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Abstract

In the present note we show that the union of r general lines and one fat line in $\mathbb{P}^3$ imposes independent conditions on forms of sufficiently high degree d, where the bound is independent of the number of lines. This extends former results of Hartshorne and Hirschowitz on unions of general lines, and of Aladpoosh on unions of general lines and one double line.

Keywords postulation problems, fat flats, Hilbert functions, Serre vanishing

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1 Introduction

Let $X \subset \mathbb{P}^n$ be a closed subscheme defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. The Hilbert function of X encodes a number of properties of X and has been classically an object of vivid research in algebraic geometry and commutative algebra. We first recall the definition.

Definition 1.1 (Hilbert function)**.** The Hilbert function of a scheme $X \subset \mathbb{P}^n(\mathbb{K})$ is

$$\mathrm{HF}_X : \mathbb{Z} \ni d \to \dim_{\mathbb{K}}[S(X)]_d \in \mathbb{Z},$$

where $S(X)$ denotes the graded homogeneous coordinate ring of X.

It is well known that the Hilbert function becomes eventually (i.e., for large d) a polynomial. We denote the Hilbert polynomial of X by HP_X. Whereas the Hilbert polynomial can be (in principle) computed algorithmically, the Hilbert function is more difficult to compute. For some varieties, like $\mathbb{P}^n$, the Hilbert function is equal to the Hilbert polynomial, but this behaviour is rare. The next simplest behaviour occurs for subschemes with bipolynomial Hilbert function.

Definition 1.2 (Bipolynomial Hilbert function)**.** Following [5] we say that X has a *bipolynomial Hilbert function* if

$$\mathrm{HF}_X(d) = \min\{\mathrm{HP}_{\mathbb{P}^n}(d), \mathrm{HP}_X(d)\} \tag{1}$$

for all $d \geqslant 1$.

In other words, X has a bipolynomial Hilbert function if $X \subset \mathbb{P}^n$ imposes the *expected* number of conditions on forms of arbitrary degree $d \geqslant 1$. Essentially by definition, a scheme consisting of general points in $\mathbb{P}^n$ has bipolynomial Hilbert function. An analogous result for X consisting of r general lines in $\mathbb{P}^n$ with $n \geqslant 3$ has been proved by Hartshorne and Hirschowitz in [10, Theorem 0.1]. A new proof has been announced recently by Aladpoosh and Catalisano [3]. Recently Carlini, Catalisano and Geramita [7] showed that if X consists of r general lines and one general fat point, then, up to a short list of exceptions in $\mathbb{P}^3$, X has bipolynomial Hilbert function, see also [2] and [4].

Aladpoosh in [1] has proved recently that also schemes consisting of general lines and one double line have bipolynomial Hilbert function, with the exception of one double line and two simple lines in $\mathbb{P}^4$ imposing dependent conditions on forms of degree 2. She also conjectured [1, Conjecture 1.2] that the same holds true for r general lines and one fat flat of arbitrary dimension. In the present note we provide evidence supporting this conjecture for a fat line of arbitrary multiplicity m. Our main result is the following.

Main Theorem. *Let $m \geqslant 1$ be a fixed integer. Then for $d \geqslant d_0(m) := 3\binom{m+1}{3}$, the Hilbert function of a subscheme $X \subset \mathbb{P}^3$ consisting of $r \geqslant 0$ general lines and one line of multiplicity m (i.e. defined by the m-th power of the ideal of a line) satisfies formula (1).*

In other words, a general fat line and an arbitrary number of general lines with multiplicity 1 impose independent conditions on forms of degree $d \geqslant d_0(m)$ (see Theorem 4.1).

It follows from the Serre Vanishing Theorem [11, Theorem 1.2.6] that for any subscheme $X \subset \mathbb{P}^n$, there exists a bound $d_0(X)$ such that X imposes independent conditions on forms of degree $d \geqslant d_0(X)$. The point here is that we obtain an explicit bound that depends only on the multiplicity of the fat line and is independent of the number of reduced lines.

We will set up the proof in a way which employs the general strategy of Hartshorne and Hirschowitz [10] and Carlini, Catalisano and Geramita [7]. This amounts to work inductively by constructing a suitable sequence of generic subschemes $Z_0, Z_1, \ldots$, along with suitable specializations Z_i' of Z_i. The starting scheme Z_0 consists of the lines in the theorem plus a number of generic points. The essential difficulty in this strategy lies in the choice of the intermediate schemes Z_i and their specializations Z_i'. In our approach this is achieved by using intermediate schemes that contain, apart from disjoint lines and points, also crosses and so-called zig-zags (see Def. 2.4).

2 Preliminaries and auxiliary results

We begin by recalling a formula for the number of conditions, $c(n, m, d)$, imposing the vanishing of a form of degree $d \geqslant m$ to the order m along a line in $\mathbb{P}^n$:

$$c(n, m, d) = \frac{m(nd + 2n + m - mn - 1)}{n(n-1)} \binom{n+m-2}{m}. \tag{2}$$

For a proof see e.g. [8, Lemma 2.1]. Note that

$$c(n, 1, d) = d + 1$$

for all $n \geqslant 1$.

In the next Lemma we present a useful formula relating some of numbers $c(n, m, d)$.

