Eternal Domination in Strong Grids

Fionn Mc Inerney, Nicolas Nisse, Stéphane Pérennes

Eternal Domination in Grids

Fionn Mc Inerney[1], Nicolas Nisse[1], and Stéphane Pérennes[1]

[1]Université Côte d'Azur, Inria, CNRS, I3S, France

Abstract

In the eternal domination game played on graphs, an attacker attacks a vertex at each turn and a team of guards must move a guard to the attacked vertex to defend it. The guards may only move to adjacent vertices on their turn. The goal is to determine the eternal domination number of a graph which is the minimum number of guards required to defend against an infinite sequence of attacks.

This paper continues the study of the eternal domination game on strong grids $P_n \boxtimes P_m$. Cartesian grids have been vastly studied with tight bounds existing for small grids such as $k \times n$ grids for $k \in \{2, 3, 4, 5\}$. It was recently proven that the eternal domination number of these grids in general is within an additive factor of $O(n + m)$ of their domination number which lower bounds the eternal domination number. Recently, Finbow et al. proved that the eternal domination number of strong grids is upper bounded by $\frac{nm}{6} + O(n + m)$. There exists a gap still between the lower and upper bound as the domination number (current lower bound) of the strong grid is $\lceil \frac{nm}{9} \rceil$. We prove that, for all $n, m \in \mathbb{N}^*$ such that $m \geq n$, $\lfloor \frac{nm}{9} \rfloor + \Omega(n+m) = \gamma^{\infty}_{all}(P_n \boxtimes P_m) = \lceil \frac{nm}{9} \rceil + O(m\sqrt{n})$.

Keywords: Eternal Domination, Combinatorial Games, Graphs, Graph Protection

1 Introduction

1.1 Background

The eternal domination game and its variants are graph protection models. The edges or vertices of the graph can be protected with our paper only considering protecting the latter. The origins of the game date back to the 1990's where the 4 papers [1],[19], [21], and [20] studied the military strategy of Emperor Constantine for defending the Roman Empire in a mathematical setting. Roughly, there are a limited number of armies at one's disposal and one wants that an army can always move to defend against an attack by invaders.

This paper considers the "all guards move" model of the eternal domination game introduced in [11]. Initially, k guards are placed on the vertices of a graph G. The variant where any number of guards may occupy the same vertex is first considered with the variant where at most one guard may occupy a vertex being studied in Section 6. Turn-by-turn an attacker chooses and attacks any vertex of G and the guards must move a guard to the attacked vertex to defend it, otherwise, they lose. Each of the guards may move to an adjacent vertex or stay idle on the guards' turn. The guards must defend against an infinite sequence of attacks to win and therefore, must always occupy a dominating set, *i.e.*, a set of vertices S such that for every vertex $v \in V(G)$, $v \in S$ or v is adjacent to a vertex $u \in S$. The goal is to find the minimum number of guards required to guarantee winning against the attacker on a graph G, denoted by $\gamma_{all}^{\infty}(G)$ [11].

The original eternal domination game, first considered in [4], had the stipulation that at most one guard could move each turn. There have been many variants of the game since then. Eternal total domination was studied in [16], where a total dominating set must be maintained by the guards each turn. A total dominating set is a dominating set S such that every vertex $u \in S$ is also dominated by another vertex $w \in S$ such that $u \neq w$. The eviction model of eternal domination was studied in [14], where a vertex containing a guard is attacked each turn, which forces the guard to move to an adjacent empty vertex with the condition that the guards must maintain a dominating set each turn. The authors of the current paper studied the Spy Game which was introduced in [5] and further studied in [6], [8], and [7]. The Spy Game is a generalization of the eternal domination game in that the attacker (spy) now moves at speed s on the graph and the guards just need to have a guard at distance at most d from him at the end of their turns in order to *control* (defend against the attacks of) the attacker (spy). For more information and results on the original eternal domination game and its variants, see the survey [17].

1.2 Recent results

The cases of paths and cycles for this variant of the game are trivial. For a path P_n on n vertices, $\gamma_{all}^{\infty}(P_n) = \lceil \frac{n}{2} \rceil$ and for a cycle C_n on n vertices, $\gamma_{all}^{\infty}(C_n) = \lceil \frac{n}{3} \rceil$ [11]. For the upper bound in paths, starting from one end of the path, one guard is assigned to every two consecutive vertices and only the guard assigned to the attacked vertex moves to it. For the lower bound in paths, the attacker attacks one end of the path and then every second vertex until the other end of the path. Any guard that previously defended against an attack is not able to defend against any of the subsequent attacks in this case. For the upper bound in cycles, a guard is placed every three vertices and all the guards rotate in the same direction to defend against attacks. The lower bound in cycles is due to the fact that $\gamma(C_n) = \lceil \frac{n}{3} \rceil$. In [15], the authors came up with a linear-time algorithm to determine $\gamma_{all}^{\infty}(T)$ for all trees T. In [3], the eternal domination game was solved for proper interval graphs. In recent years, a lot of effort was put in by several authors to determine the eternal domination number of cartesian grids, $\gamma_{all}^{\infty}(P_n \square P_m)$. Exact values were determined for $2 \times n$ cartesian grids in [12] and $4 \times n$ cartesian grids in [2]. Tight bounds for $3 \times n$ cartesian grids were obtained in [10] and improved in [9]. Finally, bounds for $5 \times n$ cartesian grids were given in [22]. The best known lower bound for $\gamma_{all}^{\infty}(P_n \square P_m)$ for values of n and m large enough, is the domination number with the latter only being recently determined in [13]. The best known upper bound for $\gamma_{all}^{\infty}(P_n \square P_m)$ was determined recently in [18], where it was shown that $\gamma_{all}^{\infty}(P_n \square P_m) \leq \gamma(P_n \square P_m) + O(n+m)$ where $\gamma(P_n \square P_m)$ is the domination number of $P_n \square P_m$.

Recently, Finbow et al. studied the eternal domination game on strong grids, $P_n \boxtimes P_m$, which are, roughly, cartesian grids where the diagonal edges exist (also known as "king" graphs). They obtained an upper bound of $\frac{nm}{6} + O(n+m)$ for the eternal domination number of $P_n \boxtimes P_m$. Note that it is trivially known that $\gamma(P_n \boxtimes P_m) = \lceil \frac{nm}{9} \rceil$ which was the best known lower bound for $\gamma_{all}^{\infty}(P_n \boxtimes P_m)$ until this paper.

Note that all these results also hold for the variant of the game where at most one guard may occupy a vertex.

1.3 Our results

The main result of this paper is that, for all $n, m \in \mathbb{N}^*$ such that $m \geq n$,

$$\lfloor \frac{nm}{9} \rfloor + \Omega(n+m) \leq \gamma_{all}^{\infty}(P_n \boxtimes P_m) \leq \lceil \frac{nm}{9} \rceil + O(m\sqrt{n})$$

.

The upper bound is proven by first, beginning with a grid of size $n\times k$ where only "horizontal" attacks are allowed and $\lceil\frac{(n-2)(k-2)}{9}\rceil + 2(k+n-2)$ guards are used. Then, a grid of size $n \times m$ is considered, which is much larger than the previous grid, *i.e.*, $m >> k$, and contains many subgrids of size $n \times k$. In this new grid, only "vertical" attacks are allowed to occur initially. The guards are placed in the $n \times k$ subgrids exactly as they were placed when only horizontal attacks were permitted and $2(\frac{k-2}{3})(m+n-2)$ guards are added. Then, in the same grid, diagonal attacks are permitted and finally, all attacks are permitted. The attacks are responded to based on their type, i.e., horizontal attacks will be responded to by, roughly, only moving the guards in the attacked row of the $n \times k$ subgrid that contains the attacked vertex. Vertical and diagonal attacks will be responded to by, roughly, only moving the guards in the $n \times k$ subgrid that contains the attacked vertex.

The lower bound is proven using a counting argument and the fact that two guards are needed in each disjoint 4×5 subgrid that includes 5 border vertices, and $\gamma^{\infty}_{all}(P_a \boxtimes P_b) \leq \gamma^{\infty}_{all}(P_n \boxtimes P_m)$ for all $a, b \in \mathbb{N}^*$ such that $a \leq n$ and $b \leq m$.

