymptotic standard deviations are equivalent up to a multiplicative constant:

$$\frac{\sigma_n}{\sigma_n'} \to \left(1 + \frac{L_\theta^2(\tau)}{H_{\theta,0}^2(\tau)}\right)^{-1/2} =: \Lambda_\theta(\tau) \leq 1 \text{ as } n \to \infty. \tag{18}$$

The behavior of $\Lambda_\theta(\tau)$ with respect to θ and τ is illustrated on Figure 1. It appears that $\Lambda_\theta(\tau)$ is an increasing function of τ and θ. As expected $\Lambda_\theta(\tau) \leq 1$ meaning that $\check{Q}_n^{(M)}(\beta_n)$ is asymptotically more efficient than [23]'s competitor, especially when θ is small.

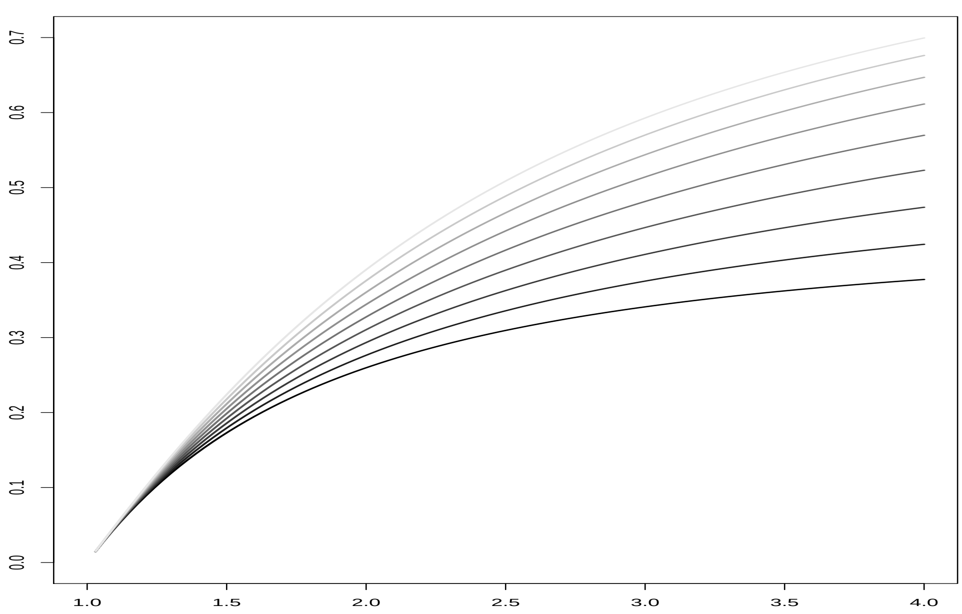

Figure 1: Ratio $\Lambda_\theta(\tau)$ between the asymptotic standard deviations σ_n and σ_n' (see equation (18)) as a function of $\tau \geq 1$ for $\theta \in \{-2, -1.5, \ldots, 2\}$. Dark lines are associated with small values of θ.

Finally, in view of Proposition 1, the unknown quantities $H_{\theta,0}(d_n)$ and $a(\ln(n/k_n))$ can be replaced by their corresponding estimators $H_{\hat{\theta}_n^{(M)},0}(d_n)$ and $\hat{a}_n^{(M)}(\ln(n/k_n))$ without changing the asymptotic distribution in Corollary 1. As mentioned before, the obtained result can then lead to asymptotic confidence intervals. Letting u_ζ the $(1+\zeta)/2$th quantile from a standard Gaussian distribution,

$$\check{Q}_n^{(M)}(\beta_n)\left[\exp\left(-\frac{\hat{a}_n^{(M)}(\ln(n/k_n))H_{\hat{\theta}_n^{(M)},0}(d_n)}{k_n^{1/2}/\ln(n/k_n)}u_\zeta\right);\exp\left(\frac{\hat{a}_n^{(M)}(\ln(n/k_n))H_{\hat{\theta}_n^{(M)},0}(d_n)}{k_n^{1/2}/\ln(n/k_n)}u_\zeta\right)\right] \quad (19)$$

is an asymptotic confidence interval for $Q(\beta_n)$ of confidence level ζ.

4. Validation on simulations

The finite-sample behavior of the quantile estimator $\check{Q}_n^{[1]}(\beta_n) := \check{Q}_n(\beta_n)$ defined in (4) is investigated on $N = 500$ simulated random samples of size $n = 5000$, in the case where $\beta_n = n^{-2} = 4.10^{-8}$.

Estimators. Three competitors are considered:

1. The first one, $\check{Q}_n^{[2]}(\beta_n)$ is deduced from (4) by letting $\hat{\theta}_{n,-}^{(M)} := 0$ in $\hat{a}_n^{(M)}(\ln(n/k_n))$ and $\hat{\theta}_n^{(M)}$, see (8) and (7). The resulting estimator $\check{Q}_n^{[2]}(\beta_n)$ should perform well for estimating extreme quantiles from distributions with associated $\theta \geq 0$.
2. Similarly, the second one is also obtained by letting $\hat{\theta}_n^{(M)} := 0$ in (4) and $\hat{\theta}_{n,-}^{(M)} := 0$ in (8). We thus obtain:

$$\check{Q}_n^{[3]}(\beta_n) := X_{n-k_n,n}\exp\left(\hat{a}_n(\ln(n/k_n))\ln\left(\frac{\ln\beta_n}{\ln(k_n/n)}\right)\right),$$

which is exactly the estimator dedicated to extreme quantiles from Weibull-tail distributions introduced in [13]. It should perform well for estimating extreme quantiles from distributions with associated $\theta = 0$.
3. Finally, the third estimator was introduced in [23]:

$$\check{Q}_n^{[4]}(\beta_n) := X_{n-\ell_n,n}\exp\left(\hat{a}_{\ell_n,n}^{[4]}L_{\hat{\theta}_{k_n,n}^{[4]}}\left(\frac{\ln(1/\beta_n)}{\nu_{\ell_n+1,n}}\right)\right),$$

with $\nu_{i,n} := \sum_{j=i}^n j^{-1}$, $\ell_n = k_n/\nu_{k_n+1,n}^2$,

$$\begin{aligned}\hat{a}_{\ell_n,n}^{[4]} &:= \frac{\hat{\gamma}_{\ell_n,n}}{\frac{1}{\ell_n}\sum_{j=1}^{\ell_n}L_{\hat{\theta}_{k_n,n}^{[4]}}\left(\frac{\nu_{j,n}}{\nu_{\ell_n+1,n}}\right)},\\ \hat{\theta}_{k_n,n}^{[4]} &:= 1+\frac{\sum_{i=1}^{k_n-1}(\ln\hat{\gamma}_{i,n}-\ln\hat{\gamma}_{k_n,n})}{\sum_{i=1}^{k_n-1}(\ln\nu_{i+1,n}-\ln\nu_{k_n+1,n})} \text{ and}\\ \hat{\gamma}_{i,n} &:= \frac{1}{i}\sum_{j=1}^{i}(\ln X_{n-j+1,n}-\ln X_{n-i,n}).\end{aligned}$$

Distribution functions. The estimators are compared on the 8 distributions described in Table 1: Gamma($a = 1.5$, s), Weibull($k = 0.5$, λ_1), Gaussian(μ_1, $\sigma = 1$), Lognormal(μ_2, $\sigma = 1$), Burr(λ_2, $c = 0.5$, $k = 0.5$), Pareto-like, super heavy-tail and finite endpoint (x^*). Note that the Pareto-like distribution is taken from [23]. The position parameters μ_1, μ_2 as well as the scaling parameters s, λ_1, λ_2 and the endpoint x^* are chosen such that the simulated data points are all larger than 1.

Results. The log ratio errors $\check{\nu}_n^{[q]} := \ln\left(\check{Q}_n^{[q]}(\beta_n)/Q_n(\beta_n)\right)$ are computed for all 4 estimators ($q = 1,\dots,4$), for each of the 500 datasets from the 8 distributions. The bias of each estimator is then estimated (on a logarithmic scale) by averaging the $\check{\nu}_n^{[q]}$ over the $N = 500$ replications. Similarly, the mean-squared error (MSE) is evaluated (on a logarithmic scale) by averaging the squared $\check{\nu}_n^{[q]}$ over the $N = 500$ replications.

The resulting bias and MSE are displayed on Figures 2–8 as functions of k_n. In terms of bias, it appears that $\check{Q}_n^{[1]}(\beta_n)$ show pretty good results with a small bias over a large range of k_n values for Gamma, Weibull, Gaussian, Lognormal, super heavy-tail and finite endpoint distributions. The bias behavior of $\check{Q}_n^{[1]}(\beta_n)$ is less satisfying on Burr and Pareto-like distributions ($\theta = 1$ in both cases) where $\check{Q}_n^{[4]}(\beta_n)$ is the best in terms of bias stability. From the MSE point of view, $\check{Q}_n^{[1]}(\beta_n)$ achieves better performances than $\check{Q}_n^{[4]}(\beta_n)$ on almost all distributions except the Pareto-like where the results are similar and the Burr distribution where $\check{Q}_n^{[4]}(\beta_n)$ is better than $\check{Q}_n^{[1]}(\beta_n)$. Similar behaviors can be observed on Figures 10–16 where the estimators $\hat{\theta}_n^{(M)}$ and $\hat{\theta}_{k_n,n}^{[4]}$ are compared.

Let us also note that assuming $\theta = 0$ improves the results only on the strict Weibull distribution, the results of $\check{Q}_n^{[3]}(\beta_n)$ being disappointing for other Weibull tail-distributions such as Gaussian or Gamma. Similarly, assuming $\theta > 0$ improves the results only on the Gamma distribution, the results of $\check{Q}_n^{[2]}(\beta_n)$ are not convincing on other distributions. This phenomenon indicates that $\hat{\theta}_{n,-}^{(M)}$ is useful even in case where $\theta > 0$, since it may temper the positive bias associated with $\hat{\theta}_{n,+}^{(M)}$.

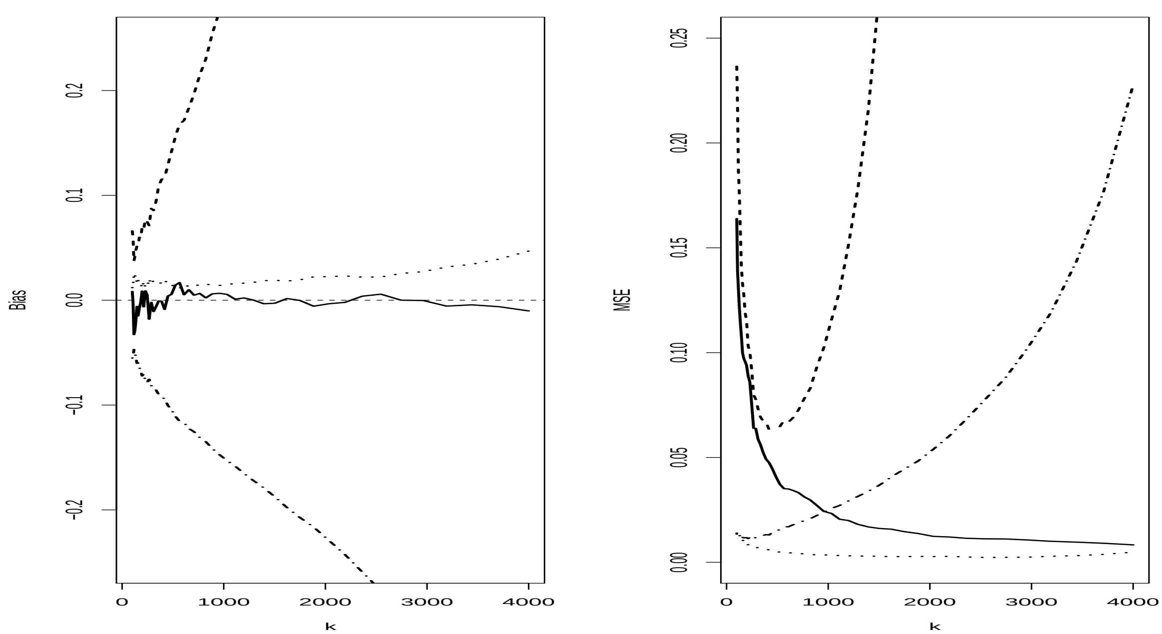

Figure 2: Results on simulated data: Gamma. Bias (left) and MSE (right) associated with $\check{Q}_n^{[1]}(\beta_n)$ (solid line), $\check{Q}_n^{[2]}(\beta_n)$ (dotted line), $\check{Q}_n^{[3]}(\beta_n)$ (dash-dotted line) and $\check{Q}_n^{[4]}(\beta_n)$ (dashed line) as functions of k_n for $\beta_n = n^{-2}$ and $n = 5000$.

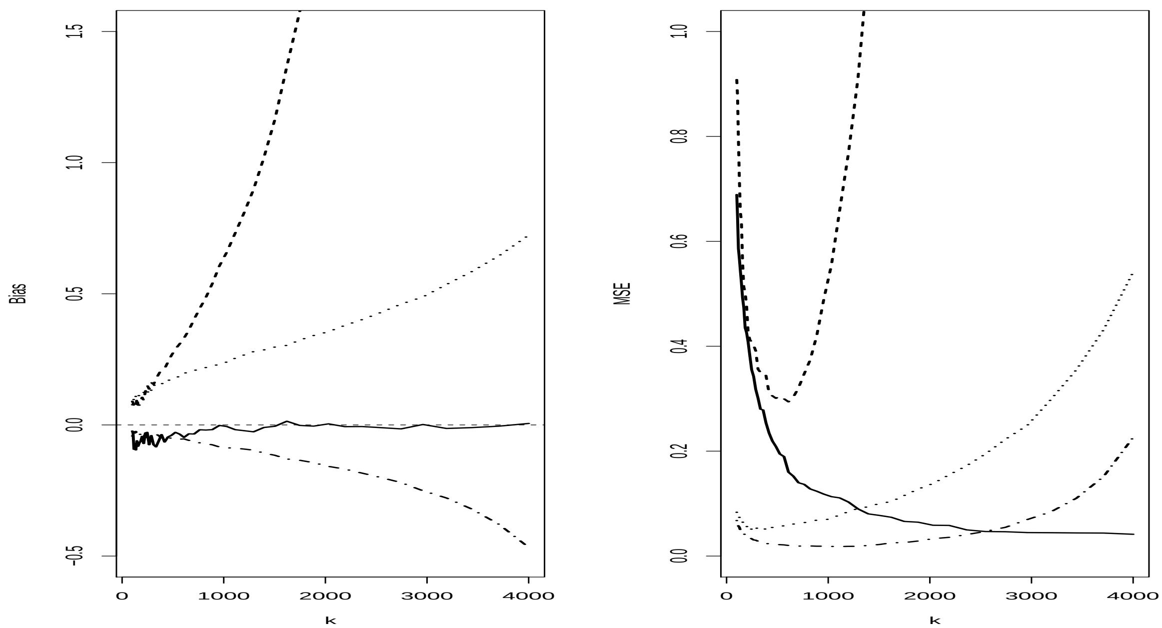

Figure 3: Results on simulated data: Weibull. Bias (left) and MSE (right) associated with $\check{Q}_n^{[1]}(\beta_n)$ (solid line), $\check{Q}_n^{[2]}(\beta_n)$ (dotted line), $\check{Q}_n^{[3]}(\beta_n)$ (dash-dotted line) and $\check{Q}_n^{[4]}(\beta_n)$ (dashed line) as functions of k_n for $\beta_n = n^{-2}$ and $n = 5000$.