Lemma 2.1. *For all positive integers n, m, d we have*

$$c(n, m, d) = c(n, m-1, d-1) + c(n-1, m, d).$$

Proof. This is a straightforward computation. □

In $\mathbb{P}^3$ the formula (2) reduces to

$$c(d,m) = c(3,m,d) = \frac{1}{6}m(m+1)(3d+5-2m).$$

Our approach to the Main Theorem uses the specialization method. This employs the semicontinuity Theorem [9, Theorem III.12.8] in the following way:

Let $f : X \to B$ be a projective morphism of noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X, flat over B. The vanishing $h^0(X_b, \mathcal{F}_b) = 0$ for some b implies then the vanishing $h^0(X_{b'}, \mathcal{F}_{b'}) = 0$ for all b' in a neighborhood of b.

In our situation, with $B = \mathbb{K}$ this means concretely that if $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_{Z_b}) = 0$ for a (special) subscheme Z_b, then $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_{Z_{b'}}) = 0$ for a (general) subscheme $Z_{b'}$ such that Z_b and $Z_{b'}$ vary in a flat family over $\mathbb{K}$.

We are going to use and generalize the notion of *sundials* following the ideas of Carlini, Catalisano and Geramita, see [6, Sections 2,3] for definitions and motivations.

Definition 2.2 (Sundials and crosses)**.** A *sundial* in $\mathbb{P}^n$ is the limiting subscheme obtained by a collision of two skew lines (hence spanning a $\mathbb{P}^3 \subset \mathbb{P}^n$) moving in a one-dimensional family. It has a nonreduced structure in the collision point which can be thought of as a vector generating the $\mathbb{P}^3$ mentioned above together with the plane spanned by the two intersecting lines.

A union of two lines in $\mathbb{P}^n$ intersecting in a single point is called a *cross*. A cross is hence a sundial with the reduced structure.

Carlini, Catalisano and Geramita proved in [5, Lemma 2.5] that there exists a flat family $g : W \to \mathbb{K}$ of schemes in $\mathbb{P}^n$, with $n \geqslant 3$ such that a general member $W_t \subset W$ is a union of two disjoint lines, whereas the special fiber W_0 is a sundial. As this property is central to our argument, we recall briefly how W can be obtained. It is enough to consider it in the projective space of dimension 3.

Construction 2.3. In $\mathbb{P}^3$ with homogeneous coordinates $(x : y : z : w)$, let M be the line defined by the ideal $I(M) = \langle z, w\rangle$ and let L_t be lines given by the ideals $I(L_t) = \langle y, z - tx\rangle$. For $t \neq 0$ let $W_t = M \cup L_t$. Intersecting the defining ideals we get $J_t = \langle yw, txw - zw, yz, txz - z^2\rangle$ as the ideal of W_t. Thus the limiting subscheme W_0 is given by the ideal $J_0 = \langle yw, zw, yz, z^2\rangle$. The primary decomposition of J_0 is

$$J_0 \ = \ \langle z, w\rangle \cap \langle y, z\rangle \cap \langle y, z^2, w\rangle,$$

so that the scheme structure at the intersection point $P = (1 : 0 : 0 : 0)$ of the lines M and L_0 is non-reduced.

It is easy to see that J_0 can be also written down as

$$J_0 \ = \ I(M) \cap I(L_0) \cap I(P)^2,$$

which is in accordance with [6, Definition 3.7].

It is a crucial point in our proof of the Main Theorem to use the following generalization of the sundial idea.

Definition 2.4 (Zig-zag)**.** A *zig-zag* of length z is the limiting subscheme obtained by a collision of an ordered set of z general lines $L_1, L_2, \ldots, L_z$ in the following way:

- the line L_2 intersects L_1;
- the line L_3 intersects L_2 in a point different from $L_2 \cap L_1$;
- and so on, i.e., L_{i+1} intersects L_i in a point different from $L_i \cap L_{i-1}$, for $i \leqslant z-1$.

We assume that there are no other intersection points among the lines but those listed above. A zig-zag of length z has thus $(z-1)$ singular points.

A *reduced zig-zag* is a zig-zag with reduced structure, i.e., it has no embedded points.

Thus a sundial is just a zig-zag of length 2 and a cross is a reduced zig-zag of length 2.

Figure 1 shows a zig-zag of length 7. Note that the lines in the figure are all skew but the ones indicated in the figure, which means there are no other intersection points but those indicated in this figure. The intersection points are embedded points as explained in Construction 2.3.

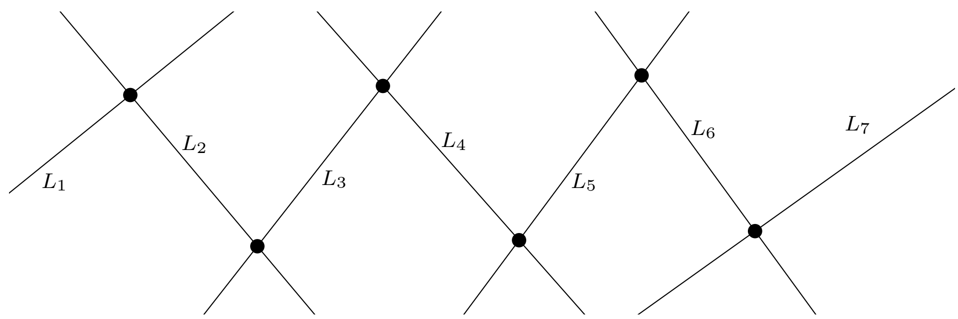

Figure 1: A zig-zag of length 7

Lemma 2.5. *For an integer $z \geqslant 2$, there exists a flat family $\{X_\lambda\}_{\lambda \in \mathbb{K}}$ of schemes in $\mathbb{P}^n$, with $n \geqslant 3$ such that a general member of $\{X_\lambda\}_{\lambda \in \mathbb{K}}$ is a union of z disjoint lines and the special fiber is a zig-zag of length z.*

Proof. The proof is a straightforward generalization of the argument in [5, Lemma 2.5]. □

Zig-zags are useful in our approach because of the following fact.

