Constrained ear decompositions in graphs and digraphs

Frédéric Havet, Nicolas Nisse

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Abstract

Ear decompositions of graphs are a standard concept related to several major problems in graph theory like the Traveling Salesman Problem. For example, the Hamiltonian Cycle Problem, which is notoriously $\mathcal{NP}$-complete, is equivalent to deciding whether a given graph admits an ear decomposition in which all ears except one are trivial (i.e. of length 1). On the other hand, a famous result of Lovász states that deciding whether a graph admits an ear decomposition with all ears of odd length can be done in polynomial time. In this paper, we study the complexity of deciding whether a graph admits an ear decomposition with prescribed ear lengths. We prove that deciding whether a graph admits an ear decomposition with all ears of length at most ℓ is polynomial-time solvable for all fixed positive integer ℓ. On the other hand, deciding whether a graph admits an ear decomposition without ears of length in $\mathcal{F}$ is $\mathcal{NP}$-complete for any finite set $\mathcal{F}$ of positive integers. We also prove that, for any $k \geq 2$, deciding whether a graph admits an ear decomposition with all ears of length $0 \mod k$ is $\mathcal{NP}$-complete.

We also consider the directed analogue to ear decomposition, which we call handle decomposition, and prove analogous results : deciding whether a digraph admits a handle decomposition with all handles of length at most ℓ is polynomial-time solvable for all positive integer ℓ; deciding whether a digraph admits a handle decomposition without handles of length in $\mathcal{F}$ is $\mathcal{NP}$-complete for any finite set $\mathcal{F}$ of positive integers (and minimizing the number of handles of length in $\mathcal{F}$ is not approximable up to $n(1-\epsilon)$); for any $k \geq 2$, deciding whether a digraph admits a handle decomposition with all handles of length $0 \mod k$ is $\mathcal{NP}$-complete. Also, in contrast with the result of Lovász, we prove that deciding whether a digraph admits a handle decomposition with all handles of odd length is $\mathcal{NP}$-complete. Finally, we conjecture that, for every set $\mathcal{A}$ of integers, deciding whether a digraph has a handle decomposition with all handles of length in $\mathcal{A}$ is $\mathcal{NP}$-complete, unless there exists $h \in \mathbb{N}$ such that $\mathcal{A} = \{1, \cdots, h\}$.

1 Introduction

The Traveling Salesman Problem (TSP) is one of the most famous and notoriously hard combinatorial optimization problem. One of its versions, known as Graph-TSP, can be defined as follows. Given a graph G, we denote by $2G$ the graph obtained from G by doubling all its edges, and a **multi-subgraph** of G is a subgraph of $2G$. A **tour** of G is a connected spanning multi-subgraph of G in which all vertices have even degree. Graph-TSP consists in finding a minimum cardinality (number of edges) tour of a given connected graph.

A relaxation of Graph-TSP is the 2-Edge-Connected Subgraph problem (2-ECSS for short). Given a connected graph G, we look for a 2-edge-connected spanning multi-subgraph with minimum number of edges. A solution of course contains two copies of each bridge, and may at first contain parallel copies of other edges too. However, the latter can always be avoided (See [12] for example.)

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Graph-TSP and 2-Edge-Connected Spanning Subgraph are $\mathcal{NP}$-hard because the 2-edge-connected spanning subgraphs of G with $|V(G)|$ edges are precisely the hamiltonian cycles. A **ρ-approximation algorithm** for a minimization problem is a polynomial-time algorithm that always computes a solution of value at most ρ times the optimum. For the above problems, a 2-approximation algorithm is trivial by taking a spanning tree and doubling all its edges.

Christofides [2] described a $\frac{3}{2}$-approximation algorithm for Graph-TSP. In the last few years, several progress were made. Oveis Gharan, Saberi, and Signh [5] gave a $(\frac{3}{2}-\epsilon)$-approximation algorithm for a tiny $\epsilon > 0$, Mömke and Svensson [9] obtained a 1.461-approximation algorithm, and Mucha [10] refined their analysis and obtained the approximation ratio of $\frac{13}{9} = 1.444\ldots$ Finally, Sebő and Vygen [12] described a $\frac{7}{5}$-approximation algorithm for Graph-TSP.

For 2-ECSS, Khuller and Vishkin [6] gave a $\frac{3}{2}$-approximation algorithm and Cheriyan, Sebő, and Szigeti [1] improved the ratio to $\frac{17}{12}$. This was further improved by Sebő and Vygen [12] who described a $\frac{4}{3}$-approximation algorithm for 2-ECSS.

1.1 Ear decomposition in graphs

An important tool in the best approximation algorithms for both problems is **ear decomposition**.

Let G be a graph and let F be a subgraph of G. An **F-ear** in G is either a cycle in G with exactly one vertex in $V(F)$ or a path having its two (different) endvertices in $V(F)$ and no internal vertex in $V(F)$, and distinct from an edge of F. A **v-ear-decomposition** of a graph G is a sequence $(H_p)_{1\leq p\leq p^*}$ such that H_1 is a $(\{v\},\emptyset)$-ear, and H_p is an $H_1\cup\cdots\cup H_{p-1}$-ear for all $2\leq p\leq p^*$, and $G=\bigcup_{p=1}^{p^*} H_p$. A vertex $v\in V$ is **included** by the ear H_p if p is the smallest index for which $v\in V(H_p)$. An **ear decomposition** is a v-ear-decomposition for some vertex v. The number of ears in any ear decomposition of G is $|E(G)|-|V(G)|+1$.

A graph has an ear decomposition if and only if it is 2-edge-connected. The **lentgh** of an ear is its number of edges. An ear is **trivial** if it has length 1. It is **even** (resp. **odd**) if its length is even (resp. odd). Maximizing the number of trivial ears is equivalent to the 2-ECSS problem because deleting trivial ears maintains 2-edge-connectivity. We show in Subsection 5.1 that minimizing the number of trivial ears is $\mathcal{NP}$-hard. Observe that if a graph has an ear decomposition with no non-trivial short ears (i.e. ears of length between 2 and ℓ for some fixed ℓ), then the spanning subgraph H obtained by removing the trivial ears satisfies $|V(H)|\leq|E(H)|\leq\frac{\ell+1}{\ell}|V(H)|$. Hence a natural idea to approximate 2-ECSS consists in finding an ear decomposition that minimizes the number of non-trivial short ears. Unfortunately, we show that this cannot work directly : we prove (Corollary 12) that for any finite set of integers $\mathcal{F}$, deciding whether a graph as an ear decomposition in which no ear has length in $\mathcal{F}$ is $\mathcal{NP}$-complete. Note however that short ears are used in the approximation algorithm given by Sebő and Vygen [12].

When the set $\mathcal{F}$ of forbidden ear lengths is infinite, the problem may become polynomial-time solvable. Lovász [8] showed that a graph G has an **odd ear decomposition** (i.e. an ear decomposition in which all ears are odd) if and only if it is **factor-critical** (that is such that $G-v$ has a perfect matching for every vertex v). This implies that one can decide in polynomial time whether a graph has an odd ear decomposition. Frank [3] gave a polynomial-time algorithm that finds an ear decomposition with the minimum number of even ears, which is an ear decomposition with the maximum number of odd ears. This algorithm plays a central role in the above-cited approximation algorithms for 2-ECSS. In contrast, we

show in Subsection 5.2 that deciding whether a graph admits an **even ear decomposition** (i.e. an ear decomposition in which all ears are even) is $\mathcal{NP}$-complete. More generally, we prove that for every fixed k, $k \geq 2$, deciding whether a graph admits an ear decomposition in which all ears have length 0 modulo k is $\mathcal{NP}$-complete.

1.2 Handle decomposition in graphs

We are also interested in the directed analogue of ear decompositions. For sake of clarity, we call it handle decomposition (which is the alternative usual name to ear decomposition).

Let D be a digraph and F a subdigraph of D. An **F-handle** H of D is either a directed cycle $(v_0, v_1, \ldots, v_{\ell-1}, v_\ell)$ with $v_0 = v_\ell$ in $V(F)$ and all other vertices not in $V(F)$, or a directed path $(v_0, v_1, \ldots, v_\ell)$ arc-disjoint with F with $v_0, v_\ell \in V(F)$ and all internal vertices not in $V(F)$. Even when H is a cycle, we call v_0 and v_ℓ are the **endvertices** of H while the vertices v_i, $1 \leq i \leq \ell - 1$ are its **internal vertices**; the vertex v_0 is the **initial vertex** of H and v_ℓ its **terminal vertex**. The **length** of a handle is the number of its arcs, here ℓ. A handle is **odd** (resp. **even**, **trivial**) if its length is odd (resp. even, 1).

Given a digraph D and a subdigraph F, an **F-handle-decomposition** of D is a sequence $(H_p)_{1 \leq p \leq p^*}$ such that H_1 is an F-handle, and H_p is an $F \cup H_1 \cup \cdots \cup H_{p-1}$-handle for all $2 \leq p \leq p^*$, and $D = F \cup \bigcup_{p=1}^{p^*} H_p$. For a subset S of vertices, an **S-handle decomposition** is an $(S, \emptyset)$-handle decomposition, and for a vertex v, a **v-handle decomposition** is a $\{v\}$-handle decomposition. A vertex $v \in V$ is **included** by the handle H_p if p is the smallest index for which $v \in V(H_p)$. A **handle decomposition** is a v-handle decomposition for some vertex v.

It is easy and well-known that a digraph admits a handle decomposition if and only if it is strongly connected. The number p^* of handles in any handle decomposition of a strongly connected digraph D is exactly $|A(D)| - |V(D)| + 1$. The value $p^* = p^*(D)$ is called the **cyclomatic number** of D. Observe that $p^*(D) = 0$ when D is a singleton and $p^*(D) = 1$ when D is a directed cycle.

There are many similarities between ear decompositions of 2-edge-connected graphs and handle decomposition of strongly connected digraphs which trace back to the introduction of these notions by Robbins [11] to establish its celebrated theorem: the graphs that have strongly connected orientations are exactly the 2-edge-connected graphs.

A handle decomposition is **odd** (resp. **even**) if each of its handles is odd (resp. even). A handle decomposition is **genuine** if it has no trivial handles.

Note that finding a handle decomposition with the maximum number of trivial handles of a given digraph D is equivalent to Minimum Spanning Strong Subdigraph (MSSS for short) which consists in finding a spanning strongly connected subdigraph of D with the minimum number of arcs, because deleting trivial handles maintains strong connectivity. This problem is well-known to be $\mathcal{NP}$-hard as it contains the well-known $\mathcal{NP}$-complete Directed Hamiltonian Cycle problem: deciding whether a strongly connected digraph has a handle decomposition with all handles except one being trivial is equivalent to deciding whether it has a directed hamiltonian cycle. Vetta [13] gave a $\frac{3}{2}$-approximation algorithm for MSSS. A digraph is **symmetric** if (v, u) is an arc whenever (u, v) is an arc. The **associated symmetric digraph** of a graph G is the symmetric digraph obtained from G by replacing each edge $\{u, v\}$ by the two arcs uv and vu. Observe that solving 2-ECSS for a graph G is equivalent to solve MSSS for its associated symmetric digraph.

Checking if there is at least one of the $\binom{|A(D)|}{k}$ sets S of k arcs such that $D \setminus S$ is strong, yields a polynomial-time algorithm to check whether a digraph D has a handle decomposition

with at least k trivial handles (when k is fixed). Recall that deciding whether a strongly connected digraph has a handle decomposition in which all handles but one are trivial. Consider the **h-subdivision** $S_h(D)$ of D, which is the digraph obtained from D by replacing each arc by a directed path of length h. There is a one-to-one correspondence between the handle decompositions of D and those of $S_h(D)$, since every path of length h replacing an arc is entirely contained in a handle. Hence D has a handle decomposition with all but one trivial handles if and only if $S_h(D)$ has a handle decomposition with all but one handles of length at most h. Therefore, for every positive integer h, it is $\mathcal{NP}$-complete to decide whether a strongly connected digraph has a handle decomposition in which all handles but one have length at most h.

In contrast, in Section 2, we study the problem of deciding whether a strongly connected digraph as a handle decomposition or an F-handle decomposition in which all handles have length at most h. If $h = 1$, the problems are clearly polynomial-time solvable. Indeed the first handle of a handle decomposition is necessarily non-trivial, so there is no handle decomposition with only trivial handles (unless if G has a single vertex). More generally, given a digraph D and a subdigraph F, one can decide in polynomial time whether D admits an F-handle decomposition with only trivial handles: it suffices to check whether $V(D) = V(F)$ or not. For each fixed $h \geq 2$, we describe polynomial-time algorithm to decide whether a digraph D has an F-handle decomposition with handles of length at most h.

