IDEALS OF THE FORM $I_1(XY)$

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ABSTRACT. In this paper we compute Gröbner bases for determinantal ideals of the form $I_1(XY)$, where X and Y are both matrices whose entries are indeterminates over a field K. We use the Gröbner basis structure to determine Betti numbers for such ideals.

1. INTRODUCTION

Let K be a field and $\{x_{ij};\, 1 \leq i \leq m,\, 1 \leq j \leq n\}$, $\{y_j;\, 1 \leq j \leq n\}$ be indeterminates over K. Let $K[x_{ij}]$ and $K[x_{ij}, y_j]$ denote the polynomial algebras over K. Let X denote an $m \times n$ matrix such that its entries belong to the ideal $\langle\{x_{ij};\, 1 \leq i \leq m,\, 1 \leq j \leq n\}\rangle$. Let $Y = (y_j)_{n\times 1}$ be the generic $n \times 1$ column matrix. Let $I_1(XY)$ denote the ideal generated by the 1×1 minors or the entries of the $m \times 1$ matrix XY. Ideals of the form $I_1(XY)$ appeared in the work of J. Herzog [9] in 1974. These ideals are closely related to the notion of Buchsbaum-Eisenbud variety of complexes. A characteristic free study of these varieties can be found in [5], where the defining equations of these varieties have been described as minors of matrices using combinatorial structure of multitableux. It has also been proved that the varieties are Cohen-Macaulay and Normal. The ideal $I_1(XY)$ is a special case of the defining ideal of a variety of complexes, when $n_0 = m$, $n_1 = n$, $n_2 = 1$, in the notation of [5]. These ideals feature once again in [18], in the study of the structure of a *universal ring* of a *universal pair* defined by Hochster. It has been proved in [18] that the set of standard monomials form a free basis for the universal ring. The initial ideal of the defining ideal is given by the set of all nonstandard monomials, which form a monomial ideal. A combination of Gröbner basis techniques and representation theory techniques yield the results in [18]. We were not aware of this work when we computed a Gröbner basis for the ideal $I_1(XY)$ using very elementary techniques. Our technique uses nothing more than the Buchberger's criterion and the description of Gröbner bases for the ideals of minors of matrices from [4] and [17].

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Given determinantal ideals I and J, the sum ideal $I + J$ is often difficult to understand and they appear in various contexts. Ideals $I_1(XY) + J$ are special in the sense that they occur in several geometric considerations like linkage and generic residual intersection of polynomial ideals, especially in the context of syzygies; see [14], [1], [3], [2], [13]. Some important classes of ideals in this category are the Northcott ideals, the Herzog ideals; see Definition 3.4 in [1] and the deviation two Gorenstein ideals defined in [10]. Northcott ideals were resolved by Northcott in [14]. Herzog gave a resolution of a special case of the Herzog ideals in [9]. These results were extended in [3]. In a similar vein, Bruns-Kustin-Miller [2] resolved the ideal $I_1(XY) + I_{\min(m,n)}(X)$, where X is a generic $m \times n$ matrix and Y is a generic $n \times 1$ matrix. Johnson-McLoud [13] proved certain properties for the ideals of the form $I_1(XY) + I_2(X)$, where X is a generic symmetric matrix and Y is either generic or generic alternating. One of the recent articles is [11] which shows connection of ideals of this form with the ideal of the dual of the quotient bundle on the Grassmannian $G(2, n)$.

Ideals of the form $I + J$ also appear naturally in the study of some natural class of curves; see [8]. While computing Betti numbers for such ideals, a useful technique is often the iterated Mapping Cone. This technique requires a good understanding of successive colon ideals between I and J, which is often difficult to compute. It is helpful if Gröbner bases for I and J are known.

In this paper our aim is to produce some suitable Gröbner bases for ideals of the form $I_1(XY)$, when Y is a generic column matrix and X is one of the following:

(1) X is a generic square matrix;
(2) X is a generic symmetric matrix;
(3) X is a generic $(n + 1) \times n$ matrix.

We have also studied $I_1(XY)$, when

(4) X is an $(m \times mn)$ generic matrix and Y is an $(mn \times n)$ generic matrix.

Our method is constructive and it would exhibit that the first two cases behave similarly. Newly constructed Gröbner bases will be used to compute the Betti numbers of $I_1(XY)$. We will see that computing Betti numbers for $I_1(XY)$ in the first two cases is not difficult, while the last two cases are not so straightforward. We will use some results from [15] and [16] which have some more deep consequences of the Gröbner basis computation carried out in this paper.

2. DEFINING THE PROBLEMS

Let K be a field and $\{x_{ij};\, 1 \leq i \leq n+1,\, 1 \leq j \leq n\}$, $\{y_j;\, 1 \leq j \leq n\}$ be indeterminates over K. Let $R = K[x_{ij}, y_j \mid 1 \leq i, j \leq n]$, $\widehat{R} = K[x_{ij}, y_j \mid 1 \leq i \leq n+1,\, 1 \leq j \leq n]$ denote polynomial K-algebras. Let $X = (x_{ij})_{n\times n}$, such that X is either generic or generic symmetric. Let $\widehat{X} = (x_{ij})_{(n+1)\times n}$ and $Y = (y_j)_{n\times 1}$ be generic matrices. We define $\mathcal{I} = I_1(XY)$ and $\mathcal{J} = I_1(\widehat{X}Y)$.

Let $g_i = \sum_{j=1}^{n} x_{ij}y_j$, for $1 \leq i \leq n$. Then, $\mathcal{I} = \langle g_1, \ldots, g_n\rangle$. Let us choose the lexicographic monomial order on R given by

(1) $x_{11} > x_{22} > \cdots > x_{nn}$;
(2) $x_{ij}, y_j < x_{nn}$ for every $1 \leq i \neq j \leq n$.

It is an interesting observation that the set $\{g_1, \ldots, g_n\}$ is a Gröbner basis for $\mathcal{I}$ with respect to the above monomial order and the elements $g_1, \ldots, g_n$ form a regular sequence as well; see Lemma 4.3 and Theorem 6.1. However, this Gröbner basis is too small in size to be of much help in applications like computing primary decomposition of $I_1(XY)$ or computing Betti numbers of ideals of the form $I_1(XY) + J$, carried out in [15] and [16] respectively. This motivated us to look for a a different Gröbner basis for $\mathcal{I}$; see Theorem 4.1. This construction gives rise to a bigger picture and naturally generalizes to a Gröbner basis for the ideal $\mathcal{J} = I_1(\widehat{X}Y)$. As an application, we compute the Betti numbers for the ideals $\mathcal{I}$ and $\mathcal{J}$; see section 6.

3. NOTATION

(i) $C_k := \{\mathbf{a} = (a_1, \cdots, a_k) \mid 1 \leq a_1 < \cdots < a_k \leq n\}$; denotes the collection of all ordered k-tuples from $\{1, \cdots, n\}$. In case of $\mathcal{J} = I_1(\widehat{X}Y)$, the set C_k would denote the collection of all ordered k-tuples $(a_1, \cdots, a_k)$ from $\{1, \cdots, n+1\}$.
(ii) Given $\mathbf{a} = (a_1, \ldots, a_k) \in C_k$;
- $X^{\mathbf{a}} = [a_1, \cdots, a_k|1, 2, \ldots, k]$ denotes the $k \times k$ minor of the matrix X, with $a_1, \ldots, a_k$ as rows and $1, \ldots, k$ as columns. Similarly, $\widehat{X}^{\mathbf{a}} = [a_1, \cdots, a_k|1, \ldots, k]$ denotes the $k \times k$ minor of the matrix $\widehat{X}$, with $a_1, \ldots, a_k$ as rows and $1, \ldots, k$ as columns.
- $S_k := \{X^{\mathbf{a}} : \mathbf{a} \in C_k\}$ and I_k denotes the ideal generated by S_k in the polynomial ring R (respectively $\widehat{R}$);
- $X^{\mathbf{a},m} := [a_1, \cdots, a_k|1, \cdots, k-1, m]$ if $m \geq k$;
- $\widetilde{X^{\mathbf{a}}} = \sum_{m\geq k}[a_1, \cdots, a_k|1, \cdots, k-1, m]y_m = \sum_{m\geq k} X^{\mathbf{a},m}y_m$;

- $\widetilde{S}_k := \{\widetilde{X^{\mathbf{a}}} : X^{\mathbf{a}} \in S_k\}$ and $\widetilde{I}_k$ denotes the ideal generated by $\widetilde{S}_k$ in the polynomial ring R (respectively $\widehat{R}$);
- $G_k = \cup_{i \geq k} \widetilde{S}_i$;
- $G = \cup_{k \geq 1} G_k$;
- $X_r^{\mathbf{a}} := [a_1, a_2, \cdots, \hat{a_r}, a_{r+1} \cdots, a_k | 1, 2, \cdots, k-1]$, if $k \geq 2$.