2 Preliminaries

Consider an $n \times m$ strong grid denoted by $P_n \boxtimes P_m$ with $m \geq n$. Cartesian coordinates are considered for the vertices of the grid, that is, a vertex in row i and column j has coordinates (i, j) where $1 \leq i \leq n$ and $1 \leq j \leq m$. Two vertices (i_1, j_1) and (i_2, j_2) are adjacent if and only if $max\{|i_2 - i_1|, |j_2 - j_1|\} = 1$. $(1, 1)$ is in the bottom-left corner and $(n, 1)$ is in the top-left corner.

Definition 1. The *border* vertices of an $n\times m$ grid are the vertices in $\{(1, j), (n, j), (i, 1), (i, m)\}$ for all $i, j \in \mathbb{N}$ such that $1 \leq i \leq n$ and $1 \leq j \leq m$. Let B be the set of all border vertices of an $n \times m$ grid.

Definition 2. The *pre-border* vertices of an $n\times m$ grid are the vertices in $\{(2, j), (n-1, j), (i, 2), (i, m-1)\}$ for all $i, j \in \mathbb{N}$ such that $2 \leq i \leq n-1$ and $2 \leq j \leq m-1$. Essentially, they are all the non-border vertices that are adjacent to the border vertices.

Definition 3. For a graph G and any vertex $v \in V(G)$, a *configuration* is a multiset of vertices $C \in V(G)$ where the number of occurrences of a vertex $v \in C$ corresponds to the number of guards at v. Thus, a configuration corresponds to the positions of the guards at one given turn.

The next lemma will be important for the strategy of the guards that will be employed in the following lemmas. The idea is that by placing enough guards on each of the border vertices of the $n\times m$ grid, a feasible "flow" of the guards may be facilitated. That is, a movement of the guards (in one turn) from a set of pre-border vertices to another set of pre-border vertices of the same size by utilizing the guards on the border vertices such that the border vertices contain the same number of guards at the end of the turn.

A guard is said to "jump" to a vertex v if the guard is currently at a vertex u such that u and v are not adjacent but there exists some movement of guards (in one turn) such that the guards occupy the same configuration (after the movement) except there is one more guard at v and one less guard at u.

The following notations will be used. Let P^* and P'^* be two vertex-disjoint paths induced by the pre-border vertices. Let $P \subseteq V(P^*)$ and $P' \subseteq V(P'^*)$ such that P (P' resp.) contains the ends of P^* (P'^* resp.).

Lemma 4. *For all $\alpha, \beta \in \mathbb{N}^*$ such that $\beta \leq \alpha$, in $P_n \boxtimes P_m$, if all the vertices in B contain α guards each, then β guards may "jump" from P to P' in one turn.*

Proof. The proof is by induction on β. The inductive hypothesis is that if all the vertices in B contain $\alpha \geq \beta$ guards each, then $\beta \leq \alpha$ guards may "jump" from P to P' in one turn such that at most β guards move off of each vertex $w \in B$ in this movement. For the base case, let us show how 1 guard can "jump" from any pre-border vertex $u \in P$ to any other pre-border vertex $v \in P'$ in one turn. There exists a path Q with endpoints u and v such that u and v are pre-border vertices in the path Q and all internal vertices of Q are in B. Additionally, the neighbour of u (v resp.) that is a border vertex and shares exactly one coordinate (in the grid) with u (v resp.) is part of the path Q. For each vertex in the path Q, if one guard moves to its neighbour in the direction of v (on a shortest path to v containing only vertices of Q and note this is possible since each vertex of B has $\alpha \geq 1$ guards), then at the end of the turn, u has one less guard, v has one more guard, and all the other vertices of Q have α guards.

Now, assume that if all the vertices in B contain α guards each, then $\beta \leq \alpha$ guards may "jump" from P to P' in one turn by the inductive hypothesis. For $\beta = \alpha + 1$ guards to jump from P to P' in one turn when there are $\alpha + 1$ guards on each vertex of B, the movements of the guards as described above in the case of 1 guard "jumping" are performed simultaneously to the movements of the guards when $\beta \leq \alpha$ guards "jump" from P to P'. Note that this is possible since, by the inductive hypothesis, at most α guards have to move off of any border vertex $w \in B$ for the movements of the guards when $\beta \leq \alpha$ guards "jump" from P to P'. □

3 Upper bound strategy

This section is devoted to proving that for all $n, m \in \mathbb{N}^*$ such that $m \geq n$, $\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \lceil \frac{nm}{9} \rceil + O(m\sqrt{n})$. The $n \times m$ strong grid will be partitioned into *blocks* which are simply subgrids of size $n \times k$. For all $1 \leq q \leq \frac{m}{k}$, the q^{th} block contains columns $(q-1)k + 1$ through qk. First, assume that $n - 2 \mod 3 = 0$, $k - 2 \mod 3 = 0$, and $m \mod k = 0$.

3.1 Horizontal attacks

Let $b \in \{1, 2, 3\}$ and for all $i, j \in \mathbb{N}^*$ such that $1 \leq i \leq \frac{n-2}{3}$ and $1 \leq j \leq \frac{k-2}{3}$, let $a_i \in \{1, 2, 3\}$, $x_i = 3(i-1) + b + 1$, and $y_{j,i} = 3(j-1) + a_i + 1$.

Horizontal Attacks. *Horizontal* attacks are attacks only on vertices "horizontally" adjacent to a guard by being to the left or right of a guard in a row where there is at least one guard occupying a non-border vertex. That is, for all $u, v \in \mathbb{N}^*$ such that $1 \leq u \leq n$ and $1 \leq v \leq k$, an attack at (u, v) implies there is a guard at $(u, v-1)$ and/or $(u, v+1)$ and at least one guard at a non-border vertex in row u.

Horizontal Configurations. Let $b \in \{1, 2, 3\}$ and for all $i \in \mathbb{N}^*$ such that $1 \leq i \leq \frac{n-2}{3}$, let $a_i \in \{1, 2, 3\}$. Let C_H be the family of horizontal configurations $C_H = \{C_H(X) | X = \{b, a_1, \ldots, a_{\frac{n-2}{3}} : b \in \{1, 2, 3\}, a_i \in \{1, 2, 3\}$ for $i = 1, \ldots, \frac{n-2}{3}\}$ in $P_n \boxtimes P_k$. For all $i, j \in \mathbb{N}^*$ such that $1 \leq i \leq \frac{n-2}{3}$ and $1 \leq j \leq \frac{k-2}{3}$, let $x_i = 3(i-1) + b + 1$ and $y_{j,i} = 3(j-1) + a_i + 1$. Let there be one guard at each vertex of $\{(x_i, y_{j,i})\}$ and one guard at each border vertex. This makes for a total of $|C_H(X)| = \frac{(n-2)(k-2)}{9} + 2(n+k) - 4$ guards. See Figure 1.

