The non-linear sewing lemma II: Lipschitz continuous formulation

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We give an unified framework to solve rough differential equations. Based on flows, our approach unifies the former ones developed by Davie, Friz-Victoir and Bailleul. The main idea is to build a flow from the iterated product of an almost flow which can be viewed as a good approximation of the solution at small time. In this second article, we give some tractable conditions under which the limit flow is Lipschitz continuous and its links with uniqueness of solutions of rough differential equations. We also give perturbation formulas on almost flows which link the former constructions.

Keywords: Rough differential equations; Lipschitz flows; Rough paths

1 Introduction

The rough path theory was introduced to deal with differential equations driven by an irregular deterministic path multidimensional x of the type

$$\mathrm{d}y_t = a + \int_s^t f(y_r)\,\mathrm{d}x_r, \tag{1}$$

where a is an initial condition and f a regular vector field. Typically, the irregularity of x is measured in α-Hölder ($\alpha \leqslant 1$) or in p-variation ($p \geqslant 1$) spaces. Such an equation is called *Rough Differential Equation (RDE)* [18, 25].

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This theory was very fruitful to study stochastic equations driven by Gaussian process which is not covered by the Itô framework, like the fractional Brownian motion [13, 26]. More generally, the rough path framework allows one to separate the probabilistic from the deterministic part in such equation and to overcome some probabilistic conditions such as using adapted or non-anticipative processes.

Recently, the ideas of the rough path theory were extended to stochastic partial differential equations (SPDE) with the works of [21, 22] which have led to significant progress in the study of some SPDE. This theory also found applications in machine learning and the recognizing of the Chinese ideograms [10, 24].

Since the seminal article [25] by T. Lyons in 1998, several approaches emerged to solve (1). They are based on two main technical arguments: fixed point theorems [20, 25] and flow approximations [2, 12, 14, 16, 19]. In particular, the rough flow theory allows one to extend work about stochastic flows, which has been developed in '80s by Le Jan-Watanabe-Kunita and others, to a non-semimartinagle setting [4].

The main goal of this article is to give a framework which unifies the approaches by flow and pursue further investigations on their properties and their relations with families of solutions to (1).

A *flow* is a family of maps $\{\psi_{t,s}\}$ from a Banach space to itself such that $\psi_{t,s}\circ\psi_{s,r} = \psi_{t,r}$ for any $r \leqslant s \leqslant t$. Typically, the map which associates the initial condition a to the solution of (1) has a flow property. The existence of a such flow heavily depends on the existence and uniqueness of the solution. However, it was proved in [7, 8] and extended to the rough path case in [6] that when non-uniqueness holds, it is possible to build a measurable flow by a selection technique. In this article we are interested by the construction of a Lipschitz flows.

The main idea to build the flow associated is to find a good approximation $\phi_{t,s}$ of $\psi_{t,s}$ when $|t-s|$ is small enough. We iterate this approximation on a subdivision $\pi = \{s \leqslant t_i \leqslant \cdots \leqslant t_j \leqslant t\}$ of $[s,t]$ by setting

$$\phi^{\pi}_{t,s} := \phi_{t,t_j} \circ \cdots \circ \phi_{t_i,s}.$$

If ϕ^{π} does converges when the mesh of π goes to zero, ϕ^{π}, the limit is necessarily a flow

This computation is similar to the ones of numerical schemes as Euler's methods of different order [11]. Moreover, this idea is found among the Trotter's formulas for bounded or unbounded linear operators which allows to compute the semi-group of the sum of two non-commutative operators only knowing the semi-groups associated to each operator [15]. This property can be used to prove the Feynman-Kac formula.

Rather than working with a particular choice for the almost flow ϕ as in [14, 19], we give here generic conditions on ϕ. We generalize the multiplicative sewing lemma of [16] and of [12], introduced to solve linear RDE to a non linear situation. In this way, we construct directly some flows. In opposite to the additive and multiplicative sewing lemma, the limit is not necessarily unique. The approximations are assumed to be Lipschitz. This is not the case for the limit.

Our framework is close to the one developed by I. Bailleul in [2, 5] with a notable difference: in the previous article [6], we have shown that a flow may exist under some weak conditions even if the iterated products ϕ^π are not uniformly Lipschitz continuous. Similarly to I. Bailleul, we have also shown that if ϕ^π is uniformly Lipschitz continuous for any partition π (UL Condition) then it converges to a Lipschitz flow, which is necessarily unique.