(iii) Suppose that $C_k = \left\{\mathbf{a}_1 < \ldots < \mathbf{a}_{\binom{n}{k}}\right\}$, where $<$ is the lexicographic ordering. Given $m \geq k$, the map

$$\sigma_m : \left\{X^{\mathbf{a}_1,m}, \ldots, X^{\mathbf{a}_{\binom{n}{k}},m}\right\} \to \left\{1, \cdots, \binom{n}{k}\right\}$$

is defined by $\sigma_m(X^{\mathbf{a}_i,m}) = i$. This is a bijective map. The map σ_k will be denoted by σ, which is the bijection from S_k to $\{1, \cdots, \binom{n}{k}\}$ given by $\sigma(X^{\mathbf{a}_i}) = \sigma_k(X^{\mathbf{a}_i,k}) = i$.

4. GRÖBNER BASIS FOR $\mathcal{I}$

We first construct a Gröbner basis for the ideal $\mathcal{I}$. A similar computation works for computing a Gröbner basis for the ideal $\mathcal{J}$, which will be discussed in the next section. Our aim in this section is to prove

Theorem 4.1. *The set G_k is a reduced Gröbner Basis for the ideal $\widetilde{I}_k$, with respect to the lexicographic monomial order induced by the following order on the variables: $y_1 > y_2 > \cdots > y_n > x_{ij}$ for all i, j, such that $x_{ij} > x_{i'j'}$ if $i < i'$ or if $i = i'$ and $j < j'$. In particular, $\mathcal{G} = G_1$ is a reduced Gröbner Basis for the ideal $\widetilde{I}_1 = \mathcal{I}$.*

We first write down the main steps involved in the proof. Let $\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}} \in G_k = \cup_{i \geq k} \widetilde{S}_i$. Then, either $X^{\mathbf{a}}, X^{\mathbf{b}} \in S_k$ or $X^{\mathbf{a}} \in S_k$, $X^{\mathbf{b}} \in S_{k'}$, for $k' > k$. Our aim is to show that $S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) \to_{G_k} 0$ and use Buchberger's criterion.

(A) By Lemma 4.2, we have $S(X^{\mathbf{a}}, X^{\mathbf{b}}) \longrightarrow_{S_k} 0$. We write $m_{\mathbf{a}} X^{\mathbf{a}} + m_{\mathbf{b}} X^{\mathbf{b}} = S(X^{\mathbf{a}}, X^{\mathbf{b}}) = \sum_{t=1}^{\binom{n}{k}} \alpha_t X^{\mathbf{a}_t} \longrightarrow_{S_k} 0$, such that $X^{\mathbf{a}_i} = X^{\mathbf{a}}$ and $X^{\mathbf{a}_j} = X^{\mathbf{b}}$, for some i and j. Therefore, by Schreyer's theorem the tuples $(\alpha_1, \ldots, \alpha_i - m_{\mathbf{a}}, \ldots, \alpha_j - m_{\mathbf{b}}, \ldots, \alpha_r)$ generate $\mathrm{Syz}(I_k)$.

(B) $\mathrm{Syz}(I_k)$ is precisely known by [6].

(C) $S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) \longrightarrow_{\widetilde{S}_k} S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) - \sum_{t=1}^{\binom{n}{k}} \alpha_t \widetilde{X}^{\mathbf{a}_t}$ by Lemma 4.8, if $X^{\mathbf{a}}, X^{\mathbf{b}} \in S_k$ and by Lemma 4.10, if $X^{\mathbf{a}} \in S_k$, $X^{\mathbf{b}} \in S_{k'}$, for $k' > k$.

(D) $S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) - \sum_{t=1}^{\binom{n}{k}} \alpha_t \widetilde{X}^{\mathbf{a}_t} = s \in \widetilde{I}_{k+1}$, by Lemma 4.8, if $X^{\mathbf{a}}, X^{\mathbf{b}} \in S_k$.

(E) $S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) - \sum_{t=1}^{\binom{n}{k}} \alpha_t \widetilde{X}^{\mathbf{a}_t} = s \in \widetilde{I}_{k'+1}$, by Lemma 4.10, if $X^{\mathbf{a}} \in S_k$, $X^{\mathbf{b}} \in S_{k'}$, for $k' > k$.
(F) $s \longrightarrow_{G_k} 0$, proved in Theorem 4.1 for both the cases.

We first prove a number of Lemmas to complete the proof through the steps mentioned above.

Lemma 4.2. *The set S_k forms a Gröbner basis of I_k with respect to the chosen monomial order on R.*

Proof. We use Buchberger's criterion for the proof. Let $\mathbf{c}, \mathbf{d} \in S_k$. Suppose that $S(X^{\mathbf{c}}, X^{\mathbf{d}}) \xrightarrow{S_k} r$. Then, $S(X^{\mathbf{c}}, X^{\mathbf{d}}) - \sum_{\mathbf{a_i} \in C_i} h_i X^{\mathbf{a_i}} = r$.

If X is generic (respectively generic symmetric), we know by [17] (respectively by [4]) that the set of all $k \times k$ minors of the matrix X forms a Gröbner basis for the ideal $I_k(X)$, with respect to the chosen monomial order. Therefore, there exists $[a_1, a_2, \cdots, a_k \mid b_1, b_2, \cdots, b_k]$, such that its leading term $\prod_{i=1}^{k} x_{a_i b_i}$ divides Lt(r). We see that if $b_k = k$, the minor belongs to the set S_k and we are done.

Let us now consider the case $b_k \geq k+1$. Let X be generic symmetric. Then, $a_k = k$ and $b_k \geq k+1$ imply that the minor belongs to the set S_k. If $a_k, b_k \geq k+1$, then $x_{a_k b_k} \mid$ Lt(r) but $x_{a_k b_k}$ doesn't divide any term of elements in S_k. Let X be generic. Then, for any a_k and under the condition $b_k \geq k+1$, then $x_{a_k b_k} \mid$ Lt(r) but $x_{a_k b_k}$ doesn't divide any term of elements in S_k. □

Lemma 4.3. *Let $h_1, h_2 \cdots, h_n \in R$ be such that with respect to a suitable monomial order on R, the leading terms of them are pairwise coprime. Then, $h_1, h_2 \cdots, h_n$ is a Gröbner basis of the ideal generated by $h_1, h_2 \cdots, h_n$ with respect to the same monomial order and they form a regular sequence in R.*

Proof. . The proof is a routine application of the division algorithm. □

Lemma 4.4. *Let $1 \leq k \leq n$. The height of the ideal I_k is $n-k+1$, in case of X.*

Proof. . Let us consider the case for X. We know that $ht(I_k) \leq n-k+1$. It suffices to find a regular sequence of that length in the ideal I_k. We claim that $\{[1 \cdots k | 1 \cdots k], [2 \cdots k+1 | 1 \cdots k], \ldots, [n-k+1 \cdots n | 1 \cdots k]\}$ forms a regular sequence. The leading term of $[a_1, a_2, \cdots, a_k \mid b_1, b_2, \cdots, b_k]$ with respect to the chosen monomial order is $\prod_{i=1}^{k} x_{a_i b_i}$. Therefore, leading terms of the above minors are mutually coprime and we are done by Lemma 4.3. □

Remark 4.5. We now assume that $X = (x_{ij})$ is a generic $n \times n$ matrix. The proof for the symmetric case is exactly the same.

Description of generators of $\mathrm{Syz}(I_k)$**.** By Lemma 4.4 we conclude that a minimal free resolution of the ideal I_k is given by the Eagon-Northcott complex. Let us describe the first syzygies of the Eagon-Northcott resolution of I_k.