Lemma 5. *Any configuration $C_H(X)$ is a dominating set of $P_n \boxtimes P_k$.*

Proof. The pre-border and border vertices are dominated by the guards on the border vertices. For all $i, j \in \mathbb{N}^*$ such that $1 \leq i \leq \frac{n-2}{3}$ and $1 \leq j \leq \frac{k-2}{3}$, the guards on the vertices $(x_i, y_{j,i})$ dominate the vertices $\{(x_i+1, y_{j,i}), (x_i-1, y_{j,i}), (x_i, y_{j,i}-1), (x_i, y_{j,i}+1), (x_i+1, y_{j,i}+1), (x_i+1, y_{j,i}-1), (x_i-1, y_{j,i}-1), (x_i-1, y_{j,i}+1)\}$. □

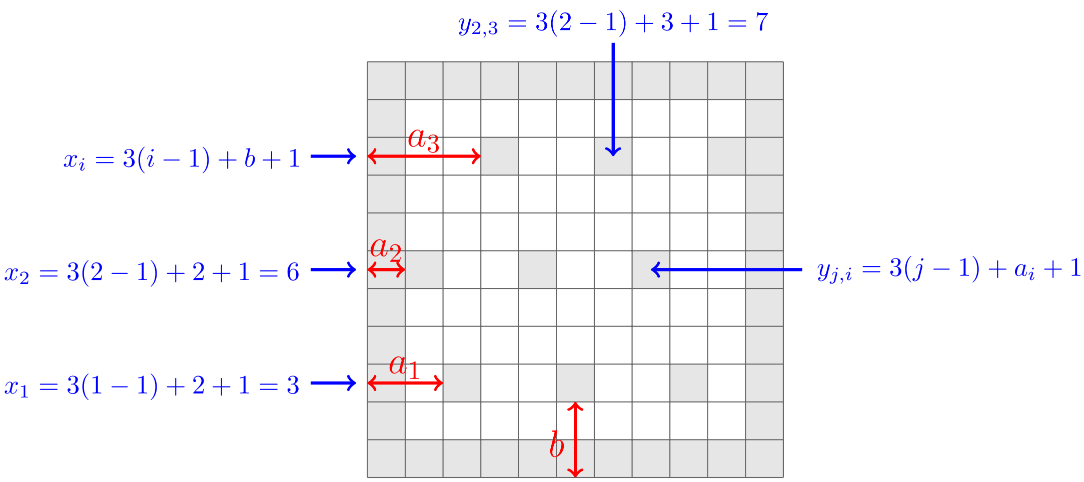

Figure 1: $P_{11} \boxtimes P_{11}$ where the squares are vertices. Example of a configuration $C_H(X)$ where $X = \{b, a_1, \ldots, a_{\frac{n-2}{3}}\} = \{2, 2, 1, 3\}$, there is one guard at each square in gray, and the white squares contain no guards.

Horizontal Attacks for Horizontal Configurations. According to the guards occupying a configuration $C_H(X)$, a horizontal attack is any attack at a vertex in $A_H \subseteq V$ defined as follows. The vertices that may be attacked are the vertices in $A_H = \{(x_i, z)\}$ for all $i, z \in \mathbb{N}^*$ such that $1 \leq i \leq \frac{n-2}{3}$ and $1 \leq z \leq k$. To simplify the upcoming proof of Lemma 6, the following equivalent definition of A_H is given which does not include the vertices occupied by guards in the previous definition. $A_H = \{(x_i, y_{j,i} - 1), (x_i, y_{j,i} + 1), (x_r, 2), (x_p, k-1)\}$ for all $i, j, r, p \in \mathbb{N}^*$ such that $1 \leq i, r, p \leq \frac{n-2}{3}$, $1 \leq j \leq \frac{k-2}{3}$, $y_{j,i} + 1 \leq m$, $y_{j,i} - 1 \geq 1$, $a_r = 3$, and $a_p = 1$. See Figure 2.

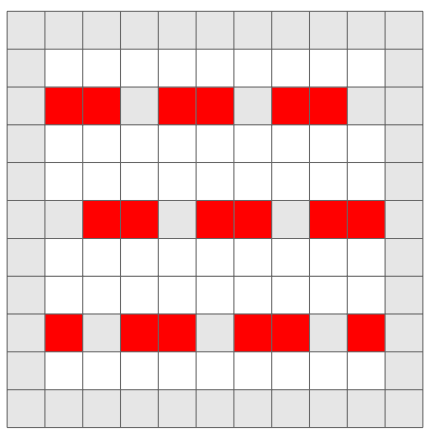

Figure 2: $P_{11} \boxtimes P_{11}$ where the squares are vertices. Example of the attackable vertices in red when only horizontal attacks are considered. The guards occupy a configuration $C_H(X)$ where $X = \{b, a_1, \ldots, a_{\frac{n-2}{3}}\} = \{2, 2, 1, 3\}$, there is one guard at each square in gray, and the white squares contain no guards.

Lemma 6. *For all $v \in A_H$, there exists a movement of the guards, in one turn, from a configuration $C_H(X)$ to a configuration $C_H(X')$ that defends against an attack at v.*

Proof. Initially, $\lceil \frac{(n-2)(k-2)}{9} \rceil + 2(n+k) - 4$ guards are in a configuration $C_H(X)$ (see Figure 1). Consider an attack at some vertex $v \in A_H$, w.l.o.g. say $(x_\ell, y_{w,\ell} - 1)$ (as the cases of attacks at $(x_\ell, y_{w,\ell} + 1)$,$(x_\ell, 2)$, and $(x_\ell, k-1)$, are symmetric to at least one of the 3 cases below) for some integers $1 \le \ell \le \frac{n-2}{3}$ and $1 \le w \le \frac{k-2}{3}$. The guards will move from a configuration $C_H(X)$ to a configuration $C_H(X')$ that defends against the attack at v, where $X' = \{b', a'_1, \ldots, a'_{\frac{n-2}{3}}\}$, *i.e.*, $v \in C_H(X')$.

Intuitively, for the guards to move from a configuration $C_H(X)$ to a configuration $C_H(X')$ that defends against this attack at v, all the guards in row x_ℓ will shift left except for perhaps the guards on the border vertices (it depends on the value of a_ℓ).

Precisely, due to the configuration of the guards, there is a guard at $(x_\ell, y_{w,\ell})$. There are three cases of how the guards will move in response to the attack since there are three possible values of a_ℓ.

Case i) $a_\ell = 2$. To defend against the attack, all the guards in row x_ℓ except those that occupy border vertices, shift one vertex to the left. That is, the guard at $(x_\ell, y_{j,\ell})$ moves to $(x_\ell, y_{j,\ell} - 1)$ for all $j \in \mathbb{N}^*$ such that $1 \le j \le \frac{k-2}{3}$. Since the positions of the other guards did not change and $y_{j,\ell} - 1 = 3(j-1) + a_\ell = 3(j-1) + 2$, the guards occupy a configuration $C_H(X')$ where $b' = b$, $a'_i = a_i$ for all $1 \le i \le \frac{n-2}{3}$ such that $i \ne \ell$, and $a'_\ell = a_\ell - 1 = 1$.

Case ii) $a_\ell = 1$. To defend against the attack, all the guards in row x_ℓ except the one at $(x_\ell, 1)$, shift one vertex to the left. That is, the guard at $(x_\ell, y_{j,\ell})$ moves to $(x_\ell, y_{j,\ell} - 1)$ for all $j \in \mathbb{N}^*$ such that $1 \le j \le \frac{k-2}{3}$. Also, the guard at $(x_\ell, 2)$ jumps to $(x_\ell, k-1)$ by Lemma 4 and since none of the border guards have to move for any other purpose. Since the positions of the other guards did not change and $y_{j,\ell} - 1 = 3(j-1) + a_\ell = 3(j-1) + 1$, the guards occupy a configuration $C_H(X')$ where $b' = b$, $a'_i = a_i$ for all $1 \le i \le \frac{n-2}{3}$ such that $i \ne \ell$, and $a'_\ell = a_\ell + 2 = 3$. See Figure 3.

Case iii) $a_\ell = 3$. To defend against the attack, all the guards in row x_ℓ except those that occupy border vertices, shift one vertex to the left. That is, the guard at $(x_\ell, y_{j,\ell})$ moves to $(x_\ell, y_{j,\ell} - 1)$ for all $j \in \mathbb{N}$ such that $1 \le j \le \frac{k-2}{3}$. Since the positions of the other guards did not change and $y_{j,\ell} - 1 = 3(j-1) + a_\ell = 3(j-1) + 3$, the guards occupy a configuration $C_H(X')$ where $b' = b$, $a'_i = a_i$ for all $1 \le i \le \frac{n-2}{3}$ such that $i \ne \ell$, and $a'_\ell = a_\ell - 1 = 2$.

□

3.2 Vertical attacks

Vertical Attacks. *Vertical* attacks are non-horizontal attacks only on vertices "vertically" adjacent to a guard by being above or below a guard not occupying a border vertex or in the same column as a guard that can jump to that vertex. That is, for all $u, v \in \mathbb{N}^*$ such that $1 \le u \le n$ and $1 \le v \le m$, an attack at (u, v) implies there is a guard at $(u-1, v)$ and/or $(u+1, v)$ and/or $(2, v)$ if $u = n-1$ and/or $(n-1, v)$ if $u = 2$.