Lemma 2.6. *Let S be a zig-zag of length z in $\mathbb{P}^3$ formed by lines $L_1, \ldots, L_z$. Let Q be a smooth quadric in $\mathbb{P}^3$ such that all singular points of S lie on Q but none of the lines in S is contained in Q. Then the colon ideal*

$$J = I_S : I_Q$$

defines the reduced zig-zag $V(J) = L_1 \cup \ldots \cup L_z$.

Proof. Again straightforward. □

Apart from the semicontinuity, the residual exact sequence and Castelnuovo's inequality are key ingredients in the proof. We discuss them now.

Definition 2.7 (Trace and residual scheme)**.** Let Y be a divisor of degree e in $\mathbb{P}^n$ and let $Z \subset \mathbb{P}^n$ be a closed subscheme. Then the subscheme $Z'' = \mathrm{Tr}_Y(Z)$ defined in Y by the ideal

$$I_{Z'',Y} = (I_Y + I_Z)\,/I_Y \subset \mathcal{O}_Y$$

is the *trace of* Z *on* Y.

The colon ideal $I_{Z'} = (I_Z : I_Y) \subset \mathcal{O}_{\mathbb{P}^n}$ defines $Z' = \mathrm{Res}_Y(Z)$, the *residual scheme of* Z *with respect to* Y.

One has the following *residual exact sequence*

$$0 \longrightarrow \mathcal{I}_{Z'}(-Y) \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{I}_{Z'',Y} \longrightarrow 0\,, \tag{3}$$

where $\mathcal{I}_W$ is the sheafification of the ideal I_W. Twisting (3) by $\mathcal{O}_{\mathbb{P}^n}(d)$ we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d-e) \otimes \mathcal{I}_{\mathrm{Res}_Y(Z)} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z \longrightarrow \mathcal{O}_Y(d) \otimes \mathcal{I}_{\mathrm{Tr}_Y(Z)} \longrightarrow 0. \tag{4}$$

Taking then the long cohomology sequence of (4) we obtain the following statement, which is called Castelnuovo's inequality, see e.g. [6, Lemma 3.3].

Lemma 2.8 (Castelnuovo's inequality)**.** *Let $Y \subset \mathbb{P}^n$ be a divisor of degree e and let $d \geqslant e$ be an integer. Let $Z \subset \mathbb{P}^n$ be a closed subscheme. Then*

$$h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z) \leqslant h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-e) \otimes \mathcal{I}_{\mathrm{Res}_Y(Z)}) + h^0(Y, \mathcal{O}_Y(d) \otimes \mathcal{I}_{\mathrm{Tr}_Y(Z)}). \tag{5}$$

We call the space $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-e) \otimes \mathcal{I}_{\mathrm{Res}_Y(Z)})$ the *residual linear system* of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z)$ with respect to Y and $H^0(Y, \mathcal{O}_Y(d) \otimes \mathcal{I}_{\mathrm{Tr}_Y(Z)})$ the *trace linear system* of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{I}_Z)$ on Y.

3 Nonspeciality of certain linear series on $\mathbb{P}^1 \times \mathbb{P}^1$

In the proof of the Main Theorem we will consider trace linear systems on a smooth quadric in $\mathbb{P}^3$. This section serves as a preparation of relevant results on linear systems on a smooth quadric in $\mathbb{P}^3$ identified with $\mathbb{P}^1 \times \mathbb{P}^1$. Special linear systems with general points of multiplicity at most 3 on $\mathbb{P}^1 \times \mathbb{P}^1$ have been classified by Lenarcik in [12]. Here we recall a part of [12, Theorem 2] relevant in our situation.

Lemma 3.1. *Let Z be the fat point scheme in $\mathbb{P}^1 \times \mathbb{P}^1$ defined by the ideal*

$$I_Z = I(P_1) \cap \ldots \cap I(P_p) \cap I(Q_1)^2 \cap \ldots \cap I(Q_q)^2,$$

where $P_1, \ldots, P_p, Q_1, \ldots, Q_q$ are general points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $0 \leqslant a \leqslant b$ be non-negative integers. The linear system

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b) \otimes \mathcal{I}_Z)$$

is special if and only if one of the following cases holds

- *$a = 0$, $p + 2q \leqslant b$ and $q \geqslant 1$,*
- *$a = 2$, $p = 0$, $b = q - 1$ and q is odd.*

Using this result, we prove now an auxiliary postulation statement for higher multiplicities:

Lemma 3.2. *Given an integer $m \geqslant 2$ let k be an integer with $k \geqslant \binom{m+1}{3}$. Then a scheme composed by 2 general points P_1, P_2 taken with multiplicity m imposes independent conditions on linear systems on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (a,b) if $a \leqslant b$, $a \geqslant k-1$, $a > 0$ and $b \geqslant 3k$.*

Proof. For $m = 2$ the assertion for arbitrary $k \geqslant \binom{m+1}{3}$ follows from Lemma 3.1. Indeed, the case $a = 0$ is excluded by our assumption and the case $a = 2$ is also excluded, since in our situation $q = 2$.