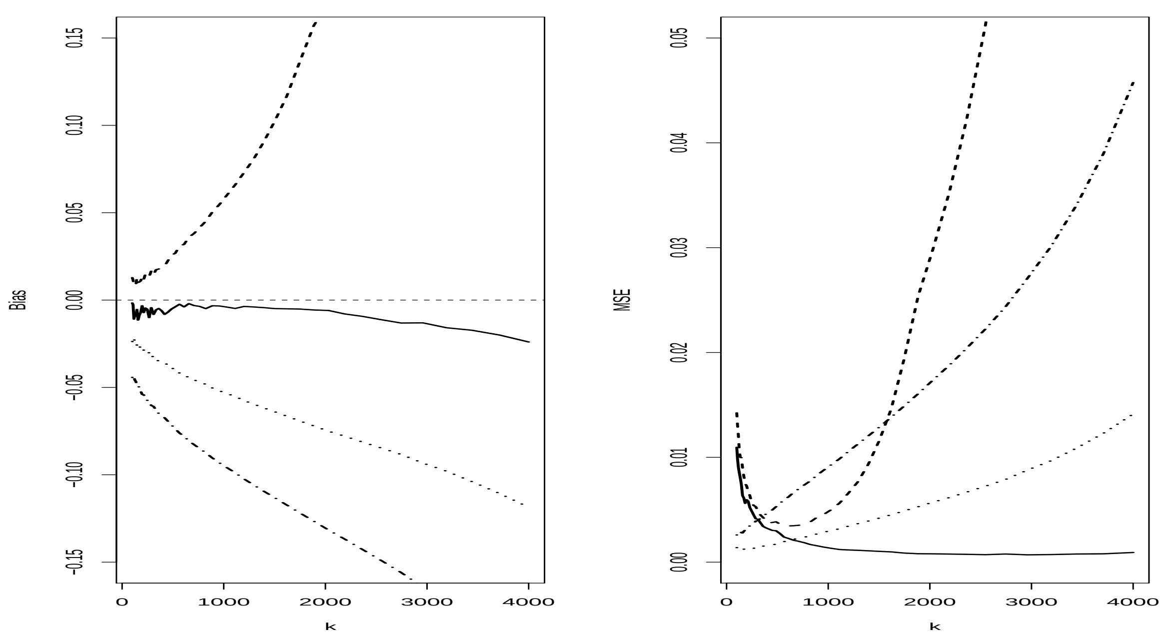

Figure 4: Results on simulated data: Gaussian. Bias (left) and MSE (right) associated with $\check{Q}_n^{[1]}(\beta_n)$ (solid line), $\check{Q}_n^{[2]}(\beta_n)$ (dotted line), $\check{Q}_n^{[3]}(\beta_n)$ (dash-dotted line) and $\check{Q}_n^{[4]}(\beta_n)$ (dashed line) as functions of k_n for $\beta_n = n^{-2}$ and $n = 5000$.

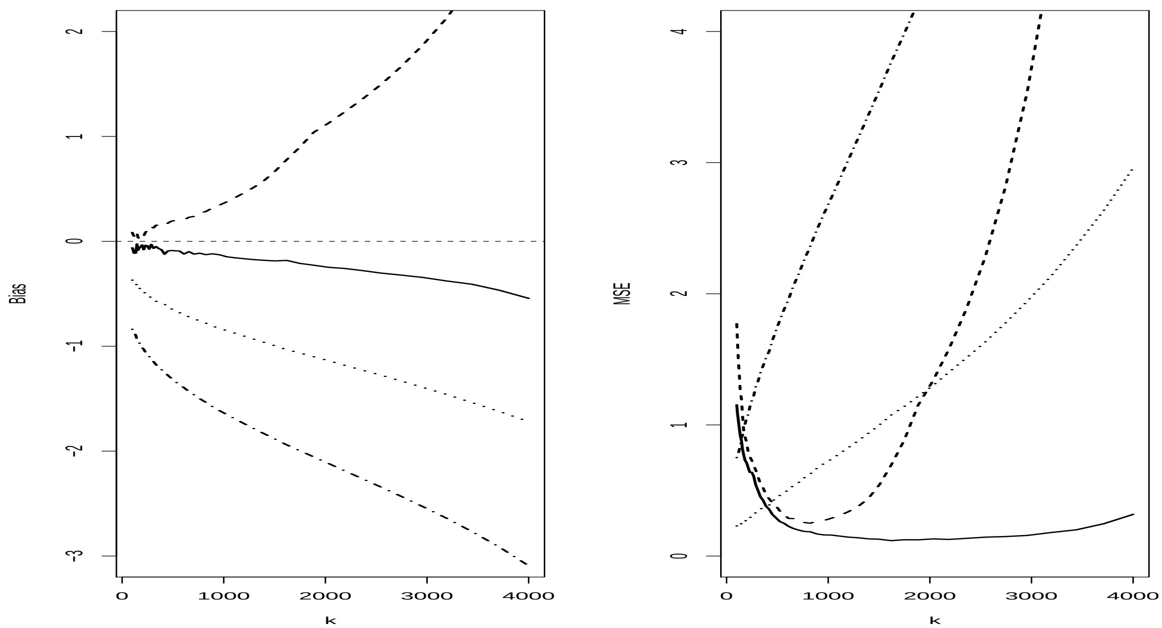

Figure 5: Results on simulated data: Lognormal. Bias (left) and MSE (right) associated with $\check{Q}_n^{[1]}(\beta_n)$ (solid line), $\check{Q}_n^{[2]}(\beta_n)$ (dotted line), $\check{Q}_n^{[3]}(\beta_n)$ (dash-dotted line) and $\check{Q}_n^{[4]}(\beta_n)$ (dashed line) as functions of k_n for $\beta_n = n^{-2}$ and $n = 5000$.

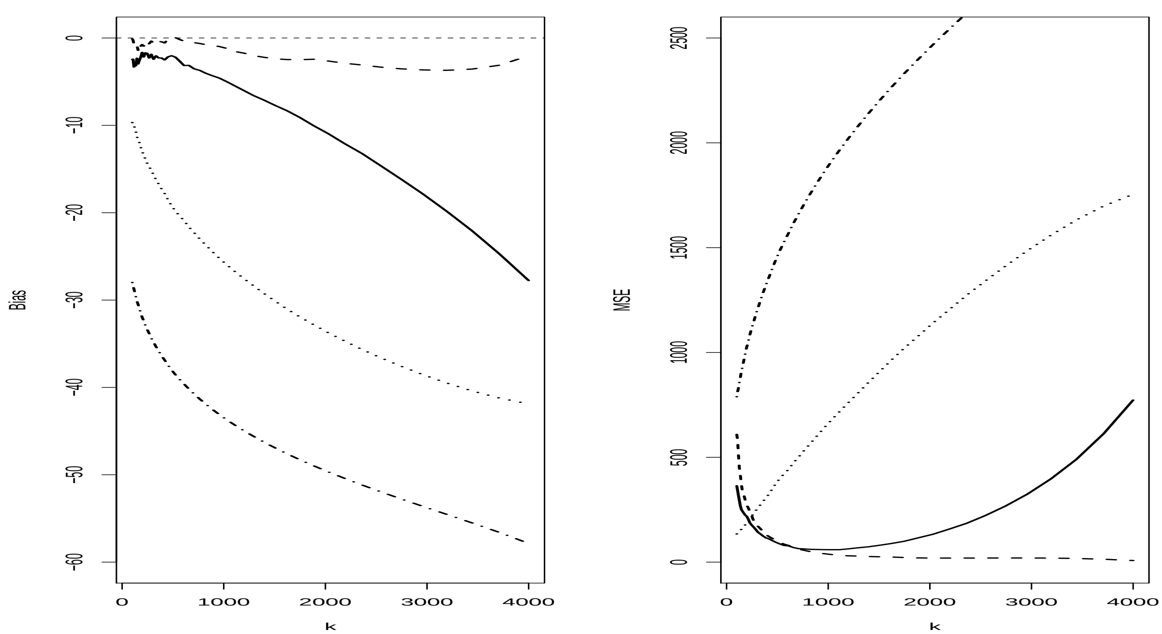

Figure 6: Results on simulated data: Burr. Bias (left) and MSE (right) associated with $\check{Q}_n^{[1]}(\beta_n)$ (solid line), $\check{Q}_n^{[2]}(\beta_n)$ (dotted line), $\check{Q}_n^{[3]}(\beta_n)$ (dash-dotted line) and $\check{Q}_n^{[4]}(\beta_n)$ (dashed line) as functions of k_n for $\beta_n = n^{-2}$ and $n = 5000$. Top: Burr, bottom: Pareto-like.

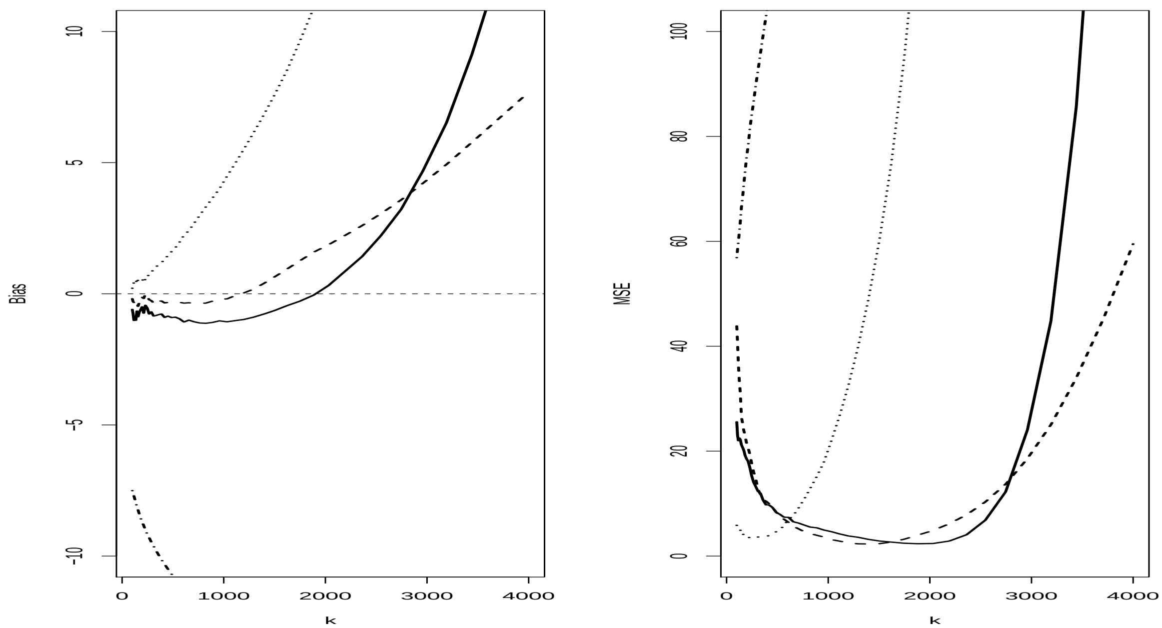

Figure 7: Results on simulated data: Pareto-like. Bias (left) and MSE (right) associated with $\check{Q}_n^{[1]}(\beta_n)$ (solid line), $\check{Q}_n^{[2]}(\beta_n)$ (dotted line), $\check{Q}_n^{[3]}(\beta_n)$ (dash-dotted line) and $\check{Q}_n^{[4]}(\beta_n)$ (dashed line) as functions of k_n for $\beta_n = n^{-2}$ and $n = 5000$.

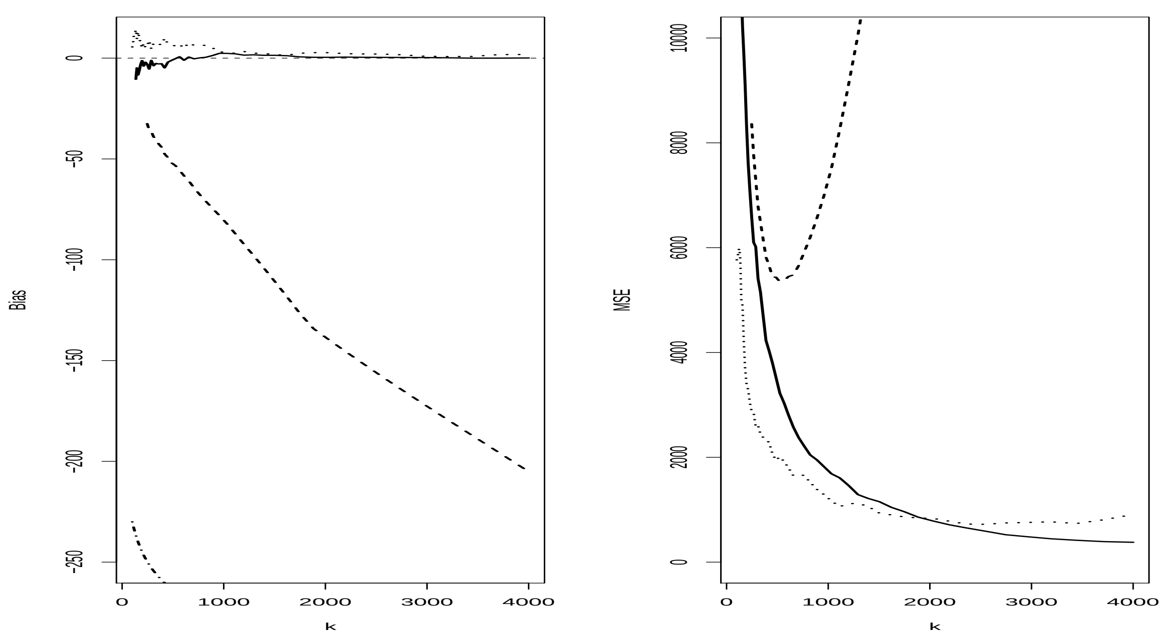

Figure 8: Results on simulated data: super heavy-tail. Bias (left) and MSE (right) associated with $\check{Q}_n^{[1]}(\beta_n)$ (solid line), $\check{Q}_n^{[2]}(\beta_n)$ (dotted line), $\check{Q}_n^{[3]}(\beta_n)$ (dash-dotted line) and $\check{Q}_n^{[4]}(\beta_n)$ (dashed line) as functions of k_n for $\beta_n = n^{-2}$ and $n = 5000$.

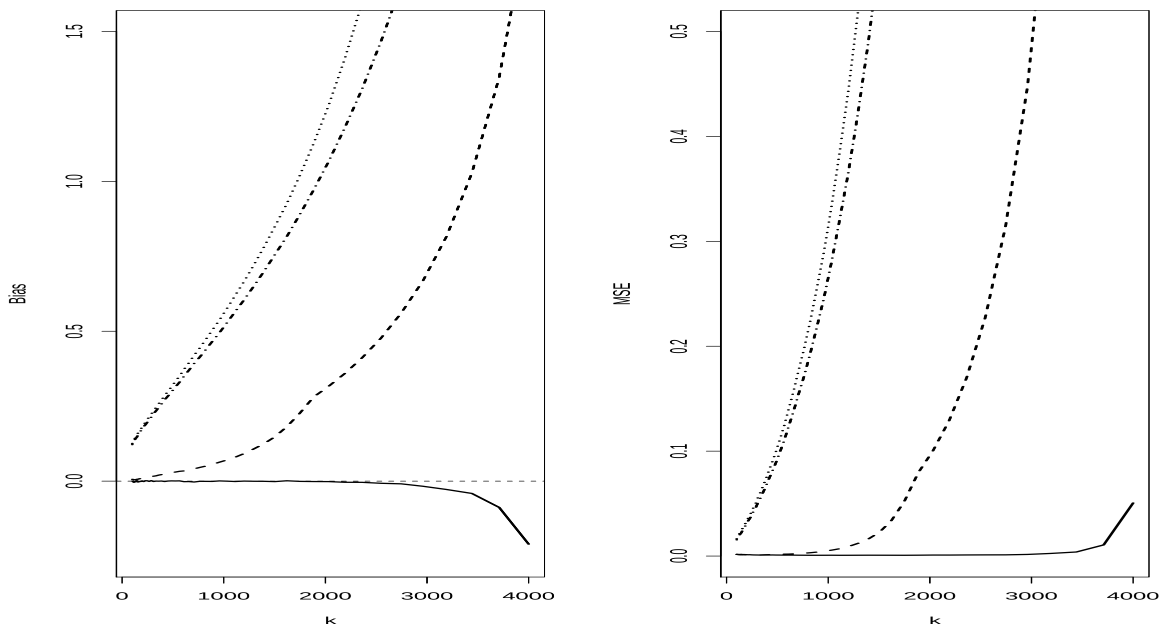

Figure 9: Results on simulated data: finite endpoint. Bias (left) and MSE (right) associated with $\check{Q}_n^{[1]}(\beta_n)$ (solid line), $\check{Q}_n^{[2]}(\beta_n)$ (dotted line), $\check{Q}_n^{[3]}(\beta_n)$ (dash-dotted line) and $\check{Q}_n^{[4]}(\beta_n)$ (dashed line) as functions of k_n for $\beta_n = n^{-2}$ and $n = 5000$.