Vertical Configurations. For all $i, q \in \mathbb{N}^*$ such that $1 \le i \le \frac{n-2}{3}$ and $1 \le q \le \frac{m}{k}$, let $a_i^q, b^q \in \{1,2,3\}$ where the superscript q corresponds to the q^{th} $n \times k$ block in the $n \times m$ grid. Let C_V be the family of vertical configurations $C_V = \{C_V(Y) | Y = \{X^1, \ldots, X^{\frac{m}{k}} : X^q = \{b^q, a_1^q, \ldots, a^q_{\frac{n-2}{3}}\},\ b^q \in \{1,2,3\}, a_i^q \in \{1,2,3\}$ for $q = 1, \ldots, \frac{m}{k}, i = 1, \ldots, \frac{n-2}{3}\}$ in $P_n \boxtimes P_m$. Note that $C_H(X^q)$ corresponds to the horizontal configuration (see subsection 3.1) of the guards in the q^{th} block of $P_n \boxtimes P_m$. For all $i, j, q \in \mathbb{N}^*$ such that $1 \le i \le \frac{n-2}{3}$, $1 \le j \le \frac{k-2}{3}$, and $1 \le q \le \frac{m}{k}$, let $x_i^q = 3(i-1) + b^q + 1$, and $y_{j,i}^q = 3(j-1) + a_i + 1$ with the y-coordinate for each of the blocks being reset to 1 for the first column of each block. Let there be one guard at each vertex of $\{(x_i^q, y_{j,i}^q)\}$, one guard at each vertex of the columns $(q-1)k+1$ and qk, and

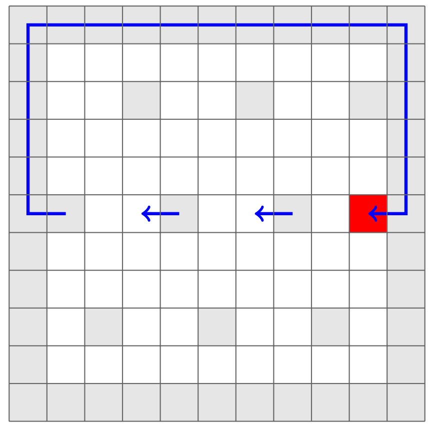

Figure 3: $P_{11} \boxtimes P_{11}$ where the squares are vertices. Example of an attack in Case ii) at the red square. The guards occupy a configuration $C_H(X)$ where $X = \{b, a_1, \ldots, a_{\frac{n-2}{3}}\} = \{2, 2, 1, 3\}$, there is one guard at each square in gray, and the white squares contain no guards. The arrows (in blue) show the movements of the guards in response to the attack.

$\frac{k-2}{3} + 1$ guards at each border vertex. For all of the $\frac{m}{k}$ blocks in total, this makes for a total of $|C_V(Y)| = \frac{m}{k}|C_H(X^q)| + 2(\frac{k-2}{3} + 1)(n + m - 2)$ guards. See Figure 4.

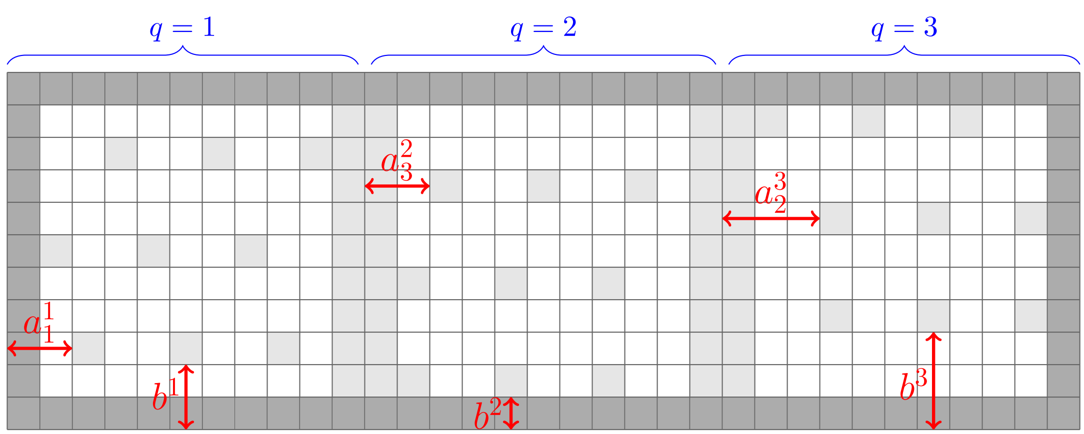

Figure 4: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 1, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards.

Lemma 7. *Any configuration $C_V(Y)$ is a dominating set of $P_n \boxtimes P_m$.*

Proof. Each of the $n \times k$ blocks is dominated by the guards within since they occupy a configuration $C_H(X)$ by Lemma 5. The pre-border and border vertices are dominated by the guards on the border vertices. $\square$

Vertical Attacks for Vertical Configurations. According to the guards occupying a configuration $C_V(Y)$, a vertical attack is any attack at a vertex in $A_V \subseteq V$ defined as follows. The vertices that may be attacked are the vertices in $A_V = \{(x_i^q - 1, y_{j,i}^q), (x_i^q + 1, y_{j,i}^q), (2, y_{j,i}^r), (n-1, y_{j,i}^p)\}$ for all $i, j, q, r, p \in \mathbb{N}^*$ such that $1 \leq i \leq \frac{n-2}{3}$, $1 \leq j \leq \frac{k-2}{3}$, $1 \leq q, r, p \leq \frac{m}{k}$, $x_i^q + 1 \leq n$, $x_i^q - 1 \geq 1$, $b^r = 3$, and $b^p = 1$. See Figure 5.

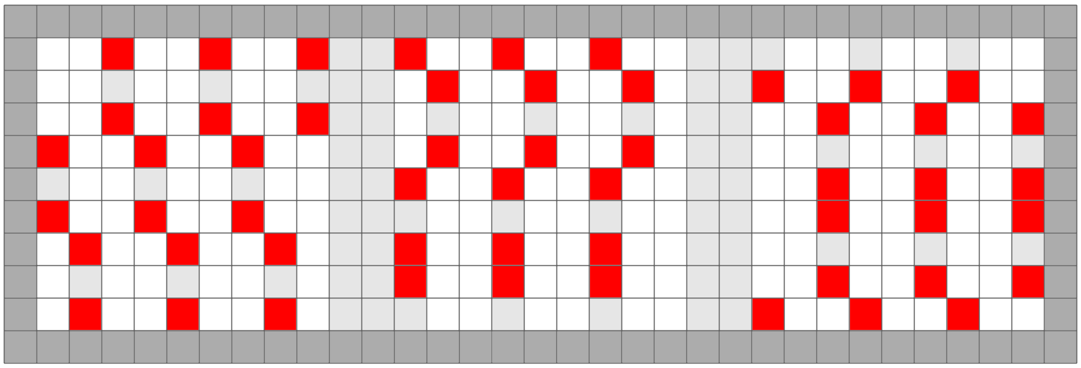

Figure 5: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of the attackable vertices in red when only vertical attacks are considered. The guards occupy a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 1, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards.

Lemma 8. *For all $v \in A_V$, there exists a movement of the guards from a configuration $C_V(Y)$ to a configuration $C_V(Y')$ that defends against an attack at v.*

Proof. Initially, $\frac{m}{k}|C_H(X^q)| + 2(\frac{k-2}{3} + 1)(n + m - 2)$ guards are in a configuration $C_V(Y)$ (see Figure 4).