The present article gives a sufficient condition, called the *4-points control*, on the almost flow ϕ that ensures the UL Condition. We then study various consequences of this condition: existence of an inverse, unique family of solutions, convergence of the Euler scheme, ... The *4-points control* can be checked on the almost flow, which is then called a *stable almost flow*. This condition is weaker that the one given by I. Bailleul in [2, 5]: there it should roughly be $\mathscr{C}^1$ with a Lipschitz continuous spatial derivative while in our case, the spatial derivative may be only Hölder continuous. The question of the existence of a Lipschitz flow without the UL condition, in relation with Stochastic Differential Equations, should be dealt with in a subsequent work.

We also study the relationship between almost flows and family of solutions to (1) in the sense of Davie [14] as they are two different objects. In particular, we show that when an almost flow is stable, when the family of solutions to the RDE is unique and Lipschitz continuous. We also relate the distance between two families of solutions with respect to the distance between two almost flows when one is stable. Again, consequences of this result will be drawn in a subsequent work.

We also give several conditions under which perturbations of almost flows, a convenient tool to construct numerical schemes, converge to the same limit flow. These perturbative arguments are the key to unify expansions that are *a priori* of different nature.

Finally, we apply our framework to recover the results of A.M. Davie [14], P. Friz & N. Victoir [17, 19] and I. Bailleul [2, 5] using various perturbation arguments. Although not done here, our framework could be applied to deal with branched rough paths, that are high-order expansions indiced by trees, which are studied in [9] and shown to fit the Bailleul's framework [3].

Outline. After introducing in Section 2 the main notations and general definitions, we recall in Section 3 the notion of almost flow which is introduced in our previous article [6]. In Section 4, we define the 4-point control as well as stable almost flow ϕ. We prove that under these conditions, ϕ^π converges to a Lipschitz flow. In Section 5, we give conditions to modify the almost flow ϕ by adding a perturbation ϵ while retains the convergence to a flow. We prove that under suitable conditions, the inverse the approximation ϕ and that ϕ^{-1} is a good approximation of the inverse of the flow. The link between the uniqueness of the solution of (1) and the existence of a flow is studied in Section 7. In Section 8, our formalism links the former approaches based on flow [2, 14, 19].

2 Notations

The following notations and hypotheses will be constantly used throughout all this article.

2.1 Controls and remainders

Let us fix $T > 0$, a time horizon. We write $\mathbb{T} := [0, T]$ as well as

$$\mathbb{T}_+^2 := \{(s,t) \in \mathbb{T}^2 \mid s \leqslant t\} \text{ and } \mathbb{T}_+^3 := \{(r,s,t) \in \mathbb{T}^3 \mid r \leqslant s \leqslant t\},$$
$$\mathbb{T}_-^2 := \{(s,t) \in \mathbb{T}^2 \mid s \geqslant t\} \text{ and } \mathbb{T}_-^3 := \{(r,s,t) \in \mathbb{T}^3 \mid r \geqslant s \geqslant t\}.$$

We also set $\mathbb{T}^3 = \mathbb{T}_+^3 \cup \mathbb{T}_-^3$.

A *control* ω is a family from $\mathbb{T}_+^2 := \{0 \leqslant s \leqslant t \leqslant T\}$ to $\mathbb{R}_+$ which is *super-additive*, that is

$$\omega_{r,s} + \omega_{s,t} \leqslant \omega_{r,t},\ \forall (r,s,t) \in \mathbb{T}_+^3,$$

and continuous close to its diagonal with $\omega_{s,s} = 0$, $s \in \mathbb{T}$. For example $\omega_{s,t} = C|t-s|$ where C is a non-negative constant.

A *remainder* is a continuous, increasing function $\varpi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for some $0 < \varkappa < 1$,

$$2\varpi\left(\frac{\delta}{2}\right) \leqslant \varkappa\varpi(\delta),\ \delta > 0. \tag{2}$$

A typical example for ϖ is $\varpi(\delta) = \delta^\theta$ for any $\theta > 1$.

Let $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ be non-decreasing function with $\lim_{T\to 0} \delta_T = 0$.

We fix $\gamma \in (0, 1]$. We also consider a continuous, increasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\eta(\omega_{s,t})\varpi(\omega_{s,t})^\gamma \leqslant \delta_T\varpi(\omega_{s,t}),\ \forall (s,t) \in \mathbb{T}_+^2. \tag{3}$$

2.2 Functions spaces

We denote by $(\mathrm{V}, |\cdot|)$ a Banach spaces. The space of continuous functions from V to V is denoted by $\mathscr{C}(\mathrm{V})$. We set $\|x\|_\infty := \sup_{t\in[0,T]}|x_t|$.

