Bispindles in strongly connected digraphs with large chromatic number

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Abstract

A (k_1+k_2)-*bispindle* is the union of k_1 (x,y)-dipaths and k_2 (y,x)-dipaths, all these dipaths being pairwise internally disjoint. Recently, Cohen et al. showed that for every $(1,1)$- bispindle B, there exists an integer k such that every strongly connected digraph with chromatic number greater than k contains a subdivision of B. We investigate generalizations of this result by first showing constructions of strongly connected digraphs with large chromatic number without any $(3,0)$-bispindle or $(2,2)$-bispindle. We then consider $(2,1)$-bispindles. Let $B(k_1,k_2;k_3)$ denote the $(2,1)$-bispindle formed by three internally disjoint dipaths between two vertices x,y, two (x,y)-dipaths, one of length k_1 and the other of length k_2, and one (y,x)-dipath of length k_3. We conjecture that for any positive integers k_1,k_2,k_3, there is an integer $g(k_1,k_2,k_3)$ such that every strongly connected digraph with chromatic number greater than $g(k_1,k_2,k_3)$ contains a subdivision of $B(k_1,k_2;k_3)$. As evidence, we prove this conjecture for $k_2=1$ (and k_1,k_3 arbitrary).

Mathematics Subject Classifications: 05C15, 05C20

1 Introduction

Throughout this paper, a *proper colouring* of a digraph is a proper colouring of its underlying graph. Similarly, the *chromatic number* of a digraph D, denoted by $\chi(D)$, is the chromatic number of its underlying graph. In a digraph D, a *dipath* is an oriented path where all the arcs are oriented in the same direction, from the initial vertex towards the terminal vertex.

A classical result due to Gallai, Hasse, Roy and Vitaver is the following.

Theorem 1 (Gallai [10], Hasse [11], Roy [13], Vitaver [14])**.** *If* $\chi(D) \geqslant k$*, then* D *contains a dipath of order* k*.*

This raises the following question.

Question 2. Which digraphs are subdigraphs of all digraphs with large chromatic number?

A famous theorem by Erdős [9] states that there exist graphs with arbitrarily high girth and arbitrarily large chromatic number. This means that if H is a digraph containing an oriented (non necessarily directed) cycle, there exist digraphs with arbitrarily high chromatic number with no subdigraph isomorphic to H. Thus the only possible candidates to generalize Theorem 1 are the *oriented trees* that are orientations of trees. Burr [6] proved that every $(k-1)^2$-chromatic digraph contains every oriented tree of order k and made the following conjecture.

Conjecture 3 (Burr [6])**.** For a digraph D, if $\chi(D) \geqslant (2k-2)$, then D contains a copy of any oriented tree T of order k.

The best known upper bound, due to Addario-Berry et al. [2], is in $(k/2)^2$. However, for oriented paths with two blocks (*blocks* are maximal directed subpaths), the best possible upper bound is known.

Theorem 4 (Addario-Berry et al. [1])**.** *Let* P *be an oriented path with two blocks on* $n > 3$ *vertices, then every digraph with chromatic number (at least)* n *contains* P*.*

The following celebrated theorem of Bondy shows that the story does not stop here.

Theorem 5 (Bondy [4])**.** *Every strongly connected digraph of chromatic number at least* k *contains a directed cycle of length at least* k*.*

The strong connectivity assumption is indeed necessary, as transitive tournaments contain no directed cycle but can have arbitrarily high chromatic number.

Observe that a directed cycle of length at least k can be seen as a subdivision of $\vec{C}_k$, the directed cycle of length k. Recall that a *subdivision* of a digraph F is a digraph that can be obtained from F by replacing each arc (u, v) by a dipath from u to v. Cohen et al. [8] conjecture that Bondy's theorem can be extended to all oriented cycles.

Conjecture 6 (Cohen et al. [8])**.** For every oriented cycle C, there exists a constant $f(C)$ such that every strong digraph with chromatic number at least $f(C)$ contains a subdivision of C.

The strongly connected connectivity assumption is also necessary in Conjecture 6 as shown by Cohen et al. [8]. This follows from the following result.

Theorem 7 (Cohen et al. [8])**.** *For any positive integers b and k, there exists an acyclic digraph $D_{k,b}$ such that any cycle in $D_{k,b}$ has at least b blocks and $\chi(D_{k,b}) > k$.*

On the other hand, Cohen et al. [8] proved Conjecture 6 for cycles with two blocks and the antidirected cycle of length 4. More precisely, denoting by $C(k, \ell)$ the cycle with two blocks, one of length k and the other of length ℓ, they proved the following result.

Theorem 8 (Cohen et al. [8])**.** *For every two positive integers k and ℓ, every strongly connected digraph with chromatic number at least $O((k + \ell)^4)$ contains a subdivision of $C(k, \ell)$.*

The bound has recently been improved to $O((k + \ell)^2)$ by Kim et al. [12].

A *p-spindle* is the union of p internally disjoint (x, y)-dipaths for some vertices x and y. Vertex x is said to be the *tail* of the spindle and y its *head*. A *$(p + q)$-bispindle* is the internally disjoint union of a p-spindle with tail x and head y and a q-spindle with tail y and head x. In other words, it is the union of p (x, y)-dipaths and q (y, x)-dipaths, all of these dipaths being pairwise internally disjoint. Note that 2-spindles are the cycles with two blocks and the $(1 + 1)$-bispindles are the directed cycles.

In this paper, we study the existence of spindles and bispindles in strongly connected digraphs with large chromatic number. First, let us give a construction of digraphs with arbitrarily large chromatic number that contain no 3-spindle and no $(2 + 2)$-bispindle.

Theorem 9. *For every positive integer k, there exists a strongly connected digraph D with $\chi(D) > k$ that contains no 3-spindle and no $(2 + 2)$-bispindle.*

Proof. Let $D_{k,4}$ be an acyclic digraph with chromatic number greater than k in which every cycle has at least four blocks. The existence of such a digraph is given by Theorem 7. Let $S = \{s_1, \dots, s_l\}$ be the set of vertices of $D_{k,4}$ with out-degree 0 and $T = \{t_1, \dots, t_m\}$ the set of vertices with in-degree 0.

Consider the digraph D obtained from $D_{k,4}$ as follows. Add to $D_{k,4}$ a dipath $P = (x_1, x_2, \dots, x_l, z, y_1, y_2, \dots, y_m)$ and the arcs (s_i, x_i) for all $i \in [l]$ and (y_j, t_j) for all $j \in [m]$. It is easy to see that D is strongly connected. Moreover, in D, every directed cycle uses the arc (x_l, z). Therefore D does not contain a (2+2)-bispindle, which has two arc-disjoint directed cycles.

Suppose now that D has a 3-spindle with tail u and head v, and let Q_1, Q_2, Q_3 be its three (u, v)-dipaths. Observe that u and v are not vertices of P, because all vertices of this dipath have either in-degree at most 2 or out-degree at most 2. In D, each oriented cycle with two blocks between vertices outside P must use the arc (x_l, z). The union of Q_1 and Q_2 form a cycle on two blocks, which means that one of the two paths, say Q_1, contains (x_l, z). But Q_2 and Q_3 also form a cycle on two blocks, but they cannot contain (x_l, z), a contradiction. □

By Theorem 9, the most we can expect in all strongly connected digraphs with large chromatic number are $(2 + 1)$-bispindles. Let $B(k_1, k_2; k_3)$ denote the $(2 + 1)$-bispindle formed by three internally disjoint paths between two vertices x, y, two (x, y)-dipaths, one of length k_1 and the other of length k_2, and one (y, x)-dipath of length k_3.

One can easily prove that every strongly connected digraph with chromatic number at least 4 contains a subdivision of $B(2, 1; 1)$.

Proposition 10. *Let D be a strongly connected digraph. If $\chi(D) \geqslant 4$, then D contains a subdivision of $B(2, 1; 1)$.*

Proof. Assume $\chi(D) \geqslant 4$. Since every strongly connected digraph contains a 2-connected strongly connected subdigraph with the same chromatic number, we may assume that D is 2-connected. Let C be a shortest directed cycle in D. It must be induced, so $\chi(D[C]) = \chi(C) \leqslant 3$. In particular, $V(D) \setminus V(C)$ is not empty.

Thus, by Proposition 5.11 in [5], there is a dipath P in D whose ends lie in C but whose internal vertices do not. Necessarily, P has length at least 2 since C is induced. Thus the union of P and C is a subdivision of $B(2, 1; 1)$. □

The bound 4 in Proposition 10 is best possible because a directed odd cycle has chromatic number 3 and contains no $B(2, 1; 1)$-subdivision.

We conjecture that Proposition 10 can be extended to any $(2 + 1)$-bispindle.

Conjecture 11. There is a function $g : \mathbb{N}^3 \to \mathbb{N}$ such that every strongly connected digraph with chromatic number at least $g(k_1, k_2, k_3)$ contains a subdivision of $B(k_1, k_2; k_3)$.

As an evidence, we prove this conjecture for $k_2 = 1$ and arbitrary k_1 and k_3. In Section 3, in order to present our method, we first investigate the case $k_2 = k_3 = 1$ and prove the following.

