A note on track-length sampling with non-exponential distributions

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Abstract

Track-length sampling is the process of sampling random intervals according to a distance distribution. It means that, instead of sampling a punctual distance from the distance distribution, track-length sampling generates an interval of possible distances. The track-length sampling process is correct if the expectation of the intervals is the target distance distribution. In other words, averaging all the sampled intervals should converge towards the distance distribution as their number increases.

In this note, we emphasize that the *distance distribution* that is used for sampling punctual distances and the *track-length distribution* that is used for sampling intervals *are not the same in general*. This difference can be surprising because, to our knowledge, track-length sampling has been mostly studied in the context of transport theory where the distance distribution is typically exponential: in this special case, the distance distribution and the track-length distribution happens to be both the same exponential distribution. However, they are not the same in general when the distance distribution is non-exponential.

We show that track-length sampling can be used with non-exponential distance distributions if they are monotonically decreasing and we derive the general expression of the track-length distribution.

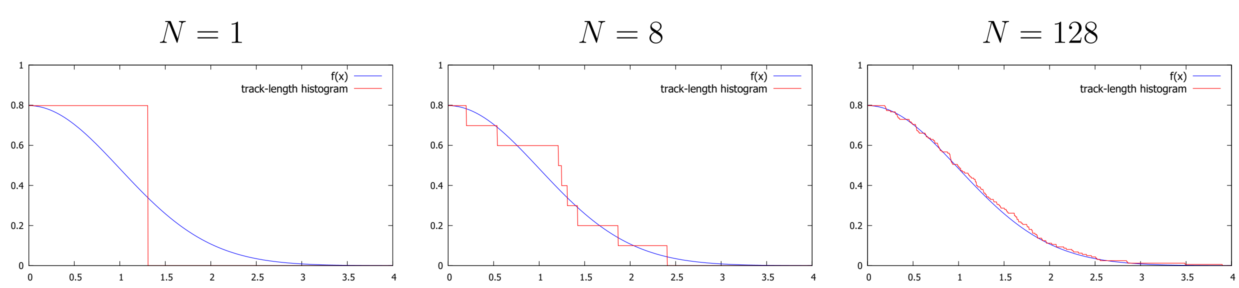

Figure 1: Track-length sampling of a truncated Gaussian distribution. The average of the sampled intervals converges towards the distance distribution as the number of intervals increases.

1 Introduction

There exists two Monte-Carlo estimators for estimating transmittance [Spa66]. On the one hand, the *collision estimator* average vertices of transport paths in a given interval. On the other hand, the *track-length estimator* averages the path edges and provide a less noisy estimate. In the context of exponential transport, the intercollision length leads to an unbiased estimate of the particles interacting flux. There, the average interaction distance in a volume equals the average number of collisions in this volume. For example, Jarosz *et al.* [JNT+11] used this notion to estimate the transmittance of photon beams using *Woodcock tracking*. Multiple intercollision samples are used together to build a piecewise constant estimate of the transmittance (as in Figure 1). However, we will show that this estimate works when the transmittance is an exponential distribution but not in the general case of non-exponential transport. There the average interaction distance distribution in a volume will not be distributed proportionally to the transmittance.

Distance distribution We note $f(x)$ the distance probability density function (PDF) that represents the histogram of distances that a particle can travel. By convention, the distribution f starts at 0 since distances are positive. A typical way for generating random distances x from f is to use the inverse-CDF method:

$$x = F^{-1}(U) \tag{1}$$

where $F(U) = \int_0^U f(u) \mathrm{d}u$ is the cumulative distribution function (CDF) and U is a uniform random number. The point samples generated in this way are such that their histogram converges towards f as the number of samples increases.

Track-length distribution We note $p(x)$ the track-length PDF from which intervals $I = [0, x]$ of possible distances can be sampled. The sampled intervals $I_n = [0, x_n]$ should be such that their average converges towards f as the number of intervals increases:

$$\lim_{N \to \infty} \frac{c}{N} \sum_{n=1}^{N} I_n(x) = f(x) \tag{2}$$

where $I_n(x)$ returns 1 if x is in interval I_n, i.e if $x < x_n$, and 0 otherwise. The constant $c = f(0)$ ensures correct normalization. We derive it in Equation (7).