Notation 1. We denote by $\mathscr{F}^+(\mathrm{V})$ the space of families $\{\phi_{t,s}\}_{(s,t)\in\mathbb{T}_+^2}$ with $\phi_{t,s} \in \mathscr{C}(\mathrm{V})$ for each $(s,t) \in \mathbb{T}_+^2$. We also set $\mathscr{F}^-(\mathrm{V})$ the space of families $\{\phi_{s,t}\}_{(s,t)\in\mathbb{T}_+^2}$ with $\phi_{s,t} \in \mathscr{C}(\mathrm{V})$ for each $(s,t) \in \mathbb{T}_+^2$ (note the reversion of the indices).

We now consider a partition $\pi = \{t_0 \leqslant \cdots \leqslant t_n\}$ of $[0,T]$ with a mesh denoted by $|\pi|$.

Notation 2 (Iterated products)**.** For $\phi \in \mathscr{F}^+(\mathrm{V})$, we write

$$\phi^\pi_{t,s} := \phi_{t,t_j} \circ \phi_{t_j,t_{j-1}} \circ \cdots \circ \phi_{t_{i+1},t_i} \circ \phi_{t_i,s},$$

where $[t_i, t_j]$ is the biggest interval of such kind contained in $[s,t]$. We say that $\phi^\pi_{t,s}$ is the *iterated product* of ϕ on a subdivision π. If no such interval exists, then $\phi^\pi_{t,s} = \phi_{t,s}$.

For $\phi \in \mathscr{F}^+(\mathrm{V})$, we define similarly

$$\phi^\pi_{s,t} := \phi_{s,t_1} \circ \phi_{t_1,t_2} \circ \cdots \circ \phi_{t_{j-1},t_j} \circ \phi_{t_j,t}.$$

For any partition π, $\phi^\pi \in \mathscr{F}^\pm(\mathrm{V})$ when $\phi \in \mathscr{F}^\pm(\mathrm{V})$. A trivial but important remark is that from the very construction,

$$\phi^\pi_{t,s} = \phi^\pi_{t,r} \circ \phi^\pi_{r,s} \text{ for any } r \in \pi.$$

In particular, $\{\phi^\pi_{t,s}\}_{(s,t)\in\mathbb{T}_\pm^2,\ s,t\in\pi}$ enjoys a (semi-)flow property (Definition 3). A natural question is then to study the limit of ϕ^π as the mesh of π decreases to 0.

Finally, for any $(r,s,t) \in \mathbb{T}_\pm^3$ we write $\phi_{t,s,r} := \phi_{t,s} \circ \phi_{s,r} - \phi_{t,r}$.

Notation 3. We extend the norm $|\cdot|$ on $\mathscr{F}^\pm(\mathrm{V})$ by

$$\|\phi\|_\varpi := \sup_{\substack{(s,t)\in\mathbb{T}_\pm^2 \\ s\neq t}} \frac{\|\phi_{t,s}\|_\infty}{\varpi(\omega_{s,t})},$$

where ω, ϖ are defined in Section 2. Possibly, $\|\phi\|_\varpi = \infty$. Actually, this norm is mainly used to consider the distance between two elements of $\mathscr{F}^\pm(\mathrm{V})$. With this norm, $(\mathscr{F}^\pm(\mathrm{V}), \|\cdot\|_\varpi)$ is a Banach space.

Definition 1. We define the equivalence relation $\sim$ on $\mathscr{F}^{\pm}(\mathrm{V})$ by $\phi \sim \psi$ if and only if there exists a constant C such that

$$\|\phi_{t,s} - \psi_{t,s}\|_\infty \leqslant C\varpi(\omega_{s,t}),\ \forall (s,t) \in \mathbb{T}^2.$$

In other words, $\phi \sim \psi$ if and only if $\|\phi - \psi\|_\varpi < +\infty$. Each quotient class of $\mathscr{F}^{\pm}(\mathrm{V})/\sim$ is called a *galaxy*, which contains elements of $\mathscr{F}^{\pm}(\mathrm{V})$ which are at finite distance from each others.

Notation 4 (Lipschitz semi-norm)**.** The Lipschitz semi-norm of a function f from a Banach space $(\mathrm{V}, |\cdot|)$ to another Banach space $(\mathrm{W}, |\cdot|')$ is

$$\|f\|_{\mathrm{Lip}} := \sup_{\substack{a,b\in \mathrm{V},\\ a\neq b}} \frac{|f(a) - f(b)|'}{|a-b|},$$

whenever this quantity is finite. And if $A \subset \mathrm{V}$ is a non-empty subset of V, we say that f is Lipschitz continuous on A when

$$\|f\|_{\mathrm{Lip},A} := \sup_{\substack{a,b\in A,\\ a\neq b}} \frac{|f(a) - f(b)|'}{|a-b|} < +\infty.$$

Notation 5 (Hölder spaces)**.** For $\gamma \in (0,1)$ and an integer r, we denote by $\mathscr{C}_b^{r+\gamma}(\mathrm{V})$ the space of bounded continuous functions from V to V with bounded derivatives up to order r and a r order derivative which is γ-Hölder continuous.