Theorem 12. *Let $k \geqslant 3$ be an integer and let D be a strongly connected digraph. If $\chi(D) > (2k - 2)(2k - 3)$, then D contains a subdivision of $B(k, 1; 1)$.*

In Section 4, using the same approach but in a more complicated way, we prove our main result:

Theorem 13. *For every positive integer k, there is a constant γ_k such that if D is a strongly connected digraph with $\chi(D) > \gamma_k$, then D contains a subdvision of $B(k, 1; k)$.*

We prove the above theorem for a huge constant γ_k. It can easily be lowered. However, we made no attempt to it here for two reasons: firstly, we would like to keep the proof as simple as possible; secondly using our method, there is no hope to get an optimal or near optimal value for γ_k.

Similar questions with χ replaced by another graph parameter can be studied. We refer the reader to [3] and [8] for more exhaustive discussions on such questions. Let us just give one result proved by Aboulker et al. [3] which can be seen as an analogue to Conjecture 11.

Theorem 14 (Theorem 28 in [3])**.** *Let k_1, k_2, k_3 be positive integers with $k_1 \geqslant k_2$. Let D be a digraph with $\delta^+(D) \geqslant 3k_1 + 2k_2 + k_3 - 5$. Then D contains a subdivision of $B(k_1, k_2; k_3)$.*

2 Definitions and preliminaries

We follow standard terminology as used in [5]. We denote by $[k]$ the set of integers $\{1, \ldots, k\}$.

Let F be a digraph. An *F-subdivision* is a subdivision of F. A digraph D is said to be *F-subdivision-free*, if it contains no F-subdivision.

The *union* of two digraphs D_1 and D_2 is the digraph $D_1 \cup D_2$ defined by $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$ and $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$. If $\mathcal{D}$ is a set of digraphs, we denote by $\bigcup \mathcal{D}$ the union of the digraphs in $\mathcal{D}$, i.e. $V(\bigcup \mathcal{D}) = \bigcup_{D \in \mathcal{D}} V(D)$ and $A(\bigcup \mathcal{D}) = \bigcup_{D \in \mathcal{D}} A(D)$.

Let P be a dipath. We denote by $s(P)$ its initial vertex and by $t(P)$ its terminal vertex. For any two vertices, a *(u,v)-dipath* or *dipath from u to v* is a dipath P with $s(P) = u$ and $t(P) = v$. For two sets X, Y of vertices, an *(X,Y)-dipath* or *dipath from X to Y* is a dipath P such that $s(P) \in X$, $t(P) \in Y$, and no internal vertex is in $X \cup Y$.

If D is a dipath or a directed cycle, then we denote by $D[a,b]$ the subdipath of D with initial vertex a and terminal vertex b. We denote by $D[a,b[$ the dipath $D[a,b] - b$, by $D]a,b]$ the dipath $D[a,b] - a$, and by $D]a,b[$ the dipath $D[a,b] - \{a,b\}$. If P and Q are two dipaths such that $V(P) \cap V(Q) = \{s(P)\} = \{t(Q)\}$, the *concatenation* of P and Q, denoted by $P \odot Q$, is the dipath $P \cup Q$.

A digraph is *connected* (resp. 2-*connected*) if its underlying graph is connected (resp. 2-connected). The *connected components* of a digraph are the connected components of its underlying graph. A digraph D is *strongly connected* or *strong* if for any two vertices x, y there is dipath from x to y. The *strong components* of a digraph are its maximal strong subdigraphs.

Let G be a graph or a digraph. A *proper k-colouring* of G is a mapping $\phi : V(G) \to [k]$ such that $\phi(u) \neq \phi(v)$ whenever u is adjacent to v. G is *k-colourable* if it admits a proper k-colouring. The *chromatic number* of G, denoted by $\chi(G)$, is the least integer k such that G is k-colourable.

A (directed) graph G is *k-degenerate* if every subgraph H of G has a vertex of degree at most k. The following three statements are well-known.

Proposition 15. *Every k-degenerate (directed) graph is $(k+1)$-colourable.*

Theorem 16 (Brooks)**.** *Let G be a connected graph. Then $\chi(G) \leqslant \Delta(G)$ unless G is a complete graph or an odd cycle.*

Lemma 17. *Let D_1 and D_2 be two digraphs. Then $\chi(D_1 \cup D_2) \leqslant \chi(D_1) \times \chi(D_2)$.*

Lemma 18. *Let D be a digraph, $D_1, \ldots, D_l$ be disjoint subdigraphs of D and D' the digraph obtained by contracting each D_i into one vertex d_i. Then $\chi(D) \leqslant \chi(D') \cdot \max\{\chi(D_i) \mid i \in [l]\}$.*

Proof. Set $k_1 = \max\{\chi(D_i) \mid i \in [l]\}$ and $k_2 = \chi(D')$. For each i, let ϕ_i be a proper colouring of D_i using colours in $[k_1]$ and let ϕ' be a proper colouring of D' using colours in $[k_2]$. Define $\phi : V(D) \to [k_1] \times [k_2]$ as follows. If x is a vertex belonging to some D_i, then $\phi(x) = (\phi_i(x), \phi'(d_i))$, else $\phi(x) = (1, \phi'(x))$. Let x and y be adjacent vertices of D. If they belong to the same subdigraph D_i, then $\phi_i(x) \neq \phi_i(y)$ and so $\phi(x) \neq \phi(y)$. If they do not belong to the same component, then the vertices corresponding to these vertices in D' are adjacent and so $\phi(x) \neq \phi(y)$. Thus ϕ is a proper colouring of D using $k_1 \cdot k_2$ colours. $\square$

The *rotative tournament on* $2k-1$ *vertices*, denoted by R_{2k-1}, is the tournament with vertex set $\{v_1, \dots, v_{2k-1}\}$ in which v_i dominates v_j if and only if $j-i$ modulo $2k-1$ belongs to $\{1, 2, \dots, k-1\}$.

Proposition 19. *Let T be a strong tournament of order $2k-1$, then T contains a $B(k,1;1)$-subdivision.*

Proof. Let T be a strong tournament of order $2k-1$. By Camion's Theorem, it has a hamiltonian directed cycle $C = (v_1, v_2, \dots, v_{2k-1}, v_1)$. If there exists an arc (v_i, v_j) with $j - i \geqslant k$ (indices are modulo $2k-1$), then the union of $C[v_i, v_j]$, (v_i, v_j) and $C[v_j, v_i]$ is a $B(k,1;1)$-subdivision. Henceforth, we may assume that $T = R_{2k-1}$. Then the union of $C[v_1, v_{k-1}] \odot (v_{k-1}, v_{k+1}, v_{k+2})$, (v_1, v_k, v_{k+2}), and $C[v_{k+2}, v_1]$ is a $B(k,1;1)$-subdivision. $\square$

We will need the following lemmas:

Lemma 20. *Let $\sigma = (u_t)_{t \in [p]}$ be a sequence of integers in $[k]$, and let l be a positive integer. If $p \geqslant l^k$, then there exists a set L of l indices such that for any $i, j \in L$ with $i < j$ the following holds : $u_i = u_j$ and $u_t > u_i$, for all $i < t < j$.*

Proof. By induction on k. The result holds trivially when $k = 1$. Assume now that $k > 1$. Let L_1 be the elements of the sequence with value 1. If L_1 has at least l elements, we are done. If not, then there is a subsequence σ' of $\left\lceil \frac{l^k - (l-1)}{l} \right\rceil = l^{k-1}$ consecutive elements in $\{2, \dots, k-1\}$. Applying the induction hypothesis to σ' yields the result. $\square$

Lemma 21. *Let $\sigma = (u_t)_{t \in [p]}$ be a sequence of integers in $[k]$. If $p > k(m-1)$, then there exists a subsequence of m consecutive integers such that the last one is the largest.*

Proof. By induction on k. The result holds trivially when $k = 1$. Let i be the smallest integer such that $u_t \leqslant k-1$ for all $t \geqslant i$. If $i > m$, then $u_{i-1} = k$, and the subsequence of the $i-1$ first elements of σ is the desired sequence. If $i \leqslant m$, apply the induction on $\sigma' = (u_t)_{i \leqslant t \leqslant p}$ which is a sequence of more than $(k-1)(m-1)$ integers in $[k-1]$, to get the result. $\square$

3 B(k,1;1)

In this section, we present a proof of Theorem 12.

Let $\mathcal{C}$ be a collection of directed cycles. It is *nice* if all cycles of $\mathcal{C}$ have length at least $2k-2$, and any two distinct cycles of $\mathcal{C}$ intersect on at most one vertex. A *component* of $\mathcal{C}$ is a connected component in the adjacency graph of $\mathcal{C}$, where vertices correspond to cycles in $\mathcal{C}$ and two vertices are adjacent if the corresponding cycles intersect. Note that if $\mathcal{S}$ is a component of $\mathcal{C}$, then $\bigcup \mathcal{S}$ is both a connected component and a strong component of $\bigcup \mathcal{C}$. Call $D_{\mathcal{C}}$ the digraph obtained from D by contracting each component of $\mathcal{C}$ into one vertex. For sake of simplicity, we denote by $D[\mathcal{S}]$ the digraph $D[\bigcup \mathcal{S}]$. Observe that this digraph contains $\bigcup \mathcal{S}$ but has more arcs.