Consider an attack at some vertex $v \in A_V$, w.l.o.g. say $(x_\ell^z - 1, y_{w,\ell}^z)$ (as the cases of attacks at $(x_\ell^z + 1, y_{w,\ell}^z)$, $(2, y_{w,\ell}^z)$, and $(n-1, y_{w,\ell}^z)$, are symmetric to at least one of the 3 cases below) for some integers $1 \leq \ell \leq \frac{n-2}{3}$, $1 \leq w \leq \frac{k-2}{3}$, and $1 \leq z \leq \frac{m}{k}$. The guards will move from a configuration $C_V(Y)$ to a configuration $C_V(Y')$ that defends against the attack at v, where $Y' = \{X^{1'}, \ldots, X^{\frac{m}{k}'}\}$, *i.e.*, $v \in C_V(Y')$.

Intuitively, for the guards to move from a configuration $C_V(Y)$ to a configuration $C_V(Y')$ that defends against this attack at v, all the guards in the block z will shift down except for perhaps the guards on the border vertices (it depends on the value of b^z).

Precisely, due to the configuration of the guards, there is a guard at $(x_\ell^z, y_{w,\ell}^z)$. There are three cases of how the guards will move in response to the attack since there are three possible values of b^z.

Case i) $b^z = 2$. To defend against the attack, all the guards in the block z that contains the attacked vertex except those that occupy border vertices of the block z, shift one vertex downwards. That is, for all $\ell, w, z \in \mathbb{N}$ such that $1 \leq \ell \leq \frac{n-2}{3}$, $1 \leq w \leq \frac{k-2}{3}$, and for the value $z \in \mathbb{N}$ such that $1 \leq z \leq \frac{m}{k}$, and the block z contains the attacked vertex, the guard at $(x_\ell^z, y_{w,\ell}^z)$ moves to $(x_\ell^z - 1, y_{w,\ell}^z)$. Since the positions of the other guards did not change and $x_\ell^z - 1 = 3(i-1) + b^z = 3(i-1) + 2$, the guards occupy a configuration $C_V(Y')$ where $X^p = X^{p'}$ for all $1 \leq p \leq \frac{m}{k}$ such that $p \neq z$, and $a_i^{z'} = a_i^z$ for all $1 \leq i \leq \frac{n-2}{3}$, but $b^{z'} = b^z - 1 = 1$.

Case ii) $b^z = 3$. To defend against the attack, all the guards in the block z that contains the attacked vertex except those that occupy border vertices of the block z, shift one vertex downwards. That is, for all $\ell, w, z \in \mathbb{N}$ such that $1 \le \ell \le \frac{n-2}{3}$, $1 \le w \le \frac{k-2}{3}$, and for the value $z \in \mathbb{N}$ such that $1 \le z \le \frac{m}{k}$, and the block z contains the attacked vertex, the guard at $(x_\ell^z, y_{w,\ell}^z)$ moves to $(x_\ell^z - 1, y_{w,\ell}^z)$. Since the positions of the other guards did not change and $x_\ell^z - 1 = 3(i-1) + b^z = 3(i-1) + 3$, the guards occupy a configuration $C_V(Y')$ where $X^p = X^{p'}$ for all $1 \le p \le \frac{m}{k}$ such that $p \ne z$, and $a_i^{z'} = a_i^z$ for all $1 \le i \le \frac{n-2}{3}$, but $b^{z'} = b^z - 1 = 2$.

Case iii) $b^z = 1$. To defend against the attack, all the guards in the block z that contains the attacked vertex except those that occupy the first and last columns and the bottom row of the border vertices of the block z, shift one vertex downwards. That is, for all $\ell, w, z \in \mathbb{N}$ such that $1 \le \ell \le \frac{n-2}{3}$, $1 \le w \le \frac{k-2}{3}$, and for the value $z \in \mathbb{N}$ such that $1 \le z \le \frac{m}{k}$, and the block z contains the attacked vertex, the guard at $(x_\ell^z, y_{w,\ell}^z)$ moves to $(x_\ell^z - 1, y_{w,\ell}^z)$. Also, the guard at $(2, y_{w,\ell}^z)$ jumps to $(2, y_{w,\ell}^z)$ by Lemma 4 (a total of $\frac{k-2}{3}$ guards jump here) and since none of the border guards have to move for any other purpose. Since the positions of the other guards did not change and $x_\ell^z - 1 = 3(i-1) + b^z = 3(i-1) + 1$, the guards occupy a configuration $C_V(Y')$ where $X^p = X^{p'}$ for all $1 \le p \le \frac{m}{k}$ such that $p \ne z$, and $a_i^{z'} = a_i^z$ for all $1 \le i \le \frac{n-2}{3}$, but $b^{z'} = b^z + 2 = 3$. See Figure 6.

□

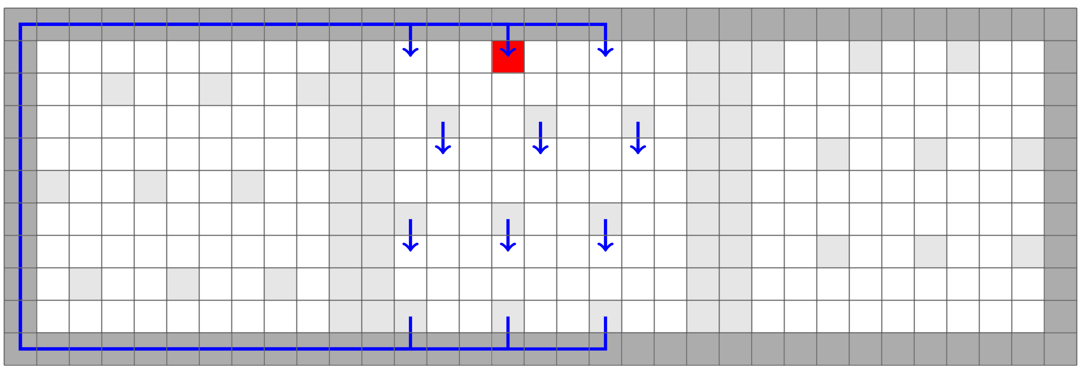

Figure 6: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of an attack in Case iii) at the red square. The guards occupy a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 1, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards. The arrows (in blue) show the movements of the guards in response to the attack.

3.3 Diagonal attacks

The same $n \times m$ grid, notations, and configurations for the guards used in section 3.2 will be used here.

Diagonal Attacks. *Diagonal* attacks are non-horizontal and non-vertical attacks only on vertices "diagonally" adjacent to a guard by being to the top-left, top-right, bottom-left or bottom-right of a guard not occupying a border vertex or at a vertex that a guard can jump to. That is, for all $u, v \in \mathbb{N}^*$ and for some $r, p \in \mathbb{N}^*$ such that $1 \le u, r \le n$, $1 \le v, p \le m$, $v \ne p$, and $u \ne r$, an attack at (u, v) implies there is a guard at $(u-1, v+1)$ and/or a guard at $(u-1, v-1)$ and/or a guard at $(u+1, v+1)$ and/or a guard at $(u+1, v-1)$) and/or a guard at $(n-1, p)$ if $u = 2$ and/or a guard at $(2, p)$ if $u = n-1$ and/or a guard at $(r, m-1)$ if $v = 2$ and/or a guard at $(r, 2)$ if $v = m-1$.

Diagonal Attacks for Vertical Configurations. According to the guards occupying a configuration $C_V(Y)$, a diagonal attack is any attack at a vertex in $A_D \subseteq V$ defined as follows. The vertices that may be attacked are the vertices in $A_D = \{(x_i^q+1, y_{j,i}^q-1), (x_i^q-1, y_{j,i}^q+1), (x_i^q-1, y_{j,i}^q-1), (x_i^q+1, y_{j,i}^q+1), (n-1, p), (2, p), (r, k-1^q), (r, 2^q)\}$ for all $i, j, q, p, r \in \mathbb{N}^*$ such that $1 \leq r \leq n$, $1 \leq p \leq m$, $1 \leq i \leq \frac{n-2}{3}$, $1 \leq j \leq \frac{k-2}{3}$, $1 \leq q \leq \frac{m}{k}$, $r \neq x_i^q$, $p \neq y_{j,1}^q$, $p \neq y_{j,\frac{n-2}{3}}^q$, $x_i + 1 \leq n$, $x_i^q - 1 \geq 1$, $x_i^q + 1 \leq n$, $y_{j,i}^q - 1 \geq 1$, and $y_{j,i}^q + 1 \leq k$.