In Section 3, we consider the opposite problem to MSSS, which consists in finding a handle decomposition with the minimum number of short handles. We first prove that deciding whether a digraph as a genuine handle decomposition $\mathcal{NP}$-complete. This implies that for any finite set of integers $\mathcal{F}$, deciding whether a digraph as a handle decomposition in which no handle has length in $\mathcal{F}$ is $\mathcal{NP}$-complete. We also show that, under the assumption $\mathcal{P} \neq \mathcal{NP}$, minimizing the number of trivial handles in a handle decomposition is not approximable within a factor $n(1 - \varepsilon)$ (where n is the number of vertices) for all $\varepsilon > 0$.

We then study the existence of odd and even handle decompositions in a digraph. Observe a graph G admits an odd ear decomposition if and only if its associated symmetric digraph admits an odd handle decomposition. Since deciding whether a graph has an odd ear decomposition was shown polynomial-time solvable by Lovász [8], deciding whether a symmetric digraph admits an odd handle decomposition is also polynomial-time solvable. In Subsection 4.1, we show that this does not extend to digraph: deciding whether a digraph admits an odd handle decomposition is $\mathcal{NP}$-complete. Then in Section 5.2, we deduce that deciding whether a digraph admits an even handle decomposition is $\mathcal{NP}$-complete. Observe that this problem is trivially polynomial-time solvable when restricted to symmetric digraphs since a symmetric digraph has an even handle decomposition if and only if it is the associated symmetric digraph of a forest (i.e. acyclic graph). Indeed, if the graph G contains a cycle, any handle decomposition of its associated symmetric digraph has a trivial handle; if G is a forest, then the directed 2-cycles corresponding to the edges of G form a handle decomposition.

We first present our result on digraphs because they are less technical than the ones on graphs but use the same kind of ideas. To lighten the notation, we abbreviate both an edge $\{u, v\}$ in a graph and an arc (u, v) in a digraph into uv. This is non ambiguous because we only deal with digraphs in Sections 2 to 4 and only with graphs in Section 5. In addition the proof of Theorem 14 on digraphs, which is left to the reader, is exactly identical to the one of Theorem 13 on graphs with this notation.

In the final section (Section 6), we give some directions for further research.

2 Handle decomposition with no long handles

Theorem 1. *Let h be a fixed integer greater than 1. There is a polyomial-time algorithm that, given a digraph D and a subdigraph F, decides whether D admits an F-handle decomposition with handles of length at most h.*

Proof. Let D/F be the digraph obtained from D by contracting F into a vertex v_F. It is easy to see that D admits an F-handle decomposition with handles of length at most h if and only if D/F admits a v_F-handle decomposition with handles of length at most h. Therefore it suffices to show a polynomial-time algorithm, that given a digraph D and a vertex v, decides whether D admits a v-handle decomposition with handles of length at most h.

Let D be a digraph and v a vertex of D. A **v-handle h-sequence** is a sequence $(H_p)_{1\leq p\leq \tilde{p}}$ such that H_1 is a $\{v\}$-handle of length at most h, and, for all $2 \leq p \leq \tilde{p}$, H_p is a $(H_1 \cup \cdots \cup H_{p-1})$-handle of length at most h. The **support** of a v-handle h-sequence $(H_p)_{1\leq p\leq \tilde{p}}$ is $\bigcup_{p=1}^{\tilde{p}} V(H_p)$. Clearly, D admits a v-handle decomposition with handles of length at most h if and only if D admits a v-handle h-sequence with support $V(D)$.

We define the order relation $\preceq$ over the v-handle h-sequences as follows:
$(H_p)_{1\leq p\leq \tilde{p}} \preceq (H'_p)_{1\leq p\leq \tilde{q}}$ if $\tilde{p} \leq \tilde{q}$ and $H_p = H'_p$ for all $1 \leq p \leq \tilde{p}$.

Claim 1.1. *Two maximal v-handle h-sequences for $\preceq$ have the same support.*

Proof. Assume for a contradiction that there are two maximal v-handle h-sequences $(H_p)_{1\leq p\leq \tilde{p}}$ and $(H'_p)_{1\leq p\leq \tilde{q}}$ with distinct supports. By symmetry, we may assume that there is a vertex x in the support of $(H'_p)_{1\leq p\leq \tilde{q}}$ that is not in the support of $(H_p)_{1\leq p\leq \tilde{p}}$. Let H'_q be the handle in which x was included. Without loss of generality, we may assume that all vertices of $\bigcup_{1\leq p<q} H'_p$ are in the support of $(H_p)_{1\leq p\leq \tilde{p}}$. Hence the endvertices of H'_q are in the support of $(H_p)_{1\leq p\leq \tilde{p}}$. Let $H_{\tilde{p}+1}$ be a subdipath of H'_q with endvertices in the support of $(H_p)_{1\leq p\leq \tilde{p}}$ and with internal vertices not in this set (for instance such a component containing x). Clearly $(H_p)_{1\leq p\leq \tilde{p}+1}$ is a v-handle h-sequence contradicting the maximality of $(H_p)_{1\leq p\leq \tilde{p}}$. ♢

Claim 1.1 implies that, to decide whether a digraph D has a v-handle decomposition with handles of length at most h, it suffices to compute a maximal v-handle h-sequence and check whether its support is $V(D)$. But a maximal v-handle h-sequence can be computed greedily: At each step p, we check whether there is a $(H_1 \cup \cdots \cup H_p)$-handle (or a $\{v\}$-handle if $p = 1$) of length at most h. This can be done in linear time using a modified shortest-path algorithm. If yes, we add it to the v-handle h-sequence and proceed to step $p+1$; otherwise we stop. □

Running the algorithm given by Theorem 1 for every vertex v yields a polynomial-time algorithm deciding whether a given digraph admits a handle decomposition with handles of length at most h.

Corollary 2. *Let h be a positive integer greater than 1. One can decide in polynomial time whether a digraph admits a handle decomposition with handles of length at most h.*

3 Handle decomposition with few short handles

Theorem 3. *Given a (strongly connected) digraph D, deciding whether D admits a genuine handle decomposition is $\mathcal{NP}$-complete. Moreover, minimizing the number of trivial handles in a handle decomposition is not approximable within $|V(D)|(1-\varepsilon)$ for all $\varepsilon > 0$ (under the assumption $\mathcal{P} \neq \mathcal{NP}$).*

Proof. The problem is clearly in $\mathcal{NP}$. To prove it is $\mathcal{NP}$-hard and not approximable, we present a reduction from 3-SAT and Min 3-SAT Deletion. 3-SAT takes a 3-CNF boolean formula and asks whether there exists a truth assignment such that all clauses are satisfied. Min 3-SAT Deletion takes a 3-CNF boolean formula and asks for the minimum number of unsatisfied clauses by a truth assignment. 3-SAT is well known to be $\mathcal{NP}$-complete [4] and Min 3-SAT Deletion is not approximable within a factor $n(1-\varepsilon)$ (where n is the number of variables) for all $\varepsilon > 0$ if $\mathcal{P} \neq \mathcal{NP}$ as shown by Klauck [7].

Let Φ be a 3-CNF boolean formula with variables $v_1, \ldots, v_n$ and clauses $C_1, \ldots, C_m$ (w.l.o.g., assume that no C_j contains both v_i and $\bar{v}_i$).

Let us first construct a digraph $D'(\Phi)$ that has an s_1-handle decomposition with at most k trivial handles if and only if Φ admits a truth assignment satisfying at least $m-k$ clauses.

Variable gadget V_i. For every $1 \leq i \leq n$, let V_i be the union of two internally disjoint directed paths $P_i = (s_i, x_i^0, x_i^1, \ldots, x_i^m, e_i)$ and $N_i = (s_i, \bar{x}_i^0, \bar{x}_i^1, \ldots, \bar{x}_i^m, e_i)$. Connect the variable gadgets as follows: for every $1 \leq i < n$, identify e_i with s_{i+1}. Let $\mathcal{V}$ denote the resulting graph.

Clause gadget K_j. For every $1 \leq j \leq m$, let K_j be the union three internally disjoint directed paths $K_j^1 = (d_j, \ell_j^1, q_j^1, f_j)$, $K_j^2 = (d_j, \ell_j^2, q_j^2, f_j)$ and $K_j^3 = (d_j, \ell_j^3, q_j^3, f_j)$. Connect the clause gadgets as follows: for every $1 \leq j < m$, identify f_i with d_{i+1}. Let $\mathcal{K}$ denote the resulting graph.

Connection between clause and variable gadgets. For every clause $C_j = \ell_1^j \vee \ell_2^j \vee \ell_3^j$ and $a \in \{0,1,2\}$, add the arc $t_{ij} = (\ell_j^a, \bar{x}_i^j)$ if $\ell_a^j = v_i$ and the arc $t_{ij} = (\ell_j^a, x_i^j)$ if $\ell_a^j = \bar{v}_i$. Note that the litteral ℓ_a^j corresponds to the vertex ℓ_j^a.

Finally, let $D'(\Phi)$ be the digragh obtained by adding the arcs (e_n, d_1) and (f_m, s_1). See Figure 1. Clearly, $D'(\Phi)$ is strongly connected.

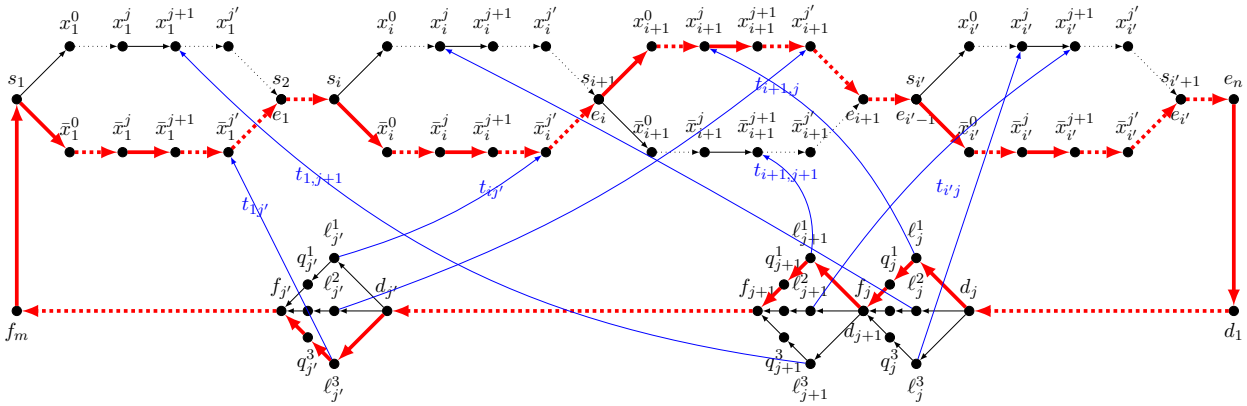

Figure 1: Example of a digraph $D'(\Phi)$ and $\Phi = \wedge_{1 \leq j \leq m} C_j$ with (as indicated by the blue arcs) $C_j = \bar{v}_{i+1} \vee \bar{v}_i \vee \bar{v}_{i'}$; $C_{j+1} = v_{i+1} \vee \bar{v}_{i'} \vee \bar{v}_1$ and $C_{j'} = v_i \vee \bar{v}_{i+1} \vee v_1$ (with $1 < i \leq i+1 < i' \leq n$ and $1 \leq j < j+1 < j' \leq m$). The bold red directed cycle containing s_1 is an example of a first s_1-handle $H_1 = X_1 \cup \cdots \cup Y_m \cup (f_m, s_1)$ that corresponds to a truth assignment ϕ where $\phi(v_1) = \mathit{false}$, $\phi(v_i) = \mathit{false}$, $\phi(v_{i+1}) = \mathit{true}$ and $\phi(v_{i'}) = \mathit{false}$ ($X_1 = N_1$, $X_i = N_i$, $X_{i+1} = P_{i+1}$ and $X_{i'} = N_{i'}$). The clause $C_{j'}$ being not satisfied by ϕ, there must be at least one trivial handle forced by $C_{j'}$ whenever $Y_{j'} = K_{j'}^1$ (in which case $t_{ij'}$ is trivial) or $Y_{j'} = K_{j'}^2$ (in which case $t_{i+1,j'}$ is trivial) or $Y_{j'} = K_{j'}^3$ (in which case $t_{1j'}$ is trivial, as in the example).

Let us first now prove that $D'(\Phi)$ has an s_1-handle decomposition with at most k trivial

handles if and only if Φ admits a truth assignment satisfying at least $m-k$ clauses.

Note that e_n is the only vertex of $\mathcal{V}$ which has out-neighbours outside of $\mathcal{V}$. Therefore, the first handle H_1 (which is directed cycle containing s_1) of any s_1-handle decomposition of $D'(\Phi)$ must be of the form

$$H_1 = X_1 \cup X_2 \cup \cdots \cup X_n \cup (e_n, d_1) \cup Y_1 \cup \cdots \cup Y_m \cup (f_m, s_1)$$

where $X_i \in \{P_i, N_i\}$ for every $1 \leq i \leq n$ and $Y_j \in \{K_j^1, K_j^2, K_j^3\}$ for every $1 \leq j \leq m$. In particular, there is a bijection between the "first half" $X_1 \cup \cdots \cup X_n$ of the possible first handles H_1 and the truth assignments ϕ (for every $1 \leq i \leq n$, if $X_i = P_i$, then $\phi(v_i) = true$ and $\phi(v_i) = false$ otherwise).