Let $\mathbf{a} = (a_1, \ldots, a_{k+1}) \in C_{k+1}$. For $1 \leq r \leq k+1$, we define $X_r^{\mathbf{a}} = [a_1, \ldots, \hat{a_r}, \ldots, a_{k+1} | 1, \ldots, k]$. Hence $X_r^{\mathbf{a}} \in S_k$. We define the map ϕ as follows.

$$\begin{array}{rcl} \{1, 2, \cdots, k\} \times C_{k+1} & \xrightarrow{\phi} & R^{\binom{n}{k}} \\ (j, \mathbf{a}) & \mapsto & \alpha \end{array}$$

$$\text{such that } \alpha(i) = \begin{cases} (-1)^{r_i+1} x_{(a_{r_i},\, j)} & \text{if } i = \sigma(X_{r_i}^{\mathbf{a}}) \text{ for some } \quad r_i; \\ 0 & \text{otherwise.} \end{cases}$$

The map σ is the bijection from S_k to $\{1, 2, \cdots, \binom{n}{k}\}$, defined before. The image of ϕ gives a complete list of generators of $\mathrm{Syz}(I_k)$.

Example 4.6. We give an example, by taking $k = 3$ and $n = 5$. Let $\sigma : S_5 \longrightarrow \{1, \cdots \binom{5}{3}\}$ be defined by,

- $[1, 2, 3 \mid 1, 2, 3] \mapsto 1$
- $[1, 2, 4 \mid 1, 2, 3] \mapsto 2$
- $[1, 2, 5 \mid 1, 2, 3] \mapsto 3$
- $[1, 3, 4 \mid 1, 2, 3] \mapsto 4$
- $[1, 3, 5 \mid 1, 2, 3] \mapsto 5$
- $[1, 4, 5 \mid 1, 2, 3] \mapsto 6$
- $[2, 3, 4 \mid 1, 2, 3] \mapsto 7$
- $[2, 3, 5 \mid 1, 2, 3] \mapsto 8$
- $[2, 4, 5 \mid 1, 2, 3] \mapsto 9$
- $[3, 4, 5 \mid 1, 2, 3] \mapsto 10$

In our example, $\phi : \{1, \cdots 3\} \times C_4 \longrightarrow R^{\binom{5}{3}}$ and $\phi(j, \mathbf{a}) \mapsto \alpha$. Let $j = 2$ and $\mathbf{a} = (1, 3, 4, 5)$. Then, $X_1^{\mathbf{a}} = [3, 4, 5 \mid 1, 2, 3]$, $X_2^{\mathbf{a}} = [1, 4, 5 \mid 1, 2, 3]$, $X_3^{\mathbf{a}} = [1, 3, 5 \mid 1, 2, 3]$, $X_4^{\mathbf{a}} = [1, 3, 4 \mid 1, 2, 3]$. Therefore, $\sigma(X_1^{\mathbf{a}}) = 10$, $\sigma(X_2^{\mathbf{a}}) = 6$, $\sigma(X_3^{\mathbf{a}}) = 5$, $\sigma(X_4^{\mathbf{a}}) = 4$. Similarly, $\alpha(4) = (-1)^{4+1} x_{52} = -x_{52}$, $\alpha(5) = (-1)^{3+1} x_{42} = x_{42}$, $\alpha(6) = (-1)^{2+1} x_{32} = -x_{32}$, $\alpha(10) = (-1)^{1+1} x_{12} = x_{12}$. Therefore, $\alpha = (0, 0, 0, -x_{52}, x_{42}, -x_{32}, 0, 0, 0, x_{12})$.

Lemma 4.7. *Let* $1 \leq k \leq n-1$ *and let* $S_k = \left\{X^{\mathbf{a}_1}, \ldots, X^{\mathbf{a}_{\binom{n}{k}}}\right\}$ *be such that* $\mathbf{a}_1 < \ldots < \mathbf{a}_{\binom{n}{k}}$ *with respect to the lexicographic ordering. Suppose that* $\alpha = (\alpha_1, \cdots, \alpha_{\binom{n}{k}}) \in Syz^1(I_k)$*, then* $\sum_{i=1}^{\binom{n}{k}} \alpha_i X^{\mathbf{a}_i} = 0$ *and* $\sum_{i=1}^{\binom{n}{k}} \alpha_i \widetilde{X^{\mathbf{a}_i}} \in \widetilde{I}_{k+1}$.

Proof. We have $\widetilde{X}^{\mathbf{a}_i} = \sum_{m \geq k} \sigma_m^{-1}(i) y_m$. Therefore

$$\sum_{i=1}^{\binom{n}{k}} \alpha_i \widetilde{X}^{\mathbf{a}_i} = \sum_i \alpha_i (\sum_{m \geq k} \sigma_m^{-1}(i) y_m) = \sum_{m \geq k} (\sum_i \alpha_i \sigma_m^{-1}(i) y_m).$$

It is enough to show that $\sum_i \alpha_i \sigma_m^{-1}(i) y_m \in \widetilde{I}_{k+1}$, for every $m \geq k$. We have $\alpha \in \mathrm{Syz}(I_k) = \langle \mathrm{Im}(\phi) \rangle$. Without loss of generality we may assume that $\alpha \in \mathrm{Im}(\phi)$. There exists $(j, \mathbf{a}_{k+1}) \in \{1, 2, \cdots k\} \times C_{k+1}$ such that $\phi(j, \mathbf{a}_{k+1}) = \alpha$. We will show that $\alpha_i \cdot \sigma_m^{-1}(i) \in I_{k+1}$ for every $m \geq k$ and each i. We have $i = \sigma(X_{r_i}^{\mathbf{a}_{k+1}})$ since $\alpha_i \neq 0$. But $\sigma_m^{-1}(i) = [a_1, \ldots, \hat{a_{r_i}}, \ldots, a_{k+1} | 1, \ldots, k-1, m]$. We have

$$[a_1, \ldots, a_{k+1} | j, 1, \ldots, k-1, m] = 0 \quad \text{for} \quad j \leq k-1 \quad \text{and}$$

$$[a_1, \ldots, a_{k+1} | k, 1, \ldots, k-1, m] = (-1)^k [a_1, \ldots, a_{k+1} | 1, \ldots, k, m] \in I_{k+1}.$$

Therefore,

$$\begin{aligned} \sum_{i=1}^{\binom{n}{k}} \alpha_i \cdot \sigma_m^{-1}(i) &= \sum_{i=1}^{\binom{n}{k}} (-1)^{r_i+1} x_{(a_{r_i}, j)} [a_1, \ldots, \hat{a_{r_i}}, \ldots, a_{k+1} | 1, \ldots, k-1, m] \\ &= [a_1, \ldots, a_{k+1} | j, 1, \ldots, k-1, m] \in I_{k+1}; \end{aligned}$$

Hence,

$$\sum_{i=1}^{\binom{n}{k}} \alpha_i \widetilde{X^{\mathbf{a}_i}} = \sum_{i=1}^{\binom{n}{k}} \alpha_i \cdot \widetilde{\sigma_m^{-1}(i)} = (-1)^k \sum_{i=1}^{\binom{n}{k}} [a_1, \ldots, a_{k+1} | 1, \ldots, k, m] y_m \in \widetilde{I}_{k+1}. \quad \square$$

Lemma 4.8. *Let* $X^{\mathbf{a}_i}, X^{\mathbf{a}_j} \in S_k = \left\{X^{\mathbf{a}_1}, \ldots, X^{\mathbf{a}_{\binom{n}{k}}}\right\}$*, for* $i \neq j$*. Then, there exist monomials* h_t *in* R *and a polynomial* $r \in \widetilde{I}_{k+1}$ *such that*

(i) $S(X^{\mathbf{a}_i}, X^{\mathbf{a}_j}) = \sum_{t=1}^{\binom{n}{k}} h_t X^{\mathbf{a}_t}$*, upon division by* S_k*;*

(ii) $S(\widetilde{X}^{\mathbf{a}_i}, \widetilde{X}^{\mathbf{a}_j}) = \sum_{t=1}^{\binom{n}{k}} h_t \widetilde{X}^{\mathbf{a}_t} + r$*, upon division by* $\widetilde{S}_k$*.*

Proof. (i) The expression follows from the observation that S_k is a Gröbner basis for the ideal I_k.