Lemma 9. *For all $v \in A_D$, there exists a movement of the guards from a configuration $C_V(Y)$ to a configuration $C_V(Y')$ that defends against an attack at v.*

Proof. Initially, $\frac{m}{k}|C_H(X^q)| + 2(\frac{k-2}{3}+1)(n+m-2)$ guards are in a configuration $C_V(Y)$ (see Figure 4).

Consider an attack at some vertex $v \in A_D$, w.l.o.g. say $(x_\ell^z - 1, y_{w,\ell}^z + 1)$ (as the cases of attacks at $(x_\ell^z+1, y_{w,\ell}^z-1), (x_\ell^z-1, y_{w,\ell}^z-1), (x_\ell^z+1, y_{w,\ell}^z+1)$, and the pre-border vertices defined in Diagonal Attacks for Vertical Configurations are symmetric) for some integers $1 \leq \ell \leq \frac{n-2}{3}$, $1 \leq w \leq \frac{k-2}{3}$, and $1 \leq z \leq \frac{m}{k}$. The guards will move from a configuration $C_V(Y)$ to a configuration $C_V(Y')$ that defends against the attack at v.

Intuitively, for the guards to move from a configuration $C_V(Y)$ to a configuration $C_V(Y')$ that defends against this attack at v, in the block z that contains the attacked vertex, the guards in row x_ℓ^z will move as they would in response to a horizontal attack and a vertical attack but simultaneously, so moving diagonally down and to the right, and the remainder of the guards in the block z will move as they would in response to a vertical attack, so moving down.

Precisely, for the movements of the guards in the block z that are not in row x_ℓ^z, refer back to the proof of Lemma 8. For the movements of the guards in row x_ℓ^z in the block z, refer back to the proofs of Lemma 6 and Lemma 8. Note that the "flow" of the guards from their pre-attack positions in row x_ℓ^z to their post-attack positions following the simultaneous horizontal and vertical movements (diagonal movements) is feasible since either a guard occupies a border vertex and is just required to shift to an adjacent border vertex as in Lemma 4 or a guard occupies a vertex (a, b) for two integers a and b and is just required to move to $(a-1, b+1)$ which exists and is adjacent to (a, b). Also, since at most $\frac{k-2}{3}+1$ guards ever move onto a border vertex of the $n \times m$ grid in response to an attack and each of these border vertices contains $\frac{k-2}{3}+1$ guards, the movements of the guards are feasible by Lemma 4. See Figure 7. □

4 Upper Bound in Strong Grids

Theorem 10. *For all $n, m \in \mathbb{N}^*$ such that $m \geq n$, $\gamma_{all}^{\infty}(P_n \boxtimes P_m) = |C_V(Y)| + O(n\sqrt{m})$.*

Proof. Initially (if necessary), the exterior rows and columns (starting from border vertices) are filled with one guard at each vertex such that the remaining subgrid of size $a \times b$ has the properties that $a-2 \mod 3 = 0$, $k = O(\sqrt{n})$, $k-2 \mod 3 = 0$, and $b \mod k = 0$. To fulfill the first property, $O(n)$ guards are required. To fulfill the second and third properties, $O(m)$ guards are required. Lastly, to fulfill the fourth property, $O(m\sqrt{n})$ guards are required. Therefore, for the remaining subgrid to have these properties, $O(m\sqrt{n})$ guards are required.

Assume that n and m, and k have the properties above now (note that the bound will hold since $a \leq n$ and $b \leq m$). That is, we assume the remaining $a \times b$ subgrid is in fact an $n \times m$ subgrid such that $n-2 \mod 3 = 0$, $k = O(\sqrt{n})$, $k-2 \mod 3 = 0$, and $m \mod k = 0$. The guards initially occupy a configuration $C_V(Y)$. By Lemma 7, they occupy a dominating set. Therefore, every vertex $u \in V(P_n \boxtimes P_m)$ is in A_H or A_V or A_D (or already contains a guard). Let the attacker attack some unoccupied vertex $v \in V(P_n \boxtimes P_m)$. If $v \in A_H$, then the guards

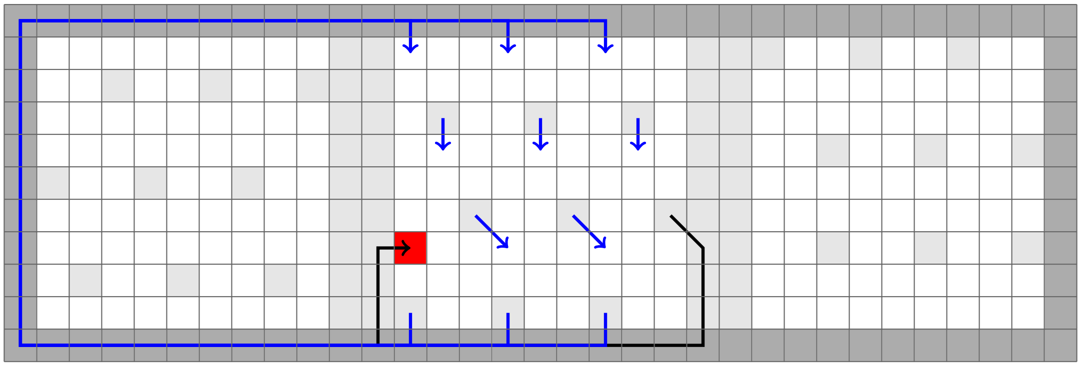

Figure 7: $P_{11} \boxtimes P_{33}$ where the squares are vertices. Example of a diagonal attack at the red square. The guards occupy a configuration $C_V(Y)$ where $k = 11$, $Y = \{X^1, X^2, X^3\}$, $X^1 = \{2, 2, 1, 3\}$, $X^2 = \{1, 1, 3, 2\}$, $X^3 = \{3, 3, 3, 1\}$, there are $\frac{k-2}{3} + 1 = 4$ guards at each square in dark gray, 1 guard at each square in light gray, and the white squares contain no guards. The arrows (in blue) show the movements of the guards in response to the attack. The arrow in black is to differentiate between the different guards jumping.

in the block that contains v will respond as in Lemma 6. If $v \in A_V$, then the guards in the block that contains v will respond as in Lemma 8. If $v \in A_D$, then the guards in the block that contains v will respond as in Lemma 9. By Lemma 6, Lemma 8, and Lemma 9, the guards occupy a configuration $C_V(Y')$ after the attack and thus, can defend against an infinite sequence of attacks. $\square$

Corollary 11. *For all $n, m \in \mathbb{N}^*$ such that $m \geq n$, $\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \lceil \frac{nm}{9} \rceil + O(m\sqrt{n})$.*

Proof. We show that $|C_V(Y)| + O(n\sqrt{m}) = \lceil \frac{nm}{9} \rceil + O(m\sqrt{n})$ and the proof follows from Theorem 10.

$$|C_H(X)| = \lceil \frac{(n-2)(k-2)}{9} \rceil + 2(n+k) - 4 = \lceil \frac{nk}{9} \rceil + O(n+k).$$

$$C_V(Y) = \lceil \frac{m}{k} \rceil C_H(X) + 2(\frac{k-2}{3})(m+n-2) = \lceil \frac{nm}{9} \rceil + O(\frac{nm}{k} + mk).$$

As in Theorem 10, $k = O(\sqrt{n})$ and the result follows. $\square$

5 Lower Bound in Strong Grids

Theorem 12. *For all $n, m \in \mathbb{N}^*$, $\gamma_{all}^{\infty}(P_n \boxtimes P_m) = \lfloor \frac{nm}{9} \rfloor + \Omega(n+m)$.*

Proof. Assume $n \mod 30 = m \mod 30 = 0$ so that n and m are divisible by both 3 and 10. The following claim shows that every 4×5 subgrid that includes 5 border vertices must have at least two guards in it or else the attacker wins.