Claim 3.1. *If every assignment of Φ satisfies at most $m-k$ clauses, then every s_1-handle decomposition of $D'(\Phi)$ has at least k trivial handles.*

Proof. Consider an s_1-handle decomposition with first handle $H_1 = X_1 \cup \cdots \cup Y_m \cup (f_m, s_1)$ and the corresponding truth assignment ϕ (by above paragraph, all assignments are then considered). We show that each clause not satisfied by ϕ forces at least one trivial handle in the decomposition. Let $C_j = \ell_1^j \vee \ell_2^j \vee \ell_3^j$ be such a clause ($1 \leq j \leq m$) and let $a \in \{1,2,3\}$ such that $Y_j = K_j^a$ (note that H_1 contains the vertex ℓ_j^a). Finally, let $1 \leq i \leq n$ such that $\ell_a^j \in \{v_i, \bar{v}_i\}$. If $\ell_a^j = v_i$ (resp., $\ell_a^j = \bar{v}_i$), since C_j is not satisfied, then $\phi(v_i) = false$ (resp., $\phi(v_i) = true$) and so $X_i = N_i$ and H_1 contains $\bar{x}_i^j$ (resp., $X_i = P_i$ and H_1 contains x_i^j). In both cases, the arc t_{ij} (recall that $t_{ij} = (\ell_j^a, \bar{x}_i^j)$ if $\ell_a^j = v_i$ and $t_{ij} = (\ell_j^a, x_i^j)$ if $\ell_a^j = \bar{v}_i$) has to be a trivial handle of the decomposition. $\Diamond$

Claim 3.2. *If there exists a truth assignment of Φ satisfying at least $m-k$ clauses, then there exists an s_1-handle decomposition of $D'(\Phi)$ with at most k trivial handles.*

Proof. Let ϕ be a truth assignment satisfying $m-k$ clauses and let $H_1 = X_1 \cup \cdots \cup Y_m \cup (f_m, s_1)$ be defined by:

- for every $1 \leq i \leq n$, $X_i = P_i$ if $\phi(v_i) = true$ and $X_i = N_i$ otherwise, and
- for every $1 \leq j \leq m$, if $C_j = \ell_1^j \vee \ell_2^j \vee \ell_3^j$ is only satisfied by ℓ_a^j ($a \in \{1,2,3\}$) then $Y_j = K_j^a$, otherwise, Y_j is chosen arbitrarily in $\{K_j^1, K_j^2, K_j^3\}$.

As for the Claim 3.1, for every clause $C_j = \ell_1^j \vee \ell_2^j \vee \ell_3^j$ that is not satisfied by ϕ, there must be at least one trivial handle. Precisely, for every $a \in \{1,2,3\}$ such that $Y_j = K_j^a$ and $1 \leq i \leq n$ such that $\ell_a^j \in \{v_i, \bar{v}_i\}$. Then the arc t_{ij} has both its ends in H_1 and so must be a trivial handle. Let us now describe how to complete the initial handle H_1 and the at most k trivial handles described above by non-trivial-handles in order to obtain an s_1-handle decomposition, which thus will have at most k trivial handles.

For $i = 1$ to $i = n$, let us build the following handles. If $\phi(v_i) = false$ (resp., $\phi(v_i) = true$), then H_1 contains N_i (resp., P_i). Let $1 \leq j_1^i < \cdots < j_{\alpha_i}^i \leq m$ be such that $\mathcal{I}_i = \{j_1^i, \ldots, j_{\alpha_i}^i\}$ is the set of integers w such that v_i (resp., $\bar{v}_i$) is a litteral of C_w. Then, iteratively for $w = \alpha_i$ down to 1, add to the decomposition the handle $H = (X_w, x_i^{j_w^i}, x_i^{j_w^i+1}, \ldots, x_i^{j_{w+1}^i})$ where $x_i^{j_{\alpha_i+1}^i} = e_i$ (resp., $H = (X_w, \bar{x}_i^{j_w^i}, \bar{x}_i^{j_w^i+1}, \ldots, \bar{x}_i^{j_{w+1}^i})$ where $\bar{x}_i^{j_{\alpha_i+1}^i} = e_i$) and $X_w = \ell_w^a$ if ℓ_w^a has already been included by a handle and $X_w = (d_w, \ell_w^a)$ otherwise. In both cases, H has length at least 2 and so, H is not trivial. Finally (after $w = 1$), add the handle $(s_i, x_i^0, \ldots, x_i^{j_1^i})$

(resp., add the handle $(s_i, \bar{x}_i^0, \ldots, \bar{x}_i^{j_1^i})$) which is of length at least 2 thanks to the vertex x_i^0 (resp., $\bar{x}_i^0$).

Then, for every $1 \leq j \leq m$ such that ℓ_j^a ($a \in \{1,2,3\}$) has not been included yet in the decomposition, let x be the (unique) out-neighbour of ℓ_j^a in $\mathcal{V}$ (note that x has already been included in the decomposition either in H_1 or by some handle of the previous phase). Add the handle (d_j, ℓ_j^a, x).

Finally, for every $1 \leq j \leq m$, let us add the handle (ℓ_j^a, q_j^a, f_j) (of length 2) for $a \in \{1,2,3\} \setminus \{a'\}$ such that $Y_j = K_j^{a'}$ is a subgraph of H_1 (i.e., if q_j^a is not yet included in the decomposition).

It can be checked that the sequence of previously defined handles is actually an s_1-handle decomposition of $D'(\Phi)$ with at most k trivial handles. ◊

Claims 3.1 and 3.2 imply that $D'(\Phi)$ has an s_1-handle decomposition with at most k trivial handles if and only if Φ admits a truth assignment satisfying at least $m-k$ clauses.

Let us now construct a digraph $D(\Phi)$ that admits a handle decomposition with at most k trivial handles if and only if $D'(\Phi)$ admits an s_1-handle decomposition with at most k trivial handles.

Let T be the digraph obtained as follows. Start with a directed triangle (a,b,c), subdivide the arc (b,c) into a directed path $(b, h_1, \ldots, h_{k+1}, c)$ and, for every $1 \leq i \leq k+1$, add the arc $\alpha_i = (a, h_i)$. The digraph $D(\Phi)$ is obtained from T and $D'(\Phi)$ by identifying b (in T) and s_1 (in $D'(\Phi)$). Clearly, $D(\Phi)$ is strongly connected.

Claim 3.3. *$D(\Phi)$ admits a handle decomposition with at most k trivial handles if and only if $D'(\Phi)$ admits an s_1-handle decomposition with at most k trivial handles.*

Proof. Let us consider any handle decomposition $\mathcal{D}$ of $D(\Phi)$ with at most k trivial handles. For purpose of contradiction, let us first assume that the first handle is a directed cycle in the subdigraph $D'(\Phi)$. In such a decomposition, the only way to include vertices a and b is with the handle $(b = s_1, h_1, \ldots, h_{k+1}, c, a, b)$ which creates the $k+1$ trivial handles (a, h_i) for every $1 \leq i \leq k+1$.

Hence, $\mathcal{D}$ has to start with a directed cycle in T (since $b = s_1$ is a cut vertex). It is easy to check that any such decomposition with the minimum number of trivial handles starts with the handles $(c, a, h_{k+1}), (a, h_k, h_{k+1}), \ldots, (a, h_1, h_2), (a, b, h_1)$, and continues with an s_1-handle decomposition of $D'(\Phi)$. ◊

Claim 3.3 concludes the proof. □

Corollary 4. *Let $\mathcal{F}$ be a finite set of positive integers. Given a (strongly connected) digraph D, deciding whether D admits a handle decomposition with no handles of length in $\mathcal{F}$ is $\mathcal{NP}$-complete. Moreover, minimizing the number of handles of length in $\mathcal{F}$ in a handle decomposition is not approximable within $|V(D)|(1-\varepsilon)$ for all $\varepsilon > 0$ (under the assumption $\mathcal{P} \neq \mathcal{NP}$).*

Proof. Reduction from the case $\mathcal{F} = \{1\}$, which is $\mathcal{NP}$-complete by Theorem 3.

Let $h = \max \mathcal{F}$. Let D be a digraph. Consider $S_h(D)$ the h-subdivision of D. There is a one-to-one correspondence between the handle decompositions of D and those of $S_h(D)$, since every path of length h replacing an arc is entirely contained in a handle. Moreover, the length of handle in $S_h(D)$ is h times the length of the corresponding handle in D. Hence D has a handle decomposition with no trivial handles if and only if $S_h(D)$ has a handle decomposition with no handles of length at most h (or equivalently no length in $\mathcal{F}$). □

4 Odd and even handle decompositions

4.1 Odd handle decompositions

Theorem 5. *Deciding whether a given (strongly connected) digraph admits an odd handle decomposition is $\mathcal{NP}$-complete.*

Proof. The problem is clearly in $\mathcal{NP}$. To prove it is $\mathcal{NP}$-hard, we present a reduction from 1-IN-3-SAT. The problem 1-IN-3-SAT takes a boolean formula Φ in 3-CNF as input and asks whether there exists a truth assignment such that every clause contains exactly one true literal. Such an assignment is a called a 1-IN-3-SAT**-assignment**. The problem 1-IN-3-SAT is well known to be $\mathcal{NP}$-hard.

Let Φ be a 3-CNF boolean formula with variables $v_1, \ldots, v_n$ and clauses $C_1, \ldots, C_m$. We define a digraph $D(\Phi)$ from the formula Φ such that $D(\Phi)$ has an odd handle decomposition if and only if Φ admits a 1-IN-3-SAT-assignment.

Variable gadget: the digraph B_q.

The digraph B_1 has four vertices $s_0, x_1, \bar{x}_1, s_1$ and the arcs $s_0x_1, s_0\bar{x}_1, x_1s_1, \bar{x}_1s_1$.
For every integer $q > 1$, the digraph B_q is defined as follows :

$$V(B_q) = \bigcup_{k=0}^{q}\{s_k\} \cup \bigcup_{k=1}^{q}\{x_k, \bar{x}_k\} \cup \bigcup_{k=2}^{q}\{a_k, \bar{a}_k\} \cup \bigcup_{k=1}^{q-1}\{b_k, \bar{b}_k\} \cup \bigcup_{k=2}^{q}\{d_k, \bar{d}_k\}$$

$$\begin{aligned} A(B_q) \;=\; & \bigcup_{k=2}^{q-1}\{s_{k-1}a_k, a_kx_k, x_kb_k, b_ks_k, s_{k-1}\bar{a}_k, \bar{a}_k\bar{x}_k, \bar{x}_k\bar{b}_k, \bar{b}_ks_k\} \\ & \cup \bigcup_{k=2}^{q}\{a_kd_k, d_k\bar{b}_{k-1}, \bar{a}_k\bar{d}_k, \bar{d}_kb_{k-1}\} \\ & \cup\{s_0x_1, x_1b_1, b_1s_1, s_0\bar{x}_1, \bar{x}_1\bar{b}_1, \bar{b}_1\bar{s}_1, s_{q-1}a_q, a_qx_q, x_qs_q, s_{q-1}\bar{a}_q, \bar{a}_q\bar{x}_q, \bar{x}_q\bar{s}_q\} \end{aligned}$$

See Figure 2.

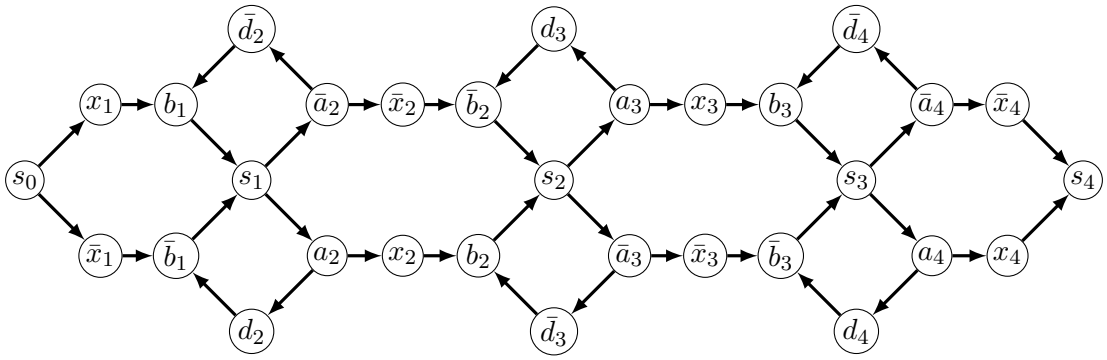

Figure 2: The digraph B_4.