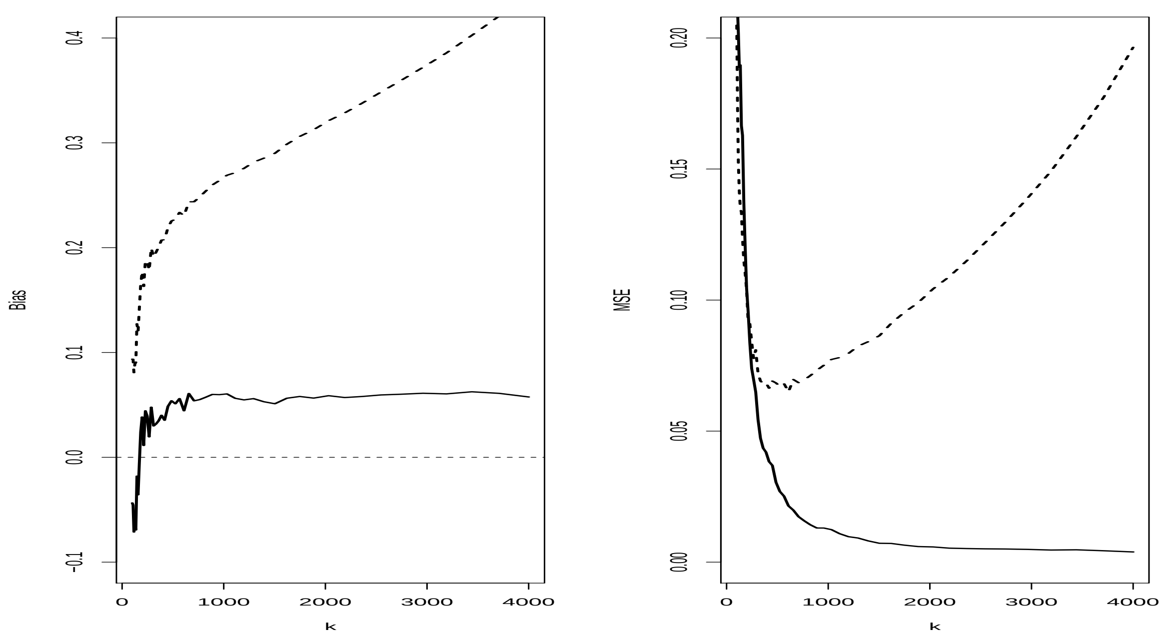

Figure 10: Results on simulated data: Gamma. Bias (left) and MSE (right) associated with $\hat{\theta}_n^{(M)}$ (solid line) and $\hat{\theta}_{k_n,n}^{[4]}$ (dashed line) as functions of k_n for $n = 5000$.

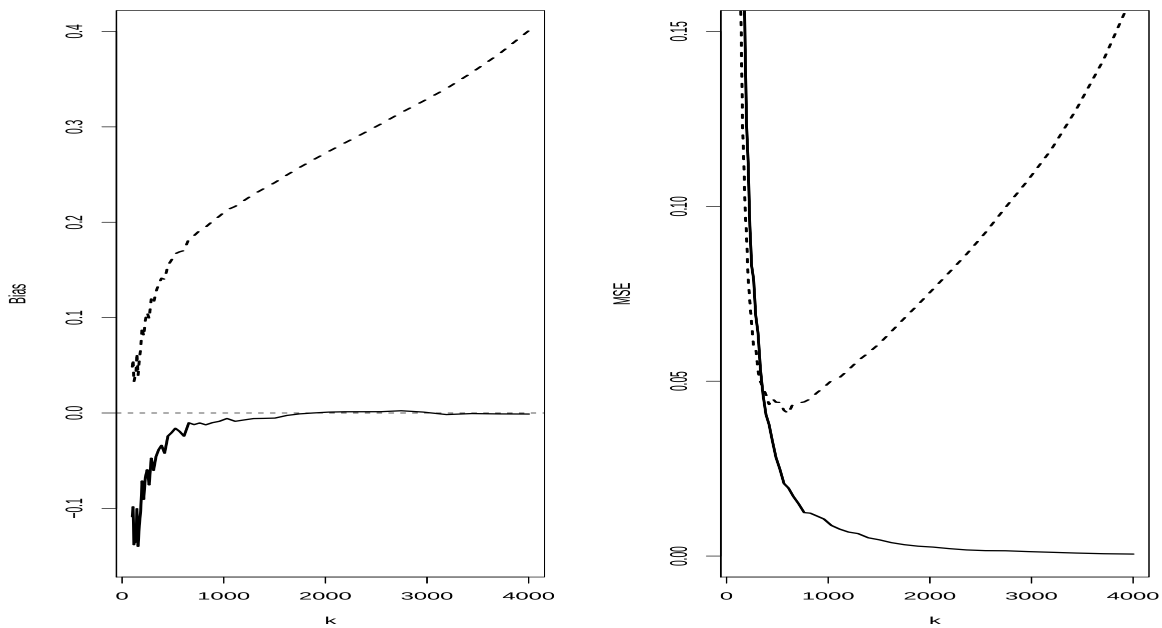

Figure 11: Results on simulated data: Weibull. Bias (left) and MSE (right) associated with $\hat{\theta}_n^{(M)}$ (solid line) and $\hat{\theta}^{[4]}_{k_n,n}$ (dashed line) as functions of k_n for $n = 5000$.

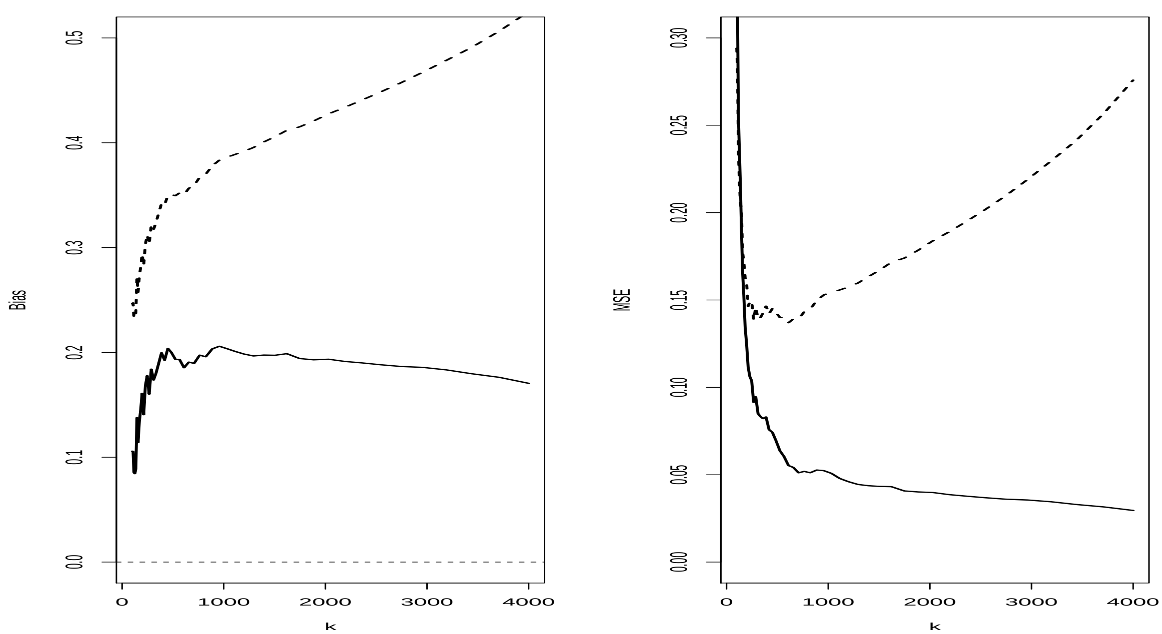

Figure 12: Results on simulated data: Gaussian. Bias (left) and MSE (right) associated with $\hat{\theta}_n^{(M)}$ (solid line) and $\hat{\theta}^{[4]}_{k_n,n}$ (dashed line) as functions of k_n for $n = 5000$.

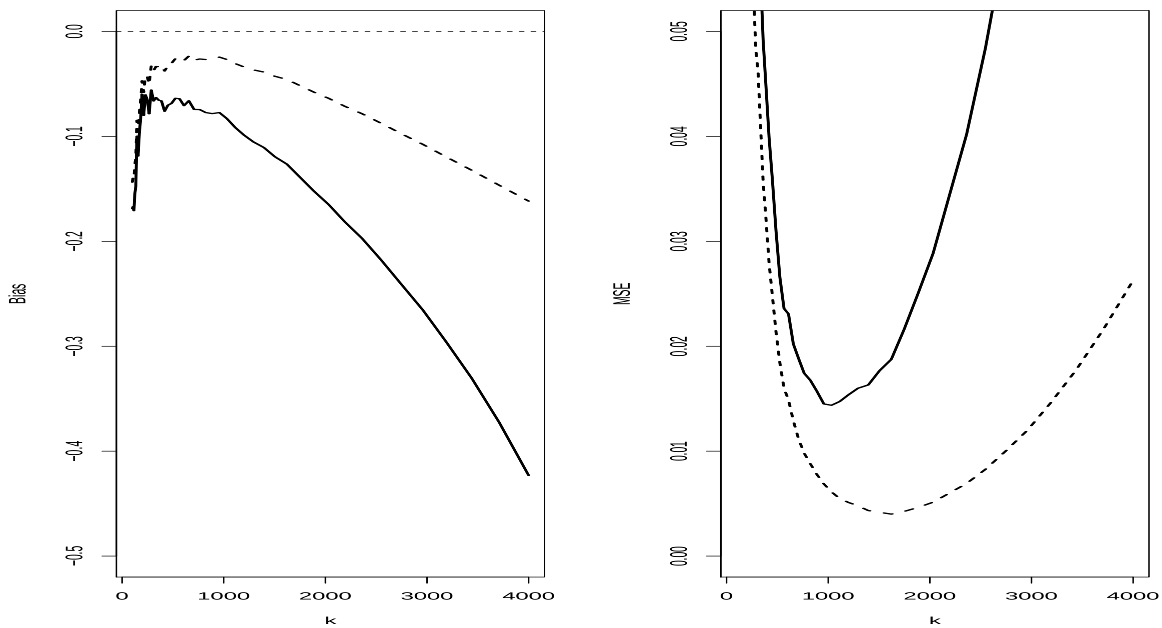

Figure 14: Results on simulated data: Burr. Bias (left) and MSE (right) associated with $\hat{\theta}_n^{(M)}$ (solid line) and $\hat{\theta}_{k_n,n}^{[4]}$ (dashed line) as functions of k_n for $n = 5000$.

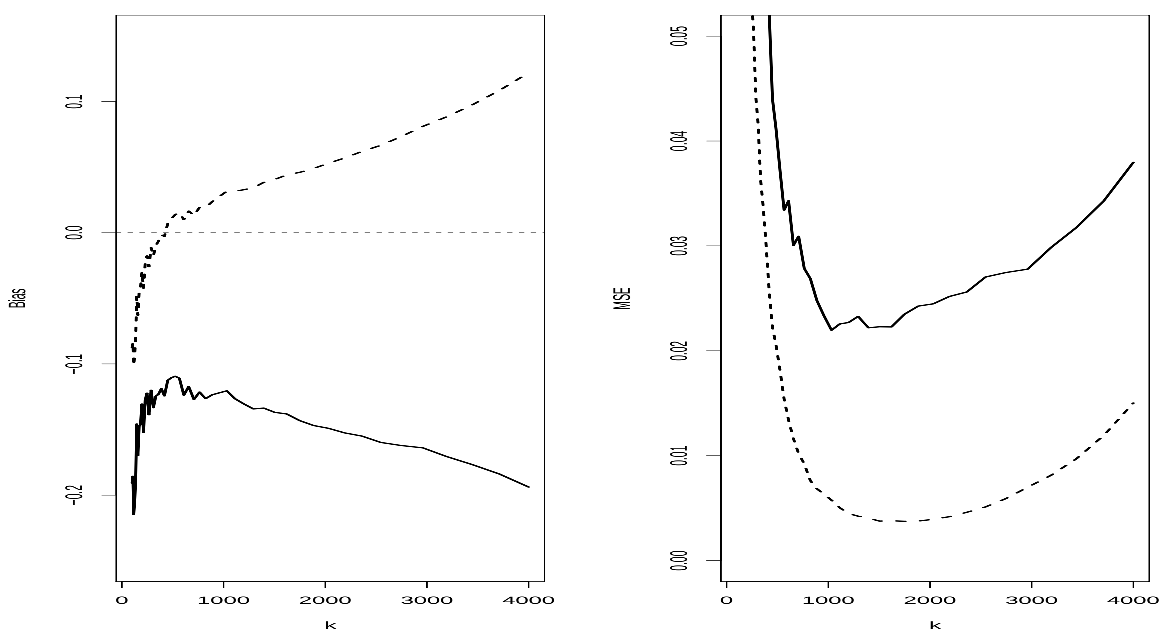

Figure 13: Results on simulated data: Lognormal. Bias (left) and MSE (right) associated with $\hat{\theta}_n^{(M)}$ (solid line) and $\hat{\theta}_{k_n,n}^{[4]}$ (dashed line) as functions of k_n for $n = 5000$.

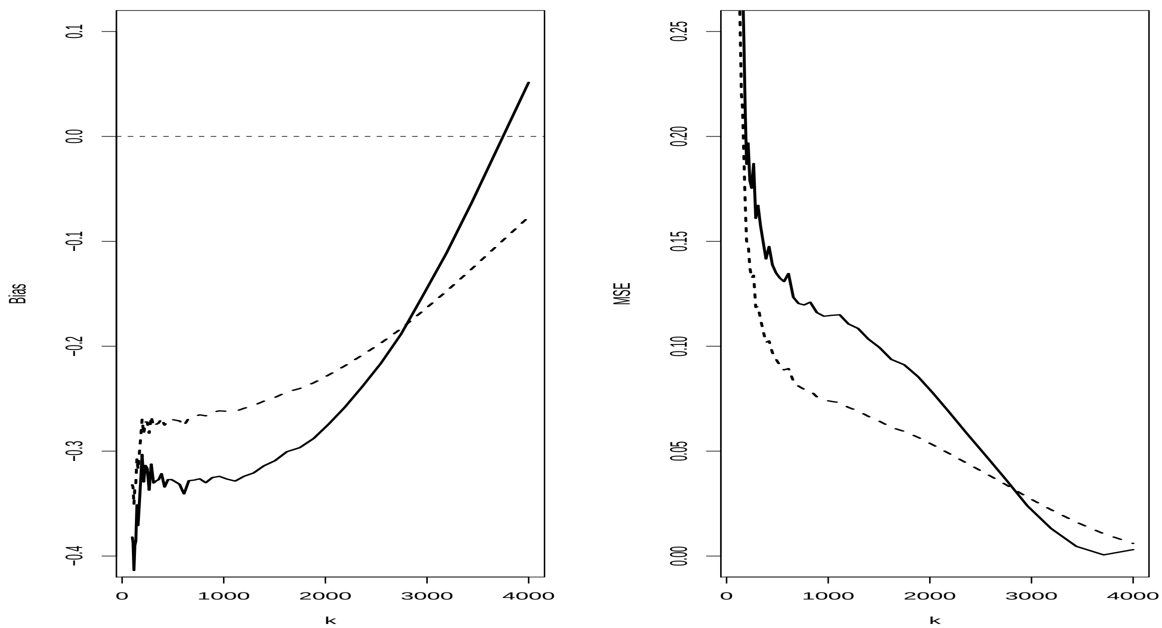

Figure 15: Results on simulated data: Pareto-like. Bias (left) and MSE (right) associated with $\hat{\theta}_n^{(M)}$ (solid line) and $\hat{\theta}_{k_n,n}^{[4]}$ (dashed line) as functions of k_n for $n = 5000$.

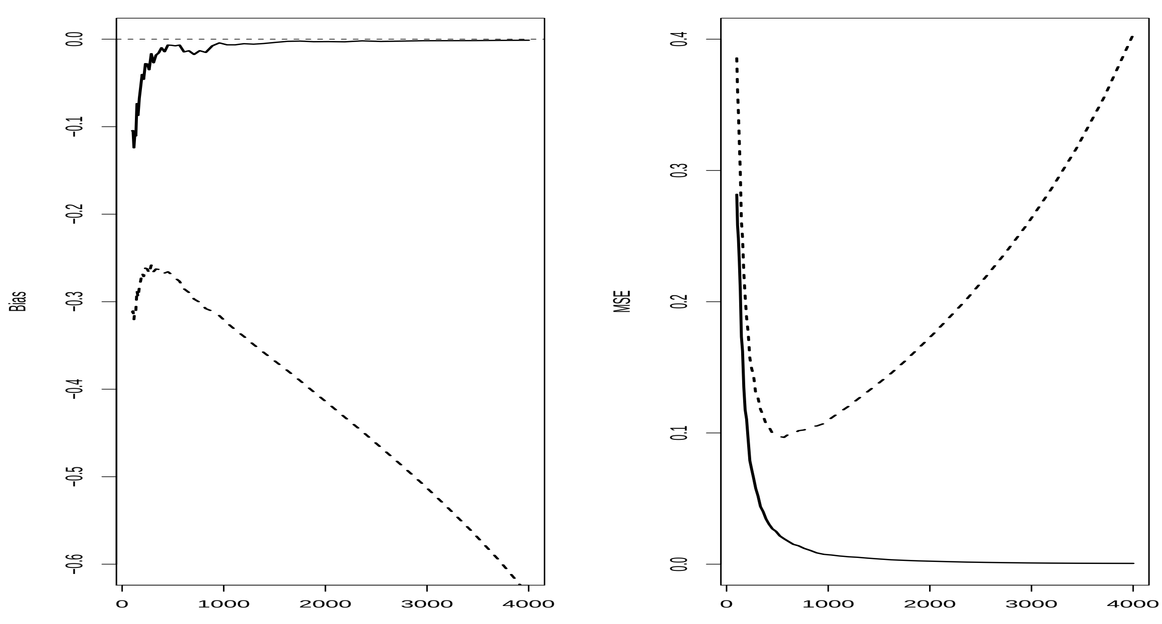

Figure 16: Results on simulated data: super heavy-tail. Bias (left) and MSE (right) associated with $\hat{\theta}_n^{(M)}$ (solid line) and $\hat{\theta}_{k_n,n}^{[4]}$ (dashed line) as functions of k_n for $n = 5000$.