We will prove that every $B(k,1;1)$-subdivision-free strong digraph D has bounded chromatic number in the following way: We take a maximal nice collection $\mathcal{C}$ of directed cycles. We will prove that for every component $\mathcal{S}$ of $\mathcal{C}$, the digraph $D[\mathcal{S}]$ has bounded chromatic number. Then we will prove that, since it contains no long directed cycle and it is strong, $D_{\mathcal{C}}$ has bounded chromatic number. Those two results allow us to conclude by Lemma 18.

We will need the following lemma:

Lemma 22. *Let $\mathcal{C}$ be a nice collection of directed cycles in a $B(k,1;1)$-subdivision-free digraph D and let C, C' be two cycles of the same component $\mathcal{S}$ of $\mathcal{C}$. There is no dipath P from C to C' whose arcs are not in $A(\bigcup \mathcal{S})$.*

Proof. By the contrapositive. We suppose that there exists such a dipath P and show that there is a $B(k,1;1)$-subdivision in D.

By definition of $\mathcal{S}$, there exists a dipath Q from C to C' in $\bigcup \mathcal{S}$. By choosing C and C' such that Q is as small as possible, then $s(Q) \neq t(P)$ and $t(Q) \neq s(P)$ (note that $s(Q)$ and $t(Q)$ can be the same vertex).

Since C has length at least $2k-2$, either $C[t(Q), s(P)]$ has length at least $k-1$ or $C[s(P), t(Q)]$ has length at least k.

- If $C[t(Q), s(P)]$ has length at least $k-1$, then the union of $Q \odot C[t(Q), s(P)] \odot P$, $C'[s(Q), t(P)]$ and $C'[t(P), s(Q)]$ is a $B(k,1;1)$-subdivision between $s(Q)$ and $t(P)$.
- If $C[s(P), t(Q)]$ has length at least k, then the union of $C[s(P), t(Q)]$, $P \odot C'[t(P), s(Q)] \odot Q$ and $C[t(Q), s(P)]$ is a $B(k,1;1)$-subdivision between $s(P)$ and $t(Q)$. ☐

Lemma 23. *Let $k \geqslant 3$ be an integer, and let $\mathcal{C}$ be a nice collection of directed cycles in a $B(k,1;1)$-subdivision-free digraph D and $\mathcal{S}$ a component of $\mathcal{C}$. Then $\chi(D[S]) \leqslant 2k-2$.*

Proof. By induction on the number of directed cycles in $\mathcal{S}$. Let C be a cycle of $\mathcal{S}$. There is no chord (x,y) of C such that $C[x,y]$ has length at least k, for otherwise there would be a $B(k,1;1)$-subdivision. Hence $D[C]$ has maximum degree at most $2k-2$. Moreover, by Proposition 19, $D[C]$ is not a tournament of order $2k-1$. Thus, by Brooks' Theorem (16), $\chi(D[C]) \leqslant 2k-2$. Let c be a proper colouring of C with $2k-2$ colours. Let $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r$ be the components of $\mathcal{S} \setminus \{C\}$. Since $\mathcal{S}$ is the union of the $\mathcal{S}_l$, $l \in [r]$, and $\{C\}$, each $\mathcal{S}_l$ has less cycles than $\mathcal{S}$. By the induction hypothesis, there exists a proper colouring c_l using $2k-2$ colours for each $D[\mathcal{S}_l]$.

Now, we claim that each $D[\mathcal{S}_l]$ intersects C in exactly one vertex. It is easy to see that C must intersect at least one cycle of each $\mathcal{S}_l$. Now suppose there exist two vertices of C, x and y, in $D[\mathcal{S}_l]$. By definition of a nice collection, they cannot belong to the same cycle of $\mathcal{S}_l$, so there exist two cycles C_i and C_j of S_l such that $x \in C_i$ and $y \in C_j$. Now $C[x, y]$ is a dipath from C_i to C_j whose arcs are not in $A(\bigcup \mathcal{S}_l)$. This contradicts Lemma 22.

Consequently, free to permute the colours of c_l, we may assume that each vertex of C receives the same colour in c and in c_l. In addition, by Lemma 22, there is no arc between different $D[\mathcal{S}_l]$ nor between $D[\mathcal{S}_l]$ and C. Hence the union of c_l and c is a proper colouring of $D[\mathcal{S}]$ using $2k - 2$ colours. □

Lemma 24. *Let $\mathcal{C}$ be a maximal nice collection of directed cycles in a $B(k, 1; 1)$-subdivision-free strong digraph D. Then $\chi(D_{\mathcal{C}}) \leqslant 2k - 3$.*

Proof. First note that since D is strong, then so is $D_{\mathcal{C}}$. Suppose $\chi(D_{\mathcal{C}}) \geqslant 2k - 2$. By Bondy's Theorem (5), there exists a directed cycle $C = (x_1, \dots, x_l, x_1)$ of length at least $2k - 2$ in $D_{\mathcal{C}}$. We derive a cycle C' in D the following way: Suppose the vertex x_i corresponds to a component $\mathcal{S}_i$ of $\mathcal{C}$: the arc (x_{i-1}, x_i) corresponds in D to an arc whose head is a vertex p_i of $\bigcup \mathcal{S}_i$, and the arc (x_i, x_{i+1}) corresponds to an arc whose tail is a vertex l_i of $\bigcup \mathcal{S}_i$. Let P_i be a dipath from p_i to l_i in $D[\mathcal{S}_i]$. Note that P_i intersects each cycle of S_i on a, possibly empty, subdipath of P_i. Then C' is the cycle obtained from C by replacing the vertices x_i by the path P_i.

C' is a cycle of D of length at least $2k-2$ because it is no shorter than C. Let C_1 be a cycle of $\mathcal{C}$. By construction of C' and $D_{\mathcal{C}}$, C' and C_1 can intersect only along a subdipath of one P_i. Suppose this dipath is more than just one vertex. Let x and y be the initial and terminal vertex, respectively, of this dipath. Then the union of $C'[x, y]$, $C_1[x, y]$ and $C_1[y, x]$ is a $B(k, 1; 1)$-subdivision, a contradiction.

So C' is a cycle of length at least $2k - 2$, intersecting each cycle of $\mathcal{C}$ on at most one vertex, and which does not belong to $\mathcal{C}$, for otherwise it would be reduced to one vertex in $D_{\mathcal{C}}$. This contradicts the fact that $\mathcal{C}$ is maximal. □

We can finally prove Theorem 12.

Proof of Theorem 12. Let $\mathcal{C}$ be a maximal nice collection of directed cycles in D. Lemmas 23, 24 and 18 give the result. □

4 B(k,1;k)

In this section, we present a proof of Theorem 13.

We prove the result by the contrapositive. We consider a $B(k, 1; k)$-subdivision-free digraph D. We shall prove that $\chi(D) \leqslant \gamma_k = 8k^2(4k^2+2)(2\cdot(4k)^{4k}+1)(2\cdot(6k^2)^{3k}+14k)$.

Our proof heavily uses the notion of k-suitable collection of directed cycles, which can be seen as a generalization of the notion of nice collection of directed cycles used to prove Theorem 12.

A collection $\mathcal{C}$ of directed cycles is *k-suitable* if all cycles of $\mathcal{C}$ have length at least $8k$, and any two distinct directed cycles $C_i, C_j \in \mathcal{C}$ intersect on a dipath $P_{i,j}$ of order at most k. We denote by $s_{i,j}$ (resp. $t_{i,j}$) the initial (resp. terminal) vertex of $P_{i,j}$.

The proof of Theorem 13 uses the same general idea as Theorem 12: take a maximal k-suitable collection of directed cycles $\mathcal{C}$; show that the digraph $D_{\mathcal{C}}$ obtained by contracting the components of $\mathcal{C}$ has bounded chromatic number, and that each component also has bounded chromatic number; conclude using Lemma 18. However, because the intersection of cycles in this collection are more complicated and because there might be arcs between directed cycles of the same component, bounding the chromatic number of the components is way more challenging. The next subsection is devoted to this.

4.1 k-suitable collections of directed cycles

Let ϕ be a colouring of a graph G. A subset of vertices or a subgraph S of G is *rainbow-coloured* by ϕ if all vertices of S have distinct colours.

Set $\alpha_k = 2 \cdot (6k^2)^{3k} + 14k$. The first step of the proof is the following lemma.

Lemma 25. *Let $\mathcal{C}$ be a k-suitable collection of directed cycles in a $B(k,1;k)$-subdivision-free digraph. There exists a proper colouring ϕ of $\bigcup \mathcal{C}$ with α_k colours, such that, each subdipath of length $7k$ of each directed cycle of $\mathcal{C}$ is rainbow-coloured.*

In order to prove this lemma, we need some definitions and preliminary results.