2 Derivation of the track-length distribution

Constraint For Equation (2) to work, we need that

$$\int_x^{\infty} p(t)\, \mathrm{d}t = \frac{1}{c}\, f(x). \tag{3}$$

where $c > 0$ is a normalization factor. By deriving Equation (3), we obtain that:

$$p(x) = -\frac{1}{c}\, f'(x). \tag{4}$$

Monotonically decreasing Equation (4) shows that p and f' have opposite signs. Since p is a PDF it should be non-negative everywhere, i.e. track-length sampling is possible only if f' is negative everywhere:

$$\boxed{\forall x \in [0, \infty),\ f'(x) \leq 0.} \tag{5}$$

This means that f is a monotonically decreasing function that starts at 0 and decreases towards 0 ($\lim_{x\to\infty} f(x) = 0$) since it integrates to 1.

Normalization coefficient Since p is a PDF, the normalization constant c must be such that

$$1 = \int_{x_0}^{\infty} p(x) = \int_{x_0}^{\infty} -\frac{1}{c} f'(x) \tag{6}$$

By solving this equation, we obtain the normalization coefficient

$$\boxed{c = f(x_0),} \tag{7}$$

that is required for scaling the averaged intervals in Equation (2).

Expression Finally, by replacing c in Equation (4) we obtain the expression of the track-length distribution:

$$\boxed{p(x) = -\frac{f'(x)}{f(x_0)}.} \tag{8}$$

3 Sampling the track-length distribution

In order to sample intervals $I = [0, x]$ we need to sample x from the track-length distribution p. A typical method to achieve this is to use the inverse-CDF method, i.e. find x such that for a given random number U

$$U = \int_0^x p(t)\, dt, \tag{9}$$

Given the definition of p from Equation (8), we obtain that

$$\int_0^x p(t)\, dt = -\frac{1}{f(0)} \int_0^x f'(t)\, dt \tag{10}$$

$$= \frac{f(0) - f(x)}{f(0)}, \tag{11}$$

and we need to solve

$$U = \frac{f(0) - f(x)}{f(0)} \tag{12}$$

whose solution is

$$\boxed{x = f^{-1}\left[f(0)\ (1 - U)\right].} \tag{13}$$

Note that the inverse f^{-1} is well-defined when f is a strictly monotonically decreasing function.

4 Examples

4.1 Exponential distribution

The exponential distance distribution is the one that is usually used for track-length sampling [Spa66]. The distribution is defined by

$$f(x) = \lambda \exp(-\lambda\, x). \tag{14}$$

By applying Equation (8) we obtain the track-length distribution:

$$p(x) = \lambda \exp(-\lambda\, x) \tag{15}$$

Using an exponential distance distribution is thus a special case where the distance distribution and the track-length distribution are the same: $f(x) = p(x)$. Hence, sampling random punctual distances x or random distance intervals $I = [0, x]$ is done in the same way. Using Equation (13), sampling either a punctual distance x or an interval $I = [0, x]$ is done by computing

$$x = -\frac{1}{\lambda} \log\left(1 - U\right). \tag{16}$$

Figure 2 shows the convergence of track-length sampling applied on an exponential distribution.