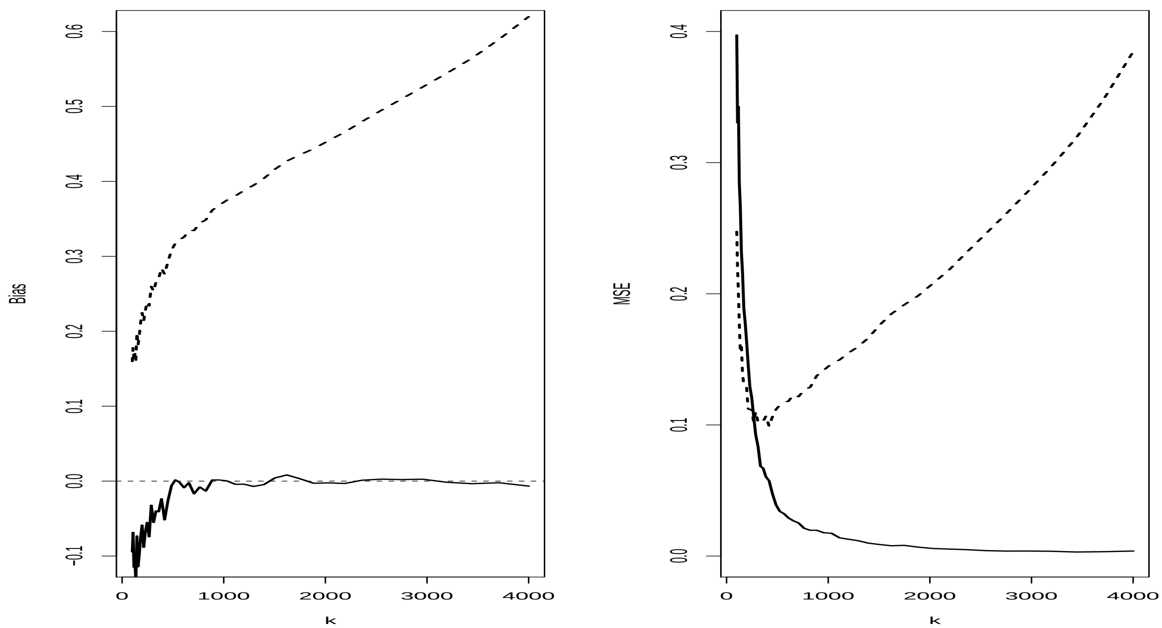

Figure 17: Results on simulated data: finite endpoint. Bias (left) and MSE (right) associated with $\hat{\theta}_n^{(M)}$ (solid line) and $\hat{\theta}_{k_n,n}^{[4]}$ (dashed line) as functions of k_n for $n = 5000$.

5. Illustration on real data

In this section, the extreme quantile estimators $\check{Q}_n^{[1]}(\beta_n)$ and $\check{Q}_n^{[4]}(\beta_n)$ are compared on the average daily river flows (in m^3/s) of the Rhône river (France). The dataset covers the period 1915–2013, and for stationarity reasons, only the winter and spring seasons were considered (from December, 1st to May, 31st), leading to $n = 18043$ measures. We focus on the extreme quantile $Q(\beta_n)$ with $\beta_n = 5.5 \times 10^{-6}$ which is exceeded with a frequency of 10^{-3} per year. Figure 18 displays the index estimates $\hat{\theta}_n^{(M)}$ and $\hat{\theta}_{k_n,n}^{[4]}$ as well as the estimates $\check{Q}_n^{[1]}(\beta_n)$ and $\check{Q}_n^{[4]}(\beta_n)$ of the extreme quantile together with their corresponding 95% asymptotic confidence intervals.

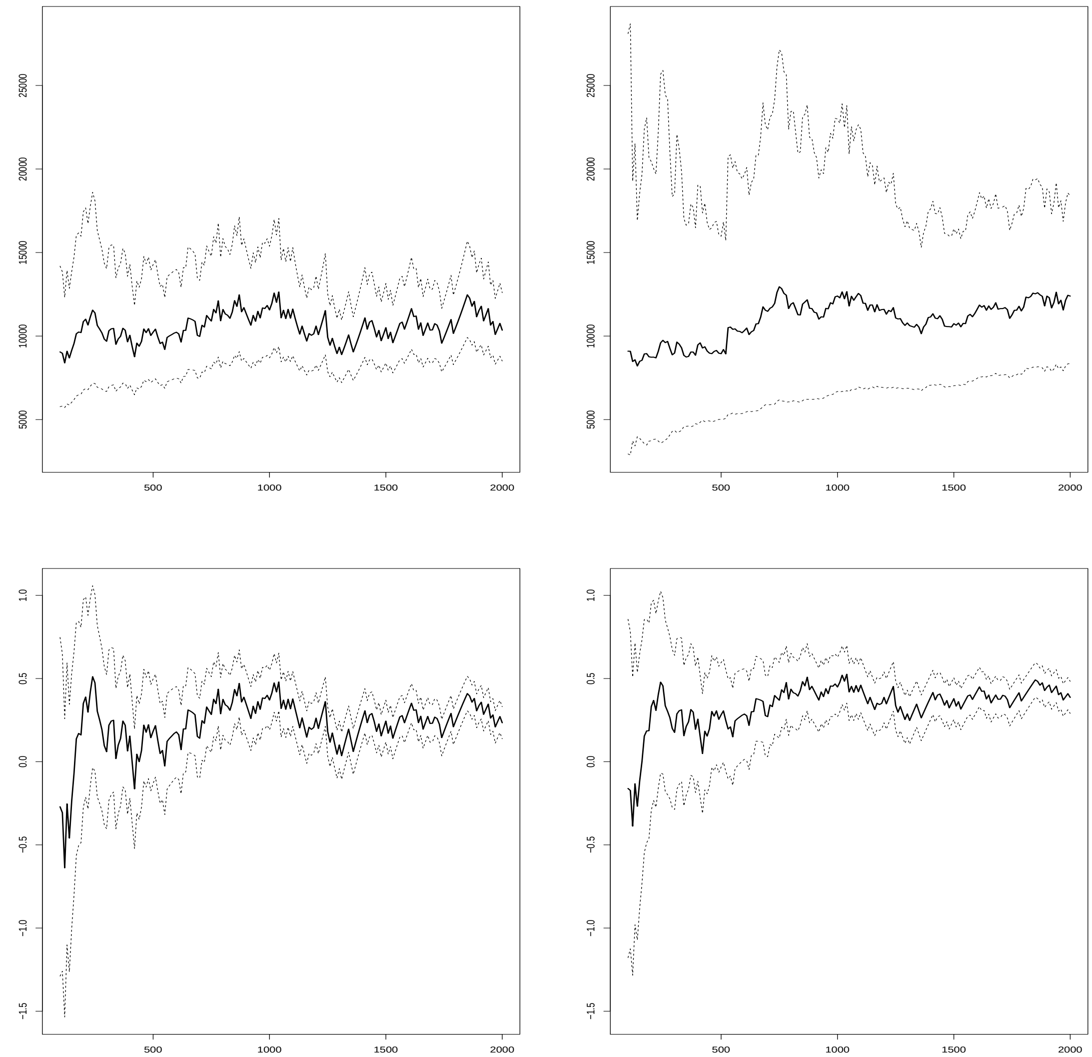

Figure 18: Results on Rhône data. Estimates $\check{Q}_n^{[1]}(\beta_n)$ (top left) and $\check{Q}_n^{[4]}(\beta_n)$ (top right) of $Q(\beta_n)$ with ($\beta_n = 5.5\ 10^{-6}$) and their corresponding index estimates $\hat{\theta}_n^{(M)}$ (bottom left) and $\hat{\theta}_{k_n,n}^{[4]}$ (bottom right) as functions of $k \in \{100, \ldots, 2000\}$. The 95% asymptotic confidence intervals are depicted by dotted lines.

In both cases, the index estimates seem fairly stable as a function of k_n, suggesting a positive value for $\theta \in [0.3, 0.4]$ associated with a log-Weibull tail-distribution. This hypothesis is confirmed by the quantile-quantile plot displayed on Figure 19. This plot is inspired from approximation (10) which suggests that the points

$$\left(\ln\left(1 + \frac{\ln(i/k_n)}{\ln(k_n/n)}\right), \ln_2 X_{n-i+1,n} - \ln_2 X_{n-k_n+1,n}\right),\ i = 1, \ldots, k_n - 1$$

should be approximately located on a line of slope $\hat{\theta}_{n,+}^{(M)}$. Following hydrologists advice, $k_n = 252$ was selected, corresponding to a flow of $2400m^3/s$. The very good fit can be interpreted as an empirical validation of the log-Weibull tail-distribution assumption.

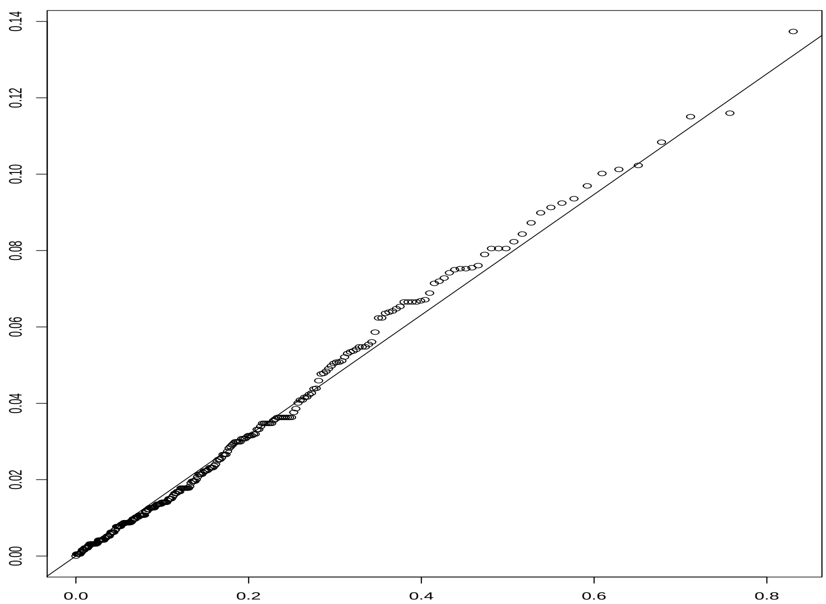

Figure 19: Results on Rhône data. Line of slope $\hat{\theta}_{n,+}^{(M)}$ superimposed to the quantile-quantile plot obtained with $k_n = 252$ (see text for details).

The behavior of extreme quantile estimates $\check{Q}_n^{[1]}(\beta_n)$ and $\check{Q}_n^{[4]}(\beta_n)$ are also similar, $\check{Q}_n^{[1]}(\beta_n)$ being more stable with respect to k_n than $\check{Q}_n^{[4]}(\beta_n)$. The first estimator $\check{Q}_n^{[1]}(\beta_n)$ points towards a constant value $Q(\beta_n) \approx 10,000 m^3/s$ while the second one $\check{Q}_n^{[4]}(\beta_n)$ exhibits a trend from 8000 to 12,000m^3/s as k_n vary from 100 to 2000. At the opposite, the widths of the 95% asymptotic confidence intervals associated with both estimators are significantly different. Indeed, the interval associated with $\check{Q}_n^{[1]}(\beta_n)$ is 10 times narrower than the one associated with $\check{Q}_n^{[4]}(\beta_n)$. This result is in accordance with (18) since here $\tau \simeq 1.24$ yields $\Lambda_\theta(\tau) \simeq 0.1$ for a large range of θ values, see Figure 1.

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6. Appendix: Proofs

Some preliminary lemmas are first provided in Paragraph 6.1, their proofs being postponed to Paragraph 6.3. Proofs of main results are given in Paragraph 6.2.

6.1. Preliminary lemmas

We first give a general tool for establishing the convergence in distribution of random vectors.

Lemma 2. *For $p \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N}$, let $W_n := (W_{n,1}, \dots, W_{n,p})^\top$ and $W := (W_1, \dots, W_p)^\top$ be two random vectors in $\mathbb{R}^p$. If there exist a sequence $\sigma_n \to 0$ and $\lambda := (\lambda_1, \dots, \lambda_p)^\top \in \mathbb{R}^p$ such that $\sigma_n^{-1}(W_n - \lambda) \xrightarrow{d} W$ then, for all $q \in \mathbb{N} \setminus \{0\}$ and all continuously differentiable functions $\varphi_1, \dots, \varphi_q$ from $\mathbb{R}^p$ to $\mathbb{R}$,*

$$\sigma_n^{-1} \left((\varphi_1(W_n), \dots, \varphi_q(W_n))^\top - (\varphi_1(\lambda), \dots, \varphi_q(\lambda))^\top \right) \xrightarrow{d} \left(W^\top \nabla\varphi_1(\lambda), \dots, W^\top \nabla\varphi_q(\lambda) \right),$$

where, for all $i \in \{1, \dots, q\}$, $\nabla\varphi_i(\lambda)$ is the gradient of φ_i evaluated at point λ.

The following lemma is the cornerstone for establishing the asymptotic normality of the quantile estimator in the intermediate case.

Lemma 3. *Let $Z_1, \dots, Z_n$ be n independent copies of a random variable Z. Denote by S_Z the survival function of Z and by $Q_Z = S_Z^{\leftarrow}$ the associated quantile function. Assume Q_Z is differentiable and that $-Q_Z'(1/\cdot)$ is regularly varying. Then, for all sequence (α_n) such that $\alpha_n \to 0$ and $n\alpha_n \to \infty$,*

$$\frac{n^{1/2}}{\alpha_n^{1/2} Q_Z'(\alpha_n)} \left(Z_{n-\lfloor n\alpha_n \rfloor, n} - Q_Z(\alpha_n) \right) \xrightarrow{d} \mathcal{N}(0, 1).$$

An elementary result on ordered statistics from standard uniform random variables is provided below.

Lemma 4. *Let $U_1, \dots, U_n$ be independent standard uniform variables. For all intermediate sequence (k_n), i.e. such that $k_n \to \infty$ and $k_n/n \to 0$, one has*

(i) $U_{k_n+1,n} \xrightarrow{\mathbb{P}} 0$.

(ii) Let $\{F_1, \ldots, F_{k_n}\}$ and $\{E_1, \ldots, E_n\}$ be two independent samples of independent standard exponential random variables. Then,

$$\left\{ \frac{\ln(U_{i+1,n}/U_{k_n+1,n})}{\ln(U_{k_n+1,n})},\ i = 0, \ldots, k_n - 1 \right\} \stackrel{d}{=} \left\{ \frac{F_{k_n-i,k_n}}{E_{n-k_n,n}},\ i = 0, \ldots, k_n - 1 \right\},$$

where $\{F_1, \ldots, F_{k_n}\}$ are independent from $E_{n-k_n,n}$.

Let us introduce some additional notations. For $J \in \mathbb{N} \setminus \{0\}$, $\zeta := (\zeta_1, \ldots, \zeta_J)^\top \in (-\infty, 1)^J$ and $t > 0$, consider the functions

$$S_n(\zeta) := \frac{1}{k_n} \sum_{i=0}^{k_n-1} \prod_{j=1}^{J} L_{\zeta_j} \left(\frac{\ln U_{i+1,n}}{\ln U_{k_n+1,n}} \right) \text{ and } \mu(t, \zeta) := \int_0^1 \prod_{j=1}^{J} L_{\zeta_j} \left(1 - \frac{\ln s}{t} \right) ds$$

and remark that, for $J = 1$ and $\zeta < 1$, $\mu(t, \zeta) = \mu_1(t, \zeta)$ and, for $J = 2$, $\mu(t, (\zeta, \zeta)) = \mu_2(t, \zeta)$. Let us also recall that, from Section 2, $\Psi_t(\zeta) = \mu_1^2(t, \zeta)/\mu_2(t, \zeta)$ for all $t > 0$ and $\zeta < 1$. The next result is of analytical nature. It provides first-order asymptotic expansions as $t \to \infty$ for functions $\mu(t, \zeta)$ and $\Psi_t(\zeta)$ locally uniformly on ζ.

Lemma 5. *(i) Let $J \in \mathbb{N} \setminus \{0\}$. For all hyper-rectangle $\mathcal{R}_J \subset (-\infty, 1)^J$, one has*

$$\lim_{t \to \infty} \sup_{\zeta \in \mathcal{R}_J} \left| t^J \mu(t, \zeta) - J! \right| = 0.$$

(ii) Denoting by Ψ_t' the first derivative of Ψ_t, one has, for all closed interval $I \subset (-\infty, 1)$,

$$\lim_{t \to \infty} \sup_{\zeta \in I} \left| t \Psi_t'(\zeta) + \frac{1}{2} \right| = 0.$$

As a consequence of Lemma 5(ii), the function Ψ_t is decreasing at least for t large enough. Lemma 6 below states Law of Large Numbers type results dedicated to particular triangular arrays of random variables.