Claim 13. *For any 4×5 subgrid that includes 5 border vertices with only one guard in it, the attacker can win in at most two turns.*

Proof. W.l.o.g. let the 4×5 subgrid include border vertices from row 1 of the $n \times m$ grid. Also, for some integer $1 \leq x \leq m-4$, let $\{(x,1), \ldots, (x+4,1)\}$ be the 5 border vertices. If there is only one guard in this subgrid, then the guard must be at $(x+2,2)$ in order to prevent the attacker from winning in one turn as otherwise, it is not possible to dominate all the vertices of the subgrid. The attacker attacks $(x+2,3)$ which forces the guard at $(x+2,2)$ to move to $(x+2,3)$ as he is the only guard adjacent to that vertex since, initially, there was only one guard in the 4×5 subgrid. Now the attacker attacks $(x+2,1)$ and wins since every guard is at distance at least 2 from this vertex after the previous moves of the guards since, initially, there was only one guard in the 4×5 subgrid. □

Since the first 5 and last 5 columns and rows are not included in the count and only one of every two 4×5 subgrids is considered, there are at least $\frac{n-10}{5} + \frac{m-10}{5}$ disjoint 4×5 subgrids that include 5 border vertices of the $n \times m$ grid such that there is a guard in the center column of the 4×5 subgrid. The fact that only one of every two 4×5 subgrids is considered guarantees that there is a guard in the center column of the 4×5 subgrid considered.

By claim 13, there must be at least two guards in each of these subgrids. Also, if one of the two guards is at $(x+2,2)$ or $(x+2,3)$, then for the 3×3 subgrid composed of the middle 3 columns and border vertices of the 4×5 subgrid to be dominated, the neighbourhoods of these two guards must intersect. Otherwise, the guard occupying the central column of this subgrid must be at $(x+2,1)$ and thus, the second guard in the subgrid can be at $(x+2,4)$ and their neighbourhoods don't intersect. In this case, however, the guard at $(x+2,1)$ only dominates 6 vertices.

Assume there are $\frac{nm}{9} + \frac{n+m}{c}$ guards for some constant $c > 0$. Each guard can dominate at most 9 vertices. By claim 13, there are at least two guards in every 4×5 subgrid that includes 5 border vertices. The total number of vertices dominated is then at most the maximum number of vertices that can be dominated per guard times the number of guards minus the number of vertices dominated by multiple guards minus 3 times the number of guards dominating only 6 vertices. Then, the total number of vertices dominated is at most

$$9(\frac{nm}{9} + \frac{n+m}{c}) - \frac{m-10}{5} - \frac{n-10}{5}.$$

If

$$9(\frac{nm}{9} + \frac{n+m}{c}) - \frac{m-10}{5} - \frac{n-10}{5} < nm,$$

then the attacker wins since the grid is not dominated.

$$\begin{gathered}
9(\frac{nm}{9} + \frac{n+m}{c}) - \frac{n-10}{5} - \frac{m-10}{5} < nm \\
\Leftrightarrow \frac{9(n+m)}{c} < \frac{n+m}{5} - 4 \\
\Leftrightarrow 45(n+m) < c(n+m-20) \\
\Leftrightarrow c > \frac{45}{1 - \frac{20}{n+m}}
\end{gathered}$$

Therefore, $\gamma^{\infty}_{all}(P_n \boxtimes P_m) > \frac{nm}{9} + \frac{n+m-20}{45}$ when $n \mod 30 = m \mod 30 = 0$.

The result follows since, trivially, $\gamma^{\infty}_{all}(P_n \boxtimes P_m) \leq \gamma^{\infty}_{all}(P_a \boxtimes P_b)$ for all $a, b \in \mathbb{N}^*$ such that $n \leq a$ and $m \leq b$.

□

6 At Most One Guard at each Vertex

This section is devoted to proving that the two main results presented thus far are also true for the variant of the eternal domination game where at most one guard may occupy a vertex. The corresponding eternal domination number for this variant will be denoted by $\gamma_{all}^{*\infty}$. This variant is considered in this section since most of the literature for the eternal domination game considers this variant.

Theorem 14. *For all $n, m \in \mathbb{N}^*$ such that $m \geq n$, $\gamma_{all}^{*\infty}(P_n \boxtimes P_m) = \lceil \frac{nm}{9} \rceil + O(m\sqrt{n})$.*

Proof. Instead of there being $\frac{k-2}{3}+1$ guards at each border vertex like in Theorem 10, these guards will be "pushed" into the interior of the grid so that the $n \times m$ grid has 1 guard on each of the vertices in the $\frac{k-2}{3}+1$ leftmost and rightmost (bottom and top resp.) columns (rows resp.). The remaining subgrid in the interior of the $n \times m$ grid will still be occupied as in Theorem 10, that is, with $\lceil \frac{(n-2(\frac{k-2}{3}+1))(m-2(\frac{k-2}{3}+1))}{9} \rceil$ guards. By adding enough guards to have one guard at each vertex of some columns and rows like in Theorem 10, it can be assumed that the remaining subgrid has the same divisibility properties as the remaining subgrid in the proof of Theorem 10. The strategy for the guards remains the same as the strategy used in Theorem 10 except for in the case when a guard or guards have to jump from one vertex to another. It will be shown how the guards move to accommodate guards jumping in the case of diagonal attacks when both the horizontal and vertical movements force guards to jump. It suffices to show this as if there exists a movement of the guards that allows this to happen, then it is feasible to do it in the case of a vertical attack or a horizontal attack.

Precisely, assume there is a diagonal attack in a block z that forces a guard to jump from $(x_i^z, k-1^z)$ to $(x_i^z-1, 2^z)$ and guards to jump from $(\frac{k-2}{3}+2, y^z_{j,\frac{k-2}{3}+2})$ to $(n-\frac{k-2}{3}-1, y^z_{j,n-\frac{k-2}{3}-1})$ for some integers $k = O(\sqrt{n})$, $1+\frac{\frac{k-2}{3}+1}{3} \leq i \leq \frac{n-2}{3} - \frac{\frac{k-2}{3}+1}{3}$, and $1 \leq z \leq \frac{m-2(\frac{k-2}{3}+1))}{k} - 1$, and all integers $1 \leq j \leq \frac{k-2}{3}$.

Intuitively, for the movements of the guards to accommodate the jumps, the guard at $(x_i^z, k-1^z)$ jumps to $(n-\frac{k-2}{3}-1, y^z_{\frac{k-2}{3},n-\frac{k-2}{3}-1})\}$ by the guards moving along the most exterior path (the one that uses the actual border of the $n \times m$ grid) like in Lemma 4. For all integers $2 \leq j \leq \frac{k-2}{3}$, the guard at $(\frac{k-2}{3}+2, y^z_{j,\frac{k-2}{3}+2})$ jumps to $(n-\frac{k-2}{3}-1, y^z_{j-1,n-\frac{k-2}{3}-1})$ by the guards moving along the $\frac{k-2}{3}-j+2^{th}$ most exterior path. Lastly, the guard at $(\frac{k-2}{3}+2, y^z_{1,\frac{k-2}{3}+2})$ jumps to $(x_i^z-1, 1^z), (x_i^z-1, 2^z)$ by taking the least exterior (most interior) path. See Figure 8.