(ii) We first note that, $\mathrm{Lt}(\widetilde{X}^{\mathbf{a}_t}) = \mathrm{Lt}(X^{\mathbf{a}_t})y_k$, for every $X^{\mathbf{a}_t} \in S_k$. Let $S(X^{\mathbf{a}_i}, X^{\mathbf{a}_j}) = cX^{\mathbf{a}_i} - dX^{\mathbf{a}_j}$, where $c = \dfrac{\mathrm{lcm}(\mathrm{Lt}(X^{\mathbf{a}_i}), \mathrm{Lt}(X^{\mathbf{a}_j}))}{X^{\mathbf{a}_i}}$ and $d = \dfrac{\mathrm{lcm}(\mathrm{Lt}(X^{\mathbf{a}_i}), \mathrm{Lt}(X^{\mathbf{a}_j}))}{X^{\mathbf{a}_j}}$

Hence,

$$\begin{aligned} S(\widetilde{X}^{\mathbf{a}_i}, \widetilde{X}^{\mathbf{a}_j}) &= c \cdot \widetilde{X}^{\mathbf{a}_i} - d \cdot \widetilde{X}^{\mathbf{a}_i} \\ &= \sum_{m \geq k} [c \cdot X^{\mathbf{a}_i, m} - d \cdot X^{\mathbf{a}_j, m}]\, y_m. \end{aligned}$$

It follows immediately that $\mathrm{Lt}(S(\widetilde{X}^{\mathbf{a}_i}, \widetilde{X}^{\mathbf{a}_j})) = y_k \mathrm{Lt}(S(X^{\mathbf{a}_i}, X^{\mathbf{a}_j}))$.

The set S_k is a Gröbner basis for the ideal I_k. Therefore, we have $\mathrm{Lt}(X^{\mathbf{a}_t}) \mid \mathrm{Lt}(S(X^{\mathbf{a}_i}, X^{\mathbf{a}_j}))$, for some t. Then, $\mathrm{Lt}(\widetilde{X}^{\mathbf{a}_t}) \mid \mathrm{Lt}(S(\widetilde{X}^{\mathbf{a}_i}, \widetilde{X}^{\mathbf{a}_j}))$ and we have $h_t = \dfrac{\mathrm{Lt}(S(X^{\mathbf{a}_i}, X^{\mathbf{a}_j}))}{\mathrm{Lt}(X^{\mathbf{a}_t})} = \dfrac{\mathrm{Lt}(S(\widetilde{X}^{\mathbf{a}_i}, \widetilde{X}^{\mathbf{a}_j}))}{\mathrm{Lt}(\widetilde{X}^{\mathbf{a}_t})}$. We can write

$$\begin{aligned} r_1 &:= S(\widetilde{X}^{\mathbf{a}_i}, \widetilde{X}^{\mathbf{a}_j}) - h_t \widetilde{X}^{\mathbf{a}_t} \\ &= \sum_{m \geq k} [c \cdot X^{\mathbf{a}_i, m} - d \cdot X^{\mathbf{a}_j, m} - h_t X^{\mathbf{a}_t, m}] y_m \\ &= \sum_{m > k} [c \cdot X^{\mathbf{a}_i, m} - d \cdot X^{\mathbf{a}_j, m} - h_t X^{\mathbf{a}_t, m}] y_m + [c \cdot X^{\mathbf{a}_i} - d \cdot X^{\mathbf{a}_j} - h_t X^{\mathbf{a}_t}] y_k \end{aligned}$$

Note that $r_1 \in \widetilde{I}_k$ and $\mathrm{Lt}(r_1) = \mathrm{Lt}(S(\widetilde{X}^{\mathbf{a}_i}, \widetilde{X}^{\mathbf{a}_j}) - h_t \widetilde{X}^{\mathbf{a}_t}) = y_k \mathrm{Lt}(S(X^{\mathbf{a}_i}, X^{\mathbf{a}_j}) - h_t X^{\mathbf{a}_t})$. We proceed as before with the polynomial $S(X^{\mathbf{a}_i}, X^{\mathbf{a}_j}) - h_t X^{\mathbf{a}_t} \in I_k$ and continue the process to obtain the desired expression involving the polynomial r.

We now show that the polynomial r is in the ideal $\widetilde{I}_{k+1}$. Let us write $H_j = h_j + d$, $H_i = h_i - c$ and $H_t = h_t$ for $t \neq i, j$. It follows from $S(X^{\mathbf{a}_i}, X^{\mathbf{a}_j}) = \sum_{t=1}^{\binom{n}{k}} h_t X^{\mathbf{a}_t}$, that $\sum_{t=1}^{\binom{n}{k}} H_t X^{\mathbf{a}_t} = 0$. Therefore, $\mathbf{H} = (H_1, \ldots, H_{\binom{n}{k}}) \in \mathrm{Syz}(I_k)$ and by Lemma 4.7 we have $\sum_{t=1}^{\binom{n}{k}} H_t \widetilde{X}^{\mathbf{a}_t} \in \widetilde{I}_{k+1}$. Hence, $r = S(\widetilde{X}^{\mathbf{a}_i}, \widetilde{X}^{\mathbf{a}_j}) - \sum_{t \neq i,j} h_t \widetilde{X}^{\mathbf{a}_t} \in \widetilde{I}_{k+1}$. □

Lemma 4.9. *(i) Let $k' > k$ and $\mathbf{a} = (a_1, \ldots, a_{k'}) \in C_{k'}$. Suppose that $X^{\mathbf{a}} = \sum_{\mathbf{b}_t \in C_k} \beta_{\mathbf{b}_t} X^{\mathbf{b}_t}$ is the Laplace expansion of $X^{\mathbf{a}}$. Then*

$$\sum_{\mathbf{b}_t \in C_k} \beta_{\mathbf{b}_t} X^{\mathbf{b}_t, i} = [a_1, \ldots, a_{k'} | 1, \ldots, k-1, i, k+1, \ldots, k'].$$

(ii) Let $k' > k$; $\mathbf{a} = (a_1, \ldots, a_{k'}) \in C_{k'}$, $\mathbf{b} = (b_1, \ldots, b_k) \in C_k$. Suppose that $X^{\mathbf{a}} = \sum_{\mathbf{p} \in C_k} \alpha_{\mathbf{p}} X^{\mathbf{p}}$ and $S(X^{\mathbf{a}}, X^{\mathbf{b}}) = cX^{\mathbf{a}} - dX^{\mathbf{b}} = \sum_{\mathbf{p} \in C_k} \beta_{\mathbf{p}} X^{\mathbf{p}}$. Then

$$c \sum_{t \geq k} [a_1, \cdots, a_{k'} | 1, \cdots, k-1, t, k+1, \cdots, k'] y_t - d\widetilde{X}^{\mathbf{b}} - \sum_{\mathbf{p} \in C_k} \beta_{\mathbf{p}} \widetilde{X}^{\mathbf{p}} \in \widetilde{I}_{k+1}.$$

Proof. (i) See [12].

(ii) We have $S(X^{\mathbf{a}}, X^{\mathbf{b}}) = cX^{\mathbf{a}} - dX^{\mathbf{b}} = \sum_{\mathbf{p} \in C_k} \beta_{\mathbf{p}} X^{\mathbf{p}}$. By rearranging terms we get $\sum_{\mathbf{p} \in C_k} (c\alpha_{\mathbf{p}} - \beta_{\mathbf{p}}) X^{\mathbf{p}} - dX^{\mathbf{b}} = 0$ and by separating out the term $(c\alpha_{\mathbf{b}} - \beta_{\mathbf{b}}) X^{\mathbf{b}}$ we get $\sum_{\mathbf{p} \neq \mathbf{b}} (c\alpha_{\mathbf{p}} - \beta_{\mathbf{p}}) X^{\mathbf{p}} + (c\alpha_{\mathbf{b}} - \beta_{\mathbf{b}} - d) X^{\mathbf{b}} = 0$. Therefore, $\sum_{\mathbf{p} \neq \mathbf{b}} (c\alpha_{\mathbf{p}} - \beta_{\mathbf{p}}) \widetilde{X}^{\mathbf{p}} + (c\alpha_{\mathbf{b}} - \beta_{\mathbf{b}} - d) \widetilde{X}^{\mathbf{b}} \in \widetilde{I}_{k+1}$, by Lemma 4.7. Hence $\sum_{t \geq k} \sum_{\mathbf{p} \neq \mathbf{b}} (c\alpha_{\mathbf{p}} - \beta_{\mathbf{p}}) X^{\mathbf{p},t} y_t + (c\alpha_{\mathbf{b}} - \beta_{\mathbf{b}} - d) \sum_{t \geq k} X^{\mathbf{b},t} y_t \in \widetilde{I}_{k+1}$. Now $\sum_{t \geq k} \sum_{\mathbf{p} \in C_k} \alpha_{\mathbf{p}} X^{p,t} = \sum_{t \geq k} [a_1, \cdots, a_{k'} | 1, \cdots, k-1, t, k+1, \cdots, k']$ by (i). Hence,

$$c \sum_{t \geq k} [a_1, \cdots, a_{k'} | 1, \cdots, k-1, t, k+1, \cdots, k'] y_t - d\widetilde{X}^{\mathbf{b}} - \sum_{\mathbf{p} \in C_k} \beta_{\mathbf{p}} \widetilde{X}^{\mathbf{p}} \in \widetilde{I}_{k+1}. \quad \square$$

Lemma 4.10. *Let $k' > k$; $\mathbf{a} = (a_1, \ldots, a_{k'}) \in C_{k'}$, $\mathbf{b} = (b_1, \ldots, b_k) \in C_k$. Suppose that $S_k = \left\{ X^{\mathbf{a}_1}, \ldots, X^{\mathbf{a}_{\binom{n}{k}}} \right\}$, such that $\mathbf{a}_1 < \ldots < \mathbf{a}_{\binom{n}{k}}$ with respect to the lexicographic ordering. Then, there exist monomials $h_t \in R$ and a polynomial $r \in \widetilde{I}_{k+1}$ such that*

(i) $S(X^{\mathbf{a}}, X^{\mathbf{b}}) = \sum_{t=1}^{\binom{n}{k}} h_t X^{\mathbf{a}_t}$, upon division by S_k.