Let P (resp. N) be the (unique) directed path of B_p going from s_0 to s_p through all vertices $x_1, \ldots, x_q$ (resp. $\bar{x}_1, \ldots, \bar{x}_q$). Note that P and N have even length. The vertices in $\{x_1, \ldots, x_q, \bar{x}_1, \ldots, \bar{x}_q\}$ are called the **variable-vertices**.

Claim 5.1. *Let q be a positive integer. Let D be a digraph containing B_q as an induced subdigraph and such that s_0 is the unique vertex of B_q having an in-neighbour in $V(D)\setminus V(B_q)$, and only the variable vertices and s_q have out-neighbours in $V(D) \setminus V(B_q)$.*

If D admits an odd handle decomposition $\mathcal{H}$ whose first handle H_p intersecting $V(B_q)$ contains s_0 and s_q, then H_p contains either P or N as subdipath. Moreover, if $q > 1$, the following hold:

- *if H_p contains P, then $\mathcal{H}$ contains the handles $(s_k, \bar{a}_{k+1}, \bar{d}_{k+1}, b_k)$ and $(a_{k+1}, d_{k+1}, \bar{b}_k, s_k)$ for every $1 \leq k < q$;*
- *otherwise, if H_p contains N, then $\mathcal{H}$ contains the handles $(s_k, a_{k+1}, d_{k+1}, \bar{b}_k)$ and $(\bar{a}_{k+1}, \bar{d}_{k+1}, b_k, s_k)$ for every $1 \leq k < q$.*

Proof. If $q = 1$, the result is obvious, so let us assume that $q > 1$. For sake of contradiction, let us assume that the first handle H_p intersecting $V(B_q)$ contains neither P nor N. Then H_p contains $(x_k, b_k, s_k, \bar{a}_{k+1}, \bar{x}_{k+1})$ (or symmetrically $(\bar{x}_k, \bar{b}_k, s_k, a_{k+1}, x_k)$) for some $1 \leq k < q$. Then, the only way to include the vertex $\bar{d}_{k+1}$ (or symmetrically d_{k+1}) is via the handle $(\bar{a}_{k+1}, \bar{d}_{k+1}, b_k)$ (or symmetrically $(a_{k+1}, d_{k+1}, \bar{b}_k)$) contradicting the fact that $\mathcal{H}$ is odd.

By symmetry, let us assume that H_p contains P. Then, for every $1 \leq k < q$, when $\bar{d}_{k+1}$ (resp. d_{k+1}) is included for the first time in $\mathcal{H}$, it must be by a handle containing $(\bar{a}_{k+1}, \bar{d}_{k+1}, b_k)$ (resp. $(a_{k+1}, d_{k+1}, \bar{b}_k)$). Since $\mathcal{H}$ is odd, this handle can only be $(s_k, \bar{a}_{k+1}, \bar{d}_{k+1}, b_k)$ (resp. $(a_{k+1}, d_{k+1}, \bar{b}_k, s_k)$). $\Diamond$

The variable beads $B(\Phi)$. For every $1 \leq i \leq n$, let q_i be the number of clauses in which the variable v_i occurs (negatively or positively).

For every $1 \leq i \leq n$, let $B^i_{q_i}$ be a copy of B_{q_i} (in what follows, the superscript i will be used to identify the corresponding variable. In particular, all vertices of $B^i_{q_i}$ are denoted as in B_{q_i} with the addition of the superscript i. Similarly, the paths P^i and N^i of $B^i_{q_i}$ correspond to the paths P and N of B_{q_i}).

Let us build the digraph $B(\Phi)$ as follows. We take the vertex-disjoint digraphs $B^i_{q_i}$ for $1 \leq i \leq n$. Then, for every $1 \leq i < n$, we identify the vertex $s^i_{q_i}$ with s^{i+1}_0. Note that, any path from s^1_0 to $s^n_{q_n}$ has even length. Finally, we add four new vertices c, y, z, w and the arcs $(s^n_{q_n}, y)$, (y, w), (w, z), (z, y), (w, c), and (c, s^1_0).

Claim 5.2. *Let D be any digraph containing $B(\Phi)$ as an induced subdigraph and such that the only vertex of $B(\Phi)$ having in-neighbours in $V(D \setminus B(\Phi))$ is c, and the only vertices of $B(\Phi)$ having out-neighbours in $V(D \setminus B(\Phi))$ are the variable-vertices of $B(\Phi)$.*

If D admits an odd handle decomposition $\mathcal{H}$, then:

(i) its first handle H_1 is (w, z, y, w), and

(ii) its second handle H_2 is from w to y, starts by (w, c, s^1_0) and follows by a directed path from s^1_0 to $s^n_{q_n}$ and y. Moreover, the restriction of H_2 to $D^i_{q_i}$ is either P^i or N^i for every $1 \leq i \leq n$.

Proof. Consider the handle including z. It must contain the dipath (w, z, y). Since this handle is odd it must be the cycle (w, z, y, w). This proves (i).

Now since the first handle is the cycle (w, z, y, w), the second handle H_2 has initial vertex w and terminal vertex y. Now by the hypothesis, c is the unique out-neighbour of w and s^1_0 is the unique out-neighbour of c. Hence H_2 starts with (w, c, s^1_0). The remaining is a directed

path from s_0^1 to $s_{q_n}^n$, which necessarily goes through each $B_{q_i}^i$. Now Claim 5.1 applied to each $B_{q_i}^i$ yields (ii). $\Diamond$

The key point in the previous claim is that, if the digraph $D(\Phi)$ (which will satisfy the hypotheses of Claim 5.2) admits an odd handle decomposition, then its second handle H_2 will define a truth assignment. Variable v_i will be assigned to *true* if P^i is a subpath of H_2 and to $false$ otherwise, i.e., if N^i is a subdipath of H_2.

Clauses gadgets. Let J be the digraph with vertex set $\{u_1, \ldots, u_5, t, c\}$ and arc set all arcs from $\{u_1, u_2, u_3\}$ to $\{u_4, u_5\}$, (u_4, u_5), (u_5, u_4), (u_4, t), (u_5, t) and (t, c).

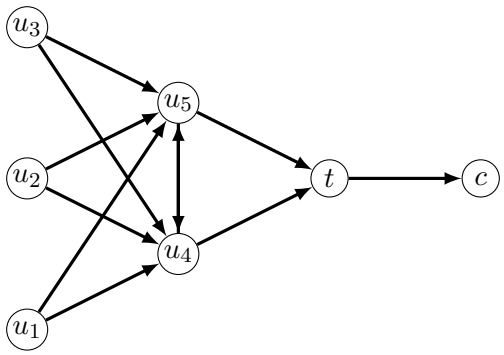

Figure 3: The digraph J.

The digraph $D(\Phi)$ is obtained from $B(\Phi)$ as follows.

For every $1 \leq j \leq m$, we add two vertex disjoint copies J^j and $(\bar{J}^j)$ of J (the super-script and the upper bar will be used to identity the copies and will be added to the vertices as well) and identify the vertices c^j and $\bar{c}^j$ with the vertex c of $B(\Phi)$. Let $C_j = \ell_{i_1} \vee \ell_{i_2} \vee \ell_{i_3}$ where, for every $\alpha \in \{1, 2, 3\}$, $\ell_{i_\alpha} \in \{v_{i_\alpha}, \bar{v}_{i_\alpha}\}$ is the literal corresponding to variable v_{i_α} in clause C_j. Moreover, for every $\alpha \in \{1, 2, 3\}$, let $k_{j,\alpha}$ be the integer such that C_j is the $k_{j,\alpha}$th clause in which variable v_{i_α} appears positively or negatively. For $\alpha \in \{1, 2, 3\}$, we do the following.

- If v_{i_α} appears positively in C_j, then add an arc from vertex $x_{k_{j,\alpha}}^{i_\alpha}$ of $B(\Phi)$ to vertex u_α^j in J^j, and identify vertex $\bar{x}_{k_{j,\alpha}}^{i_\alpha}$ of $B(\Phi)$ and vertex $\bar{u}_\alpha^j$ in $\bar{J}^j$.
- If v_{i_α} appears negatively in C_j, then add an arc from vertex $\bar{s}_{k_{j,\alpha}}^{i_\alpha}$ of $B(\Phi)$ to vertex u_α^j in J^j, and identify vertex $x_{k_{j,\alpha}}^{i_\alpha}$ of $B(\Phi)$ and the vertex $\bar{u}_\alpha^j$ in $\bar{J}^j$.

Let us now prove that $D(\Phi)$ has an odd handle decomposition if and only if Φ admits a 1-IN-3-SAT-assignment.

Assume first that there is an odd handle decomposition $\mathcal{H} = (H_p)_{1 \leq p \leq p^*}$ of $D(\Phi)$.

The digraph $D(\Phi)$ contains $B(\Phi)$ as an induced subdigraph and it satisfies the hypotheses of Claim 5.2. Therefore H_1 is the cycle (w, z, y, w) and H_2 is a directed path from w to y starting with (w, c, s_0^1) and continuing in a directed path whose intersection with each $D_{q_i}^i$ is either P^i or N^i. Let φ be the truth assignment that assigns *true* to v_i if P^i is a subdipath of H_2 and $false$ otherwise (that is when N^i is a subdipath of H_2).

By Claim 5.1, for every $1 \leq i \leq n$, $\mathcal{H}$ must contain the following handles :

- either $(s^i_k, \bar{a}^i_{k+1}, \bar{d}^i_{k+1}, b^i_k)$ and $(a^i_{k+1}, d^i_{k+1}, \bar{b}^i_k, s^i_k)$ for all $1 \leq k < q_i$ if H_2 contains P^i,
- or $(s^i_k, a^i_{k+1}, d^i_{k+1}, \bar{b}^i_k)$ and $(\bar{a}^i_{k+1}, \bar{d}^i_{k+1}, b^i_k, s^i_k)$ for all $1 \leq k < q_i$ if H_2 contains N^i.

Without loss of generality, we may assume that those handles are the H_p for $3 \leq p \leq 3m+2$. Set $D_j = \bigcup_{p=0}^{j} H_p$.

Observe now that if P^i is a subdipath of H_2, then the handle containing the arc $(\bar{x}^i_k, \bar{b}^i_k)$ (with $\bar{b}^i_{q_i} = s_{q_i}$) is necessarily the trivial handle restricted to this arc. Indeed in $D \setminus A(D_{3m+2})$, vertex $\bar{b}^i_k$ has a unique in-neighnour $\bar{x}^i_k$ which in turn has a unique in-neighbour $\bar{a}^i_k$ (or s^i_{k-1} if $k = 1$) which is a source. Similarly, if N^i is a subdipath of H_2, then the handle containing the arc (x^i_k, b^i_k) (with $b^i_{q_i} = s_{q_i}$) is necessarily the trivial handle restricted to this arc. Let $\mathcal{T}$ be the set of those $3m$ trivial handles. Without loss of generality, we may assume that the handles of $\mathcal{T}$ are the last ones in $\mathcal{H}$. Hence, setting $\tilde{p} = p^* - 3m$, we have $D_{\tilde{p}} = D \setminus \left(\bigcup_{H \in \mathcal{T}} H\right)$.

Observe now that the digraph $\bigcup_{p=3m+3}^{\tilde{p}} H_p)$ is the union of the $2m$ subdigraphs Y_j and $\bar{Y}_j$, $1 \leq j \leq m$ obtained from J^j and $\bar{J}^j$ as follows.

For every $\alpha \in \{1, 2, 3\}$:

- if $\ell_{i_\alpha} = v_{i_\alpha}$ (resp. $\ell_{i_\alpha} = \bar{v}_{i_\alpha}$) and H_2 contains P^i (resp. N^i), then the directed path $(x^{i_\alpha}_{k_{j,\alpha}}, u^j_\alpha)$ (resp. $(\bar{x}^{i_\alpha}_{k_{j,\alpha}}, u^j_\alpha)$ is added to J_j and the directed $(\bar{a}^{i_\alpha}_{k_{j,\alpha}}, u^j_\alpha)$ (resp. $(a^{i_\alpha}_{k_{j,\alpha}}, u^j_\alpha)$) is added to $\bar{J}_j$ (with $\bar{a}^{i_\alpha}_1 = a^{i_\alpha}_1 = s^{i_\alpha}_1$). Recall that in that case $\bar{x}^{i_\alpha}_{k_{j,\alpha}} = u^j_\alpha$ (resp. $x^{i_\alpha}_{k_{j,\alpha}} = u^j_\alpha$).
- if $\ell_{i_\alpha} = v_{i_\alpha}$ (resp. $\ell_{i_\alpha} = \bar{v}_{i_\alpha}$) and H_2 contains N^i (resp. P^i), then the directed path $(a^{i_\alpha}_{k_{j,\alpha}}, x^{i_\alpha}_{k_{j,\alpha}}, u^j_\alpha)$ (resp. $(\bar{a}^{i_\alpha}_{k_{j,\alpha}}, \bar{x}^{i_\alpha}_{k_{j,\alpha}}, u^j_\alpha)$) is added to J_j (with $\bar{a}^{i_\alpha}_1 = a^{i_\alpha}_1 = s^{i_\alpha}_1$), and no path is added to $\bar{J}_j$.