3 Almost flow and Uniform Lipschitz condition

In this section, we recall some notions and results introduced in [6], which are useful in next sections. As we are working on Banach spaces instead of metric spaces, we have a slightly stronger notion of almost flow than in [6].

We denote by $\mathfrak{i}$ the identity map from V to V.

Definition 2 (Almost flow)**.** An element $\phi \in \mathscr{F}^{+}(\mathrm{V})$ is an *almost flow* if for any $T > 0$ and any $(r,s,t) \in \mathbb{T}_+^3$, $a, b \in \mathrm{V}$,

$$\phi_{t,t} = \mathfrak{i}, \tag{4}$$

$$\|\phi_{t,s} - \mathfrak{i}\|_\infty \leqslant \delta_T, \tag{5}$$

$$|\phi_{t,s}(b) - \phi_{t,s}(a)| \leqslant (1+\delta_T)|b-a| + \eta(\omega_{s,t})|b-a|^\gamma, \tag{6}$$

$$\|\phi_{t,s,r}\|_\infty \leqslant M\varpi(\omega_{r,t}), \tag{7}$$

where $M \geqslant 0$ and $\phi_{t,s,r} := \phi_{t,s} \circ \phi_{s,r} - \phi_{t,r}$. If we replace $(r,s,t) \in \mathbb{T}_+^3$ by $(r,s,t) \in \mathbb{T}_-^3$, we say that ϕ is a *reverse almost flow*.

Definition 3 (Semi-flow and Flow)**.** A *semi-flow* ψ is a family of functions $(\psi_{t,s})_{(s,t)\in\mathbb{T}^2_+}$ from V to V such that $\phi_{t,t} = \mathfrak{i}$ and

$$\psi_{t,s} \circ \psi_{s,r} = \psi_{t,r} \tag{8}$$

for any $a \in \mathrm{V}$ and $(r, s, t) \in \mathbb{T}^3_+$. It is a *flow* if each $\phi_{t,s}$ is invertible with an inverse $\phi_{s,t}$ for any $(s, t) \in \mathbb{T}^2_+$ and (8) holds for any $(r, s, t) \in \mathbb{T}^3 := \mathbb{T}^3_+ \cup \mathbb{T}^3_-$.

Remark 1. A flow is invertible and for any $(s, t) \in \mathbb{T}^2$, then $\psi^{-1}_{t,s} = \psi_{s,t}$. Indeed, $\psi_{t,s} \circ \psi_{s,t} = \psi_{t,t} = \mathfrak{i}$ and $\psi_{s,t} \circ \psi_{t,s} = \psi_{s,s} = \mathfrak{i}$.

Theorem 1 ([6])**.** *Let ϕ be an almost flow (Definition 2) with $M \geqslant 0$ and δ_T, κ defined in Section 2.1 Then there exists a time horizon T small enough and a constant $L \leqslant 2M/(1 - (1 + \delta_T)\kappa - \delta_T)$ such that*

$$\|\phi^\pi_{t,s} - \phi_{t,s}\|_\infty \leqslant L\varpi(\omega_{s,t}) \tag{9}$$

for any $(s, t) \in \mathbb{T}^2_+$ and any partition π of $\mathbb{T}$.

Definition 4 (Condition UL)**.** An almost flow ϕ such that $\|\phi^\pi_{s,t}\|_{\mathrm{Lip}} \leqslant 1 + \delta_T$ for any $(s, t) \in \mathbb{T}^2_+$ whatever the partition π is said to satisfy the *uniform Lipschitz* (UL) condition.

We give a sufficient condition on an almost flow to get a Lipschitz flow in a galaxy.

Proposition 1. *Let ϕ be an almost flow which satisfies the condition UL. Then there exists a Lipschitz flow ψ with $\|\psi_{t,s}\|_{\mathrm{Lip}} \leqslant 1 + \delta_T$ for any $(s, t) \in \mathbb{T}^2_+$ such that $\phi^\pi_{t,s}(a)$ converges to $\psi_{t,s}(a)$ for any $a \in \mathrm{V}$ and any $(s, t) \in \mathbb{T}^2_+$.*

On the other hand, there could be at most one flow in a galaxy if one is Lipschitz.

Proposition 2. *Assume that there is a Lipschitz flow ψ in a galaxy G. Then ψ is the unique flow in G. Besides, for any almost flow $\phi \sim \psi$, $\phi^\pi_{s,t}(a)$ converges to $\psi_{s,t}(a)$ for any $(s, t) \in \mathbb{T}^2_+$ and $a \in \mathrm{V}$.*

We then complete the results of [6] with the following ones.