(ii) $S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) = \sum_{t=1}^{\binom{n}{k}} (h_t \widetilde{X}^{\mathbf{a}_t}) y_{k'} + r$, upon division by $\widetilde{S}_k$.

Proof. (i) The expression follows from the observation that S_k is a Gröbner basis for the ideal I_k.

(ii) Let $S(X^{\mathbf{a}}, X^{\mathbf{b}}) = cX^{\mathbf{a}} - dX^{\mathbf{b}}$, where $c = \dfrac{\text{lcm}(\text{Lt}(X^{\mathbf{a}}), \text{Lt}(X^{\mathbf{b}}))}{X^{\mathbf{a}}}$ and $d = \dfrac{\text{lcm}(\text{Lt}(X^{\mathbf{a}}), \text{Lt}(X^{\mathbf{b}}))}{X^{\mathbf{b}}}$. Then,

$$\begin{aligned} S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) &= cy_k \widetilde{X}^{\mathbf{a}} - dy_{k'} \widetilde{X}^{\mathbf{b}} \\ &= cy_k \sum_{t \geq k'} X^{\mathbf{a},t} y_t - dy_{k'} \sum_{t \geq k} X^{\mathbf{b},t} y_t \\ &= y_k y_{k'} (cX^{\mathbf{a}} - dX^{\mathbf{b}}) + \text{terms devoid of } y_k. \end{aligned}$$

We therefore have $\text{Lt}(S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}})) = y_k y_{k'} \text{Lt}(S(X^{\mathbf{a}}, X^{\mathbf{b}}))$, since y_k is the largest variable appearing in the above expression. The set S_k being a Gröbner basis for the ideal I_k, we have $\text{Lt}(X^{\mathbf{a_t}})$ dividing $\text{Lt}(S(X^{\mathbf{a_i}}, X^{\mathbf{a_j}}))$

for some t. Let $h_t = \dfrac{\mathrm{Lt}(cX^{\mathbf{a}} - dX^{\mathbf{b}})}{\mathrm{Lt}(X^{\mathbf{a}_t})}$, with $t = 1, \ldots, \binom{n}{k}$. Moreover, $\mathrm{Lt}(\widetilde{X}^{\mathbf{a}_t})$ being equal to $y_k\mathrm{Lt}(X^{\mathbf{a}_t})$, it divides $\mathrm{Lt}(S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}))$. Let

$$r_1 := S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) - \frac{\mathrm{Lt}(S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}))}{\mathrm{Lt}(\widetilde{X}^{\mathbf{a}_t})}\widetilde{X}^{\mathbf{a}_t} = S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) - y_{k'}h_t\widetilde{X}^{\mathbf{a}_t} \in \widetilde{I}_k.$$

We have

$$\begin{aligned} r_1 &= y_k y_{k'}(cX^{\mathbf{a}} - dX^{\mathbf{b}}) - y_{k'}h_t\widetilde{X}^{\mathbf{a}_t} + \text{terms devoid of } y_k \\ &= y_k y_{k'}(cX^{\mathbf{a}} - dX^{\mathbf{b}}) - y_{k'}h_t\sum_{i\geq k} X^{\mathbf{a}_t,i}y_i + \text{terms devoid of } y_k \\ &= y_k y_{k'}(cX^{\mathbf{a}} - dX^{\mathbf{b}} - h_tX^{\mathbf{a}_t}) + \text{terms devoid of } y_k \\ &= y_k y_{k'}(S(X^{\mathbf{a}}, X^{\mathbf{b}}) - h_tX^{\mathbf{a}_t}) + \text{terms devoid of } y_k. \end{aligned}$$

Hence, $\mathrm{Lt}(r_1) = \mathrm{Lt}(S(X^{\mathbf{a}}, X^{\mathbf{b}}) - h_tX^{\mathbf{a}_t}) = y_k y_{k'}\mathrm{Lt}(S(X^{\mathbf{a}}, X^{\mathbf{b}}) - h_tX^{\mathbf{a}_t})$. We proceed as before with the polynomial $S(X^{\mathbf{a}}, X^{\mathbf{b}}) - h_tX^{\mathbf{a}_t} \in I_k$ and continue the process to obtain the desired expression involving the polynomial r.

We now show that the polynomial r is in the ideal $\widetilde{I}_{k+1}$. Let us write

$$\begin{aligned} r &= S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) - \sum_{t=1}^{\binom{n}{k}}(h_t\widetilde{X}^{\mathbf{a}_t})y_{k'} \\ &= cy_k\sum_{l\geq k'} X^{\mathbf{a},l}y_l - dy_{k'}\sum_{l\geq k} X^{\mathbf{b},l}y_l - \sum_{t=1}^{\binom{n}{k}}\sum_{l\geq k} h_tX^{\mathbf{a}_t,l}y_l y_{k'} + T - T; \end{aligned}$$

where $T = c\sum_{l\geq k}[a_1, \ldots, a_{k'} \mid 1, \ldots, k-1, l, k+1, \ldots, k']y_l y_{k'}$. After a rearrangement of terms, we may write

$$\begin{aligned} r &= \left(T - \sum_{t=1}^{\binom{n}{k}}\sum_{l\geq k} h_tX^{\mathbf{a}_t,l}y_l y_{k'} - dy_{k'}\sum_{l\geq k} X^{\mathbf{b},l}y_l\right) \\ &\quad + \left(cy_k\sum_{l\geq k'} X^{\mathbf{a},l}y_l\right) - T. \end{aligned}$$

Let $T' = c\sum_{l>k}[a_1, \ldots, a_{k'} \mid 1, \ldots, k-1, l, k+1, \ldots, k']y_l y_{k'}$. Now we note, $cX^{\mathbf{a}} - dX^{\mathbf{b}} - \sum_{t=1}^{\binom{n}{k}} h_tX^{\mathbf{a}_t} = 0$. Hence $T - \sum_{t=1}^{\binom{n}{k}}\sum_{l\geq k} h_tX^{\mathbf{a}_t,l}y_l y_{k'} -$

$dy_{k'} \sum_{l \geq k} X^{\mathbf{b},l} y_l$ becomes equal to

$$T' - \sum_{t=1}^{\binom{n}{k}} \sum_{l>k} h_t X^{\mathbf{a}_t,l} y_l y_{k'} - dy_{k'} \sum_{l>k} X^{\mathbf{b},l} y_l.$$

We also have $cy_k \sum_{l \geq k'} X^{\mathbf{a},l} y_l - T = cy_k \sum_{l>k'} X^{\mathbf{a},l} y_l - T'$, since the term for $l = k'$ in $cy_k \sum_{l \geq k'} X^{\mathbf{a},l} y_l$ gets cancelled with the term appearing in T for $l = k$. Hence we write

$$\begin{aligned} r &= \left(T' - \sum_{t=1}^{\binom{n}{k}} \sum_{l>k} h_t X^{\mathbf{a}_t,l} y_l y_{k'} - dy_{k'} \sum_{l>k} X^{\mathbf{b},l} y_l \right)_1 \\ &\quad + \left(cy_k \sum_{l>k'} X^{\mathbf{a},l} y_l \right)_2 - T' \\ &= (\,)_1 + (\,)_2 - T'. \end{aligned}$$