For every three positive integers r_1, r_2, r_3, let $J(r_1, r_2, r_3)$ be the digraph obtained from J by adding three vertex-disjoint directed paths R_i, $1 \leq i \leq 3$, with initial vertex w_i, terminal vertex u_i and length r_i. Observe that, for every $1 \leq j \leq m$, the digraph Y_j is isomorphic to $J(r_1, r_2, r_3)$ for some $r_1, r_2, r_3 \in \{1, 2\}$ and that $\bar{Y}_j$ is isomorphic to $J(2 - r_1, 2 - r_2, 2 - r_3)$.

The fact that exactly one literal of each clause C_j is true follows from the following claim applied to either Y_j or $\bar{Y}_j$.

Claim 5.3. *Let r_1, r_2, r_3 be three integers. $J(r_1, r_2, r_3)$ admits an odd $\{w_1, w_2, w_3, c\}$-handle decomposition if and only if exactly one of r_1, r_2, r_3 is odd.*

Proof. Clearly, $J(r_1, r_2, r_3)$ admits an odd $\{w_1, w_2, w_3, c\}$-handle decomposition if and only if $J(r_1 + 2, r_2, r_3)$, $J(r_1, r_2 + 2, r_3)$, and $J(r_1, r_2, r_3 + 2)$ do. Therefore it suffices to prove the result for $r_1, r_2, r_3 \in \{0, 1\}$. Moreover, by symmetry, it suffices to prove that $J(0, 0, 0)$, $J(1, 1, 0)$ and $J(1, 1, 1)$ have no odd $\{w_1, w_2, w_3, c\}$-handle decomposition, and that $J(1, 0, 0)$ has one. Recall that $w_i = u_i$ for $r_i = 0$.

Observe that every odd handle adds an even number of new (i.e. internal) vertices. Hence if $J(r_1, r_2, r_3)$ admits an odd $\{w_1, w_2, w_3, c\}$-handle decomposition, then $J(r_1, r_2, r_3)$ has even order so $r_1 + r_2 + r_3$ is odd. In particular, $J(0, 0, 0)$ and $J(1, 1, 0)$ have no odd $\{w_1, w_2, w_3, c\}$-handle decomposition.

Assume for a contradiction that $J(1, 1, 1)$ admits an odd $\{w_1, w_2, w_3, c\}$-handle decomposition. By symmetry, we may assume that first handle H_1 is $(w_1, u_1, u_4, u_5, t, c)$. Then the handle including u_2 has length 2, a contradiction.

Consider now $J(1,0,0)$. The decomposition with first handle $H_1 = (w_1, u_1, u_4, u_5, t, c)$ followed by trivial handles corresponding to the remaining arcs (those of $A(J(1,0,0)) \setminus A(H_1)$ is an odd $\{w_1, w_2, w_3, c\}$-handle decomposition of $J(1,0,0)$. $\Diamond$

Reciprocally, assume that Φ admits a 1-IN-3-SAT-assignment. One can construct an odd handle decomposition of $D(\Phi)$ as follows. The first handle H_1 is (w, z, y, w). The second handle H_2 is the union of (w, c, s_0^1), the P^i for each variable v_i assigned to *true* and the N^i for each variable v_i assigned to $false$. Then, for each variable v_i assigned to *true*, we take the handles $(s_k^i, \bar{a}_{k+1}^i, \bar{d}_{k+1}^i, b_k^i)$ and $(a_{k+1}^i, d_{k+1}^i, \bar{b}_k^i, s_k^i)$, and, for each variable v_i assigned to $false$, we take the handles $(s_k^i, a_{k+1}^i, d_{k+1}^i, \bar{b}_k^i)$ and $(\bar{a}_{k+1}^i, \bar{d}_{k+1}^i, b_k^i, s_k^i)$. Thanks to Claim 5.3, we then can extend this decomposition by taking odd handles decomposition of the Y_j and $\bar{Y}_j$ because there are all isomorphic to some $J(r_1, r_2, r_3)$ with exactly one odd r_i because the truth assignment was a 1-IN-3-SAT-assignment. The odd handle decomposition finishes with trivial handles of the set $\mathcal{T}$ described above. $\square$

Observe that in the proof of Theorem 5, every handle decomposition of $D(\Phi)$ has the cycle (w, z, y, z) as first handle. Therefore we have the following.

Theorem 6. *Given a digraph D and a vertex $z \in V(D)$, deciding whether D digraph admits an odd $\{z\}$-handle decomposition is $\mathcal{NP}$-complete.*

4.2 Even handle decompositions

Let D be a digraph and let z be a vertex of D. The digraph $F(D,z)$ is the digraph defined by

$$\begin{aligned} V(F(D,z)) &= \{z\} \cup \bigcup_{v \in V(D)\setminus\{z\}} \{v^-, v^+\} \cup \bigcup_{a \in A(D)} \{x_a\}, \\ A(F(D,z)) &= \bigcup_{v \in V(D)\setminus\{z\}} \{v^- v^+\} \cup \bigcup_{a=uv \in A(D)} \{u^+ x_a, x_a v^-\}, \text{ with } z^+ = z^- = z. \end{aligned}$$

Lemma 7. *D admits an odd $\{z\}$-handle decomposition if and only if $F(D,z)$ admits an even $\{z\}$-handle decomposition.*

Proof. Assume first that D admits an odd $\{z\}$-handle decomposition $(H_p)_{1 \le p \le p^*}$. For each handle H_p, let H'_p be the handle obtained from H_p by replacing each arc uv by the directed path (u^+, x_a, v^-) and each internal vertex v by the dipath (v^-, v^+). Observe that this is well-defined as z is never an internal vertex. Clearly, $(H'_p)_{1 \le p \le p^*}$ is a $\{z\}$-handle decomposition of $F(D,z)$. Moreover, the length of H'_p is twice the length of H_p plus the number of internal vertices of H_p. Since every H_p is odd, the length of each H'_p is even. Thus $(H'_p)_{1 \le p \le p^*}$ is an even $\{z\}$-handle decomposition of $F(D,z)$.

Assume now that $F(D,z)$ admits an even $\{z\}$-handle decomposition $(H'_p)_{1 \le p \le p^*}$. Observe that for each $v \in V(D) \setminus \{z\}$, the vertices v^- and v^+ are included by the same handle since v^- is the unique in-neighbour of v^+ and v^+ is the unique out-neighbour of v^-. Hence every handle H'_p is of the form $(v_0^+, x_{v_0 v_1}, v_1^-, v_1^+, \ldots, v_{l-1}^-, v_{l-1}^+, x_{v_{l-1} v_l}, v_l^-)$ with possibly $v_0 = z$ or $v_l = z$. Let H_p be the handle $(v_0, v_1, \ldots, v_{l-1}, v_l)$. Clearly, H_p is odd since H'_p is even. Thus $(H_p)_{1 \le p \le p^*}$ is an odd $\{z\}$-handle decomposition of $F(D,z)$. $\square$

Lemma 7 and Theorem 6 directly imply the following.

Theorem 8. *Given a digraph D and a vertex $z \in V(D)$, deciding whether D digraph admits an even $\{z\}$-handle decomposition is $\mathcal{NP}$-complete.*

Theorem 9. *Deciding whether a given digraph admits an even handle decomposition is $\mathcal{NP}$-complete.*

Proof. We give a reduction from 1-IN-3-SAT. Given a 3-CNF formula Φ, we construct the digraph $D(\Phi)$ as in the proof of Theorem 5, and then consider $F(D(\Phi), z)$. One can easily check that every even handle decomposition of $F(D(\Phi), z)$ is a $\{z\}$-handle decomposition. Moreover, by Lemma 7, there is an even $\{z\}$-handle decomposition of $F(D(\Phi), z)$ if and only if there is an odd $\{z\}$-handle decomposition of $D(\Phi)$. But the proof of Theorem 5 shows that such a handle decomposition exists if and only if Φ admits a 1-IN-3-SAT-assignment. □

5 Ear decomposition in graphs

5.1 Genuine Ear Decomposition

An ear decomposition is **genuine** if it has no trivial ears.

Proposition 10. *Let G be a graph with a genuine ear decomposition $\mathcal{H}$.*

(i) Let $\mathcal{F}$ be a subset of ears of $\mathcal{H}$. If a vertex v has two neighbours u and w which are internal vertices of ears not containing v, then (u, v, w) is an ear of $\mathcal{H}$.

(i) A vertex has at most two neighbours in every ear H of $\mathcal{H}$.

(iii) For any two ears H, H' of $\mathcal{H}$, every vertex not in $H \cup H'$ has at most two neighbours in $H \cup H'$.

Proof. (i) Let H' be the ear including v. If it does not contain the edges uv and vw, then it leaves them as trivial ears. Therefore H' must contain both uv and vw. Since u and w are not included by H', necessarily $H' = (u, v, w)$.

(ii) Assume a vertex v has three neighbours in some ear H of $\mathcal{H}$. The first ear containing v (which might be H) uses two edges incident to v, leaving the third edge to be a trivial ear, a contradiction to $\mathcal{H}$ being genuine.

(iii) Assume a vertex v has three neighbours in $H \cup H'$. The first ear containing v uses two edges incident to v, leaving the third edge (which is neither in H nor H') to be a trivial ear, a contradiction to $\mathcal{H}$ being genuine. □

Theorem 11. *Given a (2-edge-connected) graph G, deciding whether G admits a genuine ear decomposition is $\mathcal{NP}$-complete.*

Proof. The problem is clearly in $\mathcal{NP}$. To prove it is $\mathcal{NP}$-hard, we present a reduction from 3-SAT. The problem 3-SAT takes a boolean formula Φ in 3-CNF and an integer $k \geq 0$ as input and asks whether there exists a truth assignment satisfying Φ. The problem 3-SAT is well known to be $\mathcal{NP}$-complete [4].

Let Φ be a 3-CNF boolean formula with variables $v_1, \ldots, v_n$ and clauses $C_1, \ldots, C_m$ (w.l.o.g., assume that no C_j contains both v_i and $\bar{v}_i$). Let us construct a graph $G(\Phi)$ that has a genuine ear decomposition if and only if Φ is satisfiable.

Let us first define the main gadget $\mathbb{J}$ built as follows. The graph $\mathbb{J}$ is obtained from two disjoint paths $(s_1, a_1, b_1, c_1, d_1, t_1)$ and $(s_2, a_2, b_2, c_2, d_2, e_2, t_2)$ by adding the edges a_1d_2, b_1e_2, b_1d_2, b_1c_2, c_1b_2, d_1d_2 and a_2e_2. See Figure 4. The vertices of $V(\mathbb{J}) \setminus \{s_1, t_1, s_2, t_2\}$ are called the *internal* vertices of $\mathbb{J}$.

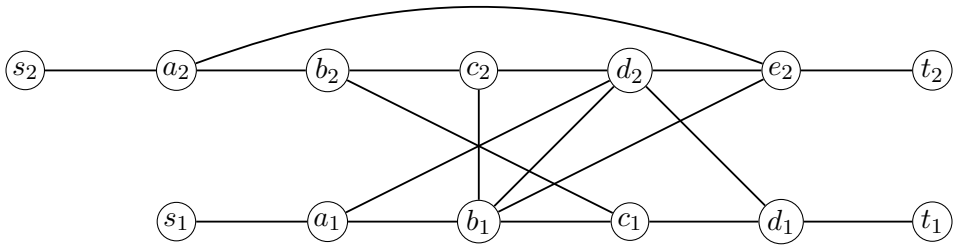

Figure 4: The gadget $\mathbb{J}$.

Claim 11.1. *Let G be any graph having $\mathbb{J}$ as an induced subgraph such that there is no edge between $\mathbb{J} \setminus \{s_1, s_2, t_1, t_2\}$ and $G \setminus \mathbb{J}$. Assume that G admits a genuine ear decomposition $\mathcal{H} = (H_p)_{1 \leq p \leq p^*}$ such that H_1 contains some vertex not in $\mathbb{J}$. Let H be the first ear of $\mathcal{H}$ including an internal vertex of $\mathbb{J}$.*

(i) If $s_1a_1 \in E(H)$, then H contains $(s_1, a_1, d_2, d_1, t_1) = E_1$.

(ii) If $s_2a_2 \in E(H)$, then H contains $(s_2, a_2, e_2, t_2) = E_2$.

(iii) H does not contain both s_1a_1 and s_2a_2.