Proposition 3. *Let ϕ be an almost flow which satisfies the condition UL. Then ϕ^π is an almost flow for any partition π.*

The prototypical example for the next result is $\varpi(\delta) = \delta^\theta$ for some $\theta > 1$. In this case, it slightly improves the rate of convergence as $\theta - 1$ with respect to the one given in [6] which is $\theta - 1 - \epsilon$ for any $\epsilon > 0$.

Proposition 4 (Rate of convergence)**.** *Let ϕ be an almost flow in the same galaxy as a Lipschitz flow ψ with $\|\psi_{t,s}\|_{\mathrm{Lip}} \leqslant (1+\delta_T)$ and $\|\psi_{t,s} - \phi_{t,s}\|_\infty \leqslant K\varpi(\omega_{s,t})$ for any $(s,t) \in \mathbb{T}_+^2$. Let us assume that ϖ is such that for a bounded function μ, $\delta^{-1}\varpi(\delta) \leqslant \mu(\delta)$ for any $\delta > 0$. Then*

$$\|\psi_{t,s}(a) - \phi^\pi_{t,s}(a)\|_\infty \leqslant KM(\pi)\omega_{0,T}(1+\delta_T) \text{ with } M(\pi) := \sup_{\substack{(r,t) \text{ successive} \\ \text{points in } \pi}} \mu(\omega_{r,t}).$$

Proof. The proof follows the one of Theorem 10.30 in [19, p. 238]. Let $\{t_i\}_{i=0,\dots,n}$ be the points of $\pi \cup \{s,t\}$ such that $t_0 = s$, $t_n = t$ and (t_i, t_{i+1}) are successive points in $\pi \cup \{s,t\}$. Set $z_k = \psi_{t,t_k}(\phi_{t_k,t}(a))$. Then

$$|\psi_{t,s}(a) - \phi^\pi_{t,s}(a)| = |z_n - z_0| \leqslant \sum_{i=0}^{n-1} |z_{i+1} - z_i|.$$

Since $\psi_{t,s}$ is Lipschitz and

$$|z_{i+1} - z_i| \leqslant (1+\delta_T)|\psi_{t_{k+1},t_k}(\phi_{t_{k+1},t_k}(a)) - \psi_{t_{k+1},t_k}(\phi_{t_{k+1},t_k}(a))|.$$

The result follows easily. □

4 Stable almost flows

In the previous section, we have recalled some results from [6] which endow the importance of Lipschitz flows. However, the UL condition is not easy to verify. In this section, we give a sufficient condition on an almost flow ϕ to ensure that it satisfies the UL condition and then that its galaxy contains a unique flow which is Lipschitz.

4.1 The 4-points control

In this section $\mathrm{V}, \mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3$ are Banach spaces and we denote by $|\cdot|$ their norms.

Definition 5 (The 4-points control)**.** A function $f : \mathrm{V}_1 \to \mathrm{V}_2$ is said to satisfy a 4-*points control* if there exists a non-decreasing, continuous function $\widehat{f} : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $\check{f} \geqslant 0$ such that

$$\begin{aligned} &|f(a) - f(b) - f(c) + f(d)| \\ &\qquad \leqslant \widehat{f}(|a-b| \vee |c-d|) \times (|a-c| \vee |b-d|) + \check{f}|a-b-c+d| \end{aligned} \tag{10}$$

for any $(a,b,c,d) \in \mathrm{V}_1$.

Any Lipschitz function f satisfies a 4-points control with $\widehat{f} = 2\|f\|_{\mathrm{Lip}}$ and $\widecheck{f} = 0$. However, for our purpose, we need to consider later more restrictive conditions on $\widecheck{f}$ and $\widehat{f}$.

Let us start with a simple example that is found in [14].

Lemma 1. *Let $f \in \mathscr{C}^{1+\gamma}(\mathrm{V}_1, \mathrm{V}_2)$, $0 < \gamma \leqslant 1$, with a bounded derivative. Then f satisfies a 4-points control with*

$$\widehat{f}(x) = 2\|\nabla f\|_\gamma x^\gamma,\ x \geqslant 0,\ \text{and}\ \widecheck{f} = \|\nabla f\|_\infty.$$

Proof. For any $a, b, c, d \in \mathrm{V}_1$,

$$\begin{aligned} f(a) - f(b) - f(c) + f(d) \\ = (a-b)\int_0^1 \nabla f(au + (1-u)b)\,\mathrm{d}u - (c-d)\int_0^1 \nabla f(cu+(1-u)d)\,\mathrm{d}u \\ = (a-b)\int_0^1 [\nabla f(au+(1-u)b) - \nabla f(cu+(1-u)d)]\,\mathrm{d}u \\ + (a-b-c+d)\int_0^1 \nabla f(cu+(1-u)d)\,\mathrm{d}u, \end{aligned}$$

which yields to

$$\begin{aligned} |f(a) - f(b) - f(c) + f(d)| \leqslant |a-b| 2\|\nabla f\|_\gamma (|a-b| \vee |c-d|)^\gamma \\ + \|\nabla f\|_\infty |a-b-c+d|. \end{aligned}$$

This concludes the proof. □

Here are a few properties of functions satisfying a 4-points control.