Clearly, the expression $(\,)_1$ belongs to $\widetilde{I}_{k+1}$, by Lemma 4.9. We note that no term of $(\,)_1$ contains y_k. So also for T'. Hence, the leading term of r is the leading term of $(\,)_2$. By an application of similar argument as above we see that the expression $(\,)_2$, after division by elements of $\widetilde{S}_k$, further reduces to

$$\begin{aligned} & - \left(\sum_{l>k'} \sum_{s \geq k'} c[a_1, \ldots, a_{k'} | 1, \ldots, k-1, s, k+1, \ldots, k'-1, l] y_l y_s \right) \\ = & - \left(\sum_{l>k'} \sum_{s>k'} c[a_1, \ldots, a_{k'} | 1, \ldots, k-1, s, k+1, \ldots, k'-1, l] y_l y_s \right) \\ & - \left(\sum_{l>k'} c[a_1, \ldots, a_{k'} | 1, \ldots, k-1, k', k+1, \ldots, k'-1, l] y_l y_{k'} \right). \end{aligned}$$

Moreover,

$$\sum_{l>k'} c[a_1, \ldots, a_{k'} | 1, \ldots, k-1, k', k+1, \ldots, k'-1, l] y_l y_{k'} + T' = 0$$

and

$$\sum_{l>k'} \sum_{s>k'} c[a_1, \ldots, a_{k'} | 1, \ldots, k-1, s, k+1, \ldots, k'-1, l] y_l y_{k'} = 0.$$

Therefore, after division by elements of $\widetilde{S}_k$, the expression $(\ )_1 + (\ )_2 - T'$ reduces to $(\ )_1$, which is in $\widetilde{I}_{k+1}$. □

Proof of Theorem 4.1. We use induction on $n-k$ to prove that G_k is a Gröbner basis for the ideal $\widetilde{I}_k$. For $n-k=0$; the set $G_k = \widetilde{S}_n$ contains only one element and hence trivially forms a Gröbner basis. We apply Buchberger's algorithm to prove our claim. Let $X^{\mathbf{a}}, X^{\mathbf{b}} \in G_k$. The following cases may arise:

- $X^{\mathbf{a}}, X^{\mathbf{b}} \in S_k$, for $\mathbf{a}, \mathbf{b} \in C_k$;
- $X^{\mathbf{a}} \in S_{k'}$ and $X^{\mathbf{b}} \in S_k$ where $k' > k$; $\mathbf{a} \in C_{k'}$ and $\mathbf{b} \in C_k$.

We have proved in Lemmas 4.8 and 4.10 that upon division by $\widetilde{S}_k$, the S-polynomial $S(\widetilde{X}^{\mathbf{a}}, \widetilde{X}^{\mathbf{b}}) \longrightarrow r$ for some $r \in \widetilde{I}_{k+1}$, in both the cases. By induction hypothesis, G_{k+1} is a Gröbner basis for $\widetilde{I}_{k+1}$. Hence r reduces to 0 modulo G_{k+1} and hence modulo G_k, since $G_{k+1} \subset G_k$.

We now show that G_k is a reduced Gröbner basis for $\widetilde{I}_k$. Let $X^{\mathbf{a}} \in S_{k'}$ and $X^{\mathbf{b}} \in S_k$ where $k' \geq k$; $\mathbf{a} \in C_{k'}$ and $\mathbf{b} \in C_k$. Then, $\widetilde{X}^{\mathbf{a}} = \sum_{i \geq k'} X^{\mathbf{a},i} y_i$ and $\widetilde{X}^{\mathbf{b}} = \sum_{i \geq k} X^{\mathbf{b},i} y_i$. If $k' > k$, then $y_{k'} | \mathrm{Lt}(\widetilde{X}^{\mathbf{a}})$ but does not divide $\mathrm{Lt}(\widetilde{X}^{\mathbf{b}})$. Hence, $\mathrm{Lt}(\widetilde{X}^{\mathbf{a}})$ does not divide $\mathrm{Lt}(\widetilde{X}^{\mathbf{b}})$. If $k' = k$, then $\mathrm{Lt}(\widetilde{X}^{\mathbf{a}}) = x_{(a_1,1)} \cdots x_{(a_k,k)} y_k$ and $\mathrm{Lt}(\widetilde{X}^{\mathbf{b}}) = x_{(b_1,1)} \cdots x_{(b_k,k)} y_k$. Therefore, $\widetilde{X}^{\mathbf{a}} | \widetilde{X}^{\mathbf{b}}$ implies that $\mathbf{a} = \mathbf{b}$. This proves that the Gröbner basis is reduced. □

5. GRÖBNER BASIS FOR $\mathcal{J}$

Theorem 5.1. *Let us consider the lexicographic monomial order induced by $y_1 > y_2 > \cdots > y_n > x_{11} > x_{12} > \cdots > x_{(n+1),(n-1)} > x_{(n+1),n}$ on $\widehat{R} = K[x_{ij}, y_j \mid 1 \leq i \leq n+1,\ 1 \leq j \leq n]$. The set G_k is a reduced Gröbner Basis for the ideal $\widetilde{I}_k$. In particular, $\mathcal{G} = G_1$ is a reduced Gröbner Basis for the ideal $\widetilde{I}_1 = \mathcal{J}$.*

Proof. The scheme of the proof is the same as that for $\mathcal{I}$, with suitable changes made for $\widehat{X}$ in the Lemmas. We only reiterate the last part of the proof where we carry out induction on $n-k$. For $n-k=0$, the set $G_k = \widetilde{S}_n = \{\Delta_1 y_n, \ldots, \Delta_{n+1} y_n\}$, where $\Delta_i = \det(\widehat{X}_i)$. We first note that $\mathrm{Lt}(\Delta_i)$ and $\mathrm{Lt}(\Delta_j)$ are coprime. Therefore,

$$\begin{aligned} S(\Delta_i y_n, \Delta_j y_n) &= \mathrm{Lt}(\Delta_j) \cdot (\Delta_i y_n) - \mathrm{Lt}(\Delta_i) \cdot (\Delta_j y_n) \\ &= \mathrm{Lt}(\Delta_j)(\mathrm{Lt}(\Delta_i) y_n + y_n p_i) - \mathrm{Lt}(\Delta_i)(\mathrm{Lt}(\Delta_j) y_n - y_n p_j) \\ &= (\mathrm{Lt}(\Delta_j) y_n) p_i - (\mathrm{Lt}(\Delta_i) y_n) p_j \\ &= (\Delta_j y_n - p_j y_n) p_i - (\Delta_i y_n - p_i y_n) p_j \\ &= \Delta_j y_n p_i - \Delta_i y_n p_j \longrightarrow_{G_n} 0. \end{aligned}$$

The rest of the proof is essentially the same as that for Theorem 4.1. □

6. BETTI NUMBERS OF $\mathcal{I}$ AND $\mathcal{J}$

Theorem 6.1. *Suppose that $X = (x_{ij})_{n\times n}$ is either a generic or a generic symmetric $n \times n$ matrix and Y a generic $n \times 1$ matrix given by $Y = (y_j)_{n\times 1}$. If X is generic, we write $g_i = \sum_{j=1}^{n} x_{ij}y_j$ and $\mathcal{I} = I_1(XY) = \langle g_1, g_2, \cdots, g_n\rangle$. If X is generic symmetric, we write $g_1 = \sum_{j=1}^{n} x_{1j}y_j$, $g_n = (\sum_{1\le k\le n} x_{kn}y_k)$ and $g_i = (\sum_{1\le k<i} x_{ki}y_k) + (\sum_{i\le k\le n} x_{ik}y_k)$ for $1 < i < n$ and $\mathcal{I} = I_1(XY) = \langle g_1, \cdots, g_n\rangle$. The generators $g_1, \ldots, g_n$ of $\mathcal{I} = I_1(XY)$ in either case form a regular sequence in the polynomial K-algebra $R = K[x_{ij},\, y_j \mid 1 \le i, j \le n]$. Moreover, $\{g_1, \ldots, g_n\}$ form a Gröbner basis for $\mathcal{I}$ in either case with respect to the lexicographic monomial order which satisfies (1) and (2) given below:*

(1) $x_{11} > x_{22} > \cdots > x_{nn}$;
(2) $x_{ij}, y_j < x_{nn}$ for every $1 \le i \ne j \le n$.