Proof. Observe that since there is no edge between $\mathbb{J}\setminus\{s_1, s_2, t_1, t_2\}$ and $G\setminus\mathbb{J}$ and H_1 contains no internal vertex of $\mathbb{J}$, the ear H must contain at least two edges among s_1a_1, s_2a_2, d_1t_1 and e_2t_2.

A vertex is **bad** if it has three neighbours in H. By Proposition 10 (i), there is no bad vertex.

(i) Assume $s_1a_1 \in E(H)$.

H cannot contain $\{a_1, b_1, d_1\}$ for otherwise d_2 is bad. H cannot contain $\{a_1, b_1, c_2\}$ for otherwise d_2 is bad. H cannot contain $\{a_1, b_1, d_2\}$ for otherwise (a_1, b_1) or (b_1, d_2) is a trivial ear. H cannot contain $\{a_1, b_1, e_2\}$ for otherwise d_2 is bad.

H cannot contain (b_1, c_1, b_2, c_2) for otherwise (b_1, c_2) is a trivial ear. H cannot contain $(a_1, b_1, c_1, b_2, a_2, e_2)$ for otherwise d_2 is bad. Assume for a contradiction that H contains $(a_1, b_1, c_1, b_2, a_2, s_2)$. By Proposition 10 (i), $\mathcal{H}$ contains the ear $H' = (b_2, c_2, b_1)$. Now d_2 has three neighbours in $H \cup H'$, which contradicts Proposition 10 (iii). Henceforth H does not contain (a_1, b_1, c_1), and so it does not contain (a_1, b_1).

Consequently H contains (s_1, a_1, d_2). It cannot contain (a_1, d_2, b_1) for otherwise (a_1, b_1) is a trivial ear. It cannot contain (a_1, d_2, c_2) or (a_1, d_2, e_2) for otherwise b_1 is bad. Thus H contains (s_1, a_1, d_2, d_1). Now H cannot contain $(s_1, a_1, d_2, d_1, c_1)$ for otherwise b_1 is bad. Hence $H \cap \mathbb{J}$ contains $(s_1, a_1, d_2, d_1, t_1) = E_1$.

(ii) Assume $s_2a_2 \in E(H)$.

H cannot contain $\{c_2, d_2, b_1\}$, for otherwise c_2d_2, c_2b_1 or b_1d_2 is a trivial ear. H cannot contain $\{c_2, d_2, e_2\}$, for otherwise b_1 is bad. H cannot contain $\{b_2, c_2, d_2, d_1\}$, for otherwise by Proposition 10-(i), (c_2, b_1, d_2) is an ear, and so c_1 contradicts Proposition 10-(iii). Henceforth H does not contain (b_2, c_2, d_2).

H cannot contain (b_2, c_2, b_1, c_1), for otherwise (b_2, c_1) is a trivial ear. H cannot contain (b_2, c_2, b_1, a_1), for otherwise d_2 is bad. Henceforth H does not contain (b_2, c_2).

H cannot contain $(a_2, b_2, c_1, d_1, d_2, e_2)$, for otherwise (a_2, e_2) is a trivial ear. H cannot contain $(a_2, b_2, c_1, d_1, d_2, b_1)$, for otherwise (b_1, c_1) is a trivial ear. H cannot contain

$(a_2, b_2, c_1, d_1, d_2, c_2)$, for otherwise (b_2, c_2) is a trivial ear. H cannot contain $(a_2, b_2, c_1, d_1, d_2, a_1)$, for otherwise b_1 is bad. Henceforth H does not contain $(a_2, b_2, c_1, d_1, d_2)$.

Assume for a contradiction that H contains $(a_2, b_2, c_1, d_1, t_1)$. The ear containing b_2c_2 cannot contain d_2 for otherwise it must be (b_2, c_2, d_2, d_1) to avoid (d_2, d_1) to be a trivial ear, and b_1 contradicts Proposition 10-(iii). But then this ear must be (b_2, c_2, b_1, c_1) and d_2 contradicts Proposition 10-(iii). Henceforth H does not contain (a_2, b_2, c_1, d_1).

H cannot contain $(a_2, b_2, c_1, b_1, e_2)$, for otherwise (a_2, e_2) is a trivial ear. H cannot contain $(a_2, b_2, c_1, b_1, d_2)$, for otherwise c_2 is bad. H cannot contain $(a_2, b_2, c_1, b_1, c_2)$, for otherwise (b_2, c_2) is a trivial ear. H cannot contain $(a_2, b_2, c_1, b_1, a_1)$, for otherwise by Proposition 10-(i), (a_1, d_2, b_1) is an ear, and so c_2 contradicts Proposition 10-(iii). Henceforth H does not contain (a_2, b_2, c_1, b_1), and so it does not contain (a_2, b_2, c_1).

Consequently, H must contain a_2e_2.

H cannot contain (a_2, e_2, d_2, c_2), for otherwise b_2 is bad. H cannot contain (a_2, e_2, d_2, a_1), for otherwise b_1 is bad. H cannot contain $(a_2, e_2, d_2, b_1, a_1)$, for otherwise (d_2, a_1) is a trivial ear. H cannot contain $(a_2, e_2, d_2, b_1, c_2)$, for otherwise (d_2, c_2) is a trivial ear. H cannot contain $(a_2, e_2, d_2, b_1, c_1, d_1)$, for otherwise (d_2, d_1) is a trivial ear. H cannot contain $(a_2, e_2, d_2, b_1, c_1, b_2)$, for otherwise c_2 is bad. H cannot contain $(a_2, e_2, d_2, d_1, c_1)$, for otherwise b_1 is bad. H cannot contain $(a_2, e_2, d_2, d_1, t_1)$, for otherwise, by Proposition 10-(i), (d_2, b_1, e_2), (b_1, c_1, d_1), (a_2, b_2, c_1), and (b_2, c_2, d_2) are ears, and so (b_1, c_2) is a trivial ear. Henceforth H does. not contain (a_2, e_2, d_2).

H cannot contain (a_2, e_2, b_1, a_1), for otherwise d_2 is bad. H cannot contain (a_2, e_2, b_1, c_2), for otherwise d_2 is bad. H cannot contain (a_2, e_2, b_1, d_2), for otherwise (d_2, e_2) is a trivial ear. H cannot contain $(a_2, e_2, b_1, c_1, d_1)$, otherwise d_2 is bad. H cannot contain $(a_2, e_2, b_1, c_1, b_2)$, for otherwise (a_2, b_2) is a trivial ear. Henceforth H does. not contain (a_2, e_2, d_2).

Consequently, H contains $(s_2, a_2, e_2, t_2) = E_2$.

(iii) Assume for a contradiction that H contains both s_1a_1 and s_2a_2. By (i) and (ii) it must contain E_1 and E_2. Thus (d_2, e_2) must be a trivial ear, a contradiction. ◊

$\mathbb{J}$ is a very important gadget in our reduction and will be used in many places. By **replacing two edges x_1y_1 and x_2y_2 by a copy of $\mathbb{J}$**, we mean removing the edges x_1y_1 and x_2y_2 and add a copy of $\mathbb{J}$ with s_1, t_1, s_2, t_2 identified with x_1, y_1, x_2, y_2, respectively, all other vertices of the copy of $\mathbb{J}$ being new vertices.

We are now ready to construct the graph $G(\Phi)$.

Variable gadgets. For every $1 \leq i \leq n$, let V_i be the union of two internally disjoint paths $P_i = (z_i, x_i^0, x_i^1, \ldots, x_i^m, z_i', z_{i+1})$ and $\overline{P}_i = (z_i, \bar{x}_i^0, \bar{x}_i^1, \ldots, \bar{x}_i^m, \bar{z}_i', z_{i+1})$. Hence, for every $1 < i \leq n$, the vertex z_i belongs both to the gadget corresponding to variable v_{i-1} and to the gadget corresponding to variable v_i. Finally, for every $1 \leq i \leq n$, let us replace the edges $x_i^m z_i'$ and $\bar{x}_i^m \bar{z}_i'$ by a copy $\mathbb{J}^i$ of $\mathbb{J}$. The vertices of $\mathbb{J}^i$ are identified by the superscript ^{i}: for instance, the vertex d_2 of the copy $\mathbb{J}^i$ of $\mathbb{J}$ will be denoted by d_2^i.

Clause gadgets. For every $1 \leq j \leq m$, let K_j be the union three internally disjoint paths $K_j^a = (f_j, \ell_j^a, q_j^a, g_j^a, h_j^a, f_{j+1})$, $0 \leq a \leq 2$. Hence, for every $1 < j \leq m$, the vertex f_j belongs both to the gadget corresponding to clause C_{j-1} and to the gadget corresponding to clause C_j. Finally, for every $1 \leq j \leq m$, and for every $0 \leq a \leq 2$, we replace the edges $q_j^a g_j^a$ and g_j^{a+1}, h_j^{a+1} by a copy $\mathbb{J}^{j/a}$ of $\mathbb{J}$ (superscript are modulo 3). The vertices of $\mathbb{J}^{j/a}$ are identified by the superscript $^{j/a}$: for instance, the vertex d_2 of the copy $\mathbb{J}^{j/a}$ of $\mathbb{J}$ will be denoted by $d_2^{j/a}$).

Connection between clause and variable gadgets. For every $1 \leq j \leq m$, let us consider the clause $C_j = \ell_0^j \vee \ell_1^j \vee \ell_2^j$. For every $a \in \{0,1,2\}$, let $1 \leq i \leq n$ such that $\ell_a^j \in \{v_i, \bar{v}_i\}$.

If $\ell_a^j = v_i$ (resp., if $\ell_a^j = \bar{v}_i$), then we replace the edges $\bar{x}_i^{j-1}\bar{x}_i^j$ (resp., $x_i^{j-1}x_i^j$) and $\ell_j^a q_j^a$ by a copy $\mathbb{J}^{j,i}$ of $\mathbb{J}$. The vertices of $\mathbb{J}^{j,i}$ are identified by the double supserscript ^{j,i}: for instance, the vertex d_2 of the copy $\mathbb{J}^{j,i}$ of $\mathbb{J}$ will be denoted by $d_2^{j,i}$.

Close the graph. Add a vertex r and the edges $z_{n+1}r$ and $f_1 r$. Add a **starter subgraph** $\mathbb{S}$ with vertex set $\{\alpha, \beta, \gamma, \delta\}$ and edge set $\{\alpha\beta, \alpha\gamma, \beta\gamma, \beta\delta, \gamma\delta\}$, and add the edges αz_1 and $f_{m+1}\delta$.

The resulting graph is $G(\Phi)$ (note that $G(\Phi)$ is 2-edge-connected). Let us show that $G(\Phi)$ has a genuine ear decomposition if and only if Φ is satisfiable.

Suppose first that there is a genuine ear decomposition $\mathcal{H} = (H_p)_{p=1}^{p^*}$ of $G(\Phi)$.

Claim 11.2. *Either $H_1 = (\alpha, \beta, \gamma, \alpha)$ and $H_2 = (\beta, \delta, \gamma)$ or $H_1 = (\delta, \beta, \gamma, \delta)$ and $H_2 = (\beta, \alpha, \gamma)$.*

Proof. Assume for a contradiction that H_1 does not contain the edge $\beta\gamma$. If β and γ are included by a same ear H_p, then H_p must either have $(\alpha, \beta, \gamma, \delta)$ as a subpath, or have $(\alpha, \gamma, \beta, \delta)$ as a subpath, or be the cycle $(\alpha, \beta, \delta, \gamma, \alpha)$. In all cases, there will be trivial ears ((α, γ) and (δ, β) in the first case, (α, β) and (δ, γ) in the second one, (β, γ) in the third one), a contradiction. Hence, β and γ are not included by a same ear. By symmetry, we may assume that β is included first. The ear including β must contain (α, β, δ) as subpath. Hence, the ear including γ must be (α, γ, δ) or (α, γ, β) or (δ, γ, β), leaving (γ, β) or (γ, δ) or (γ, α) as trivial ear, a contradiction.

Hence,H_1 contains the edge $\beta\gamma$ and so, to avoid any trivial ear, it is either $(\alpha, \beta, \gamma, \alpha)$ or $(\delta, \beta, \gamma, \delta)$. Then $H_2 = (\beta, \delta, \gamma)$ or $H_2 = (\beta, \alpha, \gamma)$ for otherwise there would be a trivial ear. $\diamondsuit$

Let us study the properties of H_3 the third ear of $\mathcal{H}$.

Let Q_i be the path obtained from P_i as follows: for all $1 \leq j \leq m$, if $\bar{v}_i$ is a literal of C_j, replace the edge $x_i^{j-1}x_i^j$ by the path $E_1^{j,i}$; replace $x_i^m z_i'$ by the path E_1^i. Let $\overline{Q}_i$ be the path obtained from $\overline{P}_i$ as follows: for all $1 \leq j \leq m$, if v_i is a literal of C_j, replace the edge $\bar{x}_i^{j-1}\bar{x}_i^j$ by the path $E_1^{j,i}$; replace $x_i^m z_i'$ by the path E_2^i.