Lemma 26. *Let $\mathcal{C}$ be a k-suitable collection of directed cycles in a $B(k,1;k)$-subdivision-free digraph. Let C_1, C_2, C_3 be three pairwise-intersecting directed cycles of $\mathcal{C}$, and let v belong to $V(C_2) \cap V(C_3) \setminus V(C_1)$. Then exactly one of the following holds:*

(i) $C_2[t_{1,2}, v]$ and $C_3[t_{1,3}, v]$ have both length less than $3k$;

(ii) $C_2[v, s_{1,2}]$ and $C_3[v, s_{1,3}]$ have both length less than $3k$.

Proof. Observe first that since C_2 has length at least $8k$ and $P_{1,2}$ has length at most $k-1$, the sum of the lengths of $C_2[t_{1,2}, v]$ and $C[v, s_{1,2}]$ is at least $7k+1$. Similarly, the sum of the lengths of $C_2[t_{1,3}, v]$ and $C[v, s_{1,3}]$ is at least $7k+1$. In particular, if (i) holds, then (ii) does not hold and vice-versa.

Suppose for a contradiction that both (i) and (ii) do not hold. By symmetry and the above inequalities, we may assume that both $C_2[t_{1,2}, v]$ and $C_3[v, s_{1,3}]$ have length more than $3k$. But $v \notin V(C_1)$, so $v \notin V(P_{1,3})$. Thus $C_3[v, t_{1,3}]$ has also length at least $3k$.

If there is a vertex in $V(C_1) \cap V(C_2) \cap V(C_3)$, then $C_3[v, t_{1,3}]$ would have length less than $2k$ (since it would be contained in $P_{2,3} \cup P_{1,3}$ and each of those paths has length less than k), a contradiction. Hence $V(C_1) \cap V(C_2) \cap V(C_3) = \emptyset$. In particular, $P_{1,2}$, $P_{1,3}$, and $P_{2,3}$ are disjoint.

The dipath $C_2[s_{1,2}, t_{2,3}]$ has length at least $3k$ because it contains $C_2[t_{1,2}, v]$. Moreover, the dipath $C_3[t_{2,3}, s_{1,3}]$ has length at least $2k$ because $C_3[v, s_{1,3}]$ has length at least $3k$ and $C_3[v, t_{2,3}]$ has length less than k. Thus $C_3[t_{2,3}, s_{1,3}] \odot C_1[s_{1,3}, s_{1,2}]$ has length at least $2k$. Consequently, the union of $C_2[s_{1,2}, t_{2,3}]$, $C_2[t_{2,3}, s_{1,2}]$, and $C_3[t_{2,3}, s_{1,3}] \odot C_1[s_{1,3}, s_{1,2}]$ is a $B(k,1;k)$-subdivision, a contradiction. $\square$

Let $\mathcal{C}$ be a k-suitable collection of directed cycles. For every set of vertices or digraph S, we denote by $\mathcal{C} \cap S$ the set of directed cycles of $\mathcal{C}$ that intersect S.

Let $C_1 \in \mathcal{C}$. For each $C_j \in \mathcal{C} \cap C_1$ such that $C_j \neq C_1$, let Q_j be the subdipath of C_j containing all the vertices that are at distance at most $3k$ from $P_{1,j}$ in the cycle underlying C_j. Then the dipaths $C_j[s(Q_j), s_{1,j}]$ and $C_j[t_{1,j}, t(Q_j)]$ have length $3k$. Set $Q_j^- = C[s(Q_j), s_{1,j}[$ and $Q_j^+ = C]t_{1,j}, t(Q_j)]$.

Set $I(C_1) = C_1 \cup \bigcup_{C_j \in \mathcal{C} \cap C_1} Q_j$, $I^+(C_1) = \bigcup_{C_j \in \mathcal{C} \cap C_1} Q_j^+$ and $I^-(C_1) = \bigcup_{C_j \in \mathcal{C} \cap C_1} Q_j^-$. Observe that Lemma 26 implies directly the following.

Corollary 27. *Let $\mathcal{C}$ be a k-suitable collection of directed cycles and let $C_1 \in \mathcal{C}$.*

(i) $I^+(C_1)$ and $I^-(C_1)$ are vertex-disjoint digraphs.

(ii) $I^-(C_1) \cap C_j = Q_j^-$ and $I^+(C_1) \cap C_j = Q_j^+$, for all $C_j \in \mathcal{C} \cap C_1$.

Lemma 28. *Let $\mathcal{C}$ be a k-suitable collection of directed cycles in a $B(k,1;k)$-subdivision-free digraph D. Let C_1 be a directed cycle of $\mathcal{C}$ and let A be a connected component of $\bigcup \mathcal{C} - I(C_1)$. All vertices of $\bigcup(\mathcal{C} \cap A) - A$ belong to a unique directed cycle C_A of $\mathcal{C}$.*

Proof. Suppose it is not the case. Then there are two distinct directed cycles C_2, C_3 of $\mathcal{C} \cap A$ that intersect with C_1. Observe that there is a sequence of distinct directed cycles $C_2 = C_1^*, C_2^*, \dots, C_q^* = C_3$ of $\mathcal{C} \cap A$ such that $C_j^* \cap C_{j+1}^* \neq \emptyset$ because A is a connected component of $\bigcup \mathcal{C} - I(C_1)$. Free to consider the first $C_j^* \neq C_2$ in this sequence such that $V(C_j^*) \not\subseteq A$ in place of C_3, we may assume that all C_j^*, $2 \leqslant j \leqslant q-1$, have all their vertices in A. In particular, there exists a (C_3, C_2)-dipath Q_A in $D[A]$.

Let $R_3 = C_1[t_{1,2}, t_{1,3}] \odot Q_3$. Clearly, R_3 has length at least $3k$. Let v be the last vertex in $Q_2 \cap R_3$ along Q_2. (This vertex exists since $t_{1,2} \in Q_2 \cap R_3$.) Since there is a (C_3, C_2)-dipath in $D[A]$, by Corollary 27, $C_3[t(Q_3), s(Q_A)]$ is in $D[A]$. Thus there exists a $(t(Q_3), C_2)$-dipath R_A in $D[A]$. Let w be its terminal vertex. By definition of A, w is in $C_2[t(Q_2), s(Q_2)]$, therefore $C_2[w, v]$ has length at least $3k$ since it contains $C_2[s(Q_2), s_{1,2}]$. Consequently, both $C_2[v, t(Q_2)]$ and $R_3[v, t(Q_3)]$ have length less than k for otherwise the union of $C_2[w, v]$, $C_2[v, w]$ and $R_3[v, t(Q_3)] \odot R_A$ would be a $B(k,1;k)$-subdivision. In particular, $v \neq t(Q_2)$. This implies that $s_{2,3} \in V(Q_2 \cap R_3)$. Moreover, $Q_2[s_{2,3}, t(Q_2)]$ has length less than $2k$ because $Q_2[s_{2,3}, v]$ is a subdipath of $P_{2,3}$ and so has length less than k. Therefore $C_2[t_{1,2}, s_{2,3}] = Q_2[t_{1,2}, s_{2,3}]$ has length at least k because Q_2 has length at least $3k$. It follows that the union of $C_2[s_{2,3}, t_{1,2}]$, $C_2[t_{1,2}, s_{2,3}]$ and $R_3[t_{1,2}, s_{2,3}]$ is a $B(k,1;k)$-subdivision, a contradiction. □

Lemma 29. *Let $\mathcal{C}$ be a k-suitable collection of directed cycles in a $B(k,1;k)$-subdivision-free digraph. For any directed cycle $C_1 \in \mathcal{C}$, the digraph $I^+(C_1)$ has no directed cycle.*

Proof. Suppose for a contradiction that $I^+(C_1)$ contains a directed cycle C'. Clearly, it must contain arcs from at least two Q_j^+.

Assume that C' contains several vertices of Q_j^+. Necessarily, there must be two vertices x, y of $Q_j^+ \cap C'$ such that no vertex of $C']x, y[$ is in C_j and y is before x in Q_j^+. Therefore

$C'[x,y] \odot Q^+[y,x]$ is also a directed cycle in $I^+(C_1)$. Free to consider this cycle, we may assume that $C' \cap Q_j^+$ is a dipath.

Doing so, for all j, we may assume that $C' \cap Q_j^+$ is a dipath for every $C_j \in \mathcal{C} \cap C_1$. Without loss of generality, we may assume that there are directed cycles $C_2, \dots, C_p$ such that

- C' is in $Q_2^+ \cup \cdots \cup Q_p^+$;
- for all $2 \leqslant j \leqslant p$, $C' \cap Q_j^+$ is a dipath P_j^+ with initial vertex a_j and terminal vertex b_j;
- the a_j and the b_j appear according to the following order around C': $(a_2, b_p, a_3, b_2, \dots, a_p, b_{p-1}, a_2)$ with possibly $a_{j+1} = b_j$ for some $1 \leqslant j \leqslant p$ where $a_{p+1} = a_2$.

For $2 \leqslant j \leqslant p$, set $B_j = C_j[b_j, a_j]$. Note that B_j has length at least $4k$, because Q_2^+ has length less than $3k$.