Lemma 2. *Let f, g satisfying a 4-points control with g Lipschitz continuous.*

(i) *The function f is locally Lipschitz continuous.*
(ii) *If $f, g : \mathrm{V}_1 \to \mathrm{V}_2$, then for any $\lambda, \mu \in \mathbb{R}$, $\lambda f + \mu g$ satisfies a 4-points control.*
(iii) *If $f : \mathrm{V}_1 \to \mathrm{V}_2$ and $g : \mathrm{V}_3 \to \mathrm{V}_1$, then $f \circ g : \mathrm{V}_3 \to \mathrm{V}_2$ satisfies a 4-points control.*
(iv) *If $g : \mathrm{V} \to \mathrm{V}$ with $\|g\|_{\mathrm{Lip}} < 1$, then $\mathfrak{i} + g$ is invertible and $k := (\mathfrak{i} + g)^{-1}$ is Lipschitz and satisfies a 4-points control with $\widehat{k}(x) = \widehat{g}(\|k\|_{\mathrm{Lip}} x)\|k\|_{\mathrm{Lip}}$ for $\|k\|_{\mathrm{Lip}} \leqslant 1/(1 - \|g\|_{\mathrm{Lip}})$, and $\widecheck{k} = 1/(1 - \widecheck{g})$.*

Proof. For showing (i), we choose $a = d$ and $b = c$ in (10) and then,

$$|f(a) - f(b)| \leqslant \left[\widehat{f}(|a-b|) + \widecheck{f}\right]|a-b|,$$

which proves that f is locally Lipschitz continuous.

Moreover, we can choose $\widehat{\lambda f + \mu g} = |\lambda|\widehat{f} + |\mu|\widehat{g}$ and $\widecheck{\lambda f + \mu g} = |\lambda|\widecheck{f} + |\mu|\widecheck{g}$ to obtain a 4-points control on $\lambda f + \mu g$. This proves (ii).

To show (iii), we use the fact that g is Lipschitz according to (i). With $h = f \circ g$,

$$
\begin{aligned}
&|h(a) - h(b) - h(c) + h(d)| \\
&\qquad \leqslant \widehat{f}(|g(a) - g(b)| \vee |g(c) - g(d)|) \, [|g(a) - g(c)| \vee |g(b) - g(d)|] \\
&\qquad\quad + \widecheck{f}\widehat{g}(|a - b| \vee |c - d|) \, [|a - c| \vee |b - d|] + \widecheck{f}\widecheck{g}|a - b - c + d| \\
&\qquad\qquad \leqslant \widehat{f}(\|g\|_{\mathrm{Lip}}|a - b| \vee |c - d|)\|g\|_{\mathrm{Lip}}(|a - c| \vee |b - d|) \\
&\qquad\qquad\quad + \widecheck{f}\widehat{g}(|a - b| \vee |c - d|)[|a - c| \vee |b - d|] + \widecheck{f}\widecheck{g}|a - b - c + d|.
\end{aligned}
$$

This proves that h satisfies the 4-points control.

It remains to show (iv). From the Lipschitz inverse function theorem [1, p. 124], $\mathfrak{i} + g$ is invertible with an inverse k that satisfies $\|k\|_{\mathrm{Lip}} \leqslant (1 - \|g\|_{\mathrm{Lip}})^{-1}$. Besides,

$$
\begin{aligned}
|a - b - c + d + (g(a) - g(b) - g(c) + g(d))| \geqslant{} & (1 - \widecheck{g})|a - b - c + d| \\
& - \widehat{g}(|a - b| \vee |c - d|) \times [|a - c| \vee |b - d|] \,,
\end{aligned}
$$

which yields to

$$
\begin{aligned}
(1 - \widecheck{g})|k(a) - k(b) - k(c) + k(d)| \leqslant{} & |a - b - c + d| \\
& + \widehat{g}(\|k\|_{\mathrm{Lip}}|a - b| \vee |c - d|)\|k\|_{\mathrm{Lip}} \, [|a - c| \vee |b - d|] \,.
\end{aligned}
$$

Hence, for $x \in \mathbb{R}_+$, $\widehat{k} = \widehat{g}(\|k\|_{\mathrm{Lip}} x)\|k\|_{\mathrm{Lip}}$ and $\widecheck{k} = (1 - \widecheck{g})^{-1}$. Therefore, k satisfies a 4-points control. □