We proceed by induction on m and k. Let $m > 2$ and $k \geqslant \binom{m+1}{3}$ be fixed and assume that the assertion holds for all $m' < m$ and all $k' \geqslant \binom{m'+1}{3}$. Let $s = (a+1)(b+1) - 2\binom{m+1}{2}$ and let $Q_1, \ldots, Q_s$ be s general points on $\mathbb{P}^1 \times \mathbb{P}^1$. It is enough to show that there is no divisor of bidegree (a,b) which passes through $Q_1, \ldots, Q_s$ and additionally through the points P_1, P_2 with multiplicity m. It suffices to prove this claim for a particular position of points $Q_1, \ldots, Q_s$.

To this end let C be a smooth curve of bidegree $(1,1)$ passing through P_1 and P_2. Thus C is a smooth rational curve. Let $t = a+b-2m+1$. By above assumptions this is a non-negative integer. We specialize now the points $Q_1, \ldots, Q_t$ onto the curve C leaving the points $Q_{t+1}, \ldots, Q_s$ as general points on $\mathbb{P}^1 \times \mathbb{P}^1$, hence not on C. Assume to the contrary that there is a divisor Γ such that $\operatorname{mult}_{P_i} \Gamma \geqslant m$ for $i = 1,2$ and $\operatorname{mult}_{Q_j} \Gamma \geqslant 1$ for $j = 1, \ldots, s$. Then C must be a component of Γ, because $(\Gamma \cdot C) = a + b$. But the trace of Γ on C has at least 2 points of multiplicity m and other t reduced points with $2m + t = a + b + 1$. The residual divisor $\Gamma' = \Gamma - C$ has bidegree $(a-1, b-1)$, passes through the points P_1 and P_2 with multiplicity $m-1$ and also passes through the points $Q_{t+1}, \ldots, Q_s$. Since $s - t = ab - 2\binom{m}{2}$, the existence of Γ' is excluded by our induction assumption. This concludes the proof. □

4 The proof of the Main Theorem

In this section we will prove the Main Theorem, which is equivalent to the following statement.

Theorem 4.1 (Maximal rank property)**.** *For a subscheme $X \subset \mathbb{P}^3$ consisting of a general line of multiplicity m and an arbitrary number r of general lines, the restriction map*

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(X, \mathcal{O}_X(d))$$

has maximal rank for all $d \geqslant d_0(m) = 3\binom{m+1}{3}$.

As pointed out in the introduction, we will employ the general strategy of Hartshorne and Hirschowitz [10, Theorem 1.1]. Specifically, we will proceed inductively along a suitable sequence of subschemes $Z_0, Z_1, \ldots$, for which we choose suitable specializations $Z_0', Z_1', \ldots$. Subschemes consisting of general lines, a fat line and additional points are unfortunately too simple for the specialization process. Our idea is to allow instead some intermediate schemes $Z = Z(m,r,s,q,z)$ consisting of one general line of multiplicity m, r general lines, s general crosses, q general points and a reduced zig-zag of length z (along with particular specializations Z' of Z, which will be introduced in Definition 4.4).

We now set up some notation that will be useful for the remainder of the paper. We denote by

$$\mathcal{L}(k, \varepsilon; m, r, s, q, z) = \mathcal{L}(d; Z) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d) \otimes \mathcal{I}_Z)$$

the linear system of polynomials in $\mathbb{P}^3$ of degree $d = 3k + \varepsilon$, with $\varepsilon \in \{0, 1, 2\}$ vanishing along the subscheme Z. One defines the *virtual dimension* of $\mathcal{L}(d; Z)$ to be:

$$\mathrm{vdim}(\mathcal{L}(d; Z)) = \binom{d+3}{3} - c(d, m) - r(d + 1) - s(2d + 1) - q - (zd + 1).$$

Similarly we will write

$$\Lambda((a, b); p, p_d, p_m, m) = \Lambda((a, b); \Omega) = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a, b) \otimes \mathcal{I}_\Omega)$$

to indicate the linear system on $\mathbb{P}^1 \times \mathbb{P}^1$ of polynomials of bidegree (a, b) vanishing along the subscheme $\Omega = \Omega(p, p_d, p_m, m)$ consisting of p general points, p_d general double points and p_m general points of multiplicity m. In our considerations p_m is either 0 or 2, depending on whether we specialize the fat line onto the quadric or not. We define the *virtual dimension* of $\Lambda((a, b); \Omega)$ as

$$\mathrm{vdim}(\Lambda((a, b); \Omega)) = (a + 1)(b + 1) - p - 3p_d - \binom{m+1}{2} p_m.$$

Given $m \geqslant 1$ and $d \geqslant d_0(m) = 3\binom{m+1}{3}$ there exist unique integers $r(d, m) \geqslant 0$ and $0 \leqslant q(d, m) \leqslant d$ such that

$$\mathrm{HP}_{\mathbb{P}^3}(d) = c(d, m) + r(d, m)(d + 1) + q(d, m). \tag{6}$$

So $\mathrm{HP}_{\mathbb{P}^3}(d)$ is the virtual number of conditions that one m-fold line, $r(d, m)$ generic ordinary lines, and $q(d, m)$ generic points impose.