Proof. The monomial order chosen is lexicographic order induced by the ordering among the variables given by (1) and (2). It is clear from the expressions of g_i that their leading terms are pairwise coprime. Therefore, the proof follows from Lemma 4.3. □

Corollary 6.2. *$\mathcal{I}$ is minimally resolved by the Koszul complex $\mathbb{G}$ and the i-th Betti number of $\mathcal{I}$ is $\binom{n}{i}$.*

Theorem 6.3. *Suppose that $\widehat{X} = (x_{ij})_{(n+1)\times n}$ is a generic $(n+1)\times n$ matrix and Y a generic $n \times 1$ matrix given by $Y = (y_j)_{n\times 1}$. Let $g_i = \sum_{j=1}^{n+1} x_{ij}y_j$ and $\mathcal{J} = I_1(\widehat{X}Y) = \langle g_1, \cdots, g_{n+1}\rangle$. The total Betti numbers of the ideal $\mathcal{J}$ are $\beta_0 = 1, \beta_1 = n + 1$, $\beta_{n+1} = n$, $\beta_{k+1} = \binom{n}{k} + \binom{n}{k-1} + \binom{n}{k+1}$ for $1 \le k < n$.*

We first discuss the scheme of the proof below. We will use the following observations to compute the total Betti numbers of $\mathcal{J}$.

Step 1. The minimal graded free resolution of $\mathcal{I} = \langle g_1, \cdots, g_n\rangle$ is given by the Koszul Resolution.
Step 2. We prove that $\langle g_1, \cdots, g_n : g_{n+1}\rangle = \langle g_1, \cdots, g_n, \Delta\rangle$; where $\Delta = \det(X)$. This proof requires the fact that $\langle g_1, \cdots, g_n, \Delta\rangle$ is a prime ideal, which has been proved in Theorem 5.4 in [15].
Step 3. We prove that $\langle g_1, \cdots g_n : \Delta\rangle = \langle y_1, y_2, \cdots, y_n\rangle$.

Step 4. We construct a graded free resolution of $\langle g_1, \cdots, g_n, \Delta\rangle$ using mapping cone between resolutions of $\langle g_1, \cdots, g_n\rangle$ and $\langle y_1, \cdots, y_n\rangle$. We extract a minimal free resolution from this resolution.

Step 5. Finally, we construct a graded free resolution of $\langle g_1, \cdots, g_n, g_{n+1}\rangle$ using mapping cone between free resolutions of $\langle g_1, \cdots, g_n, \Delta\rangle$ and $\langle g_1, \cdots, g_n\rangle$. We extract a minimal free resolution from this resolution.

Remark 6.4. We need detailed information about the ideal $\langle g_1, \cdots, g_n, \Delta\rangle$, where $\Delta = \det(X)$. We need the fact that this ideal is a prime ideal, which has been proved in Theorem 5.4 in [15]. We also need a minimal free resolution for this ideal, which has been proved below in Lemma 6.10. We came to know much later that $\langle g_1, \cdots, g_n, \Delta\rangle$ was defined in [14]. It is known as the generic Northcott ideal and a minimal free resolution can be found in [14]. However, we give a different proof here using our Gröbner basis computation, which also shows the linking of nested complete intersection ideals. Moreover, Northcott's resolution can perhaps be used to prove that $\langle g_1, \cdots, g_n, \Delta\rangle$ is a prime ideal, although our proof in [15] is absolutely different and uses the result in [7].

Lemma 6.5. $\Delta y_i = \sum_{j=1}^{n} A_{ji} g_j$, *where* A_{ji} *is the cofactor of* x_{ji} *in* X.

Proof. We have

$$\Delta y_i = \sum_{j=1}^{n} A_{ji} x_{ji} y_i = \sum_{j=1}^{n} A_{ji} \left(\sum_{k=1}^{n} x_{jk} y_k \right) - \sum_{j=1}^{n} A_{ji} \left(\sum_{k \neq i} x_{jk} y_k \right) = \sum_{j=1}^{n} A_{ji} g_j,$$

since $\sum_{j=1}^{n} A_{ji} \left(\sum_{k \neq i} x_{jk} y_k \right) = \sum_{k \neq i} \left(\sum_{j=1}^{n} A_{ji} x_{jk} \right) y_k = 0$. □

Lemma 6.6. $\langle g_1, \cdots, g_n, \Delta\rangle \subseteq \langle g_1, \cdots, g_n : g_{n+1}\rangle$.

Proof. We have $g_i \in \langle g_1, \cdots, g_n : g_{n+1}\rangle$, for every $1 \leq i \leq n$. Moreover, $y_i \Delta \in \langle g_1, \cdots, g_n\rangle$, by Lemma 6.5. Hence, $g_{n+1}\Delta \in \langle g_1, \cdots, g_n\rangle$. □

Lemma 6.7. $\langle g_1, \cdots, g_n : g_{n+1}\rangle = \langle g_1, \cdots, g_n, \Delta\rangle$

Proof. We have proved that $\langle g_1, \cdots, g_n, \Delta\rangle \subseteq \langle g_1, \cdots, g_n : g_{n+1}\rangle$ in Lemma 6.6. We now prove that $\langle g_1, \cdots, g_n : g_{n+1}\rangle \subseteq \langle g_1, \cdots, g_n, \Delta\rangle$. Let $z \in \langle g_1, \cdots, g_n : g_{n+1}\rangle$. Then $z g_{n+1} \in \langle g_1, \cdots, g_n\rangle \subset \langle g_1, \cdots, g_n, \Delta\rangle$. It is easy to see that $g_{n+1} \notin \langle g_1, \cdots, g_n, \Delta\rangle$. Therefore, $z \in \langle g_1, \cdots, g_n, \Delta\rangle$, since $\langle g_1, \cdots, g_n, \Delta\rangle$ is a prime ideal by Theorem 5.4 in [15]. □

Lemma 6.8. $\langle g_1, \cdots, g_n : \Delta\rangle = \langle y_1, \cdots, y_n\rangle$

Proof. We have $y_i\Delta \in \langle g_1, \cdots, g_n\rangle$ by Lemma 6.5; which implies that $\langle y_1, \cdots, y_n\rangle \subset \langle g_1, \cdots, g_n : \Delta\rangle$. Let $z \in \langle g_1, \cdots, g_n : \Delta\rangle$. Then $z\Delta \in \langle g_1, \cdots, g_n\rangle \subseteq \langle y_1, \cdots, y_n\rangle$. Therefore, $z \in \langle y_1, \cdots, y_n\rangle$, since $\Delta \notin \langle y_1, \cdots, y_n\rangle$ and $\langle y_1, \cdots, y_n\rangle$ is a prime ideal. □

Mapping Cones. The resolution for $\langle y_1, \cdots, y_n\rangle$ is given by the Koszul complex $\mathbb{F}_\cdot$. We now give a resolution of $\langle g_1, \cdots, g_n, \Delta\rangle$ by the mapping cone technique. We know that $\langle g_1, \cdots, g_n : \Delta\rangle = \langle y_1, \cdots, y_n\rangle$, by Lemma 6.8. We first construct a connecting homomorphism $\phi_\cdot : \mathbb{F}_\cdot \longrightarrow \mathbb{G}_\cdot$. Let ϕ_0 denote the multiplication by Δ. In order to make the map ϕ_0 a degree zero map, we set the grading as $\mathbb{F}_0 \cong (R(-n))^1$ and $\mathbb{G}_0 = (R(0))^1$. Since $\mathbb{F}_\cdot$ and $\mathbb{G}_\cdot$ are both Koszul resolutions, we set the grading as $\mathbb{G}_i \cong (R(-2i))^{\binom{n}{i}}$ and $\mathbb{F}_i \cong (R(-n-i))^{\binom{n}{i}}$. Now we see that, $i \neq n$ implies that $-2i \neq -n-i$. Hence the image of ϕ_i for $i \neq n$ is contained in the maximal ideal. We have $\mathbb{F}_i = \mathbb{G}_i$, only for $i = n$. If we can show that the map ϕ_n is not the zero map, then this will be the only free part of the resolution which we can cancel out for obtaining the minimal resolution.

Lemma 6.9. *The map ϕ_n is not the zero map.*

Proof. We refer to [8]. If ϕ_n is the zero map, then $\phi_0(R) \subseteq \delta_1(\mathbb{G}_1)$, where $\delta_\cdot$ denotes the differential of $\mathbb{G}$. The image of δ_1 is the ideal $\langle g_1, \cdots, g_n\rangle$, which does not contain $\phi_0(1) = \Delta$. The map ϕ_n is not the zero map. □

Therefore, the above discussion proves the following Lemma.