Consider the closed directed walk

$$W = C_p[a_2, b_p] \odot B_p \odot C_{p-1}[a_p, b_{p-1}] \odot \cdots \odot B_3 \odot C_2[a_3, b_2] \odot B_2.$$

W contains a directed cycle C_W. Wihtout loss of generality, we may assume that this cycle is of the form

$$C_W = B_q[v, a_q] \odot C_{q-1}[a_q, b_{q-1}] \odot \cdots \odot B_3 \odot C_2[a_3, b_2] \odot B_2[b_2, v]$$

for some vertex $v \in B_2 \cap B_q$. (The case when W is a directed cycle corresponds to $q = p+1$ and $B_2 = B_{p+1}$.)

Note that necessarily, $q \geqslant 4$, for B_3 does not intersect B_2, for otherwise $b_3 = b_2$ since the intersection of C_2 and C_3 is a dipath.

Observe that $C_W[b_2, v] = C_2[b_2, v]$ or $C_W[v, a_4]$ has length at least k. Indeed, if $q = p+1$, then it follows from the fact that B_2 has length as least $4k$; if $5 \leqslant q \leqslant p$, then it comes from the fact that B_4 is a subdipath of $C_W[v, a_r]$; if $q = 4$, then it follows from Lemma 26 applied to C_3, C_2, C_4 in the role of C_1, C_2, C_3 respectively. In both cases, $C_W[b_2, a_4]$ has length at least k.

Furthermore, $C_W[a_4, b_2]$ has length at least k because it contains B_3. Therefore the union of $C_W[b_2, a_4]$, $C_W[a_4, b_2]$ and $C'[b_2, a_4] = C_3[b_3, a_4]$ is a $B(k,1;k)$-subdivision, a contradiction. □

Lemma 30. *Let $\mathcal{C}$ be a k-suitable collection of directed cycles in a $B(k,1;k)$-subdivision-free digraph.*

Let ϕ be a partial colouring of a directed cycle $C_1 \in \mathcal{C}$ such that only a path of length at most $7k$ is coloured and this path is rainbow-coloured. Then ϕ can be extended into a colouring of $I(C_1)$ using α_k colours, such that every subdipath of length at most $7k$ of C_1 is rainbow-coloured and Q_j is rainbow-coloured, for every $C_j \in \mathcal{C} \cap C_1$.

Proof. We can easily extend ϕ to C_1 using $14k$ colours (including the at most $7k$ already used colours) so that every subdipath of C_1 of length $7k$ is rainbow-coloured.

We shall now prove that there exists a colouring ϕ^+ of $I^+(C_1)$ with $(6k^2)^{3k}$ (new) colours so that Q_j^+ is rainbow-coloured for every $C_j \in \mathcal{C} \cap C_1$, and a colouring ϕ^- of $I^-(C_1)$ with $(6k^2)^{3k}$ (other new) colours so that Q_j^- is rainbow-coloured for every $C_j \in \mathcal{C} \cap C_1$. The union of the three colourings ϕ, ϕ^+, and ϕ^- is clearly the desired colouring of $I(C_1)$. (Observe that a vertex of $I(C_1)$ is coloured only once because C_1, $I^+(C_1)$ and $I^-(C_1)$ are disjoint by Corollary 27.)

It remains to prove the existence of ϕ^+ and ϕ^-. By symmetry, it suffices to prove the existence of ϕ^+. To do so, we consider an auxiliary digraph D_1^+. For each $C_j \in \mathcal{C} \cap C_1$, let T_j^+ be the transitive tournament whose hamiltonian dipath is Q_j^+. Let $D_1^+ = \bigcup_{C_j \in \mathcal{C} \cap C_1} T_j^+$. The arcs of $A(T_j^+) \setminus A(Q_j^+)$ are called *fake arcs*. Clearly, ϕ^+ exists if and only if D_1^+ admits a proper $(6k^2)^{3k}$-colouring. Henceforth it remains to prove the following claim.

Claim 31. $\chi(D_1^+) \leqslant (6k^2)^{3k}$.

Subproof. To each vertex v in $I^+(C_1)$ we associate the set $\mathrm{Dis}(v)$ of the lengths of the $C_j[t_{1,j}, v]$ for all directed cycles $C_j \in \mathcal{C} \cap C_1$ containing v such that $C_j[t_{1,j}, v]$ has length at most $3k$.

Suppose for a contradiction that $\chi(D_1^+) \leqslant (6k^2)^{3k}$. By Theorem 1, D_1^+ admits a dipath of length $(6k^2)^{3k}$. Replacing all fake arcs (u, v) in some $A(T_j^+)$, by $Q_j^+[u, v]$ we obtain a directed walk P in $I^+(C_1)$ of length at least $(6k^2)^{3k}$. By Lemma 29, P is necessarily a dipath. Set $P = (v_1, \dots, v_p)$. We have $p \geqslant (6k^2)^{3k}$.

For $1 \leqslant i \leqslant p$, let $m_i = \min \mathrm{Dis}(v_i)$. Lemma 20 applied to $(m_i)_{1 \leqslant i \leqslant p}$ yields a set L of $6k^2$ indices such that for any $i < j \in L$, $m_i = m_j$ and $m_k > m_i$, for all $i < k < j$. Let $l_1 < l_2 < \cdots < l_{6k^2}$ be the elements of L and let $m = m_{l_1} = \cdots = m_{l_{6k^2}}$.

For $1 \leqslant j \leqslant 6k^2 - 1$, let $M_j = \max \bigcup_{l_j \leqslant i < l_{j+1}} \mathrm{Dis}(v_i)$. By definition $M_j \leqslant 3k$. Applying Lemma 21 to $(M_j)_{1 \leqslant j \leqslant 6k^2}$, we get a sequence of size $2k$ $M_{j_0+1}, \dots, M_{j_0+2k}$ such that M_{j_0+2k} is the greatest. For sake of simplicity, we set $\ell_i = j_0 + i$ for $1 \leqslant i \leqslant 2k$. Let f be the smallest index not smaller than ℓ_{2k} for which $M_{\ell_{2k}} \in \mathrm{Dis}(v_f)$.

Let j_1 be an index such that $C_{j_1}[t_{1,j_1}, v_{\ell_1}]$ has length m and set $P_1 = C_{j_1}[t_{1,j_1}, v_{\ell_1}]$. Let j_2 be an index such that $C_{j_2}[t_{1,j_2}, v_{\ell_k}]$ has length m and set $P_2 = C_{j_2}[t_{1,j_2}, v_{\ell_k}]$. Let j_3 be an index such that $C_{j_3}[t_{1,j_3}, v_f]$ has length $M_{\ell_{2k}}$ and set $P_3 = C_{j_3}[v_f, s_{1,j_3}]$ (some vertices of P_3 are not in $I^+(C_1)$).

Note that any internal vertex x of P_1 or P_2 has an integer in $\mathrm{Dis}(x)$ which is smaller than m and every internal vertex y of P_3 has an integer in $\mathrm{Dis}(y)$ which is greater than $M_{\ell_{2k}}$, or does not belong to $I^+(C_1)$. Hence, P_1, P_2 and P_3 are disjoint from $P[v_{\ell_1}, v_f]$.

We distinguish between the intersection of P_1, P_2 and P_3:

- Suppose P_3 does not intersect $P_1 \cup P_2$.
  - Assume first that P_1 and P_2 are disjoint. If $s(P_1)$ is in $C_1[t(P_3), s(P_2)]$, then the union of $P_1 \odot P[v_{\ell_1}, v_{\ell_k}]$, $P[v_{\ell_k}, v_f] \odot P_3 \odot C_1[t(P_3), s(P_1)]$ and $C_1[s(P_1), s(P_2)] \odot P_2$ is a $B(k, 1; k)$-subdivision, a contradiction. If $s(P_1)$ is in $C_1[s(P_2), t(P_3)]$, then

the union of $C_1[s(P_2), s(P_1)] \odot P_1 \odot P[v_{\ell_1}, v_{\ell_k}]$, $P[v_{\ell_k}, v_f] \odot P_3 \odot C_1[t(P_3), s(P_2)]$, and P_2 is a $B(k,1;k)$-subdivision, a contradiction.

– Assume now P_1 and P_2 intersect. Let u be the last vertex along P_2 on which they intersect. The union of $P_1[u, v_{\ell_1}] \odot P[v_{\ell_1}, v_{\ell_k}]$, $P[v_{\ell_k}, v_f] \odot P_3 \odot C[t(P_3), s(P_1)] \odot P_1[s(P_1), u]$, and $P_2[u, v_{\ell_k}]$ is a $B(k,1;k)$-subdivision, a contradiction.

- Assume P_3 intersects $P_1 \cap P_2$. Let v be the first vertex along P_3 in $P_1 \cap P_2$ and let u be the last vertex of $P_1 \cap P_2$ along P_2. The union of $P_1[u, v_{\ell_1}] \odot P[v_{\ell_1}, v_{\ell_k}]$, $P[v_{\ell_k}, v_f] \odot P_3[v_f, v] \odot P_1[v, u]$, and $P_2[u, v_{\ell_k}]$ is a $B(k,1;k)$-subdivision, a contradiction.