Remark 4.2. Concretely, we have

$$r(d, m) = \left\lfloor \frac{1}{d+1} \left(\binom{d+3}{3} - \frac{1}{6} m(m + 1)(3d + 5 - 2m) \right) \right\rfloor$$

and

$$q(d, m) = \binom{d+3}{3} - \frac{1}{6} m(m + 1)(3d + 5 - 2m) - (d + 1)r(d, m).$$

In particular,

- for $d = 3k$
$$r(d, m) = \frac{3}{2}k^2 + \frac{5}{2}k + 1 - \binom{m+1}{2} \quad \text{and} \quad q(d, m) = 2\binom{m+1}{3},$$
- for $d = 3k + 1$
$$r(d, m) = \frac{3}{2}k^2 + \frac{7}{2}k + 2 - \binom{m+1}{2} \quad \text{and} \quad q(d, m) = 2\binom{m+1}{3},$$
- for $d = 3k + 2$
$$r(d, m) = \frac{3}{2}k^2 + \frac{9}{2}k + 3 - \binom{m+1}{2} \quad \text{and} \quad q(d, m) = k + 1 + 2\binom{m+1}{3}.$$

The following theorem (to be proved in Subsection 4.1) implies the Main Theorem.

Theorem 4.3. *Let $d \geqslant d_0(m) = 3\binom{m+1}{3}$ and let $Z = Z(m, r(d,m), 0, q(d,m), 0)$, or $Z = Z(m, r(d,m)+1, 0, 0, 0)$. Let further Q be some smooth quadric. Then there exists a sequence $Z = Z_0, Z_1, \ldots, Z_u$ of schemes $Z_i = Z(m_i, r_i, s_i, q_i, z_i)$ together with specializations Z'_i such that the following statements hold for each $i = 0, \ldots, u-1$*

(1) $Z_{i+1} = \mathrm{Res}_Q(Z'_i)$;

(2) $h^0(Q, \mathcal{O}_Q(d-2i) \otimes I_{\mathrm{Tr}_Q(Z'_i)}) = 0$,

and such that Z_u satisfies the conditions

(i) $Z_u = Z(m_u, r(d-2u, m_u), 0, q(d-2u, m_u), 0)$, *or*
$Z_u = Z(m_u, r(d-2u, m_u)+1, 0, 0, 0)$;

(ii) $d - 2u \geqslant d_0(m_u)$;

(iii) $m_u \in \{m-1, m-2, 1, 0\}$.

Proof of Theorem 4.1. We proceed by induction on m. The base case $m = 1$ has been proved for all $d \geqslant 0 = d_0(1)$ in [10] and the base case $m = 2$ by Aladpoosh [1] for all $d \geqslant 3 = d_0(2)$.

Consider now $m \geqslant 3$. For $d \geqslant d_0(m) \geqslant 4$ it suffices to prove the bijectivity of the restriction map in the case of schemes $Z = Z(m, r(d,m), 0, q(d,m), 0)$, and the injectivity in the case of schemes $Z = Z(m, r(d,m)+1, 0, 0, 0)$. This amounts in either case to proving the identity

$$\dim(\mathcal{L}(d; Z)) = 0\,.$$

Theorem 4.3 together with Castelnuovo's inequality yields

$$\begin{aligned}\dim(\mathcal{L}(d;Z)) &\leqslant \dim(\mathcal{L}(d-2u; Z_u)) + \sum_{i=0}^{u-1} h^0(Q, \mathcal{O}_Q(d-2i) \otimes I_{\mathrm{Tr}_Q(Z'_i)}) \\ &= \dim(\mathcal{L}(d-2u; Z_u))\,.\end{aligned}$$

The latter must be zero since Z_u satisfies the induction hypothesis. □

4.1 Proof of Theorem 4.3

In order to prove Theorem 4.3, we will need the next lemma describing which schemes result from certain specializations.

Definition 4.4. Let Q be a smooth quadric in $\mathbb{P}^3$. We denote by $R(\delta, \ell, \ell_s, \ell_z, t, t_s, t_z)$ the specialization Z' of $Z = Z(m, r, s, q, z)$ given by assuming the following lines to be disjoint lines belonging to the same ruling of Q:

- δ m-fold lines (here δ will be either 0 or 1);
- ℓ ordinary lines;
- ℓ_s lines from ℓ_s crosses (one line from each cross);
- $\ell_z = \lfloor \frac{z}{2} \rfloor$ lines from the reduced zig-zag of length z,

and assuming furthermore

- t among the q points to be general points on Q,
- $2t_s$ of the r lines to form t_s sundials whose intersection with Q is a zero-dimensional scheme containing the singular points of the sundials,
- t_z+1 of the lines to form one zig-zag whose zero-dimensional intersection with Q contains all t_z singular points.

Lemma 4.5. *Let Z' be the specialization $R(\delta,\ell,\ell_s,\ell_z,t,t_s,t_z)$ of the scheme $Z = Z(m,r,s,q,z)$. Then*

$$\mathrm{Res}_Q(Z') = Z(m-\delta, r-\ell+\ell_s+(z-\ell_z)-2t_s-(t_z+1), s-\ell_s+t_s, q-t, t_z+1)$$

and

$$\mathrm{Tr}_Q(Z') = D + \Omega(2r-2\ell-2\ell_z-3\ell_s-2t_s-2t_z+t+4s+z+\gamma, t_s+t_z, 2-2\delta, m),$$

where D is a divisor on Q consisting of δ lines of multiplicity m, with $\delta \in \{0,1\}$, and $\ell+\ell_s+\ell_z$ reduced lines, all contained in the same ruling on Q.