Lemma 6. *Let (t_m) be a sequence such that $\log(m)/t_m \to 0$ as $m \to \infty$ and let $F_1, \ldots, F_m$ be independent copies of a standard exponential random variable.*
(i) For all $\delta > 0$ and $\zeta \in (-\infty, 1)^J$ with $J \in \mathbb{N} \setminus \{0\}$, one has

$$\frac{t_m^{J-1}}{m} \sum_{i=1}^{m} F_i \left(1 + \frac{F_i}{t_m} \right)^{\zeta_1 - 1} \prod_{j=2}^{J} L_{\zeta_j} \left(1 + \frac{F_i}{\delta t_m} \right) \xrightarrow{\mathbb{P}} J!/\delta^{J-1}.$$

(ii) For all $\delta > 0$, $(\xi_1, \ldots, \xi_4) \in (-\infty, 1)^4$ with $\xi_3 > \xi_4$ and $J_i \in \mathbb{N}$, $i \in \{1, 2, 3\}$, one has for $J = J_1 + J_2 + 2J_3$ that

$$\frac{t_m^J}{m} \sum_{i=1}^{m} L_{\xi_1}^{J_1} \left(1 + \frac{F_i}{\delta t_m} \right) L_{\xi_2}^{J_2} \left(1 + \frac{F_i}{\delta t_m} \right) \left[L_{\xi_3} \left(1 + \frac{F_i}{\delta t_m} \right) - L_{\xi_4} \left(1 + \frac{F_i}{\delta t_m} \right) \right]^{J_3} \xrightarrow{\mathbb{P}} \frac{J!}{\delta^J} \left(\frac{\xi_3 - \xi_4}{2} \right)^{J_3}.$$

Finally, Lemmas 7 and 8 are the key tools for establishing the joint asymptotic normality of the random pair $(M_n^{(1)}, M_n^{(2)})$.

Lemma 7. *Let (k_n) be an intermediate sequence such that $k_n \to \infty$ and $\ln(k_n)/\ln(n) \to 0$ as $n \to \infty$. For $J_1, J_2 \in \mathbb{N} \setminus \{0\}$ and for all $\zeta^{(1)} \in (-\infty, 1)^{J_1}$, $\zeta^{(2)} \in (-\infty, 1)^{J_2}$, the random vector*

$$k_n^{1/2} \left\{ \frac{S_n(\zeta^{(1)})}{\mu(\ln(n/k_n), \zeta^{(1)})} - 1, \frac{S_n(\zeta^{(2)})}{\mu(\ln(n/k_n), \zeta^{(2)})} - 1 \right\}$$

converges in distribution to a centered Gaussian random vector with covariance matrix

$$\begin{pmatrix} (2J_1)!/(J_1!)^2 - 1 & (J_1+J_2)!/(J_1!J_2!) - 1 \\ (J_1+J_2)!/(J_1!J_2!) - 1 & (2J_2)!/(J_2!)^2 - 1 \end{pmatrix}.$$

Lemma 8. *Let (k_n) be an intermediate sequence such that $k_n \to \infty$ and $\ln(k_n)/\ln(n) \to 0$ as $n \to \infty$. For all $(\xi_1, \dots, \xi_4) \in (-\infty, 1)^4$ with $\xi_3 > \xi_4$ and $J_i \in \mathbb{N}$, $i \in \{1, 2, 3\}$, one has for $J = J_1 + J_2 + 2J_3$ that*

$$\frac{[\ln(n/k_n)]^J}{k_n} \sum_{i=0}^{k_n-1} L_{\xi_1}^{J_1}\left(\frac{\ln U_{i+1,n}}{\ln U_{k_n+1,n}}\right) L_{\xi_2}^{J_2}\left(\frac{\ln U_{i+1,n}}{\ln U_{k_n+1,n}}\right)\left[L_{\xi_3}\left(\frac{\ln U_{i+1,n}}{\ln U_{k_n+1,n}}\right) - L_{\xi_4}\left(\frac{\ln U_{i+1,n}}{\ln U_{k_n+1,n}}\right)\right]^{J_3}$$

converges in probability to $J!\left(\frac{\xi_3-\xi_4}{2}\right)^{J_3}$.

6.2. Proofs of main results

Proof of Theorem 1. Let $\{Z_i := \ln(X_i),\ i = 1, \dots, n\}$. These random variables are independent with common quantile function $Q_Z(u) = V(\ln(1/u))$, $u \in (0, 1)$. Under **(A1)**, Q_Z is differentiable with first derivative verifying $-Q_Z'(1/x) = xV'(\ln(x)) \in \mathcal{RV}(1)$. One can thus apply Lemma 3 to obtain

$$\frac{(n\alpha_n)^{1/2}}{V'(\ln(\alpha_n^{-1}))}\left(Z_{n-\lfloor n\alpha_n\rfloor,n} - Q_Z(\alpha_n)\right) \xrightarrow{d} \mathcal{N}(0, 1).$$

Now, since $a(x) \sim xV'(x)$ as $x \to \infty$ in view of [15, Corollary 1.1.10], it follows that

$$\frac{(n\alpha_n)^{1/2}\ln(\alpha_n^{-1})}{a(\ln(\alpha_n^{-1}))}\left(Z_{n-\lfloor n\alpha_n\rfloor,n} - Q_Z(\alpha_n)\right) \xrightarrow{d} \mathcal{N}(0, 1).$$

The result is then proved by remarking that $Z_{n-\lfloor n\alpha_n\rfloor,n} = \ln(X_{n-\lfloor n\alpha_n\rfloor,n})$ and $Q_Z = \ln Q$. ■

Proof of Proposition 1. Let us first show that

$$\frac{\hat{a}_n(\ln(n/k_n))}{a(\ln(n/k_n))} \xrightarrow{\mathbb{P}} 1.$$

In view of

$$\sigma_n^{-1}\frac{L_\theta(d_n)}{H_{\theta,0}(d_n)}\left(\frac{\hat{a}_n(\ln(n/k_n))}{a(\ln(n/k_n))} - 1\right) \xrightarrow{d} \Lambda,$$

it is sufficient to prove that $\sigma_n^{-1}L_\theta(d_n)/H_{\theta,0}(d_n) \to \infty$ as $n \to \infty$.
Let us first assume that $d_n \to 1$ as $n \to \infty$. Since $L_\theta(1+u) \sim u$ and $H_{\theta,0}(1+u) \sim u^2/2$ as $u \to 0$, $L_\theta(d_n)/H_{\theta,0}(d_n) \sim 2/(d_n - 1)$ and $\sigma_n^{-1}L_\theta(d_n)/H_{\theta,0}(d_n) \to \infty$ as $n \to \infty$. Second, if $d_n \to d \in (1, \infty)$ then $L_\theta(d_n)/H_{\theta,0}(d_n) \to L_\theta(d)/H_{\theta,0}(d) > 0$ and the result is proved. Finally, if $d_n \to \infty$, remarking that, as $t \to \infty$,

$$\frac{L_\theta(t)}{H_{\theta,0}(t)} \sim \begin{cases} 1/\ln(t) & \text{if } \theta > 0, \\ 2/\ln(t) & \text{if } \theta = 0, \\ -\theta & \text{if } \theta < 0 \end{cases} \tag{20}$$

implies $\sigma_n^{-1}L_\theta(d_n)/H_{\theta,0}(d_n) \to \infty$ by assumption.

Let us now prove the second part of Proposition 1. The following equality holds:

$$H_{\hat{\theta}_n,0}(d_n) - H_{\theta,0}(d_n) = (\hat{\theta}_n - \theta) \int_1^{d_n} s^{\theta-1} \ln^2(s) \frac{\exp((\hat{\theta}_n - \theta)\ln(s)) - 1}{(\hat{\theta}_n - \theta)\ln(s)} ds. \tag{21}$$

Since for all $s \in (1, d_n)$, $|(\hat{\theta}_n - \theta)\ln(s)| \leq |\hat{\theta}_n - \theta| \ln(d_n) = O_{\mathbb{P}}(\sigma_n \ln d_n) = o_{\mathbb{P}}(1)$ by assumption, it is easy to check that

$$H_{\hat{\theta}_n,0}(d_n) - H_{\theta,0}(d_n) = (\hat{\theta}_n - \theta) \int_1^{d_n} s^{\theta-1} \ln^2(s) ds (1 + o_{\mathbb{P}}(1)),$$

or equivalently,

$$\frac{H_{\hat{\theta}_n,0}(d_n)}{H_{\theta,0}(d_n)} - 1 = \sigma_n^{-1}(\hat{\theta}_n - \theta) \times \frac{\sigma_n}{H_{\theta,0}(d_n)} \int_1^{d_n} s^{\theta-1} \ln^2(s) ds (1 + o_{\mathbb{P}}(1)).$$

The three situations $d_n \to 1$, $d_n \to d > 1$ and $d_n \to \infty$ are again considered separately. First, since

$$\int_1^{1+u} s^{\theta-1} \ln^2(s) ds \sim \frac{u^3}{3} \text{ and } H_{\theta,0}(u) \sim \frac{u^2}{2},$$

as $u \to 0$, one has for $d_n \to 1$ that

$$\frac{H_{\hat{\theta}_n,0}(d_n)}{H_{\theta,0}(d_n)} - 1 \sim \sigma_n^{-1}(\hat{\theta}_n - \theta) \times \frac{2}{3}\sigma_n(d_n - 1) \xrightarrow{\mathbb{P}} 0.$$

The case $d_n \to d$ is straightforward. Finally, when $d_n \to \infty$,

$$\frac{1}{H_{\theta,0}(d_n)} \int_1^{d_n} s^{\theta-1} \ln^2(s) ds \sim \begin{cases} \ln(d_n) & \text{if } \theta > 0, \\ 3\ln(d_n)/2 & \text{if } \theta = 0, \\ -2/\theta & \text{if } \theta < 0. \end{cases}$$

Collecting conditions $\sigma_n \ln(d_n) \to 0$ and $\sigma_n^{-1}(\hat{\theta}_n - \theta) \xrightarrow{d} \Theta$ concludes the proof. ■

Proof of Theorem 2. Let us start with the expansion:

$$\frac{\sigma_n^{-1}}{a(\ln(n/k_n)) H_{\theta,0}(d_n)} \ln \frac{\check{Q}_n(\beta_n)}{Q(\beta_n)} = T_{1,n} + T_{2,n} + T_{3,n} + T_{4,n},$$

with

$$\begin{aligned}
T_{1,n} &= \frac{\sigma_n^{-1}}{a(\ln(n/k_n)) H_{\theta,0}(d_n)} \left(\ln X_{n-k_n,n} - \ln Q(k_n/n)\right), \\
T_{2,n} &= \frac{\hat{a}_n(\ln(n/k_n))}{a(\ln(n/k_n)) H_{\theta,0}(d_n)} \sigma_n^{-1} \left(L_{\hat{\theta}_n}(d_n) - L_\theta(d_n)\right), \\
T_{3,n} &= \frac{L_\theta(d_n)}{H_{\theta,0}(d_n)} \sigma_n^{-1} \left(\frac{\hat{a}_n(\ln(n/k_n))}{a(\ln(n/k_n))} - 1\right) \text{ and} \\
T_{4,n} &= \frac{\sigma_n^{-1}}{H_{\theta,0}(d_n)} \left(\frac{\ln Q(k_n/n) - \ln Q(\beta_n)}{a(\ln(n/k_n))} + L_\theta(d_n)\right).
\end{aligned}$$

Clearly, under **(A3)**, $T_{1,n} \xrightarrow{d} \Omega$ and $T_{3,n} \xrightarrow{d} \Lambda$. Next, remark that Proposition 1 entails that the asymptotic distribution of $T_{2,n}$ is the same as the one of

$$\frac{\sigma_n^{-1}}{H_{\theta,0}(d_n)} \left(L_{\hat{\theta}_n}(d_n) - L_\theta(d_n)\right).$$

Furthermore, similarly to (21) in the proof of Proposition 1, one can show that $L_{\hat{\theta}_n}(d_n) - L_\theta(d_n) = (\hat{\theta}_n - \theta) H_{\theta,0}(d_n)(1 + o_{\mathbb{P}}(1))$. As a consequence, $T_{2,n} \xrightarrow{d} \Theta$. Finally, since $\ln Q(\alpha) = V(\ln(1/\alpha))$, it follows that

$$H_{\theta,0}(d_n)\sigma_n T_{4,n} = \frac{V(\ln(n/k_n)) - V(\ln(\beta_n))}{a(\ln(n/k_n))} + L_\theta(d_n).$$

Let us consider separately the three cases $d_n \to 1$, $d_n \to d > 1$ and $d_n \to \infty$.
First, if $d_n \to 1$, the second order condition **(A2)** entails $H_{\theta,0}(d_n)\sigma_n T_{4,n} \sim H_{\theta,\rho}(d_n)\tilde{A}(\ln(n/k_n))$. Since for all $\rho \leq 0$, $H_{\theta,\rho}(1+u) \sim H_{\theta,0}(1+u) \sim u^2/2$ as $u \to 0$, it follows that $T_{4,n} \sim \sigma_n^{-1}\tilde{A}(\ln(n/k_n)) = o(1)$ by assumption.
Next, if $d_n \to d > 1$, conditions **(A2)** and $\sigma_n^{-1}\tilde{A}(\ln(n/k_n)) \to 0$ imply that $T_{4,n} \to 0$ as $n \to \infty$.
Finally, when $d_n \to \infty$, [15, Lemma 4.3.5] entails that

$$T_{4,n} = \mathcal{O}\left\{ \frac{L_\theta(d_n)}{H_{\theta,0}(d_n)} \sigma_n^{-1}\tilde{A}(\ln(n/k_n)) \right\} = o(1),$$

using (20). To conclude, if $d_n \to d \in [1, \infty]$, $T_{4,n} \to 0$ as $n \to \infty$ and the result is proved. ■

Proof of Theorem 3. For $i = 1, \ldots, n$, let $U_i := S(X_i)$ so that $\{U_1, \ldots, U_n\}$ is a set of independent standard uniform random variables. Let $\delta > 0$ and $r(\cdot) := a(\cdot)/V(\cdot)$. For $s \in (\alpha^\delta, 1)$, let us plug $x := \ln(1/\alpha)$ and $t := 1 + \ln s/\ln\alpha$ in (12). Consequently, as $\alpha \to 0$,

$$r^{-1}(\ln(1/\alpha)) \ln \frac{\ln Q(s\alpha)}{\ln Q(\alpha)} = L_{\theta_-}\left(1 + \frac{\ln s}{\ln \alpha}\right) + A(\ln(1/\alpha)) H_{\theta_-,\rho'}\left(1 + \frac{\ln s}{\ln\alpha}\right) + o[A(\ln 1/\alpha)], \quad (22)$$

uniformly in $s \in (\alpha^\delta, 1)$. As a consequence of Lemma 4(i, ii), one may apply (22) with α replaced by $U_{k_n+1,n}$ and s replaced by $U_{i+1,n}/U_{k_n+1,n}$ to get, for n large enough,

$$\begin{aligned} r^{-1}(\ln U_{k_n+1,n}^{-1}) \frac{M_n^{(1)}}{\mu_1(\ln(n/k_n), \theta_-)} - 1 \quad &= \quad \frac{S_n(\theta_-)}{\mu_1(\ln(n/k_n), \theta_-)} - 1 \\ &+ \quad \frac{A(\ln U_{k_n+1,n}^{-1})}{k_n \mu_1(\ln(n/k_n), \theta_-)} \sum_{i=0}^{k_n-1} H_{\theta_-,\rho'}\left(\frac{\ln U_{i+1,n}}{\ln U_{k_n+1,n}}\right) \\ &+ \quad o_{\mathbb{P}}\left(\frac{A(\ln U_{k_n+1,n}^{-1})}{\mu_1(\ln(n/k_n), \theta_-)}\right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} & r^{-2}(\ln U_{k_n+1,n}^{-1}) \frac{M_n^{(2)}}{\mu_2(\ln(n/k_n), \theta_-)} - 1 = \frac{S_n((\theta_-, \theta_-))}{\mu_2(\ln(n/k_n), \theta_-)} - 1 \\ + \quad & \frac{2A(\ln U_{k_n+1,n}^{-1})}{k_n \mu_2(\ln(n/k_n), \theta_-)} \sum_{i=0}^{k_n-1} L_{\theta_-}\left(\frac{\ln U_{i+1,n}}{\ln U_{k_n+1,n}}\right) H_{\theta_-,\rho'}\left(\frac{\ln U_{i+1,n}}{\ln U_{k_n+1,n}}\right) \\ + \quad & \frac{A^2(\ln U_{k_n+1,n}^{-1})}{k_n \mu_2(\ln(n/k_n), \theta_-)} \sum_{i=0}^{k_n-1} H^2_{\theta_-,\rho'}\left(\frac{\ln U_{i+1,n}}{\ln U_{k_n+1,n}}\right) + o_{\mathbb{P}}\left(\frac{A^2(\ln U_{k_n+1,n}^{-1})}{\mu_2(\ln(n/k_n), \theta_-)}\right) \\ + \quad & \frac{S_n(\theta_-)}{\mu_2(\ln(n/k_n), \theta_-)} o_{\mathbb{P}}[A(\ln U_{k_n+1,n}^{-1})] + \frac{o_{\mathbb{P}}[A^2(\ln U_{k_n+1,n}^{-1})]}{k_n \mu_2(\ln(n/k_n), \theta_-)} \sum_{i=0}^{k_n-1} H_{\theta_-,\rho'}\left(\frac{\ln U_{i+1,n}}{\ln U_{k_n+1,n}}\right). \end{aligned}$$