Let L_j^a be the union of the five paths (f_j, l_j^a), $E_2^{j,i}$ for the integer i such that $\ell_j^a \in \{v_i, \bar{v}_i\}$, $E_1^{j/a}$, $E_2^{j/a-1}$, and (h_j^a, f_{j+1}) (Indices a are modulo 3).

Claim 11.3. *(i) For every $1 \leq i \leq n$, H_3 contains either Q_i or $\overline{Q}_i$ (but not both).*

(ii) For every $1 \leq j \leq m$, H_3 contains L_j^a for some $a \in \{0,1,2\}$.

Proof. The proof can be sketched as follows. By Claim 11.2, H_3 must be a path from α to δ. It first goes to z_1 and enters the variable gagdets. In each variable gagdet, H_3 can choose to go along Q_i or $\bar{Q}_i$, but once it has chosen one of these paths, the gagdets $\mathbb{J}^{j,i}$ force H_3 to continue on it until it reaches x_i^m or $\bar{x}_i^m$. Then $\mathbb{J}^i$ forces H_3 to go to z_{i+1} and not to U-turn on the opposite path ($\overline{Q}_i$ or Q_i) of V_i. Hence H_3 enters the next variable gagdet. And so on, H_3 visits each variable gadget V_i, each time going through Q_i or $\overline{Q}_i$ (but not both) until it reaches z_{n+1}. Since it cannot U-turn, H_3 must go to r and then f_1, where it enters the clause gagdget. Now as above, in the clause gagdet K_j, it must follow one of the path L_j^a and cannot make any U-turn on L_j^{a+1} or L_j^{a+2}, because of the gadgets $\mathbb{J}^{j/a}$ and $\mathbb{J}^{j/a-1}$. Thus H_3 must go through each clause gadget going each time through exactly one of the L_j^a.

Let us now give a detailed proof.

Let us prove (i) by induction on i.

Let us first prove that H_3 contains either the edge $z_i x_i^0$ or the edge $z_i \bar{x}_i^0$ (but not both). If $i = 1$, then Claim 11.2 implies that H_3 is a path from α to δ in $G(\Phi) - \{\beta, \gamma\}$. Its first edge must be αz_1. Its second edge is either $z_1 x_1^0$ or $z_1 \bar{x}_1^0$. If $i > 1$, then by the induction hypothesis, it contains either Q_{i-1} or $\overline{Q}_{i-1}$. In the first (resp. second) case, by Claim 11.1 (iii) (applied to $\mathbb{J}^{i-1}$ since $E_2^{i-1} \subseteq \overline{Q}_{i-1}$ and $E_1^{i-1} \subseteq Q_{i-1}$), it does not contain the edge $\bar{z}'_{i-1} e_2^{i-1}$ (resp. $z'_{i-1} d_1^{i-1}$). So H_3 must contain $z_I x_i^0$ or $z_i \bar{x}_i^0$.

Assume H_3 contains $z_i x_i^0$. Either $\bar{v}_i$ is not a literal of C_1 and the edge $x_i^0 x_i^1$ has not been replaced and H_3 must contain it, or $\bar{v}_i$ is a literal of C_1 and the edge $x_i^0 x_i^1$ has been replaced (with another edge) in a copy of $\mathbb{J}^{1,i}$ and, by Claim 11.1 (i), H_3 contains the path $E_1^{1,i}$. And so on, by induction on j, we show that H_3 contains the edge $x_i^{j-1} x_i^j$ or the path $E_1^{j,i}$ for all $1 \leq j \leq m$. Now by Claim 11.1 (i) applied to $\mathbb{J}^i$, H_3 contains the the path E_1^i. Hence H_3 contains Q_i.

Similarly, one proves that if H_3 contains $z_i \bar{x}_i^0$ then H_3 contains $\overline{Q}_i$. This prove (i)

Let us now prove (ii) by induction on j.

Let us first prove that H_3 contains one of the edges $f_j \ell_j^a$, $a \in \{0, 1, 2\}$. If $j = 1$, since H_3 contains either Q_m or $\overline{Q}_m$ (and so E_1^m or E_2^m), as above we get that H_3 must contain $z_{n+1} r$ and thus also $r f_1$. Then it continues by $f_1 \ell_1^a$ for some $a \in \{0, 1, 2\}$. If $j \neq 1$, then by the induction hypothesis, H_3 contains $L_{j-1}^{a'}$ for some $a' \in \{0, 1, 2\}$. Suppose for a contradiction that H_3 continues with the edge $f_j h_{j-1}^{a'+1}$, then it must also contains $h_{j-1}^{a'+1} g_{j-1}^{a'+1}$, a contradiction to Claim 11.1 (iii) applied to $\mathbb{J}^{j-1/a'}$. Suppose for a contradiction that H_3 continues with the edge $f_j h_{j-1}^{a'-1}$, then by Claim 11.1 applied to $\mathbb{J}^{j-1/a'+1}$, it must contain $E_2^{j-1/a'+1}$ which ends in $g_{j-1}^{a'-1}$, and must continue by $g_{j-1}^{a'-1} s_1^{j-1/a'-1}$. This contradicts Claim 11.1 applied to $\mathbb{J}^{j-1/a'-1}$. Hence H_3 must contain one of the $f_j \ell_j^a$, $a \in \{0, 1, 2\}$.

Now let i be the integer such that $\ell_j^a \in \{v_i, \bar{v}_i\}$. Claim 11.1 applied to $\mathbb{J}^{j,i}$, and then successively to $\mathbb{J}^{j/a}$ and $\mathbb{J}^{j/a-1}$ yields that H_3 contains L_j^a. $\Diamond$

Let ϕ be the truth assignment defined by $\phi(v_i) = true$ if H_3 contains Q_i and $\phi(v_i) = false$ otherwise. This assignment is well-defined by Claim 11.3 (i).

Let us check that it satisfies Φ. Consider a clause C_j. By Claim 11.3 (i), there is $a \in \{0, 1, 2\}$ such that H_3 contains L_j^a. Let i be the integer such that $\ell_j^a \in \{v_i, \bar{v}_i\}$. If $\ell_j^a = v_i$ (resp. $\ell_j^a = \bar{v}_i$), then Claim 11.1 applied to $\mathbb{J}^{j,i}$ implies that H_3 does not contain $E_1^{j,i}$ (since L_j^a contains $E_2^{j,i}$), and so H_3 does not contain $\overline{Q}_i$ (resp. Q_i). Hence H_3 must contain Q_i (resp. $\overline{Q}_i$). Hence $\phi(\ell_j^a) = true$. Consequently, C_j is satisfied.

Reciprocally, assume that there is a truth assignment ϕ satisfying Φ. Let us contruct a genuine ear decomposition $\mathcal{H} = (H_p)_{p=1}^{p^*}$ of $G(\Phi)$. $H_1 = (\alpha, \beta, \gamma, \alpha)$ and $H_2 = (\beta, \delta, \gamma)$

Let us now contruct H_3. For each $1 \leq i \leq n$, let $R_i = Q_i$ if $\phi(v_i) = true$ and $R_i = \bar{Q}_i$ if $\phi(v_i) = false$. For every clause $C_j = \ell_0^j \vee \ell_1^j \vee \ell_2^j$, $1 \leq j \leq m$, let S_j be a path L_j^a for some $a \in \{0, 1, 2\}$ such that $\phi(l_j^a) = true$. (Such an a exists, because Φ is satisfied). Let $H_3 = (\alpha, z_1) \cup \bigcup_{i=1}^n R_i \cup (z_{n+1}, r, f_1) \cup \bigcup_{j=1}^m S_j \cup (f_{m+1}, \delta)$.

By our choice of H_3, the intersection of each copy of $\mathbb{J}$ with H_3 is either the path E_1 or the path E_2. We can then extend the ear decomposition in order to cover all internal vertices of $\mathbb{J}$ as follows: If $H_3 \cup \mathbb{J} = E_1$, then add the ears (a_1, b_1, d_2), (b_1, e_2, d_2), (b_1, c_2, d_2), (b_1, c_1, d_1),

(c_1, b_2, c_2) and (b_2, a_2, e_2). If $H_3 \cup \mathbb{J} = E_2$, then add the ears (e_2, b_1, d_2, e_2), (d_2, a_1, b_1), (b_1, c_2, d_2), (c_2, b_2, a_2), (b_1, c_1, b_2) and (c_1, d_1, d_2).

It is then simple matter to extend the ear decomposition into a genuine ear decomposition of G. □

The h-**subdivision** of a graph G, denoted by $S_h(G)$, is the graph obtained by replacing each edge of G by a path of length h.

Corollary 12. *For every fixed finite set $\mathcal{F}$ of positive integers, it is $\mathcal{NP}$-complete to decide whether a graph G has an ear decomposition with no ear of length in $\mathcal{F}$.*

Proof. Let $h = \max(\mathcal{F})$. Let G be a graph. There is a one-to-one correspondence between the ear decompositions of G and those of $S_h(G)$, since every path of length h replacing an arc is entirely contained in an ear. Moreover, the length of an ear in $S_h(G)$ is exactly h times the length of its corresponding ear in G. Therefore an ear decomposition of $S_h(G)$ has no ear of length less than h, and it has an ear decomposition with no ear of length h (and thus with no ear of length in $\mathcal{F}$) if and only if G has a genuine ear decomposition. Hence by Theorem 11, deciding whether a graph G has an ear decomposition with no ear of length in $\mathcal{F}$ is $\mathcal{NP}$-complete. □

5.2 Even ear decomposition

Let k be an integer greater than 1. A **modulo-k-ear-decomposition** is an ear decomposition such that every ear has length divisible by k.

Theorem 13. *Let k be an integer greater than 1. Given a (2-edge-connected) graph G, deciding whether G admits a modulo-k-ear-decomposition is $\mathcal{NP}$-complete.*

Proof. The proof is very similar to the one of Theorem 11, but the gadget $\mathbb{J}$ is replaced by the gadget $\mathbb{M}$ which is constructed as follows. We take 16 distinct vertices $s_1, a_1, b_1, c_1, d_1, e_1, f_1, t_1$ and $s_2, a_2, b_2, c_2, d_2, e_2, f_2, t_2$; for $i = 1, 2$, we add the edges $s_i a_i$, $c_i b_i$, $d_i c_i$, $f_i t_i$ and a path, denoted by $P(x, y)$, of length $k-1$ from x and y for each pair $(x, y) \in \{(a_i, b_i), (b_i, c_i), (c_i, d_i), (e_i, f_i)\}$; finally, we add a path, denoted by $Q(x, y)$, of length k between x and y for each pair $(x, y) \in \{(b_1, c_2), (c_2, e_1), (b_2, c_1), (c_1, e_2), (d_1, e_1), (d_2, e_2)\}$.

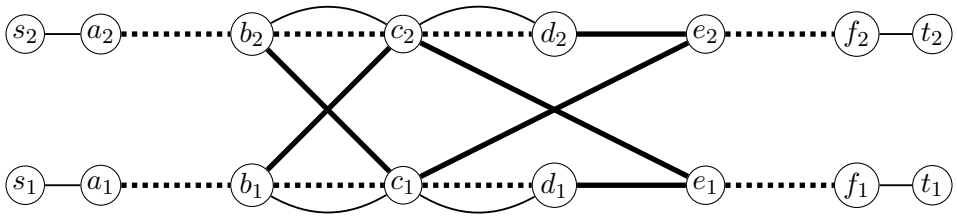

Figure 5: The gadget $\mathbb{M}$. Bold dotted lines represent paths of length $k-1$ and bold full lines represent paths of length k.

Claim 13.1. *Let G be any graph having $\mathbb{M}$ as subgraph such that there is no edge between $\mathbb{M} - \{s_1, s_2, t_1, t_2\}$ and $G - \mathbb{M}$. Assume that G admits a modulo-k-ear-decomposition $\mathcal{H}$, with first ear containing some vertex not in $\mathbb{M}$, and let H be the first ear of $\mathcal{H}$ including a vertex in $\mathbb{M} - \{s_1, s_2, t_1, t_2\}$.*

(i) If $s_1a_1 \in E(H)$, then $H \cap \mathbb{M} = (s_1, a_1) \cup P(a_1, b_1) \cup Q(b_1, c_2) \cup Q(c_2, e_1) \cup P(e_1, f_1) \cup (f_1, t_1) = E_1$. Moreover, $\mathcal{H}$ must contain the ears $P(b_1, c_1) \cup (c_1, b_1)$, $P(c_1, d_1) \cup (d_1, c_1)$, $P(b_2, c_2) \cup (c_2, b_2)$, $P(c_2, d_2) \cup (d_2, c_2)$, $Q(b_2, c_1)$, $Q(d_1, e_1)$, and without loss of generality, $Q(c_1, e_2)$ and an ear starting with $Q(d_2, e_2) \cup P(e_2, f_2) \cup (f_2, t_2)$.