Here $\gamma = \begin{cases} 0, & \text{if } \ell_z = 0, \\ 1, & \text{if } \ell_z > 0. \end{cases}$

Proof. The argument amounts just in counting the various objects to which the specialization applies and which it results. We omit the simple details. □

Now we turn to the proof of Theorem 4.3. The particular sequence of subschemes differs according to the divisibility of d by 3. In order to simplify the notation we denote the relevant linear series by

$$\begin{aligned} B(k,\varepsilon,m) &= \mathcal{L}(k,\varepsilon;m,r(3k+\varepsilon,m),0,q(3k+\varepsilon,m),0); \\ I(k,\varepsilon,m) &= \mathcal{L}(k,\varepsilon;m,r(3k+\varepsilon,m)+1,0,0,0). \end{aligned}$$

As the proof is technically quite involved, we start by outlining the overall course of the argument: The proof consists in a number of cases, in each of which an initial system $\mathcal{L}_0$ will yield a residual system $\mathcal{L}_u$ through a sequence of reduction steps with intermediate systems

$$\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_u,$$

according to the plan shown in Table 1.

The intermediate systems $\mathcal{L}_i$ will be constructed as residual systems from suitably chosen schemes Z'_i as

$$\mathcal{L}_{i+1} := \mathcal{L}(d-2i-2; \mathrm{Res}_Q(Z'_i)),$$

giving on the quadric the trace systems

$$\Lambda_{i+1} := \Lambda((d-2i, d-2i); \mathrm{Tr}_Q(Z'_i)).$$

4.1.1 The bijective cases

With $d = 3k+\varepsilon$, the initial system in every case is

$$\mathcal{L}(k,\varepsilon;m,r(3k+\varepsilon,m),0,q(3k+\varepsilon,m),0).$$

The system $\mathcal{L}_0$	yields through a sequence of length u	the system $\mathcal{L}_u$
$B(k,0,m)$	1	$B(k-1,1,m-1)$
$B(k,1,m)$	2	$B(k-1,0,m-1)$
$B(k,2,m)$	1	$B(k,0,m-1)$
$I(k,0,m)$	2	$I(k-2,2,m-2)$
$I(k,1,m)$	1	$I(k-1,2,m-1)$
$I(k,2,3\ell)$	$3\ell-1$	$B(k-2\ell+1,1,1)$
$I(k,2,3\ell+1)$	$3\ell+1$	$B(k-2\ell,0,0)$
$I(k,2,3\ell+2)$	$3\ell+1$	$B(k-2\ell,0,1)$

Table 1: Initial systems and systems resulting from reduction steps

Case $B(k,0,m)$**.** We specialize only once, and we pick

$$Z_0' = R(1, 2k+1-m, 0, 0, m(m-1), 0, 0).$$

By Lemma 4.5, we obtain the trace system

$$H^0(\mathcal{O}_Q(d)\otimes I_{\mathrm{Tr}_Q(Z')}) = \Lambda((d, d-(2k+1)); 2r-2(2k+1-m)+m(m-1), 0, 0, m)$$

which is of virtual dimension

$$(3k+1)k - (2r(3k,m) - 2(2k+1-m) + m(m-1)) = (3k+1)k - (3k+1)k = 0.$$

As only reduced general points are involved, this system is non-special, so its actual dimension is also zero. This shows that condition (2) in Theorem 4.3 is fulfilled. The residual system is

$$\begin{aligned} \mathcal{L}_1 &= \mathcal{L}(k-1, 1; m-1, r(3(k-1)+1, m-1), 0, q(3(k-1)+1, m-1), 0) \\ &= B(k-1, 1, m-1) \end{aligned}$$

by Lemma 4.5. Note that the subscheme $Z_1 := \mathrm{Res}_Q(Z_0')$ satisfies conditions (i)–(iii) of Theorem 4.3.

Case $B(k,1,m)$**.** In this case we use two specializations. First set

$$Z_0' = R(1, 2k+1-m, 0, 0, m(m-1), 2k, 0),$$

resulting in the residual system

$$\mathcal{L}_1 = \mathcal{L}(k-1, 2; m-1, \frac{3}{2}k^2 - \frac{5}{2}k + 1 - \frac{1}{2}m^2 + \frac{1}{2}m, 2k, \frac{1}{3}m^3 - m^2 + \frac{2}{3}m, 0)$$

and the trace system

$$\Lambda_1 = \Lambda((k, 3k+1); 3k^2 - k + 2, 2k, 0, m),$$

which is, by Lemma 3.1, a zero-dimensional system. Then we set

$$Z_1' = R(0, 1, 2k, 0, 0, 0, 0)$$

and obtain the residual system

$$\mathcal{L}_2 = \mathcal{L}(k-1, 0; m-1, r(3(k-1), m-1), 0, q(3(k-1), m-1), 0) = B(k-1, 0, m-1),$$

and the trace system

$$\Lambda_2 = \Lambda((k-2, 3k-1); 3k^2 - 3k - m^2 + m, 0, 2, m-1).$$

Applying Lemma 3.2, we obtain $\dim(\Lambda_2) = 0$.

Case $B(k, 2, m)$**.** In this case we use the specialization

$$Z_0' = R(1, 2k + 2 - m, 0, 0, k + 1 + m(m-1), 0, 0).$$

We obtain

$$\mathcal{L}_1 = \mathcal{L}(k, 0; m-1, r(3k, m-1), 0, q(3k, m-1), 0) = B(k, 0, m-1)$$

and

$$\Lambda_1 = \Lambda((k, 3k+2); 3k^2 + 6k + 3, 0, 0, m)$$

which is of dimension 0.