- Assume now that P_3 intersects $P_1 \cup P_2$ but not $P_1 \cap P_2$. Let v be the first vertex along P_3 in $P_1 \cup P_2$.

  – If $v \in P_2$, let u be the last vertex on $P_2 \cap P_3$ along P_3. Observe that $P_3[v, u]$ is also a subdipath of P_2 and therefore contains no vertex of P_1. Furthermore, there is a dipath Q from u to v_{ℓ_1} in $P_3[u, t(P_3)] \cup C_1 \cup P_1$. Hence, the union of $P[v_{\ell_k}, v_f] \odot P_3[v_f, v]$, $Q \odot P[v_{\ell_1}, v_{\ell_k}]$, and $P_2[u, v_{\ell_k}]$ is a $B(k,1;k)$-subdivision, a contradiction.

  – If $v \in P_1$, let u be the last vertex on $P_1 \cap P_3$ along P_3. Observe that $P_3[v, u]$ is also a subdipath of P_1 and therefore contains no vertex of P_2. Furthermore, there is a dipath Q from u to v_{ℓ_k} in $P_3[u, t(P_3)] \cup C_1 \cup P_2$. The union of $P[v_{\ell_k}, v_f] \odot P_3[v_f, u]$, $P_1[u, v_{\ell_1}] \odot P[v_{\ell_1}, v_{\ell_k}]$ and Q is a $B(k,1;k)$-subdivision, a contradiction. $\lozenge$

Claim 31 shows the existence of ϕ^+ and completes the proof of Lemma 30. $\square$

We are now ready to prove Lemma 25. In fact, we prove the following stronger statement.

Lemma 32. *If there exists a partial colouring ϕ such that one of the directed cycle C_1 has a path of length less than $7k$ which is rainbow-coloured, then we can extend this colouring to all $D[\mathcal{C}]$ using less than α_k colours such that, on each directed cycle, every subdipath of length $7k$ is rainbow-coloured.*

Proof. By induction on the number of directed cycles in $\mathcal{C}$. Consider a rainbow-colouring of a subdipath of length less than $7k$ of a directed cycle $C_1 \in \mathcal{C}$. By Lemma 30, we can extend this colouring to a colouring ϕ_1 of $I(C_1)$ at most α_k colours. Note that the non-coloured vertices of $\bigcup \mathcal{C}$ are in one of the connected components of $\bigcup \mathcal{C} - I(C_1)$. Let A be a connected component of $\bigcup \mathcal{C} - I(C_1)$. The coloured (by ϕ_1) vertices of $\mathcal{C} \cap A$ are those of $(\mathcal{C} \cap A) - A$. Hence, by Lemma 28, they all belong to some directed cycle C_j and so to the dipath Q_j which has length at most $7k$. Hence, by the induction hypothesis, we can extend ϕ_1 to A. Doing this for each component, we extend ϕ_1 to the whole $\bigcup \mathcal{C}$. $\square$

Set $\beta_k = k(4k^2+2)(2 \cdot (4k)^{4k} + 1)\alpha_k$. The second step of the proof is the following lemma.

Lemma 33. *Let $\mathcal{C}$ be a k-suitable collection of directed cycles in a $B(k,1;k)$-subdivision-free digraph D. For every component $\mathcal{S}$ of $\mathcal{C}$, we have $\chi(D[\mathcal{S}]) \leqslant \beta_k$.*

Proof. We define a sort of Breadth-First-Search for $\mathcal{S}$. Let C_0 be a directed cycle of $\mathcal{S}$ and set $L_0 = \{C_0\}$. For every directed cycle C_s of $\mathcal{S} \cap C_0$, we put C_s in level L_1 and say that C_0 is the *father* of C_s. We build the levels L_i inductively until all directed cycles of $\mathcal{S}$ are put in a level : L_{i+1} consists of every directed cycle C_l not in $\bigcup_{j \leqslant i} L_j$ such that there exists a directed cycle in L_i intersecting C_l. For every $C_l \in L_{i+1}$, we choose one of the directed cycles in L_i intersecting it to be its *father.* Henceforth every directed cycle in L_{i+1} has a unique father even though it might intersect many directed cycles of L_i. A directed cycle C is an *ancestor* of C' if there is a sequence $C = C_1, \dots, C_q = C'$ such that C_i is the father of C_{i+1} for all $i \in [q-1]$.

For a vertex x of $\bigcup \mathcal{S}$, we say that x *belongs to* level L_i if i is the smallest integer such that there exists a directed cycle in L_i containing x. Observe that the vertices of each directed cycle C_l of $\mathcal{S}$ belong to consecutive levels, that is there exists i such that $V(C_l) \subseteq L_i \cup L_{i+1}$.

To bound the chromatic number of $D[\mathcal{S}]$, we partition its arc set in (A_0, A_1, A_2), where

- A_0 is the set of arcs of $D[\mathcal{S}]$ which ends belong to the same level, and
- A_1 is the set of arcs of $D[\mathcal{S}]$ which ends belong to different levels i and j with $|i-j| < k$.
- A_2 is the set of arcs of $D[\mathcal{S}]$ which ends belong to different levels i and j with $|i-j| \geqslant k$.

For $i \in \{0,1,2\}$, let D_i be the spanning subdigraph of $D[\mathcal{S}]$ with arc set A_i. We shall now bound the chromatic numbers of D_0, D_1 and D_2.

Claim 34. $\chi(D_1) \leqslant k$.

Subproof. Let ϕ_1 be the colouring that assigns to all vertices of level L_i the colour i modulo k, it is easy to see that ϕ_1 is a proper colouring of D_1. $\Diamond$

Let C_l be a directed cycle of L_i, $i \geqslant 1$ and $C_{l'}$ its father.

Let p_l^+ and r_l^+ be the vertices such that $C_l[t_{l,l'}, p_l^+]$ and $C_l[p_l^+, r_l^+]$ have length k. Let p_l^- and r_l^- be the vertices such that $C_l[p_l^-, s_{l,l'}]$ and $C_l[r_l^-, p_l^-]$ have length k. Let R_l^- be the set of vertices of $C_l]r_l^-, s_{l,l'}[$, P_l^- the set of vertices of $C_l]p_l^-, s_{l,l'}[$, R_l^+ the set of vertices of $C_l]t_{l,l'}, r_l[$, P_l^+ the set of vertices of $C_l]t_{l,l'}, p_l^+[$, and finally let R_l' be the set of vertices belonging to L_i in $C_l \setminus \{R_l^+ \cup R_l^-\}$.

Claim 35. *Let x be a vertex in L_i with $i \geqslant 1$. Let C_l and C_m be two directed cycles of L_i containing x. Then either $x \in P_l^+$ and $x \in P_m^+$, or $x \in P_l^-$ and $x \in P_m^-$.*

Subproof. Suppose for a contradiction that $x \in P_l^+$ and $x \notin P_m^+$. Let $C_{l'}$ and $C_{m'}$ be the fathers of C_l and C_m respectively (they can be the same directed cycle). By definition of the L_j's, there exists a dipath P from $t_{l,l'}$ to $s_{m,m'}$ only going through $C_{l'}$, $C_{s'}$ and their ancestors. In particular P is disjoint from $C_l - C_{l'}$ and $C_s - C_{s'}$. Observe that $C_l[s_{l,l'}, t_{l,m}]$ has length at most $3k$ because it is contained in the union of $P_{l,l'}$, $P_{l,m}$, and $C_l[t_{l,l'}, x]$ which has length at most k because $x \in P_l^+$. Hence $C_l[t_{l,m}, s_{l,l'}]$ has length at least k. Moreover $C_m[s_{m,m'}, t_{l,m}]$ contains $C_m[t_{m,m'}, x]$ which has length at least k because $x \notin P_m^+$. Thus the union of $C_l[t_{l,m}, s_{l,l'}] \odot P$, $C_m[t_{l,m}, s_{m,m'}]$, and $C_m[s_{m,m'}, t_{l,m}]$ is a $B(k,1;k)$-subdivision, a contradiction. The case where $x \in P_l^-$ and $x \notin P_m^-$ is symmetrical and the case where x does not belong to $P_l^- \cup P_l^+ \cup P_m^- \cup P_m^+$ is identical. $\Diamond$

Claim 35 implies that each level L_i may be partitioned into sets X_i^+, X_i^- and X_i', where X_i^+ (resp. X_i^-) is the set of vertices x of L_i such that every $x \in R_l^+$ (resp. $x \in R_l^-$) for every directed cycle C_l of L_i containing x and X_i' is set of vertices in L_i but not in $X_i^+ \cup X_i^-$. Set $X^+ = V(C_0) \cup \bigcup_{i \geqslant 1} X_i^+$, $X^- = \bigcup_{i \geqslant 1} X_i^-$ and $X' = \bigcup_{i \geqslant 1} X_i'$. Clearly (X^+, X^-, X') is a partition of $V(D[\mathcal{S}])$.

Claim 36. $\chi(D_2) \leqslant 4k^2 + 2$.