(ii) If $s_2a_2 \in E(H)$, then $H \cap \mathbb{M} = (s_2, a_2) \cup P(a_2, b_2) \cup Q(b_2, c_1) \cup Q(c_1, e_2) \cup P(e_2, f_2) \cup (f_2, t_2) = E_2$. Moreover, $\mathcal{H}$ must contain the ears $P(b_1, c_1) \cup (c_1, b_1)$, $P(c_1, d_1) \cup (d_1, c_1)$, $P(b_2, c_2) \cup (c_2, b_2)$, $P(c_2, d_2) \cup (d_2, c_2)$, $Q(b_1, c_2)$, $Q(d_2, e_2)$, and without loss of generality, $Q(c_2, e_1)$ and an ear starting with $Q(d_1, e_1) \cup P(e_1, f_1) \cup (f_1, t_1)$.

Proof. Observe that the k-cycles $P(b_1, c_1) \cup (c_1, b_1)$, $P(c_1, d_1) \cup (d_1, c_1)$, $P(b_2, c_2) \cup (c_2, b_2)$, $P(c_2, d_2) \cup (d_2, c_2)$, must be ears. Indeed, the first ear containing an arc of one of these cycles either is the whole cycle, or intersects the cycle in a path of lentgh 1 or $k-1$. But in the latter case, it leaves an ear of length $k-1$ or 1, respectively, a contradiction.

(i) Assume that $s_1a_1 \in E(H)$. By the above observation, $H \cap \mathbb{M}$ must be $(s_1, a_1) \cup P(a_1, b_1) \cup Q(b_1, c_2) \cup Q(c_2, d_1) \cup P(e_1, f_1) \cup (f_1, t_1) = E_1$. Moreover, $Q(b_2, c_1)$, $Q(d_1, e_1)$ must be ears. Now there are tow ears containing e_2, one H with e_2 as internal vertex and one H' with e_2 as endvertex. If $H = Q(c_1, e_2) \cup Q(d_2, e_2)$, then replacing H and H' by $Q(d_2, e_2) \cup H'$ and $Q(c_1, e_2)$ we obtain another modulo-k-ear-decomposition. If $H' = Q(d_2, e_2)$, then H starts with $Q(c_1, e_2) \cup P(e_2, f_2) \cup (f_2, t_2)$, thus replacing H and H' by $Q(d_2, e_2) \cup (H \setminus Q(c_1, e_2)$ and $Q(c_1, e_2)$ we obtain another modulo-k-ear-decomposition. Otherwise H starts with $Q(d_2, e_2) \cup P(e_2, f_2) \cup (f_2, t_2)$ and $H' = Q(c_1, e_2)$.

The proof of (ii) is identical to the one of (i). By symmetry, just switch the subscripts 1 and 2. ◊

Let Φ be a 3-CNF boolean formula with variables $v_1, \ldots, v_n$ and clauses $C_1, \ldots, C_m$ (w.l.o.g., assume that no C_j contains both v_i and $\bar{v}_i$). We construct a graph $G_1(\Phi)$ in a similar way as the the graph $G(\phi)$, but the gagdets $\mathbb{J}$ are replaced by gadgets $\mathbb{M}$ and the starter subgraph $\mathbb{S}$ is now the union of two paths between α and δ, one of length 1 and one of length $k-1$, still connected to the graph via the edges αz_1 and $f_{m+1}\delta$. Finally, from $G_1(\Phi)$, we construct a graph $G_2(\Phi)$, by replacing each edge that is neither in $\mathbb{S}$ nor in copies of $\mathbb{M}$ by a path of length $k-1$.

The proof is then similar to the one of Theorem 11. So it is left to the reader. The fact that we subdivided all edges neither in $\mathbb{S}$ nor in copies of $\mathbb{M}$ allows to extend the first three ears into a modulo-k-ear-decomposition of G_2. □

In the same way, as Theorem 13, one can show the following.

Theorem 14. *Let k be an integer greater than 1. Given a (strongly-connected) digraph D, deciding whether G admits a modulo-k-handle-decomposition is $\mathcal{NP}$-complete.*

Proof. The proof is exactly the same as the one of Theorem 13. Just replace 'edge' by 'arc', 'path' by 'directed path', and 'ear' by 'handle'. □

6 To go further

Let $\mathcal{A}$ be a set of positive integers. We denote by $\overline{\mathcal{A}}$ the set $\mathbb{N} \setminus \mathcal{A}$, and for any positive integer k we set $k\mathcal{A} = \{k \times a \mid a \in \mathcal{A}\}$. An **$\mathcal{A}$-ear-decomposition** of a graph is an ear decomposition in which all ears have length in $\mathcal{A}$. Similarly, an **$\mathcal{A}$-handle-decomposition**

of a digraph is a handle decomposition in which all handles have length in $\mathcal{A}$. Note that an $\mathcal{A}$-ear-decomposition (resp. $\mathcal{A}$-handle-decomposition) of a graph (resp. digraph) can be seen as an ear decomposition (resp. handle decomposition) with no ear (resp. handle) with length in $\overline{\mathcal{A}}$.

In view of all our results, it is natural to consider the following problems.

$\mathcal{A}$-Ear-Decomposition
Input: A graph G.
Question: Does G admit an $\mathcal{A}$-ear-decomposition ?

$\mathcal{A}$-Handle-Decomposition
Input: A digraph D.
Question: Does D admit an $\mathcal{A}$-handle-decomposition ?

It would be nice to characterize the graphs (resp. digraphs) for which $\mathcal{A}$-Ear-Decomposition (resp. $\mathcal{A}$-Handle-Decomposition) is polynomial-time solvable.

Below is an easy lemma, that might be useful in proving such a characterization.

Lemma 15. *Let $\mathcal{A}$ be a set of positive integers.*

(i) If $1 \in \mathcal{A}$ and $\mathcal{A}$-Ear-Decomposition is $\mathcal{NP}$-complete, then $\mathcal{A}$-Handle-Decomposition is $\mathcal{NP}$-complete (even wen restricted to symmetric digraph).

(ii) If $\overline{\mathcal{F}}$-Ear-Decomposition is $\mathcal{NP}$-complete, then for every $\mathcal{A}$ such that $k\overline{\mathcal{F}} \subseteq \mathcal{A} \subseteq \overline{k\mathcal{F}}$ $\mathcal{A}$-Ear-Decomposition is $\mathcal{NP}$-complete for all positive integer k.

(iii) If $\overline{\mathcal{F}}$-Handle-Decomposition is $\mathcal{NP}$-complete, then $\overline{k\mathcal{F}}$-Handle-Decomposition is $\mathcal{NP}$-complete for all positive integer k.

Proof. (i) Assume $1 \in \mathcal{A}$. Let G be a 2-edge-connected graph. Let $\overleftrightarrow{G}$ be the symmetric digraph associated to G. Let us describe a correspondence between the $\mathcal{A}$-ear-decompositions of G and the $\mathcal{A}$-handle-decompositions of $\overleftrightarrow{G}$.

From an ear decomposition $\mathcal{H}$ of G, we can obtain a handle decomposition $\overrightarrow{\mathcal{H}}$ of $\overleftrightarrow{G}$ by replacing each ear H of $\mathcal{H}$ by several handles of $\overrightarrow{\mathcal{H}}$: a directed orientation $\vec{H}$ of H, and all reverse trivial handles, that are the (v,u) for all $uv \in A(\vec{H})$. Clearly, if $\mathcal{H}$ is an $\mathcal{A}$-Ear-Decomposition of G, then $\overrightarrow{\mathcal{H}}$ is a $\mathcal{A}$-Handle-Decomposition of $\overleftrightarrow{G}$.

Reciprocally, consider a handle decomposition $\overrightarrow{\mathcal{H}}$ of $\overleftrightarrow{G}$. Free to reorder the handles, we may assume that each non-trivial handles is followed by its reverse trivial handles, and that the remaining trivial handles are grouped by set of opposite handles. Then, replacing each non-trivial handle $\vec{H}$ together with its reverse handle, by the ear H obtained from $\vec{H}$ by forgetting the orientation, and replacing each pair of opposite trivial handles (u,v), (v,u) by the ear (u,v), we obtain an ear decomposition of G. Clearly, if $\overrightarrow{\mathcal{H}}$ is a $\mathcal{A}$-Handle-Decomposition of $\overleftrightarrow{G}$, then $\mathcal{H}$ is an $\mathcal{A}$-Ear-Decomposition of G.

(ii) By considering the k-subdivision of a graph. The proof is similar to Corollary 12.

(iii) By considering the k-subdivision of a digraph. The proof is similar to Corollary 4. □

In the exact same way as what we did for digraphs in Section 2, one can show the undirected analogue to Corollary 2.

Theorem 16. *Let h be a positive integer. One can decide in polynomial time whether a graph admits an ear decomposition with ears of length at most h.*

Theorem 16 states that if $\mathcal{A} = \{1, \ldots, h\}$ for some positive integer h, then $\mathcal{A}$-Ear-Decomposition is polynomial-time solvable. However, we believe that, under the assumption $\mathcal{P} \neq \mathcal{NP}$, this is the only case of a finite $\mathcal{A}$ such that $\mathcal{A}$-Ear-Decomposition is polynomial-time solvable.

Conjecture 17. Let $\mathcal{A}$ be a finite set of positive integers. $\mathcal{A}$-Ear-Decomposition is $\mathcal{NP}$-complete unless there is a positive integer h such that $\mathcal{A} = \{1, \ldots, h\}$.

The above conjecture cannot be generalized to all sets $\mathcal{A}$ including the infinite ones because Odd-Ear-Decomposition is polynomial-time solvable. In contrast, we believe that there is no infinite set of integers $\mathcal{A}$ such that $\mathcal{A}$-Handle-Decomposition is polynomial-time solvable, and that the directed analogue of Conjecture 17 can be generalized to all sets of integers.

Conjecture 18. Let $\mathcal{A}$ be a set of positive integers. $\mathcal{A}$-Handle-Decomposition is $\mathcal{NP}$-complete unless there is a positive integer h such that $\mathcal{A} = \{1, \ldots, h\}$.

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References

[1] Joseph Cheriyan, András Sebö, and Zoltán Szigeti. Improving on the 1.5-approximation of a smallest 2-edge connected spanning subgraph. *SIAM J. Discrete Math.*, 14(2):170–180, 2001.

[2] Nicos Christofides. Worst-case analysis of a new heuristic for the travelling salesman problem. Technical Report 388, Graduate School of Industrial Administration, Carnegie Mellon University, 1976.

[3] András Frank. Conservtive weightings and ear-decompositions of graphs. *Combinatorica*, 13:65–81, 1993.

[4] M.R. Garey, D.S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theor. Comput. Sci.*, 1(3):237–267, 1976.

[5] Shayan Oveis Gharan, Amin Saberi, and Mohit Singh. A randomized rounding approach to the traveling salesman problem. In *Proceedings of the 2011 IEEE 52nd Annual Symposium on Foundations of Computer Science*, FOCS '11, pages 550–559, Washington, DC, USA, 2011. IEEE Computer Society.

[6] Samir Khuller and Uzi Vishkin. Biconnectivity approximations and graph carvings. *Journal of the ACM*, 41(2):214–235, March 1994.

[7] Hartmut Klauck. On the hardness of global and local approximation. In Rolf Karlsson and Andrzej Lingas, editors, *Algorithm Theory — SWAT'96*, pages 88–99, Berlin, Heidelberg, 1996. Springer Berlin Heidelberg.

[8] László Lovász. A note on factor-critical graphs. *Studia Sci. Math. Hungar.*, 7:279–280, 1972.

[9] Tobias Mömke and Ola Svensson. Approximating graphic TSP by matchings. In Rafail Ostrovsky, editor, *IEEE 52nd Annual Symposium on Foundations of Computer Science, FOCS 2011, Palm Springs, CA, USA, October 22-25, 2011*, pages 560–569. IEEE Computer Society, 2011.

[10] Marcin Mucha. $\frac{13}{9}$-approximation for Graphic TSP. *Theory of Computing Systems*, 55(4):640–657, Nov 2014.

[11] H.E. Robbins. A theorem on graphs with an application to a problem on traffic control. *Amer. Math. Mon.*, 46:281–283, 1939.

[12] András Sebő and Jens Vygen. Shorter tours by nicer ears: 7/5-approximation for the graph-TSP, 3/2 for the path version, and 4/3 for two-edge-connected subgraphs. *Combinatorica*, Jul 2014.

[13] Adrian Vetta. Approximating the minimum strongly connected subgraph via a matching lower bound. In *Proc. 12th Annual ACM-SIAM Symposium on Discrete Algorithms (Washington, DC, 2001)*, pages 417–426. SIAM, 2001.