4.1.2 The injective cases

With $d = 3k + \varepsilon$, the initial state in every case now is

$$\mathcal{L}_0 = \mathcal{L}(k, \varepsilon; m, r(3k+\varepsilon, m) + 1, 0, 0, 0).$$

Case $I(k, 0, m)$**.** We have $\mathcal{L}_0 = \mathcal{L}(k, 0; m, r(3k, m) + 1, 0, 0, 0)$ so that

$$\operatorname{vdim}(\mathcal{L}_0) = -3k - 1 + \frac{1}{3}m(m-1)(m+1) < 0$$

for $d = 3k \geqslant d_0(m) = 3\binom{m+1}{3}$. We apply the specialization

$$Z_0' \quad = \quad R(1, 2k + 1 - m, 0, 0, 0, 0, m(m-1) - 2).$$

By Lemma 4.5 the trace system is

$$\Lambda_1 \quad = \quad \Lambda((3k, k-1); 2(r(3k, m) + 1 - (2k + 1 - m) - m(m-1) + 2), m(m-1) - 2, 0, m).$$

It is easy to see that Λ_1 has non-positive virtual dimensions for $d \geqslant d_0(m)$, and thus the actual dimension is zero by Lemma 3.1. Hence the quadric is a component of $\mathcal{L}_0$ and the residual system is

$$\mathcal{L}_1 = \mathcal{L}(k-1, 1; m-1, \frac{3}{2}k^2 + \frac{1}{2}k + 2 - \frac{3}{2}m^2 + \frac{3}{2}m, 0, 0, m^2 - m - 1).$$

The second specialization is

$$Z_1' \quad = \quad R(1, 2k + 1 - m - (\frac{1}{2}m(m-1) - 1), 0, \frac{1}{2}m(m-1) - 1, 0, 0, 0).$$

Therefore, Lemma 4.5 implies that the trace system is now

$$\Lambda_2 \quad = \quad \Lambda((3k-2,k-2);3k^2-3k+2-2m^2+4m,0,0,m-1).$$

By Lemma 3.1 again, its dimension is zero, so that the quadric turns out to be also a fixed component of $\mathcal{L}_1$. Note that we have the identity

$$r(3k,m)+1-(2k+1-m)-(2k+1-m)=r(3(k-2)+2,m-2)+1.$$

Hence the final residual system is

$$\mathcal{L}_2=\mathcal{L}(k-2,2;m-2,r(3(k-2)+2,m-2)+1,0,0,0)=I(k-2,2,m-2).$$

Case $I(k,1,m)$**.** Here $\mathcal{L}_0=\mathcal{L}(k,1;m,r(3k+1,m)+1,0,0,0)$, which has virtual dimension

$$\mathrm{vdim}(\mathcal{L}_0)=-3k-2+\frac{1}{3}m(m-1)(m+1)<0.$$

We apply the specialization

$$Z_0'=R(1,2k+2-m,0,0,0,0,0)$$

which by the identity

$$r(3k+1,m)+1-(2k+2-m)=r(3(k-1)+2,m-1)+1$$

yields

$$\mathcal{L}_1=\mathcal{L}(k-1,2;m-1,r(3(k-1)+2,m-1)+1,0,0,0)=I(k-1,2,m-1)$$

as the residual system and

$$\Lambda_1=\Lambda((k-1,3k+1);3k^2+3k+2-m^2+m,0,0,m)$$

as the trace system. Its virtual dimension is

$$\mathrm{vdim}(\Lambda_1)=-k-2+m^2-m<0,$$

so $\dim(\Lambda_1)=0$.

Case $I(k,2,m)$**.** In this case we will employ the novel machinery introduced in Sect. 2, because specializing to lines on a conic and sundials seems not to suffice. Apart from using zig-zags this case is also considerably more complicated in that it decomposes into three subcases, each of which in turn requires a number of reduction steps. The division into subcases is determined by the divisibility of m by 3, which is a direct consequence of the chosen specializations. The particular choice of reductions is far from obvious and can in fact be considered to be the core innovation of this note.

We have $\mathcal{L}_0=\mathcal{L}(k,2;m,r(3k+2,m)+1,0,0,0)$ and

$$\mathrm{vdim}(\mathcal{L}_0)=-2k-2+\frac{1}{3}m(m-1)(m+1)<0.$$

In each case the first specialization is

$$Z_0'=R(1,2k+2-m,0,0,0,0,k+m(m-1)-1).$$

It results in

$$\mathcal{L}_1 = \mathcal{L}(k, 0; m-1, \frac{3}{2}(k^2+k-m^2+m)+2, 0, 0, k+m^2-m).$$

In subsequent steps we work, for $p = 2, \dots, m-1$, with the specializations

$$Z'_{p-1} = R(1, 2k+2-m-\left\lfloor \frac{p-1}{3} \right\rfloor - \left\lfloor \frac{t_{z_{p-1}}+1}{2} \right\rfloor, 0, \left\lfloor \frac{t_{z_{p-1}}+1}{2} \right\rfloor, 0, 0, t_{z_p}),$$

where the number t_{z_p} depends on the divisibility of p by 3:

$$t_{z_p} = \begin{cases} k + pm(m-p) + \frac{1}{3}p(p-1)(p+1) - 2p + 1 & \text{if } p \equiv 1, 2 \\ pm(m-p) + \frac{1}{3}p(p-1)(p+1) - 2p + 2\frac{p}{3} & \text{if } p \equiv 0 \end{cases} \pmod 3.$$

The definition of the numbers t_{z_p} is governed by the requirement that the trace system on Q is of the virtual (and hence by Lemma 3.1 also the actual) dimension 0.

We conclude the proof by explaining in detail the final reductions in each of the subcases.