The reason for introducing the 4-points control lies in its good behavior with respect to composition. More precisely, if f satisfies a 4-points control while g and h are Lipschitz continuous and bounded,

$$
\begin{aligned}
\|f \circ g - f \circ h\|_{\mathrm{Lip}} &\leqslant \widehat{f}(\|g - h\|_\infty)\|g\|_{\mathrm{Lip}} \vee \|h\|_{\mathrm{Lip}} + \check{f}\|g - h\|_{\mathrm{Lip}}, \\
\|f \circ g - f \circ h\|_\infty &\leqslant \left(\widehat{f}(0) + \check{f}\right) \|g - h\|_\infty.
\end{aligned}
$$

4.2 Definition of a stable almost flow

Definition 6. A family $\phi \in \mathscr{F}^+(V)$ is said to satisfy a ϖ-*compatible* 4-*points control* if there exists a family of functions $(\widehat{\phi}_{t,s})_{(s,t)\in\mathbb{T}_+^2}$ and constants $(\widecheck{\phi}_{t,s})_{(s,t)\in\mathbb{T}^2}$

such that for any $[s,t] \subset [0,T]$ the estimation (10) holds. Moreover, for any $[s,t] \subset [u,v]$, $(\widehat{\phi}_{t,s})_{(s,t)\in\mathbb{T}_+^2}$ is said ϖ-compatible if

$$\widehat{\phi}_{s,t} \leqslant \widehat{\phi}_{u,v},$$
$$\widehat{\phi}_{t,s}(\alpha\varpi(\omega_{s,t})) \leqslant \phi^{\circledast}(\alpha)\varpi(\omega_{s,t}),$$

where $\alpha, \phi^{\circledast}(\alpha)$ are non-negative constants with $\phi^{\circledast}(\alpha)$ which can depend on α.

Definition 7 (Stable almost flow)**.** We say that an almost flow ϕ with $\gamma = 1$ in (6) is a *stable almost flow* if

- it satisfies a ϖ-compatible 4-points control with

$$\check{\phi}_{t,s} \leqslant 1 + \delta_T,$$

- there a constant $C \geqslant 0$ such that for $(r,s,t) \in \mathbb{T}_+^3$,

$$\|\phi_{t,s,r}\|_{\mathrm{Lip}} \leqslant C\varpi(\omega_{r,t}), \tag{11}$$

  where $\phi_{t,s,r} = \phi_{t,s} \circ \phi_{s,r} - \phi_{t,r}$.

We denote the family of stable almost flow $\mathcal{SA}_{\delta_T,\varpi}(V)$. If we replace assumption $(r,s,t) \in \mathbb{T}_+^3$ by $(r,s,t) \in \mathbb{T}_-^3$, we say that ϕ is a *reverse stable almost flow.*

4.3 Non linear sewing lemma for stable almost flow

The following proposition justifies Definition 7.

Theorem 2. *If $\phi \in \mathcal{SA}_{\delta_T,\varpi}(\mathrm{V})$ is a stable almost flow then for any partition π,*

$$\|\phi^\pi_{t,s} - \phi_{t,s}\|_{\mathrm{Lip}} \leqslant L\varpi(\omega_{s,t}), \forall (s,t) \in \mathbb{T}_+^2, \tag{12}$$

where L is a constant that depends on T, $T \mapsto \delta_T$, $\varkappa$, ω, $\check{\phi}$, $\phi^{\circledast}$ and C in (11)*. In particular, the almost flow ϕ satisfies the condition UL, up to changing δ_T.*

Remark 2. When ϕ is a stable almost flow, we assume that $\gamma = 1$ in Definition 2. This implies that (6) becomes

$$|\phi_{t,s}(b) - \phi_{t,s}(a)| \leqslant (1+\delta_T)|b-a|. \tag{13}$$

Proof. Let us choose a partition π. Let $r \in \mathbb{T}$ be fixed and $(s,t) \in \mathbb{T}_+^2$, such that $r \leqslant s$,

$$U^\pi_{s,t} := \|\phi^\pi_{t,r} - \phi_{t,s} \circ \phi^\pi_{s,r}\|_{\mathrm{Lip}}.$$

For any $0 \leqslant r \leqslant s \leqslant t \leqslant u \leqslant T$,

$$U^\pi_{s,u} \leqslant U^\pi_{t,u} + \|\phi_{u,t} \circ \phi^\pi_{t,r} - \phi_{u,t} \circ \phi_{t,s} \circ \phi^\pi_{s,r}\|_{\mathrm{Lip}} + \|\phi_{u,t} \circ \phi_{t,s} \circ \phi^\pi_{s,r} - \phi_{u,s} \circ \phi^\pi_{s,r}\|_{\mathrm{Lip}}. \tag{14}$$