Subproof. Since $X^+ \cup X^- \cup X' = V(D_2)$, we have $\chi(D_2) \leqslant \chi(D_2[X^+ \cup X']) + \chi(D_2[X^- \cup X'])$. We shall prove that $\chi(D_2[X^+ \cup X']) \leqslant 2k^2 + 1$ and $\chi(D_2[X^- \cup X']) \leqslant 2k^2 + 1$, which imply the result.

Let x and y be two adjacent vertices of $D_2[X^+ \cup X']$. Let L_i be the level of x and L_j be the level of y. Without loss of generality, we may assume that $j \geqslant i + k$. Let C_x be the directed cycle of L_i such that $x \in C_x$ and C_y the directed cycle of L_j such that $y \in C_y$. By considering ancestors of C_x and C_y, there is a shortest sequence of directed cycles $C_1, \dots, C_p$ such that $C_1 = C_x$ and $C_p = C_y$ and for all $l \in [p-1]$, either C_l is the father of C_{l+1} or C_{l+1} is the father of C_l. In particular C_{p-1} is the father of C_p. Since $y \in X^+ \cup X'$, then $C[y, t_{p-1,p}]$ has length at least k.

Assume that (x, y) is an arc. In $\bigcup_{l=1}^{p-1} C_l$, there is a dipath P from $t_{p-1,p}$ to x. This dipath has length at least $k-1$ because it must go through all levels $L_{i'}$, $i \leqslant i' \leqslant j-1$ because the vertices of any directed cycle of $\mathcal{S}$ are in two consecutive levels. Hence the union of $P \odot (x, y)$, $C_p[t_{p-1,p}, y]$, and $C_p[y, t_{p-1,p}]$ is a $B(k,1;k)$-subdivision, a contradiction. Hence (y, x) is an arc.

Suppose that C_x is not an ancestor of C_y. In particular, C_2 is the father of C_1 and there exists a path P from $t_{1,2}$ to y in $\bigcup_{l=2}^{p-1} C_l$ of length at least $k-1$ and internally disjoint from C_1. Hence the union of $P \odot yx$, $C_1[x, t_{1,2}]$ and $C_1[t_{1,2}, x]$ is a subdivsion of $B(k,1;k)$. Hence C_x is an ancestor of C_y.

In particular, C_l is the father of C_{l+1} for all $l \in [p-1]$. Let P be the dipath from $t_{1,2}$ to y in $\bigcup_{l=2}^{p} C_l$. It has length at least $k-1$ because it must go through all levels L_i, $1 \leqslant i \leqslant p-1$. $C_1[x, t_{1,2}]$ has length less than k, for otherwise the union of $P \odot yx$, $C_1[x, t_{1,2}]$ and $C_1[t_{1,2}, x]$ would be a subdivision of $B(k,1;k)$.

To summarize, the only arcs of $D_2[X^+ \cup X']$ are arcs (y, x) such that C_x is an ancestor of C_y and $C_1[x, t_{1,2}]$ has length less than k with $C_1 \dots C_p$ the sequence of directed cycles such that $C_1 = C_x$ to $C_p = C_y$ and C_l is the father of C_{l+1} for all $l \in [p-1]$. In particular, $D_2[X^+ \cup X']$ is acyclic.

Let y be a vertex of $D_2[X^+ \cup X']$. Let L_p be the level of y and let $C_0, \dots, C_p$ be the sequence of directed cycles such that C_{l-1} is the father of C_l for all $l \in [p]$. For $0 \leqslant l \leqslant p-1$, let R_l be the subdipath of C_l of length $k-1$ terminating at $t_{l,l+1}$. By the above property, the out-neighbbours of y are in $\bigcup_{l=0}^{p-1} R_l$. Suppose for a contradiction that y has out-degree at least $2k^2+1$. Then there are $2k+1$ distinct indices $l_1 < \cdots < l_{2k+1}$ such that for all $i \in [2k+1]$, C_{l_i} contains an out-neighbour X_i of y. Let P be the shortest dipath from x_1 to y in $\bigcup_{l=l_1}^{p} C_l$. This dipath intersects all directed cycles C_l $l_1 \leqslant l \leqslant p$. Let z be the first vertex of P along $C_{l_{k+1}}[x_{k+1}, t_{l_{k+1}, l_{k+2}}]$. Vertex z belongs to either $L_{l_{k+1}-1}$ or $L_{l_{k+1}}$. Thus $P[x_1, z]$ and $P[z, y]$ have length at least $k-1$ and k respectively since P goes through all levels from L_{l_1} to L_p. Hence the union of $(y, x_1) \odot P[x_1, z]$, $(y, x_{k+1}) \odot C_{l_{k+1}}[x_{k+1}, z]$, and $P[z, y]$ is a $B(k,1;k)$-subdivision, a contradiction. Therefore $D_2[X^+ \cup X']$ has maximum out-degree at most $2k^2$.

$D_2[X^+ \cup X']$ is acyclic and has maximum out-degree at most $2k^2$. Therefore it is $2k^2$-degenerate, and so $\chi(D_2[X^+ \cup X']) \leqslant 2k^2+1$. By symmetry, we have $\chi(D_2[X^- \cup X']) \leqslant 2k^2+1$. $\Diamond$

To bound $\chi(D_0)$ we partition the vertex set according to a colouring ϕ of $\bigcup \mathcal{S}$ given by Lemma 25. For every colour $c \in [\alpha_k]$, let $X^+(c)$ be the set $X^+ \cap \phi^{-1}(c)$ of vertices of X^+ coloured c, and $X^-(c)$ the set $X^- \cap \phi^{-1}(c)$ of vertices of X^- coloured c. Similarly, let $X_i^+(c) = X_i^+ \cap \phi^{-1}(c)$ and $X_i^-(c) = X_i^- \cap \phi^{-1}(c)$. We denote by $D_0^+(c)$ (resp. $D_0^-(c)$, $D_0'(c)$) the subdigraph of D_0 induced by the vertices of $X^+(c)$, (resp. $X^-(c)$, $X'(c)$).

Claim 37. *$\chi(D_0'(c)) = 1$ for all $c \in [\alpha_k]$.*

Subproof. We need to prove that $D_0'(c)$ has no arc. Suppose for a contradiction that (x, y) is an arc of $D_0'(c)$. By definition of D_0, the vertices x and y are in a same level L_i. Let C_l and C_m be two directed cycles of L_i such that $x \in C_l$ and $y \in C_m$.

If $C_l = C_m$, then both $C_l[x, y]$ and $C_l[y, x]$ have length at least $7k$ because the subdipaths of length $7k$ of C_l are rainbow-coloured by ϕ. Hence the union of those paths and (x, y) is a $B(k,1;k)$-subdivision, a contradiction. Henceforth, C_l and C_m are distinct directed cycles.

Suppose first that C_l and C_m intersect. By Claim 35, $s_{l,m}$ belongs to P_l^-, P_l^+ or L_{i-1}, and by construction of R_l', $C_l[x, s_{l,m}]$ and $C_l[s_{l,m}, x]$ are both longer than k. Therefore they form with $(x, y) \odot C_m[y, s_{l,m}]$ a $B(k,1;k)$-subdivision, a contradiction.

Suppose now that C_l and C_m do not intersect. Let C_l' and C_m' be the fathers of C_l and C_m respectively. Let P be the dipath from $s_{m,m'}$ to $s_{l,l'}$ in $\bigcup_{j<i} L_j$. Then the union of $C_l[s_{l,l'}, x]$, $(x, y) \odot C_m[y, s_{m,m'}] \odot P$, and $C_l[x, s_{l,l'}]$ is a $B(k,1;k)$-subdivision, a contradiction. $\Diamond$

Claim 38. *$\chi(D_0^+(c)) \leqslant (4k)^{4k}$ for all $c \in [\alpha_k]$.*

Subproof. Set $p = (4k)^{4k}$. Suppose for a contradiction that there exists c such that $\chi(D_0^+(c)) > p$. Observe that $D_0^+(c)$ is the disjoint union of the $D[X_i^+(c)]$. Thus there exists a level L_{i_0} such that $\chi(D[X_i^+(c)]) > p$. Moreover $i_0 > 0$, because the vertices of

C_0 coloured c form a stable set. By Theorem 1, there exists a dipath $P = (v_0, \dots, v_p)$ of length p in $D[X_i^+(c)]$.

Suppose that P contains two vertices x and y of a same directed cycle C of $\mathcal{S}$. Without loss of generality, we may assume that $P]x,y[$ contains no vertices of C. Now both $C[x,y]$ and $C[y,x]$ have length at least $7k$ because the subdipaths of length $7k$ of C are rainbow-coloured by ϕ. Thus the union of $C[x,y]$, $P[x,y]$ and $C[y,x]$ is a $B(k,1;k)$-subdivision, a contradiction. Hence P intersects every directed cycle of $\mathcal{S}$ at most once.

For every $v \in V(P)$, let $\mathrm{Len}(v)$ be the set of lengths of $C_l[t_{l,l'}, v]$ for all directed cycles $C_l \in L_{i_0}$ containing v and whose father is $C_{l'}$.