From Rényi's representation, $\ln(1/U_{k_n+1,n})/\ln(n/k_n) \xrightarrow{\mathbb{P}} 1$ and since $|A|$ is regularly varying, it follows that

$$\left|\frac{A(\ln(1/U_{k_n+1,n}))}{A(\ln(n/k_n))}\right| \xrightarrow{\mathbb{P}} 1 \tag{23}$$

as $n \to \infty$. Now, since for all $t > 0$,

$$H_{\theta_-,\rho'}(t) = \begin{cases} 1/\rho'(L_{\theta_-+\rho'}(t) - L_{\theta_-}(t)) & \text{if } \rho' \neq 0, \\ L_{\theta_-}(t)L_0(t) - \theta_-^{-1}(L_{\theta_-}(t) - L_0(t)) & \text{if } \rho' = 0 \text{ and } \theta_- \neq 0, \\ L_0^2(t)/2 & \text{if } \rho' = \theta_- = 0, \end{cases}$$

Lemma 5(i), Lemma 7, Lemma 8 and condition $k_n A^2(\ln(n/k_n))/\ln^2(n/k_n) \to 0$ yield

$$r^{-1}(\ln U_{k_n+1,n}^{-1})\frac{M_n^{(1)}}{\mu_1(\ln(n/k_n),\theta_-)} - 1 = \frac{S_n(\theta_-)}{\mu_1(\ln(n/k_n),\theta_-)} - 1 + o_{\mathbb{P}}\left(k_n^{-1/2}\right),$$

and

$$r^{-2}(\ln U_{k_n+1,n}^{-1})\frac{M_n^{(2)}}{\mu_2(\ln(n/k_n),\theta_-)} - 1 = \frac{S_n((\theta_-,\theta_-))}{\mu_2(\ln(n/k_n),\theta_-)} - 1 + o_{\mathbb{P}}\left(k_n^{-1/2}\right).$$

Note that Lemma 7 can be applied since $k_n A^2(\ln(n/k_n))/\ln^2(n/k_n) \to 0$ implies $\ln(k_n)/\ln(n) \to 0$, see [13, Lemma 1]. As a consequence, the random vector

$$k_n^{1/2}\left(r^{-1}(\ln U_{k_n+1,n}^{-1})\frac{M_n^{(1)}}{\mu(\ln(n/k_n),\theta_-)} - 1, r^{-2}(\ln U_{k_n+1,n}^{-1})\frac{M_n^{(2)}}{\mu_2(\ln(n/k_n),\theta_-)} - 1\right) \tag{24}$$

converges in distribution to a centered Gaussian random vector (P_1, P_2) with covariance matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Let us investigate the asymptotic distribution of $k_n^{1/2}(\hat{\theta}_{n,+}^{(M)} - \theta_+)$ where we recall that

$$\hat{\theta}_{n,+}^{(M)} = \frac{M_n^{(1)}}{\mu_1(\ln(n/k_n),0)},$$

see (5). From [15, Eq. 3.5.13], $r(\cdot)$ is regularly varying and thus $\ln(1/U_{k_n+1,n})/\ln(n/k_n) \xrightarrow{\mathbb{P}} 1$ implies $r(\ln(1/U_{k_n+1,n}))/r(\ln(n/k_n)) \xrightarrow{\mathbb{P}} 1$. Since $\mu_1(\ln(n/k_n),\theta_-) = \mu_1(\ln(n/k_n),0)$ for $\theta > 0$ and $\mu_1(\ln(n/k_n),\theta_-) \sim \mu_1(\ln(n/k_n),0)$ for $\theta \leq 0$ from Lemma 5(i), convergence in distribution (24) yields in both cases

$$k_n^{1/2}\left(\hat{\theta}_{n,+}^{(M)} - \theta_+\right) = r(\ln(n/k_n))P_{1,n} + k_n^{1/2}\left\{r(\ln(1/U_{k_n+1,n})) - \theta_+\right\}(1+o(1)),$$

where $P_{1,n} \xrightarrow{d} P_1$. Now, [15, Eq. B.3.46] ensures that $(r(x) - \theta_+)/A(x) \to \lambda \in \mathbb{R}$ as $x \to \infty$ and thus, taking into account of (23),

$$k_n^{1/2}\left(\hat{\theta}_{n,+}^{(M)} - \theta_+\right) = r(\ln(n/k_n))P_{1,n} + \mathcal{O}_{\mathbb{P}}(k_n^{1/2}A(\ln(n/k_n)) = \theta_+ P_{1,n} + \mathcal{O}_{\mathbb{P}}(k_n^{1/2}A(\ln(n/k_n))), \tag{25}$$

since $r(\ln(n/k_n)) \to \theta_+$ as $n \to \infty$. Now, using convergence in distribution (24) and a Taylor expansion yield

$$\frac{1}{\Psi_{\ln(n/k_n)}(\theta_-)}\frac{(M_n^{(1)})^2}{M_n^{(2)}} = 1 + k_n^{-1/2}(2P_{1,n} - P_{2,n}) + o_{\mathbb{P}}(k_n^{-1/2}),$$

where $(P_{1,n}, P_{2,n}) \xrightarrow{d} (P_1, P_2)$. From Lemma 5(i), $\Psi_{\ln(n/k_n)}(\theta_-) \to 1/2$ as $n \to \infty$, and thus

$$2k_n^{1/2} \left(\frac{(M_n^{(1)})^2}{M_n^{(2)}} - \Psi_{\ln(n/k_n)}(\theta_-) \right) \xrightarrow{d} 2P_1 - P_2, \tag{26}$$

where it is easily seen that $2P_1 - P_2 \sim \mathcal{N}(0,1)$. Now let $\sigma_n := k_n^{-1/2} \ln(n/k_n) \to 0$. For all $z \in \mathbb{R}$ and n large enough,

$$\mathbb{P}\left(\sigma_n^{-1} \left(\hat{\theta}_{n,-}^{(M)} - \theta_- \right) \le z \right) = \mathbb{P}\left(\frac{(M_n^{(1)})^2}{M_n^{(2)}} \ge \Psi_{\ln(n/k_n)} \left(\theta_- + \sigma_n z \right) \right),$$

since for n large enough, $\Psi_{\ln(n/k_n)}$ is decreasing. Hence,

$$\mathbb{P}\left(\sigma_n^{-1} \left(\hat{\theta}_{n,-}^{(M)} - \theta_- \right) \le z \right) = \mathbb{P}\left(2k_n^{1/2} \left(\frac{(M_n^{(1)})^2}{M_n^{(2)}} - \Psi_{\ln(n/k_n)}(\theta_-) \right) \ge z_{n,k_n} \right),$$

with $z_{n,k_n} := 2k_n^{1/2}(\Psi_{\ln(n/k_n)}\left(\theta_- + \sigma_n z\right) - \Psi_{\ln(n/k_n)}\left(\theta_-\right))$. The mean-value theorem entails that

$$z_{n,k_n} = 2k_n^{1/2} \sigma_n z \Psi'_{\ln(n/k_n)} \left(\theta_- + \tau_n \sigma_n z \right),$$

where $\tau_n \in (0,1)$. We thus have that $z_{n,k_n} \to -z$ as $n \to \infty$ from Lemma 5(ii) and replacing σ_n by its expression. Taking into account of convergence (26) leads to

$$\frac{k_n^{1/2}}{\ln(n/k_n)} \left(\hat{\theta}_{n,-}^{(M)} - \theta_- \right) \xrightarrow{d} 2P_1 - P_2 \sim \mathcal{N}(0,1). \tag{27}$$

Collecting (25) and (27) concludes the proof. ■

Proof of Theorem 4. Keeping in mind the notations introduced in the proof of Theorem 3, the following expansion holds

$$\frac{\hat{a}_n^{(M)}(\ln(n/k_n))}{a(\ln(n/k_n))} = \mathcal{F}_{1,n} \times \mathcal{F}_{2,n} \times \mathcal{F}_{3,n},$$

with

$$\mathcal{F}_{1,n} := \frac{r^{-1}(\ln(1/U_{k_n+1,n}))M_n^{(1)}}{\mu_1(\ln(n/k_n), \theta_-)}, \; \mathcal{F}_{2,n} := \frac{a(\ln(1/U_{k_n+1,n}))}{a(\ln(n/k_n))} \text{ and } \mathcal{F}_{3,n} := \frac{\mu_1(\ln(n/k_n), \theta_-)}{\mu_1(\ln(n/k_n), \hat{\theta}_{n,-}^{(M)})}.$$

First, (24) entails that

$$k_n^{1/2}(\mathcal{F}_{1,n} - 1) \xrightarrow{d} P_1. \tag{28}$$

Let us now consider $\mathcal{F}_{2,n}$. From [15, Theorem 2.3.6 and Corollary 2.3.5], there exist a function a_0 with, as $t \to \infty$

$$\frac{a_0(t)}{a(t)} = 1 + \mathcal{O}(\tilde{A}(t))$$

and a function A_0 with $A_0(t) = \mathcal{O}(\tilde{A}(t))$ as $t \to \infty$ such that, for all $\varepsilon > 0$, $\delta > 0$ and n large enough,

$$A_0^{-1}(\ln(n/k_n)) \left(\frac{a_0(\ln(1/U_{k_n+1,n}))}{a_0(\ln(n/k_n))} - \left(\frac{\ln(1/U_{k_n+1,n})}{\ln(n/k_n)} \right)^\theta \right) = \left(\frac{\ln(1/U_{k_n+1,n})}{\ln(n/k_n)} \right)^\theta L_\rho \left(\frac{\ln(1/U_{k_n+1,n})}{\ln(n/k_n)} \right) + R_n,$$

where
$$|R_n| \le \varepsilon \max\left\{\left(\frac{\ln(1/U_{k_n+1,n})}{\ln(n/k_n)}\right)^{\theta+\rho+\delta}, \left(\frac{\ln(1/U_{k_n+1,n})}{\ln(n/k_n)}\right)^{\theta+\rho-\delta}\right\}.$$
Hence, since $|\tilde{A}|$ is a regularly varying function and $nU_{k_n+1,n}/k_n \xrightarrow{\mathbb{P}} 1$,
$$\begin{aligned}\mathcal{F}_{2,n} &= \left\{\left(\frac{\ln(1/U_{k_n+1,n})}{\ln(n/k_n)}\right)^{\theta} + \mathcal{O}\{\tilde{A}(\ln(n/k_n))\}\left(\frac{\ln(1/U_{k_n+1,n})}{\ln(n/k_n)}\right)^{\theta} L_\rho\left(\frac{\ln(1/U_{k_n+1,n})}{\ln(n/k_n)}\right)\right. \\ &+ \left.\mathcal{O}\{\tilde{A}(\ln(n/k_n))R_n\}\right\}\{1+\mathcal{O}\{\tilde{A}(\ln(n/k_n))\}\}. \qquad (29)\end{aligned}$$
Let us now consider the expansion
$$k_n^{1/2}(\mathcal{F}_{2,n}-1) = k_n^{1/2}\left(\mathcal{F}_{2,n} - \left(\frac{\ln(1/U_{k_n+1,n})}{\ln(n/k_n)}\right)^{\theta}\right) + k_n^{1/2}\left(\left(\frac{\ln(1/U_{k_n+1,n})}{\ln(n/k_n)}\right)^{\theta} - 1\right) =: T_{1,n}+T_{2,n}.$$
Since $\ln(U_{k_n+1,n})/\ln(k_n/n) \xrightarrow{\mathbb{P}} 1$, (29) entails that
$$T_{1,n} = \mathcal{O}\{k_n^{1/2}\tilde{A}(\ln(n/k_n))\} = o_{\mathbb{P}}(1),$$
by assumption. Next, Lemma 3 yields $\xi_n := k_n^{1/2}(\ln(1/U_{k_n+1,n}) - \ln(n/k_n)) \xrightarrow{d} \mathcal{N}(0,1)$ and thus
$$T_{2,n} = k_n^{1/2}\left(\left(1+\frac{\xi_n}{k_n^{1/2}\ln(n/k_n)}\right)^{\theta} - 1\right) = o_{\mathbb{P}}(1).$$
To sum up, we have shown that
$$k_n^{1/2}(\mathcal{F}_{2,n}-1) \xrightarrow{\mathbb{P}} 0. \qquad (30)$$
Let us finally focus on $\mathcal{F}_{3,n}$. The mean-value theorem entails that
$$\mu_1\left(\ln(n/k_n), \hat{\theta}_{n,-}^{(M)}\right) - \mu_1\left(\ln(n/k_n), \theta_-\right) = (\hat{\theta}_{n,-}^{(M)} - \theta_-)\dot{\mu}\left(\ln(n/k_n), \theta_{n,-}^*\right),$$
where $\theta_{n,-}^* = \theta_- + \tau(\hat{\theta}_{n,-}^{(M)} - \theta_-)$ for some random value $\tau \in (0,1)$ and
$$\dot{\mu}(t,x) = \frac{\partial}{\partial x}\mu(t,x).$$
It has been shown in the proof of Lemma 5(ii) that $\mathcal{I}_1(t,x) = t^2\dot{\mu}(t,x) \to 1$ as $t \to \infty$, uniformly on all closed intervals included in $(-\infty, 1)$. Hence, under the assumptions of Theorem 4, $\theta_{n,-}^* \xrightarrow{\mathbb{P}} \theta_-$ from Theorem 3 and
$$\begin{aligned}&k_n^{1/2}\ln(n/k_n)\left\{\mu_1\left(\ln(n/k_n), \hat{\theta}_{n,-}^{(M)}\right) - \mu_1\left(\ln(n/k_n), \theta_-\right)\right\} \\ =\ &(\ln(n/k_n))^2\dot{\mu}\left(\ln(n/k_n), \theta_{n,-}^*\right)\frac{k_n^{1/2}}{\ln(n/k_n)}\left(\hat{\theta}_{n,-}^{(M)} - \theta_-\right) \xrightarrow{d} 2P_1 - P2,\end{aligned}$$
from (27). Lemma 5(i) yields
$$k_n^{1/2}(\mathcal{F}_{3,n}-1) \xrightarrow{d} P_2 - 2P_1. \qquad (31)$$
Collecting (28), (30) and (31), Lemma 2 leads to
$$k_n^{1/2}\left(\frac{\hat{a}_n^{(M)}(\ln(n/k_n))}{a(\ln(n/k_n))} - 1\right) \xrightarrow{d} P_2 - P_1 \sim \mathcal{N}(0,2),$$
which is the expected result. ■

Proof of Corollary 1. It is sufficient to show that condition **(A3)** is satisfied by the estimators $\hat{\theta}_n^{(M)}$ and $\hat{a}_n^{(M)}(\ln(n/k_n))$ with $\sigma_n := k_n^{-1/2}\ln(n/k_n)$ and $(\Omega, \Theta, \Lambda) = (0, \Theta, 0)$ where Θ follows a standard Gaussian distribution. First, from Theorem 1,

$$\frac{k_n^{1/2}}{\ln(n/k_n)} \frac{\ln(X_{n-k_n,n}) - \ln Q(k_n/n)}{a(\ln(n/k_n))H_{\theta,0}(d_n)} = \mathcal{O}_{\mathbb{P}}\left(\frac{1}{\ln^2(n/k_n)H_{\theta,0}(d_n)}\right).$$

Clearly, when $d_n \to d \in (1, \infty]$, $\ln^2(n/k_n)H_{\theta,0}(d_n) \to \infty$. When $d_n \to 1$,

$$\ln^2\left(\frac{n}{k_n}\right) H_{\theta,0}(d_n) \sim \frac{1}{2}\ln^2\left(\frac{n}{k_n}\right)(d_n - 1)^2 = \ln^2\left(\frac{k_n}{n\beta_n}\right) \to \infty, \tag{32}$$

since $n\beta_n \to c \geq 0$. As a consequence, if $d_n \to d \in [1, \infty]$,

$$\sigma_n^{-1}\frac{\ln(X_{n-k_n,n}) - \ln Q(k_n/n)}{a(\ln(n/k_n))H_{\theta,0}(d_n)} \xrightarrow{\mathbb{P}} 0. \tag{33}$$