To precisely describe the movements of the guards to accommodate the jumps, the paths for the guards to follow like in Lemma 4 (except here there is only one guard moving along the path between the vertex being jumped from and the vertex being jumped to) are the following:

$$\{(x_i^z, k-1^z), \ldots, (1, k-1^z), \ldots, (1,1), \ldots, (n,1), \ldots, (n, y^z_{\frac{k-2}{3},n-\frac{k-2}{3}-1}), \ldots, (n-\frac{k-2}{3}-1, y^z_{\frac{k-2}{3},n-\frac{k-2}{3}-1})\},$$

$$\{(\tfrac{k-2}{3}+2, y^z_{j,\frac{k-2}{3}+2}), \ldots, (\tfrac{k-2}{3}-j+2, y^z_{j,\frac{k-2}{3}+2}), \ldots, (\tfrac{k-2}{3}-j+2, \tfrac{k-2}{3}-j+2), \ldots, (n-\tfrac{k-2}{3}+j-1, \tfrac{k-2}{3}-j+2), \ldots, (n-\tfrac{k-2}{3}+j-1, y^z_{j-1,n-\frac{k-2}{3}-1}), (n-\tfrac{k-2}{3}-1, y^z_{j-1,n-\frac{k-2}{3}-1})\}$$

for all integers $2 \leq j \leq \frac{k-2}{3}$, and

$$\{(\tfrac{k-2}{3}+2, y^z_{1,\frac{k-2}{3}+2}), (\tfrac{k-2}{3}+1, y^z_{1,\frac{k-2}{3}+2}), \ldots, (\tfrac{k-2}{3}+1, 1^z), \ldots, (x_i^z-1, 1^z), (x_i^z-1, 2^z)\}.$$

All these paths are disjoint and this completes the proof. □

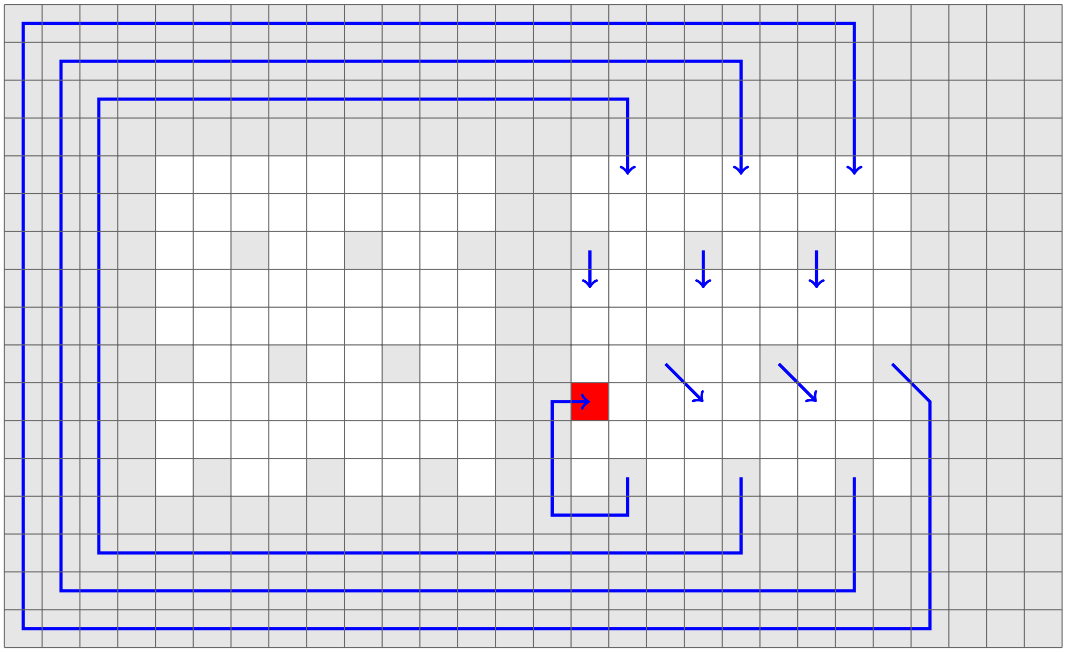

Figure 8: $P_{17} \boxtimes P_{28}$ where the squares are vertices. Example of a diagonal attack at the red square when at most one guard may occupy a vertex. There is 1 guard at each square in light gray, and the white squares contain no guards. The arrows (in blue) show the movements of the guards in response to the attack.

Theorem 15. *For all* $n, m \in \mathbb{N}^*$, $\gamma_{all}^{*\infty}(P_n \boxtimes P_m) = \lfloor \frac{nm}{9} \rfloor + \Omega(n+m)$.

Proof. The proof follows directly from the proof of Theorem 12 as the fact that multiple guards could occupy the same vertex was not used in the proof of Theorem 12. □

7 Further Work

Our results in the strong grid leave the open problem of tightening the bounds. Also, for which other grid graphs can our techniques used in obtaining the upper bound be applied?

References

[1] J. Arquilla and H. Fredricksen. "graphing" an optimal grand strategy. *Military Operations Research*, 1(3):3–17, 1995.

[2] I. Beaton, S. Finbow, and J.A. MacDonald. Eternal domination numbers of $4 \times n$ grid graphs. *J. Comb. Math. Comb. Comput.*, 85:33–48, 2013.

[3] A. Braga, C. Souza, and O. Lee. The eternal dominating set problem for proper interval graphs. *Information Processing Letters*, 115, 2015.

[4] A. Burger, E. J. Cockayne, W. R. Gründlingh, C. M. Mynhardt, J. H. van Vuuren, and W. Winterbach. Infinite order domination in graphs. *J. Comb. Math. Comb. Comput.*, 50:179–194, 2004.

[5] N. Cohen, M. Hilaire, N. A. Martins, N. Nisse, and S. Pérennes. Spy-game on graphs. In *8th International Conference on Fun with Algorithms, FUN 2016*, pages 10:1–10:16, 2016.

[6] N. Cohen, F. Mc Inerney, N. Nisse, and S. Pérennes. Study of a combinatorial game in graphs through linear programming. In *28th International Symposium on Algorithms and Computation (ISAAC 2017)*, 2017. RR, `https://hal.archives-ouvertes.fr/hal-01462890`.

[7] N. Cohen, F. Mc Inerney, N. Nisse, and S. Pérennes. Study of a combinatorial game in graphs through linear programming. Technical report, INRIA, 2017. RR, `https://hal.archives-ouvertes.fr/hal-01462890`.

[8] N. Cohen, N. A. Martins, F. Mc Inerney, N. Nisse, S. Pérennes, and R. Sampaio. Spy-game on graphs: complexity and simple topologies. *To appear in Theoretical Computer Science*, 2017.

[9] A. Z. Delaney and M. E. Messinger. Closing the gap: Eternal domination on $3 \times n$ grids. *To appear in Contributions to Discrete Mathematics*, 2015.

[10] S. Finbow, M. E. Messinger, and M. F. van Bommel. Eternal domination in $3 \times n$ grids. *Australas. J. Combin.*, 61:156–174, 2015.

[11] W. Goddard, S. M. Hedetniemi, and S. T. Hedetniemi. Eternal security in graphs. *J. Comb. Math. Comb. Comput.*, 52:160–180, 2005.

[12] J. L. Goldwasser, W. F. Klostermeyer, and C. M. Mynhardt. Eternal protection in grid graphs. *Util. Math.*, 91:47–64, 2013.

[13] D. Gonçalves, A. Pinlou, M. Rao, and S. Thomassé. The domination number of grids. *SIAM J. Discrete Math.*, 25(3):1443–1453, 2011.

[14] W. F. Klostermeyer, M. Lawrence, and G. MacGillivray. Dynamic dominating sets: the eviction model for eternal domination. Manuscript, 2014.

[15] W. F. Klostermeyer and G. MacGillivray. Eternal dominating sets in graphs. *J. Comb. Math. Comb. Comput.*, 68, 2009.

[16] W. F. Klostermeyer and C. M. Mynhardt. Eternal total domination in graphs. *Ars Combin.*, 68:473–492, 2012.

[17] W. F. Klostermeyer and C. M. Mynhardt. Protecting a graph with mobile guards. *Applicable Analysis and Discrete Mathematics*, 10, 2014.

[18] I. Lamprou, R. Martin, and S. Schewe. Perpetually dominating large grids. In *10th International Conference on Algorithms and Complexity (CIAC 2017)*, pages 393–404, 2017.

[19] C. S. Revelle. Can you protect the roman empire? *Johns Hopkins Magazine*, 50(2), 1997.

[20] C. S. Revelle and K. E. Rosing. Defendens imperium romanum: A classical problem in military strategy. *Amer. Math. Monthly*, 107:585–594, 2000.

[21] I. Stewart. Defend the roman empire! *Scientific American*, pages 136–138, 1999.

[22] C. M. van Bommel and M. F. van Bommel. Eternal domination numbers of $5 \times n$ grid graphs. *J. Comb. Math. Comb. Comput.*, 97:83–102, 2016.