Subcase $I(k, 2, m = 3\ell)$. In this case we consider the sequence $Z_0, Z_1, \dots, Z_{m-2}$ defined above and use as a final step $Z_{m-1} = \mathrm{Res}_Q(Z')$ for

$$Z' = R(1, 2k+2-m-(\ell-1) - \left\lfloor \frac{t_{z_{m-2}}+1}{2} \right\rfloor + 1, 0, \left\lfloor \frac{t_{z_{m-2}}+1}{2} \right\rfloor, 0, 0, 0).$$

The final residual system is

$$\mathcal{L}_{m-1} \quad = \quad \mathcal{L}(k-2\ell+1, 1; 1, r(3k+2, 3\ell) + \frac{21}{2}\ell^2 - \frac{23}{2}\ell + 3 - 6k\ell + 2k, 0, 0, 0).$$

Since

$$r(3k+2, 3\ell) + \frac{21}{2}\ell^2 - \frac{23}{2}\ell + 3 - 6k\ell + 2k = r(3(k-2\ell+1)+1, 1)$$

and $q(3(k-2\ell+1)+1, 1) = 0$ we have

$$\mathcal{L}_{m-1} \quad = \quad B(k-2\ell+1, 1, 1).$$

The final trace system is

$$\Lambda_{m-1} = \Lambda((k-2\ell, 3k-6\ell+6); 2r(3k+2, 3\ell) - 12k\ell - 16\ell + 21\ell^2 + 3k + 3 - 9\ell^3, 0, 0, 2),$$

which has virtual dimension $-2k - 2 + \frac{1}{3}m(m-1)(m+1) < 0$.

Subcase $I(k, 2, m = 3\ell+1)$. Consider the sequence $Z_0, Z_1, \dots, Z_{m-1}$ defined above and use as a final step $Z_m = \mathrm{Res}_Q(Z')$ for

$$Z' = R(1, 2k+2-m-\ell - \left\lfloor \frac{t_{z_{m-1}}+1}{2} \right\rfloor + 1, 0, \left\lfloor \frac{t_{z_{m-1}}+1}{2} \right\rfloor, 0, 0, 0).$$

The final residual system is

$$\mathcal{L}_m = \mathcal{L}(k-2\ell, 0; 0, r(3k+2, 3\ell+1) + \frac{21}{2}\ell^2 - \frac{1}{2}\ell - 1 - 6k\ell - 2k, 0, 0, 0)$$

which thanks to the identities

$$r(3k+2, 3\ell+1) + \frac{21}{2}\ell^2 - \frac{1}{2}\ell - 1 - 6k\ell - 2k = r(3(k-2\ell), 0)$$

and $q(3(k-2\ell), 0) = 0$ equals the system $B(k-2\ell, 0, 0)$, as required. The final trace system is

$$\Lambda_m = \Lambda((k-2\ell-1, 3k-6\ell+2); 2r(3k+2, 3\ell+1) - 12k\ell - 4k - 9\ell^3 + 12\ell^2 + \ell - 2, 0, 0, 1).$$

Also in this case we have

$$\operatorname{vdim}(\Lambda_m) = -2k - 2 + \frac{1}{3}m(m-1)(m+1) < 0.$$

Subcase $I(k, 2, m = 3\ell+2)$**.** Use as in the first subcase the sequence $Z_0, Z_1, \ldots, Z_{m-2}$ defined above and use as a final step $Z_{m-1} = \operatorname{Res}_Q(Z')$ for

$$Z' = \left(1, 2k+2-m-\ell - \left\lfloor \frac{t_{z_{m-2}}+1}{2} \right\rfloor + 1, 0, \left\lfloor \frac{t_{z_{m-2}}+1}{2} \right\rfloor, 0, 0, 0\right).$$

The final residual system is

$$\mathcal{L}_{m-1} = \mathcal{L}(k-2\ell, 0; 1, r(3k+2, 3\ell+2) + \frac{21}{2}\ell^2 + \frac{5}{2}\ell - 6k\ell - 2k, 0, 0, 0)$$

with

$$r(3k+2, 3\ell+2) + \frac{21}{2}\ell^2 + \frac{5}{2}\ell - 6k\ell - 2k = r(3(k-2\ell), 1)$$

and $q(3(k-2\ell), 1) = 0$, so we have $\mathcal{L}_{m-1} = B(k-2\ell, 0, 1)$. The final trace system is

$$\Lambda_{m-1} = \Lambda((k-2\ell-1, 3k-6\ell+2); 2r(3k+2, 3\ell+2) - 12k\ell - 4k - 9\ell^3 + 3\ell^2 - 2\ell, 0, 0, 2).$$

Its dimension is zero since

$$\operatorname{vdim}(\Lambda_{m-1}) = -2k - 2 + \frac{1}{3}m(m-1)(m+1) < 0.$$

□

5 Final remarks

We have developed a software to handle calculations necessary here. See [13] for a Maple core file and a file containing an explicit example explaining how to use our program. The software has proved indispensable in order to manipulate sets of data and to discover general patterns leading to suitable specializations. Using this software we were not able to find any systems in the range $d < d_0(m)$ for which the maximal rank statement in Theorem 4.1 would fail. We therefore expect that the statement holds in these cases as well:

Conjecture 5.1 (Maximal Rank Conjecture)**.** *The restriction maps in Theorem 4.1 have maximal rank for all* $d \geqslant 1$.

We hope that with some modifications, the software mentioned above might prove useful in similar situations, in particular might help to advance towards the proof of Aladpoosh's Conjecture. We also expect that our results can be generalized to projective spaces of arbitrary dimension. This is a subject of ongoing research.

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