For $1 \leqslant i \leqslant p$, let $m_i = \min \mathrm{Len}(v_i)$. By Claim 35, $\mathrm{Len}(v_i) \subseteq [2k]$. Lemma 20 applied to $(m_i)_{1 \leqslant i \leqslant p}$ yields a set L of $4k^2$ indices such that for any $i < j \in L$, $m_i = m_j$ and $m_k > m_i$, for all $i < k < j$. Let $l_1 < l_2 < \cdots < l_{4k^2}$ be the elements of L and let $m = m_{l_1} = \cdots = m_{l_{4k^2}}$.

For $1 \leqslant j \leqslant 4k^2 - 1$, let $M_j = \max \bigcup_{l_j \leqslant i < l_{j+1}} \mathrm{Len}(v_i)$. By definition $M_j \leqslant 2k$. Applying Lemma 21 to $(M_j)_{1 \leqslant j \leqslant 4k^2}$, we get a sequence of size $2k$ $M_{j_0+1}, \dots, M_{j_0+2k}$ such that M_{j_0+2k} is the greatest. For sake of simplicity, we set $\ell_i = j_0 + i$ for $1 \leqslant i \leqslant 2k$. Let f be the smallest index not smaller than ℓ_{2k} for which $M_{\ell_{2k}} \in \mathrm{Len}(v_f)$.

Let j_1 and j_1' be indices such that $v_{\ell_1} \in C_{j_1}$, C_{j_1} is in L_{i_0}, $C_{j_1'}$ is the father of C_{j_1} and $C_{j_1}[t_{j_1',j_1}, v_{\ell_1}]$ has length m. Set $P_1 = C_{j_1}[t_{j_1',j_1}, v_{\ell_1}]$. Let j_2 and j_2' be indices such that $v_{\ell_k} \in C_{j_2}$, C_{j_2} is in L_{i_0}, $C_{j_2'}$ is the father of C_{j_2} and $C_{j_2}[t_{j_2',j_2}, v_{\ell_k}]$ has length m. Set $P_2 = C_{j_2}[t_{j_2',j_2}, v_{\ell_k}]$. Let j_3 and j_3' be indices such that $v_f \in C_{j_3}$, C_{j_3} is in L_i, $C_{j_3'}$ is the father of C_{j_3} and $C_{j_3}[t_{j_3',j_3}, v_f]$ has length $M_{\ell_{2k}}$. Set $P_3 = C_{j_3}[v_f, s_{j_3',j_3}]$. Note that any internal vertex x of P_1 or P_2 has an integer in $\mathrm{Len}(x)$ which is smaller than m and every internal vertex y of P_3 either has an integer in $\mathrm{Len}(y)$ which is greater than $M_{\ell_{2k}}$, or does not belong to $X^+(c)$. Hence, P_1, P_2 and P_3 are disjoint from $P[v_{\ell_1}, v_f]$.

We distinguish cases according to the intersection between P_1, P_2 and P_3: Let P_4 be a shortest dipath in $\cup_{i<i_0} L_i$ from t_{j_1',j_1} to t_{j_2',j_2} and P_5 be a shortest dipath in $\cup_{i<i_0} L_i$ from s_{j_3',j_3} to t_{j_2',j_2}

- Suppose P_3 does not intersect $P_1 \cup P_2$.
  - Suppose P_1 and P_2 are disjoint. Let v be the last vertex of P_4 in $P_4 \cap P_5$. The union of $P_5[v, t_{j_1',j_1}] \odot P_1 \odot P[v_{\ell_1}, v_{\ell_k}]$, $P_4[v, t_{j_2',j_2}] \odot P_2$, and $P[v_{\ell_k}, v_f] \odot P_3 \odot P_5[s_{j_3',j_3}, v]$ is a $B(k,1;k)$-subdivision, a contradiction.
  - Assume now P_1 and P_2 intersect. Let u be the last vertex along P_2 on which they intersect. The union of $P_1[u, v_{\ell_1}] \odot P[v_{\ell_1}, v_{\ell_k}]$, $P_2[u, v_{\ell_k}]$, and $P[v_{\ell_k}, v_f] \odot P_3 \odot P_5 \odot P_1[t_{j_1',j_1}, u]$ is a $B(k,1;k)$-subdivision, a contradiction.
- Assume P_3 intersects $P_1 \cap P_2$. Let v be the first vertex along P_3 in $P_1 \cap P_2$ and let u be the last vertex of $P_1 \cap P_2$ along P_2. The union of $P_1[u, v_{\ell_1}] \odot P[v_{\ell_1}, v_{\ell_k}]$, $P_2[u, v_{\ell_k}]$, and $P[v_{\ell_k}, v_f] \odot P_3[v_f, v] \odot P_1[v, u]$ is a $B(k,1;k)$-subdivision, a contradiction.
- Assume now that P_3 intersects $P_1 \cup P_2$ but not $P_1 \cap P_2$. Let v be the first vertex along P_3 in $P_1 \cup P_2$.

– If $v \in P_2$, let u be the last vertex of $P_2 \cap P_3$ along P_3. Observe that $P_3[v,u]$ is also a subdipath of P_2 and therefore contains no vertex of P_1. Hence, the union of $P_3[u, s_{j'_3,j_3}] \odot P_5 \odot P_1 \odot P[v_{\ell_1}, v_{\ell_k}]$, $P_2[u, v_{\ell_k}]$, and $P[v_{\ell_k}, v_f] \odot P_3[v_f, v]$ is a $B(k,1;k)$-subdivision, a contradiction.

– If $v \in P_1$, let u be the last vertex of $P_1 \cap P_3$ along P_3. Observe that $P_3[v,u]$ is also a subdipath of P_1 and therefore contains no vertex of P_2. Hence the union of $P_1[u, v_{\ell_1}] \odot P[v_{\ell_1}, v_{\ell_k}]$, $P_3[u, s_{j'_3,j_3}] \odot P_6 \odot P_2$, and $P[v_{\ell_k}, v_f] \odot P_3[v_f, u]$, is a $B(k,1;k)$-subdivision, a contradiction. $\Diamond$

Similarly to Claim 38, one proves that $\chi(D_0^-(c)) \leqslant (4k)^{4k}$ for all $c \in [\alpha_k]$. Hence, $\chi(D_0(c)) \leqslant \chi(D_0^+(c)) + \chi(D_0^-(c)) + \chi(D'_0(c)) \leqslant 2 \cdot (4k)^{4k} + 1$. Thus

$$\chi(D_0) \leqslant (2 \cdot (4k)^{4k} + 1)\alpha_k.$$

Via Lemma 17, this equation and Claims 34 and 36 yield

$$\chi(D) \leqslant \chi(D_0) \times \chi(D_1) \times \chi(D_2) \leqslant k(4k^2+2)(2 \cdot (4k)^{4k} + 1)\alpha_k = \beta_k. \qquad \square$$

4.2 Proof of Theorem 13

Consider a maximal k-suitable collection $\mathcal{C}$ of directed cycles in D. Recall that $D_{\mathcal{C}}$ is the digraph obtained by contracting every component of $\mathcal{C}$ into one vertex. For each connected component $\mathcal{S}_i$ of $\mathcal{C}$, we call s_i the new vertex created.

Claim 39. *$\chi(D_{\mathcal{C}}) \leqslant 8k$.*

Proof. First note that since D is strong so is $D_{\mathcal{C}}$.

Suppose for a contradiction that $\chi(D_{\mathcal{C}}) > 8k$. By Theorem 5, there exists a directed cycle $C = (x_1, x_2, \ldots, x_l, x_1)$ of length at least $8k$. For each vertex x_j that corresponds to an s_i in D, the arc (x_{j-1}, x_j) corresponds in D to an arc whose head is a vertex p_i of $\mathcal{S}_i$ and the arc (x_j, x_{j+1}) corresponds to an arc whose tail is a vertex l_i of $\mathcal{S}_i$. Let P_j be the dipath from p_i to l_i in $\bigcup \mathcal{C}$. Note that this dipath intersects the elements of $\mathcal{S}_i$ only along a subdipath. Let C' be the directed cycle obtained from C where we replace all contracted vertices x_j by the dipath P_j. First note that C' has length at least $8k$. Moreover, a directed cycle of $\mathcal{C}$ can intersect C' only along one P_j, because they all correspond to different strong components of $\bigcup \mathcal{C}$. Thus C' intersects each directed cycle of $\mathcal{C}$ on a subdipath. Moreover this subdipath has length less than k for otherwise D would contain a $B(k,1;k)$-subdivision. So C' is a directed cycle of length at least $8k$ which intersects every directed cycle of $\mathcal{C}$ along a subdipath of length less than k. This contradicts the maximality of $\mathcal{C}$. $\square$

Using Lemma 18 with Claim 39 and Lemma 33, we get that $\chi(D) \leqslant 8k \cdot \beta_k$. This proves Theorem 13 for $\gamma_k = 8k \cdot \beta_k = 8k^2(4k^2+2)(2 \cdot (4k)^{4k} + 1)(2 \cdot (6k^2)^{3k} + 14k)$.

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