Next, Theorem 3 entails that

$$\sigma_n^{-1}\left(\hat{\theta}_n^{(M)} - \theta\right) \xrightarrow{d} \mathcal{N}(0, 1). \tag{34}$$

Finally, from Theorem 4,

$$\frac{k_n^{1/2}}{\ln(n/k_n)} \frac{L_\theta(d_n)}{H_{\theta,0}(d_n)} \left(\frac{\hat{a}_n^{(M)}(\ln(n/k_n))}{a(\ln(n/k_n))} - 1\right) = \mathcal{O}_{\mathbb{P}}\left(\frac{L_\theta(d_n)}{\ln(n/k_n)H_{\theta,0}(d_n)}\right).$$

If $d_n \to 1$, since $H_{\theta,0}(d_n)/L_\theta(d_n) \sim (d_n - 1)/2$,

$$\frac{L_\theta(d_n)}{\ln(n/k_n)H_{\theta,0}(d_n)} \to 0, \tag{35}$$

as shown in (32). When $d_n \to d > 1$, it is clear that (35) holds. Finally, when $d_n \to \infty$, (20) entails that $L_\theta(d_n)/H_{\theta,0}(d_n) \to -\theta_-$ and thus (35) also holds. To sum up, when $d_n \to d \in [1, \infty]$,

$$\sigma_n^{-1}\frac{L_\theta(d_n)}{H_{\theta,0}(d_n)}\left(\frac{\hat{a}_n^{(M)}(\ln(n/k_n))}{a(\ln(n/k_n))} - 1\right) \xrightarrow{\mathbb{P}} 0, \tag{36}$$

and the conclusion follows from (33), (34) and (36). ■

6.3. Proofs of auxiliary results

Proof of Lemma 2. Let $\varphi : \mathbb{R}^p \mapsto \mathbb{R}$ be a continuously differentiable function. It suffices to show that

$$\sigma_n^{-1}(\varphi(W_n) - \varphi(\lambda)) \xrightarrow{d} W^\top \nabla\varphi(\lambda).$$

Conclusion of the proof will be then straightforward by applying the Cramér-Wold device. The multivariate version of the mean-value theorem leads to

$$\sigma_n^{-1}(\varphi(W_n) - \varphi(\lambda)) = \sigma_n^{-1}(W_n - \lambda)^\top \nabla\varphi(\lambda_n^*),$$

where $\lambda_n^* := (\lambda_{n,1}^*, \ldots, \lambda_{n,p}^*)^\top$ with for all $i \in \{1, \ldots, p\}$, $\lambda_{n,i}^* = \lambda_i + \tau_i(W_{n,i} - \lambda_i)$ where $\tau_i \in (0, 1)$. By assumption, $\lambda_{n,i}^* \xrightarrow{\mathbb{P}} \lambda_i$ and the continuous mapping theorem entails that $\nabla\varphi(\lambda_n^*) \xrightarrow{\mathbb{P}} \nabla\varphi(\lambda)$ and the proof is completed. ■

Proof of Lemma 3. We start with a result due to Smirnov [20] and that can be found for instance in [15, Lemma 2.2.3]. Let (α_n) be a sequence such that $\alpha_n \to 0$ and $n\alpha_n \to \infty$. If $U_1, \dots, U_n$ are independent random variables from a standard uniform distribution,

$$\frac{n^{1/2}}{\alpha_n^{1/2}} \left(U_{\lfloor n\alpha_n \rfloor+1,n} - \alpha_n \right) \xrightarrow{d} \mathcal{N}(0,1). \tag{37}$$

Since $Z_{n-\lfloor n\alpha_n \rfloor,n} \stackrel{d}{=} Q_Z(U_{\lfloor n\alpha_n \rfloor+1,n})$, our aim is to show that

$$\frac{n^{1/2}}{\alpha_n^{1/2} Q_Z'(\alpha_n)} \left(Q_Z(U_{\lfloor n\alpha_n \rfloor+1,n}) - Q_Z(\alpha_n) \right) \xrightarrow{d} \mathcal{N}(0,1).$$

Since Q_Z is a differentiable function, the mean-value theorem leads to

$$Q_Z(U_{\lfloor n\alpha_n \rfloor+1,n}) - Q_Z(\alpha_n) = \left(U_{\lfloor n\alpha_n \rfloor+1,n} - \alpha_n \right) Q_Z'(\alpha_n^*),$$

where $\alpha_n^* := \alpha_n + \tau(U_{\lfloor n\alpha_n \rfloor+1,n} - \alpha_n)$ with $\tau \in (0,1)$. From (37) and since $n\alpha_n \to \infty$,

$$\frac{U_{\lfloor n\alpha_n \rfloor+1,n} - \alpha_n}{\alpha_n} \xrightarrow{\mathbb{P}} 0,$$

and thus $\alpha_n^* = \alpha_n(1 + o_{\mathbb{P}}(1))$. Since $-Q_Z'(1/\cdot)$ is regularly varying,

$$\frac{n^{1/2}}{\alpha_n^{1/2} Q_Z'(\alpha_n)} \left(Q_Z(U_{\lfloor n\alpha_n \rfloor+1,n}) - Q_Z(\alpha_n) \right) = \frac{n^{1/2}}{\alpha_n^{1/2}} \left(U_{\lfloor n\alpha_n \rfloor+1,n} - \alpha_n \right) (1 + o_{\mathbb{P}}(1)) \xrightarrow{d} \mathcal{N}(0,1),$$

and the proof is completed. ■

Proof of Lemma 4. (i) The proof is based on Rényi's representation of standard uniform ordered statistics:

$$U_{k_n+1,n} \stackrel{d}{=} \frac{T_{k_n+1}}{T_{n+1}},$$

where for $j \in \mathbb{N} \setminus \{0\}$, T_j is the sum of j independent standard exponential random variables. The law of large numbers shows that $U_{k_n+1,n} \stackrel{\mathbb{P}}{\sim} k_n/n$ and the conclusion follows.
(ii) Remarking that

$$\left\{ \frac{\ln(U_{i+1,n}/U_{k_n+1,n})}{\ln(U_{k_n+1,n})},\ i = 0, \dots, k_n - 1 \right\} \stackrel{d}{=} \left\{ \frac{E_{n-i,n} - E_{n-k_n,n}}{E_{n-k_n,n}},\ i = 0, \dots, k_n - 1 \right\},$$

the result is then a consequence of the following Rényi's representation:

$$\{E_{j,n},\ j = 1, \dots, n\} \stackrel{d}{=} \left\{ \sum_{r=1}^{j} \frac{F_r}{n - r + 1},\ j = 1, \dots, n \right\}.$$

(iii) It is clear that

$$0 \le \max_{i \in \{0,\dots,k_n-1\}} \frac{F_{k_n-i,k_n}}{E_{n-k_n,n}} \le \frac{F_{k_n,k_n}}{E_{n-k_n,n}}.$$

Using the facts that $k_n^{1/2}(E_{n-k_n,n} - \ln(n/k_n)) \xrightarrow{d} \mathcal{N}(0,1)$ and that $F_{k_n,k_n} - \ln k_n$ converges in distribution to a Gumbel random variable entails

$$\frac{F_{k_n,k_n}}{E_{n-k_n,n}} \stackrel{\mathbb{P}}{\sim} \frac{\ln k_n}{\ln(n/k_n)} \to 0,$$

since $\log(k_n)/\log(n) \to 0$ as $n \to \infty$ and the conclusion follows. ■

Proof of Lemma 5. (i) For $j = 1, \ldots, J$, let

$$R_j(t,s) := tL_{\zeta_j}\left(1 + \frac{\ln(1/s)}{t}\right) - \ln(1/s).$$

Since $2|R_j(t,s)| \le (1-\zeta_j)\ln^2(1/s)/t$, denoting by $\underline{\zeta} := \min\{\zeta_1, \ldots, \zeta_J\}$, one has

$$-\frac{1-\underline{\zeta}}{2}\frac{\ln^2(1/s)}{t} \le R_j(t,s) \le \frac{1-\underline{\zeta}}{2}\frac{\ln^2(1/s)}{t}.$$

Hence

$$\int_0^1 \left(\ln(1/s) - \frac{1-\underline{\zeta}}{2}\frac{\ln^2(1/s)}{t}\right)^J ds \le t^J \mu(t,\zeta) \le \int_0^1 \left(\ln(1/s) + \frac{1-\underline{\zeta}}{2}\frac{\ln^2(1/s)}{t}\right)^J ds.$$

As a consequence,

$$t^J\mu(t,\zeta) - J! \ge \sum_{j=0}^{J-1}(-1)^{J-j}(2J-j)!C_J^j\left(\frac{1-\underline{\zeta}}{2}\right)^{J-j}\frac{1}{t^{J-j}} \to 0,$$

uniformly for any hyper-rectangle included in $(-\infty, 1)^J$. Similarly,

$$t^J\mu(t,\zeta) - J! \le \sum_{j=0}^{J-1}(2J-j)!C_J^j\left(\frac{1-\underline{\zeta}}{2}\right)^{J-j}\frac{1}{t^{J-j}} \to 0,$$

uniformly locally and the proof is completed.
(ii) It is easily seen that

$$t(t^2\mu_2(t,x))^2\Psi_t'(x) = 2(t\mu_1(t,x))(t^2\mu_2(t,x))\mathcal{I}_1(t,x) - (t\mu_1(t,x))^2\mathcal{I}_2(t,x),$$

with

$$\begin{aligned}
\mathcal{I}_1(t,x) &= t^2\dot{\mu}_1(t,x) = \int_0^1 t^2\dot{L}_x\left(1+\frac{\ln(1/s)}{t}\right) ds \text{ and} \\
\mathcal{I}_2(t,x) &= t^2\dot{\mu}_2(t,x) = 2\int_0^1 t^2\dot{L}_x\left(1+\frac{\ln(1/s)}{t}\right) tL_x\left(1+\frac{\ln(1/s)}{t}\right) ds,
\end{aligned}$$

and where the following notations have been introduced

$$\dot{L}_x(u) := \frac{\partial}{\partial x}L_x(u) \text{ and } \dot{\mu}_b(t,x) := \frac{\partial}{\partial x}\mu_b(t,x),\ b \in \{1,2\}.$$

The first step consists in studying the quantities $L_x(1+u)$ and $\dot{L}_x(1+u)$ for $u \ge 0$ and $x < 1$. A Taylor expansion leads to

$$L_x(1+u) = u + \frac{x-1}{2}u^2 + R_x(u), \tag{38}$$

where

$$0 \le R_x(u) \le \frac{(x-1)(x-2)}{6}u^3. \tag{39}$$

Next, an integration by part entails

$$\dot{L}_x(1+u) = \frac{1}{x}\left(\ln(1+u)(1+u)^x - L_x(1+u)\right). \tag{40}$$

Let us note that when $x = 0$,

$$\dot{L}_0\,(1+u) = \lim_{x\to 0} \dot{L}_x\,(1+u) = \frac{1}{2}\ln^2(1+u).$$

Using (38), (40) and remarking that $R_x(u) = \ln(1+u) - u + u^2/2$ yield

$$\dot{L}_x\,(1+u) = \frac{u^2}{2} + \bar{R}_x(u), \tag{41}$$

where

$$\begin{aligned} \bar{R}_x(u) &= \frac{x-2}{2}u^3 - \frac{x-1}{4}u^4 + u(R_x(u)+R_0(u)) - \frac{1}{2}R_x(u)u^2 + \frac{R_0(u)-R_x(u)}{x} \\ &+ \frac{x-1}{2}R_0(u)u^2 + R_x(u)R_0(u). \end{aligned} \tag{42}$$

Taking account of

$$\lim_{x\to 0} \frac{R_0(u)-R_x(u)}{x} = -\dot{K}_0(1+u) + \frac{u^2}{2} = -\frac{1}{2}\ln^2(1+u) + \frac{u^2}{2},$$

it follows that

$$\bar{R}_0(u) = \lim_{x\to 0} \bar{R}_x(u) = \dot{L}_0(1+u) - \frac{u^2}{2} = \frac{u^4}{8} + \frac{R_0^2(u)}{2} - \frac{u^3}{2} + uR_0(u) - \frac{u^2 R_0(u)}{2}.$$

The second step is to focus on the integral $\mathcal{I}_1(t,x)$. From (41), it can be rewritten as

$$\mathcal{I}_1(t,x) = 1 + \int_0^1 t^2 \bar{R}_x\left(\frac{\ln(1/s)}{t}\right) ds.$$

Now, using (42),

$$\begin{aligned} \int_0^1 t^2 \bar{R}_x\left(\frac{\ln(1/s)}{t}\right) ds &= \frac{3(x-2)}{t} - \frac{6(x-1)}{t^2} + \int_0^1 \ln(1/s)t\left(R_x\left(\frac{\ln(1/s)}{t}\right) + R_0\left(\frac{\ln(1/s)}{t}\right)\right) ds \\ &- \frac{1}{2}\int_0^1 \ln^2(1/s)R_x\left(\frac{\ln(1/s)}{t}\right) ds + \frac{1}{x}\int_0^1 t^2\left(R_0\left(\frac{\ln(1/s)}{t}\right) - R_x\left(\frac{\ln(1/s)}{t}\right)\right) ds \\ &+ \frac{x-1}{2}\int_0^1 \ln^2(1/s)R_0\left(\frac{\ln(1/s)}{t}\right) ds + \int_0^1 t^2 R_x\left(\frac{\ln(1/s)}{t}\right) R_0\left(\frac{\ln(1/s)}{t}\right) ds. \end{aligned}$$

It is clear that the first two terms converge to 0 as $t \to \infty$ uniformly on $x \in I$. Considering the third term, one has by (39) that

$$0 \le \int_0^1 \ln(1/s)t\left(R_x\left(\frac{\ln(1/s)}{t}\right) + R_0\left(\frac{\ln(1/s)}{t}\right)\right) ds \le \frac{4(x-1)(x-2)}{t^2} + \frac{8}{t^2}.$$

As a consequence, the third term also converges to 0 as $t \to \infty$ uniformly on $x \in I$. A similar proof can be done for the fifth, sixth and seventh terms. Considering the fourth term, let us remark that the function $x \to x^{-1}(R_0(u) - R_x(u))$ is decreasing for all $u > 0$. Thus, for all $x \in I =: [x_1, x_2]$,

$$\mathcal{J}_{x_2}(t) \le \frac{1}{x}\int_0^1 t^2\left(R_0\left(\frac{\ln(1/s)}{t}\right) - R_x\left(\frac{\ln(1/s)}{t}\right)\right) ds \le \mathcal{J}_{x_1}(t)$$

where we have introduced

$$\mathcal{J}_x(t) := \frac{1}{x}\int_0^1 t^2\left(R_0\left(\frac{\ln(1/s)}{t}\right) - R_x\left(\frac{\ln(1/s)}{t}\right)\right) ds$$

for $x \neq 0$ and

$$\mathcal{J}_0(t) := \lim_{x\to 0} \mathcal{J}_x(t) = -\frac{1}{2}\int_0^1 t^2 \ln^2\left(1+\frac{\ln(1/s)}{t}\right) ds + 1.$$

From (39), one can show that $\mathcal{J}_x(t) \to 0$ as $t \to \infty$ uniformly on $x \in I$. Finally $\mathcal{I}_1(t,x) \to 1$ as $t \to \infty$, uniformly on $x \in I$. A similar proof can be done to show that $\mathcal{I}_2(t,x) \to 6$ as $t \to \infty$, uniformly on $x \in I$. Moreover, Lemma 5(i) implies that $t(t^2\mu_2(t,x))^2\Psi_t'(x) \to -2$ and that $(t^2\mu_2(t,x))^2 \to 4$ as $t \to \infty$ uniformly on $x \in I$. The result is thus proved: $\Psi_t'(x) \to -1/2$ as $t \to \infty$ uniformly on $x \in I$. ■

Proof of Lemma 6. (i) For $i = 1, \dots, m$, let

$$Y_{m,i} := t_m^{J-1} F_i \left(1+\frac{F_i}{t_m}\right)^{\zeta_1-1} \prod_{j=2}^{J} L_{\zeta_j}\left(1+\frac{F_i}{\delta t_m}\right).$$

Let $\underline{\zeta} := \min\{\xi_1, \xi_2\}$. The following inequalities hold for all $i = 1, \dots, m$:

$$\frac{F_i}{\delta t_m} + \frac{\underline{\zeta}-1}{2}\frac{F_i^2}{\delta^2 t_m^2} \le \frac{F_i}{\delta t_m} + \frac{\zeta_j-1}{2}\frac{F_i^2}{\delta^2 t_m^2} \le L_{\zeta_j}\left(1+\frac{F_i}{\delta t_m}\right) \le \frac{F_i}{\delta t_m}.$$