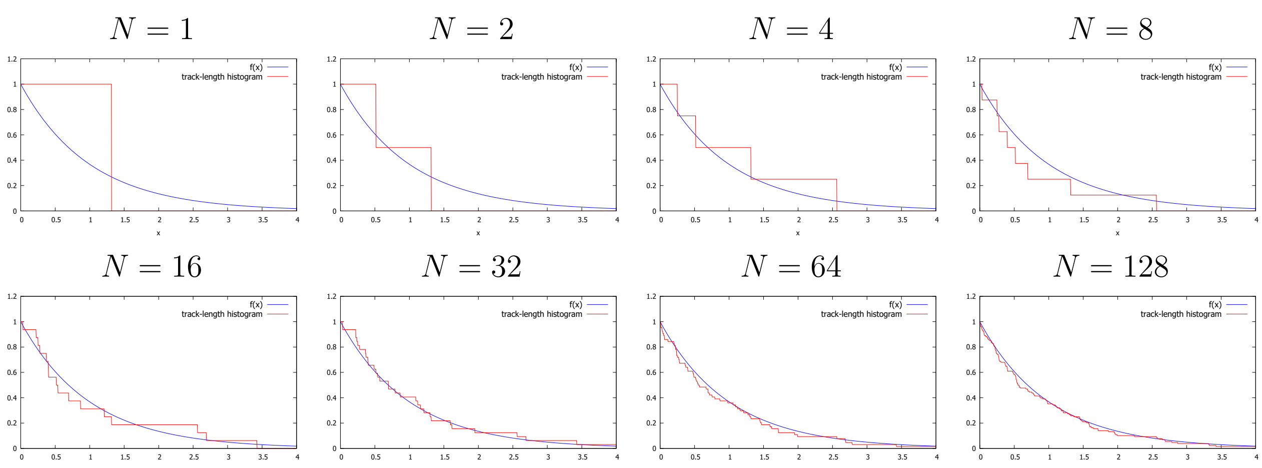

Figure 2: Track-length sampling of an exponential distribution.

4.2 Truncated Gaussian distribution

Now we look at a non-exponential distribution, for instance a truncated Gaussian distribution:

$$f(x) = \frac{2}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{17}$$

By applying Equation (8) we obtain:

$$p(x) = 2\,x\,\exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{18}$$

Note that in this case, the distance distribution and the track-length distribution are different: $f(x) \neq p(x)$. Hence, sampling random punctual distances x or random distance intervals $I = [0, x]$ are not done in the same way. Using Equation (13), we obtain that random intervals $I = [0, x]$ can be sampled by computing:

$$x = \sqrt{-2\log\left[f(x_0)\,(1-U)\,\frac{\sigma\sqrt{2\pi}}{2}\right]}. \tag{19}$$

Figure 3 shows the convergence of track-length sampling applied on a truncated Gaussian distribution.

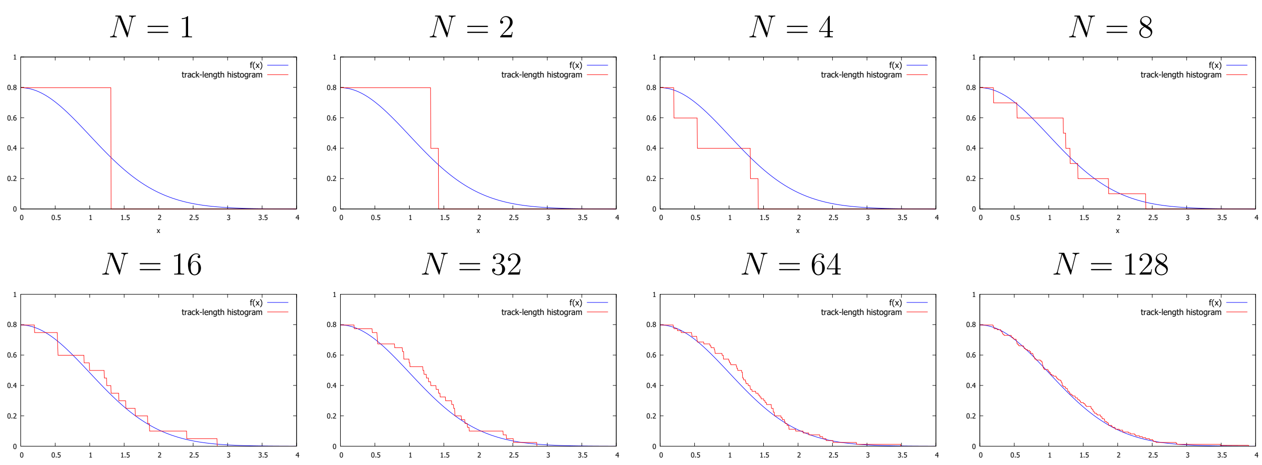

Figure 3: Track-length sampling of a truncated Gaussian distribution.

References

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