Lemma 6.10. *Hence a minimal graded free resolution of $\langle g_1, \cdots, g_n, \Delta\rangle$ is given by $\mathbb{M}_\cdot$, such that $\mathbb{M}_i \cong (R(-n-i+1))^{\binom{n}{i-1}} \oplus (R(-2i))^{\binom{n}{i}}$ for $0 < i < n$, $\mathbb{M}_0 \cong R(0)$ and $\mathbb{M}_n \cong (R(-2n))^n$.*

(Proof of Theorem 6.3.) We now find the Betti numbers for the ideal $\langle g_1, \cdots, g_{n+1}\rangle$ by constructing the mapping cone between the resolutions $\mathbb{M}_\cdot$ and the resolution $\mathbb{G}_\cdot$ of $\langle g_1, \cdots, g_n\rangle$. The connecting map ψ_0 is multiplication by g_{n+1}. Hence to make it degree zero we set, $\mathbb{G}_0 = (R(2))^1$ and $\mathbb{G}_i \cong (R(2-2i))^{\binom{n}{i}}$ for $i > 0$. Here we note that $2-2i \neq -2i$ and $-n-i+1 \neq 2-2i$ for $1 \leq i \leq n$. Hence, for each $1 \leq i \leq n$, the image of ψ_i is contained in the maximal ideal. This shows that the resolution obtained by the mapping cone between $\mathbb{M}_\cdot$ and $\mathbb{G}_\cdot$ is minimal. Hence the total Betti numbers of $\mathcal{J}$ are:

$\beta_0 = 1, \beta_1 = n+1$;
$\beta_{n+1} = n$;
$\beta_{k+1} = \binom{n}{k} + \binom{n}{k-1} + \binom{n}{k+1}$ for $1 \leq k < n$. □

Corollary 6.11. *The ring $R/\mathcal{I}$ is Cohen-Macaulay and the ring $\hat{R}/\mathcal{J}$ is not Cohen-Macaulay.*

Proof. The polynomial ring R is Cohen-Macaulay and $g_1, \ldots, g_n$ is a regular sequence therefore the ring $R/\mathcal{I}$ is Cohen-Macaulay.

We have seen that $\operatorname{projdim}_{\widehat{R}} \widehat{R}/\mathcal{J} = n+1$. Therefore, by the Auslander-Bauchsbaum formula $\operatorname{depth}_{\widehat{R}} \widehat{R}/\mathcal{J} = n(n+1)+n-(n+1) = n^2+n-1$. We have proved in Lemma 5.5 in [15] that $\langle y_1, \ldots, y_n \rangle$ is a minimal prime over $\mathcal{J}$. Therefore, $\dim \widehat{R}/\mathcal{J} \geq \dim \widehat{R}/\langle y_1, \ldots, y_n \rangle = n^2+n$; hence the ring $\widehat{R}/\mathcal{J}$ is not Cohen-Macaulay. □

7. $I_1(XY)$, WHERE X IS $m \times mn$ GENERIC MATRIX AND Y IS $mn \times n$ GENERIC MATRIX

Finally, we consider the case when $X = (x_{ij})_{m \times mn}$ is a generic matrix of size $m \times mn$ and $Y = (y_{ij})_{mn \times n}$ is generic matrix of size $mn \times n$. We define $\mathfrak{I} = I_1(XY)$. Let $g_{ij} = \sum_{t=1}^{mn} x_{it} y_{tj}$, with $1 \leq i \leq m$, $1 \leq i \leq n$. Then, $\mathfrak{I} = \langle \{ g_{ij} \mid 1 \leq i \leq m,\ 1 \leq i \leq n \} \rangle$. In this section we construct a Gröbner basis for the ideal $\mathfrak{I}$ with respect to a suitable monomial order and use that to show that the generators g_{ij}, with $1 \leq i \leq m$, $1 \leq i \leq n$ form a regular sequence. We first set a few notations before we prove the main results.

- $X = \begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix}$, where $A_s = \begin{pmatrix} x_{1(m(s-1)+1)} & \cdots & x_{1(ms)} \\ \vdots & \vdots & \vdots \\ x_{m(m(s-1)+1)} & \cdots & x_{m(ms)} \end{pmatrix}$ is the $m \times m$ matrix for every $1 \leq s \leq n$.
- $[X]_s = \begin{pmatrix} A_s & A_1 & \cdots & \widehat{A_s} & \cdots & A_n \end{pmatrix}$, for every $1 \leq s \leq n$.
- $[Y]_s = \begin{pmatrix} y_{(m(s-1)+1)s} \\ \vdots \\ y_{(ms)s} \\ y_{1s} \\ \vdots \\ y_{(mn)s} \end{pmatrix}$, for every $1 \leq s \leq n$.

We will use Theorem 4.1 for constructing a Gröbner basis for the ideal $\mathfrak{I}$. A very important reason behind considering this class of ideals is that we get some nice examples of transversal intersection of ideals. Two results that would be useful for our purpose are the following:

Lemma 7.1. *Let $>$ be a monomial ordering on R. Let I and J be ideals in R, such that $m(I)$ and $m(J)$ denote unique minimal generating sets for their leading ideals Lt(I) and Lt(J) respectively. Then, $I \cap J = IJ$ if the set of variables occurring in the set $m(I)$ is disjointed from the the set of variables occurring in the set $m(J)$.*

Proof. See Lemma 3.6 in [16]. □

Lemma 7.2. *Let I and J be graded ideals in a graded ring R, such that $I \cap J = I \cdot J$. Suppose that $\mathbb{F}_\cdot$ and $\mathbb{G}_\cdot$ are minimal free resolutions of I and J respectively. Then $\mathbb{F}_\cdot \otimes \mathbb{G}_\cdot$ is a minimal free resolution for the graded ideal $I + J$.*

Proof. See Lemma 3.7 in [16]. □

Theorem 7.3. *Let us choose the lexicographic monomial order on R induced by $y_{11} > y_{21} > \cdots > y_{(mn)1} > y_{(m+1)2} > y_{(m+2)2} > \cdots > y_{(2m)2} > y_{12} > \cdots y_{(mn)2} > \cdots > y_{(m(n-1)+1)n} > y_{(m(n-1)+2)n} > \cdots > y_{((mn)n} > y_{1n} > \cdots y_{(m(n-1))n} > x_{11} > x_{12} > \cdots > x_{m(mn)}$. Let $\mathcal{G}_s$ be the reduced Gröbner Basis of the ideal $I_1([X]_s[Y]_s)$ for $1 \leq s \leq n$, obtained by Theorem 4.1. Then $\mathfrak{G}_t = \cup_{s=1}^{t} \mathcal{G}_s$ is a reduced Gröbner Basis for the ideal $P_t = \sum_{s=1}^{t} I_1([X]_s[Y]_s)$ for $1 \leq t \leq n$. In particular, $\mathfrak{G}_n$ is a reduced Gröbner Basis for the ideal $P_n = \mathfrak{I} = I_1(XY)$.*

Proof. We have $P_t = \sum_{s=1}^{t} I_1([X]_s[Y]_s)$, and we observe that if $p \in \mathcal{G}_s$ and $q \in \mathcal{G}_t$ for $1 \leq s < t \leq n$, then $\gcd(\text{Lt}(p), \text{Lt}(q)) = 1$. Therefore the S-polynomial of p, q reduces to zero after applying division upon $\mathfrak{G}_t$. □

Theorem 7.4. *Let us denote $P_t = \sum_{s=1}^{t} I_1([X]_s[Y]_s)$, for $1 \leq t \leq n-1$. Then $P_t \cap I_1([X]_{t+1}[Y]_{t+1}) = P_t \cdot I_1([X]_{t+1}[Y]_{t+1})$. Hence the elements $g_{ij} = \sum_{t=1}^{mn} x_{it}y_{tj}$, $1 \leq i \leq m$, $1 \leq i \leq n$ form a regular sequence and the Koszul complex resolves $R/\mathfrak{I}$ as an R-module minimally.*

Proof. If $p \in \mathcal{G}_s$ and $q \in \mathcal{G}_t$, for $1 \leq s < t \leq n$. Then $\gcd(\text{Lt}(p), \text{Lt}(q)) = 1$, therefore by theorem 7.3 and lemma 7.1, we have $P_t \cap I_1([X]_{t+1}[Y]_{t+1}) = P_t \cdot I_1([X]_{t+1}[Y]_{t+1})$.

By Theorem 6.1 the generators of the ideal P_1 form a regular sequence and also the generators of the ideal $I_1([X]_s[Y]_s)$ form a regular sequence for each $1 \leq s \leq n$. Hence the Koszul complex resolve R/P_1 and $R/I_1([X]_s[Y]_s)$ minimally. Now $P_t \cap I_1([X]_{t+1}[Y]_{t+1}) = P_t \cdot I_1([X]_{t+1}[Y]_{t+1})$. Hence, by application of lemma 7.1 we can conclude that the Koszul complex resolves $R/\mathfrak{I}$ minimaly. □

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