Since $\max\{F_1, \dots, F_m\} - \ln(m)$ converges in distribution to a Gumbel random variable,

$$0 \le \max_{1\le i\le m} \frac{F_i}{t_m} = \frac{\ln(m)}{t_m}\left(1+\mathcal{O}_{\mathbb{P}}\left(\frac{1}{\ln(m)}\right)\right) = o_{\mathbb{P}}(1), \tag{43}$$

by assumption. It follows that uniformly in $i \in \{1, \dots, m\}$,

$$Y_{m,i} = \delta^{1-J} F_i^J \left(1 + o_{\mathbb{P}}(1)\right).$$

The law of large numbers entails that

$$\frac{1}{m}\sum_{i=1}^{m} F_i^J \xrightarrow{\mathbb{P}} \mathbb{E}(F_1^J) = J!,$$

and the conclusion follows.
(ii) Since for all $\xi < 1$ and $u > 0$,

$$L_\xi(1+u) = u - \frac{\xi-1}{2}u^2 + R_\xi(u),$$

with $0 \le R_\xi(u) \le (\xi-1)(\xi-2)u^3/6$ and taking into account of (43), it follows that uniformly in $i \in \{1, \dots, m\}$

$$L_{\xi_3}\left(1+\frac{F_i}{\delta t_m}\right) - L_{\xi_4}\left(1+\frac{F_i}{\delta t_m}\right) = \frac{\xi_3-\xi_4}{2}\left(\frac{F_i}{\delta t_m}\right)^2 (1+o_{\mathbb{P}}(1)).$$

The rest of the proof follows the same lines as the one of (i) and is thus omitted. ■

Proof of Lemma 7. Using the Cramér-Wold device, it suffices to obtain the asymptotic distribution of

$$T_n := k_n^{1/2}\left\{\beta_1 \frac{S_n(\zeta^{(1)})}{\mu(\ln(n/k_n), \zeta^{(1)})} + \beta_2 \frac{S_n(\zeta^{(2)})}{\mu(\ln(n/k_n), \zeta^{(2)})} - (\beta_1+\beta_2)\right\},$$

where $(\beta_1, \beta_2) \in \mathbb{R}^2$. Let us introduce the random processes indexed by $t > 0$

$$W_{i,n}(t) := \frac{\beta_1}{\mu(\ln(n/k_n), \zeta^{(1)})} \prod_{j=1}^{J_1} L_{\zeta_j^{(1)}} \left(1 + \frac{F_i}{t}\right) + \frac{\beta_2}{\mu(\ln(n/k_n), \zeta^{(2)})} \prod_{j=1}^{J_2} L_{\zeta_j^{(2)}} \left(1 + \frac{F_i}{t}\right),$$

for $i = 1, \ldots, k_n$ and where $F_1, \ldots, F_{k_n}$ are independent standard exponential random variables. Lemma 4(ii) yields

$$T_n \stackrel{d}{=} k_n^{-1/2} \sum_{i=1}^{k_n} \left\{W_{i,n}(E_{n-k_n,n}) - \mathbb{E}\left(W_{i,n}(\ln(n/k_n))\right)\right\},$$

where $E_{n-k_n,n}$ is the $(n-k_n)$th ordered statistic associated to a sample $E_1, \ldots, E_n$ of standard exponential random values independent of $F_1, \ldots, F_{k_n}$. Let us consider the following expansion $T_n =: T_{n,1} + T_{n,2}$ with

$$T_{n,1} := k_n^{-1/2} \sum_{i=1}^{k_n} \bar{W}_{i,n}(\ln(n/k_n))$$

where $\bar{W}_{i,n}(\ln(n/k_n)) := W_{i,n}(\ln(n/k_n)) - \mathbb{E}\left(W_{i,n}(\ln(n/k_n))\right)$ and

$$T_{n,2} := k_n^{-1/2} \sum_{i=1}^{k_n} \left\{W_{i,n}(E_{n-k_n,n}) - W_{i,n}(\ln(n/k_n))\right\}.$$

The asymptotic normality of the random term $T_{n,1}$ is obtained by Lyapunov's theorem. Let us observe that

$$s_n^2 := \operatorname{Var}\left(\sum_{i=1}^{k_n} \bar{W}_{i,n}(\ln(n/k_n))\right) = k_n \left\{\mathbb{E}\left(W_{1,n}^2(\ln(n/k_n))\right) - (\beta_1 + \beta_2)^2\right\}.$$

Straightforward calculations then lead to

$$\begin{aligned}\mathbb{E}\left(W_{1,n}^2(\ln(n/k_n))\right) &= \beta_1^2 \frac{\mu(\ln(n/k_n), (\zeta^{(1)}, \zeta^{(1)}))}{\mu^2(\ln(n/k_n), \zeta^{(1)})} + \beta_2^2 \frac{\mu(\ln(n/k_n), (\zeta^{(2)}, \zeta^{(2)}))}{\mu^2(\ln(n/k_n), \zeta^{(2)})} \\ &+ 2\beta_1\beta_2 \frac{\mu(\ln(n/k_n), (\zeta^{(1)}, \zeta^{(2)}))}{\mu(\ln(n/k_n), \zeta^{(1)})\mu(\ln(n/k_n), \zeta^{(2)})}.\end{aligned}$$

As a direct consequence of Lemma 5(i), one has

$$\lim_{n\to\infty} \mathbb{E}\left(W_{1,n}^2(\ln(n/k_n))\right) = \frac{(2J_1)!}{(J_1!)^2}\beta_1^2 + \frac{(2J_2)!}{(J_2!)^2}\beta_2^2 + 2\frac{(J_1+J_2)!}{J_1!J_2!}\beta_1\beta_2, \tag{44}$$

and thus $s_n^2 \sim c(\beta_1, \beta_2)k_n$ as $n \to \infty$ where the constant $c(\beta_1, \beta_2)$ is given by

$$c(\beta_1, \beta_2) := ((2J_1)!/(J_1!)^2 - 1)\beta_1^2 + ((2J_2)!/(J_2!)^2 - 1)\beta_2^2 + 2((J_1+J_2)!/(J_1!J_2!) - 1)\beta_1\beta_2.$$

Let us now check Lyapunov's condition *i.e.* that

$$\frac{1}{k_n^2} \sum_{i=1}^{k_n} \mathbb{E}\left(\bar{W}_{i,n}^4(\ln(n/k_n))\right) = \frac{1}{k_n}\mathbb{E}\left(\bar{W}_{1,n}^4(\ln(n/k_n))\right) \to 0, \tag{45}$$

as $n \to \infty$. By similar arguments as the ones leading to (44), one can show that

$$\mathbb{E}\left(\bar{W}_{1,n}^4(\ln(n/k_n))\right) = \sum_{l=1}^{4} (-1)^l C_4^l \mathbb{E}\left(W_{1,n}^l(\ln(n/k_n))\right) \mathbb{E}^{4-l}\left(W_{1,n}(\ln(n/k_n))\right)$$

converges to a constant as $n \to \infty$ and thus (45) holds. As a conclusion

$$T_{n,1} \xrightarrow{d} \mathcal{N}\left(0, c(\beta_1, \beta_2)\right). \tag{46}$$

It remains to prove that $T_{n,2} \xrightarrow{\mathbb{P}} 0$. For $i = 1, \ldots, k_n$, let $\dot{W}_{i,n}(\cdot)$ be the first derivative of the random function $W_{i,n}(\cdot)$. The mean-value theorem entails that

$$W_{i,n}(E_{n-k_n,n}) - W_{i,n}(\ln(n/k_n)) = \dot{W}_{i,n}(E^*_{n,i})(E_{n-k_n,n} - \ln(n/k_n)),$$

where for $i = 1, \ldots, k_n$, $E^*_{n,i} = \ln(n/k_n) + \Theta_{n,i}(E_{n-k_n,n} - \ln(n/k_n))$ with $\Theta_{n,i}$ a random variable in $(0,1)$. Recalling that $k_n^{1/2}(E_{n-k_n,n} - \ln(n/k_n)) \xrightarrow{d} \mathcal{N}(0,1)$, in order to show that $T_{n,2} \xrightarrow{\mathbb{P}} 0$, it suffices to prove that

$$\frac{1}{k_n}\sum_{i=1}^{k_n} \dot{W}_{i,n}(E^*_{n,i}) \xrightarrow{\mathbb{P}} 0. \tag{47}$$

First, simple calculations show that for all $t > 0$,

$$\begin{aligned} -\frac{1}{k_n}\sum_{i=1}^{k_n} \dot{W}_{i,n}(t) &= \frac{\beta_1}{t^2\mu(\ln(n/k_n), \zeta^{(1)})}\sum_{l=1}^{J_1}\left(\frac{1}{k_n}\sum_{i=1}^{k_n} F_i\left(1+\frac{F_i}{t}\right)^{\zeta_l^{(1)}-1}\prod_{j\neq l} L_{\zeta_j^{(1)}}\left(1+\frac{F_i}{t}\right)\right) \\ &+ \frac{\beta_2}{t^2\mu(\ln(n/k_n), \zeta^{(2)})}\sum_{l=1}^{J_2}\left(\frac{1}{k_n}\sum_{i=1}^{k_n} F_i\left(1+\frac{F_i}{t}\right)^{\zeta_l^{(2)}-1}\prod_{j\neq l} L_{\zeta_j^{(2)}}\left(1+\frac{F_i}{t}\right)\right). \end{aligned}$$

Hence, we have to deal with random terms proportional to $T_{3,n}(t,t)$ where

$$T_{3,n}(t_1, t_2) := \frac{1}{t_2^2\mu(\ln(n/k_n), \zeta)}\frac{1}{k_n}\sum_{i=1}^{k_n} F_i\left(1+\frac{F_i}{t_1}\right)^{\zeta_1 - 1}\prod_{j=2}^{J} L_{\zeta_j}\left(1+\frac{F_i}{t_2}\right)$$

with $J \in \mathbb{N}\setminus\{0\}$ and $\zeta \in \mathbb{R}^J$. To prove (47) let us check that

$$\bar{T}_{3,n} := \frac{1}{(E^*_{n,i})^2\mu(\ln(n/k_n), \zeta)}\frac{1}{k_n}\sum_{i=1}^{k_n} F_i\left(1+\frac{F_i}{E^*_{n,i}}\right)^{\zeta_1-1}\prod_{j=2}^{J} L_{\zeta_j}\left(1+\frac{F_i}{E^*_{n,i}}\right) \xrightarrow{\mathbb{P}} 0. \tag{48}$$

For all $\eta > 0$, let $0 < \varepsilon < \min\{\eta_+, \eta_-\}$ where $\eta_+ = 1 - ((1+\eta/2)/(1+\eta))^{1/2}$ and $\eta_- = ((1-\eta/2)/(1-\eta))^{1/2} - 1$. Let us also introduce the Borel set

$$A_{n,\varepsilon} = \left\{\left|\frac{E_{n-k_n,n} - \ln(n/k_n)}{\ln(n/k_n)}\right| \leq \varepsilon\right\}. \tag{49}$$

For all $\eta > 0$, remark that

$$\mathbb{P}\left(|\bar{T}_{n,3}| > \eta\right) \leq \mathbb{P}(A^{\mathrm{C}}_{n,\varepsilon}) + \mathbb{P}\left(\{|\bar{T}_{n,3}| > \eta\} \cap A_{n,\varepsilon}\right) \tag{50}$$

and recall that, for $i = 1, \ldots, k_n$, $E^*_{n,i} = \ln(n/k_n) + \Theta_{i,n}(E_{n-k_n,n} - \ln(n/k_n))$. Since $\Theta_{n,i} \in (0,1)$ it is clear that under $A_{n,\varepsilon}$, one has

$$(1-\varepsilon)\ln(n/k_n) \leq E^*_{n,i} \leq (1+\varepsilon)\ln(n/k_n)$$

for all $i = 1, \ldots, k_n$. Hence

$$T_{n,3}\left((1-\varepsilon)\ln(n/k_n), (1+\varepsilon)\ln(n/k_n)\right) \leq \bar{T}_{n,3} \leq T_{n,3}\left((1+\varepsilon)\ln(n/k_n), (1-\varepsilon)\ln(n/k_n)\right),$$

and thus

$$\mathbb{P}\left(\{|\bar{T}_{n,3}|>\eta\}\cap A_{n,\varepsilon}\right) \leq \mathbb{P}\left(T_{n,3}\left((1+\varepsilon)\ln(n/k_n),(1-\varepsilon)\ln(n/k_n)\right)>\eta\right) + \mathbb{P}\left(T_{n,3}\left((1-\varepsilon)\ln(n/k_n),(1+\varepsilon)\ln(n/k_n)\right)<-\eta\right).$$

Applying Lemma 6(i) and the fact that from Lemma 5(i),

$$\frac{\ln^{1-J}(n/k_n)}{\ln^2(n/k_n)\mu(\ln(n/k_n),\zeta)} \sim \frac{1}{J!}\frac{1}{\ln(n/k_n)} \to 0,$$

it follows that $T_{n,3}\left((1\pm\varepsilon)\ln(n/k_n),(1\mp\varepsilon)\ln(n/k_n)\right) \xrightarrow{\mathbb{P}} 0$. As a consequence,

$$\mathbb{P}\left(\{|\bar{T}_{n,3}|>\eta\}\cap A_{n,\varepsilon}\right) \to 0 \tag{51}$$

as $n\to\infty$. Furthermore, since $(E_{n-k_n,n}-\ln(n/k_n))/\ln(n/k_n) = \mathcal{O}_{\mathbb{P}}(k_n^{-1/2}\ln^{-1}(n/k_n)) = o_{\mathbb{P}}(1)$, one has that $\mathbb{P}(A_{n,\varepsilon}) \xrightarrow{\mathbb{P}} 1$ as $n\to\infty$. Collecting this last result, (50) and (51) implies (48) and thus (47) and the conclusion follows. ■

Proof of Lemma 8. Let $E_1,\dots,E_n,F_1,\dots,F_n$ be a sample of $2n$ independent standard exponential random variables and let $E_{n-k_n,n}$ be the $(n-k_n)$th ordered statistic associated with the sample $E_1,\dots,E_n$. Let us also introduce the random variable defined for all $t>0$ by

$$W_n(t) := \frac{(\ln(n/k_n))^J}{k_n}\sum_{i=1}^{k_n} L_{\xi_1}^{J_1}\left(1+\frac{F_i}{t}\right)L_{\xi_2}^{J_2}\left(1+\frac{F_i}{t}\right)\left(L_{\xi_3}\left(1+\frac{F_i}{t}\right)-L_{\xi_4}\left(1+\frac{F_i}{t}\right)\right)^{J_3}.$$

According to Lemma 4(ii), we have to prove that $W_n(E_{n-k_n,n}) \xrightarrow{\mathbb{P}} K := J!((\xi_3-\xi_4)/2)^{J_3}$. For all $\varepsilon>0$, let us consider the Borel set

$$\mathcal{A}_{n,\varepsilon} = \left\{\left|\frac{E_{n-k_n,n}}{\ln(n/k_n)}-1\right|\leq\varepsilon\right\},$$

introduced in (49). Remarking that the function $x\mapsto L_{\xi_1}^{J_1}(x)L_{\xi_2}^{J_2}(x)(L_{\xi_3}(x)-L_{\xi_4}(x))^{J_3}$ is increasing on $(1,\infty)$ leads to

$$\mathbb{P}\left\{|W_n(E_{n-k_n,n})-K|>\varepsilon\right\} \leq \mathbb{P}\left\{W_n((1+\varepsilon)\ln(n/k_n))>K+\varepsilon\right\} + \mathbb{P}\left\{W_n((1-\varepsilon)\ln(n/k_n))<K-\varepsilon\right\}+1-\mathbb{P}(\mathcal{A}_{n,\varepsilon}). \tag{52}$$

Using the inequality $K+\varepsilon-K/(1+\varepsilon)^J\geq\varepsilon$ yields

$$\mathbb{P}\left\{W_n((1+\varepsilon)\ln(n/k_n))>K+\varepsilon\right\} \leq \mathbb{P}\left\{W_n((1+\varepsilon)\ln(n/k_n))-\frac{K}{(1+\varepsilon)^J}>\varepsilon\right\}.$$

Since $\ln(k_n)/\ln(n)\to 0$, one can apply Lemma 6(ii) with $m=k_n$ and $t_m=\ln(n/k_n)$ to obtain

$$\lim_{n\to\infty}\mathbb{P}\left\{W_n((1+\varepsilon)\ln(n/k_n))>K+\varepsilon\right\}=0. \tag{53}$$

In the same way, one has

$$\lim_{n\to\infty}\mathbb{P}\left\{W_n((1-\varepsilon)\ln(n/k_n))<K-\varepsilon\right\}=0. \tag{54}$$

Finally, since $E_{n-k_n,n}/\ln(n/k_n) \xrightarrow{\mathbb{P}} 1$, we conclude the proof by collecting (52) to (54). ■