With the 4-points control on $\phi_{u,t}$,

$$\|\phi_{u,t} \circ \phi^\pi_{t,r} - \phi_{u,t} \circ \phi_{t,s} \circ \phi^\pi_{s,r}\|_{\mathrm{Lip}} \\ \leqslant \widehat{\phi}_{u,t}\left(\|\phi^\pi_{t,r} - \phi_{t,s} \circ \phi^\pi_{s,r}\|_\infty\right) \times \left(\|\phi^\pi_{t,r}\|_{\mathrm{Lip}} \vee (1+\delta_T)\|\phi^\pi_{s,r}\|_{\mathrm{Lip}}\right) + \check{\phi}_{u,t} U^\pi_{s,t}.$$

According to (9) (Theorem 1) for T small enough,

$$\|\phi^\pi_{t,r} - \phi_{t,s} \circ \phi^\pi_{s,r}\|_\infty \leqslant C\varpi(\omega_{s,t}).$$

Since $\widehat{\phi_{u,t}}$ is ϖ-compatible and with the control (6) with $\gamma = 1$ of the Definition 2 of almost flow,

$$\|\phi_{u,t} \circ \phi^\pi_{t,r} - \phi_{u,t} \circ \phi_{t,s} \circ \phi^\pi_{s,r}\|_{\mathrm{Lip}} \\ \leqslant \phi^{\circledast}(C)\varpi(\omega_{r,t})\left(\|\phi^\pi_{t,r}\|_{\mathrm{Lip}} \vee (1+\delta_T)\|\phi^\pi_{s,r}\|_{\mathrm{Lip}}\right) + \check{\phi}_{u,t} U^\pi_{s,t}.$$

For bounding the last term of (14), (11) yields

$$\|\phi_{u,t} \circ \phi_{t,s} \circ \phi^\pi_{s,r} - \phi_{u,s} \circ \phi^\pi_{s,r}\|_{\mathrm{Lip}} \leqslant C\varpi(\omega_{s,u})\|\phi^\pi_{s,r}\|_{\mathrm{Lip}}.$$

Assuming that r, s, t and u belong to π and combining these inequalities and the fact the ϕ is stable (see Definition 7),

$$U^\pi_{s,u} \leqslant U^\pi_{t,u} + (1+\delta_T)U^\pi_{s,t} + L^\pi(\phi^{\circledast}(C)(2+\delta_T) + C)\varpi(\omega_{s,u})$$

where

$$L^\pi := \sup_{\substack{(s,t)\in\mathbb{T}^2_+\\ s,t\in\pi}} \|\phi^\pi_{t,s}\|_{\mathrm{Lip}}.$$

For two successive points s and $t \geqslant s$ of π (See Definition 13), $\phi^\pi_{t,r} = \phi_{t,s} \circ \phi^\pi_{s,r}$ so that $U^\pi_{s,t} = 0$.

We assume that T is small enough so that $\varkappa(1 + \delta_T^2 + \delta_T) < 1$. From the Davie Lemma (Lemma 9 in Appendix),

$$U^\pi_{s,t} \leqslant L^\pi \alpha_T \varpi(\omega_{s,t}) \text{ with } \alpha_T := (2+\delta_T)\frac{(\phi^{\circledast}(C)(2+\delta_T) + C)}{1 - \varkappa(1+\delta_T^2 + \delta_T)}. \tag{15}$$

In particular, for $r = s$, $U_{r,t} = \|\phi^\pi_{t,r} - \phi_{t,r}\|_{\text{Lip}}$.

Let us bound L^π. For this, with (15) and (6) with $\gamma = 1$,

$$\begin{aligned} L^\pi &\leqslant \sup_{(s,t)\in\mathbb{T}^2_+} \|\phi^\pi_{t,s} - \phi_{t,s}\|_{\text{Lip}} + \max_{(s,t)\in\mathbb{T}^2_+} \|\phi_{t,s}\|_{\text{Lip}} \\ &\leqslant L^\pi \alpha_T \varpi(\omega_{0,T}) + 1 + \delta_T. \end{aligned}$$

For T small enough so that $\alpha_T\varpi(\omega_{0,T}) \leqslant 1/2$, L^π is uniformly bounded. Injecting this control of L^π in (15),

$$\|\phi^\pi_{t,r} - \phi_{t,r}\|_{\text{Lip}} \leqslant K\varpi(\omega_{r,t}),\ \forall (r,t) \in \mathbb{T}^2_+,\ r,t \in \pi, \tag{16}$$

for some constant K that does not depend on the partition π.

It remains to establish (16) for any pair of time $(r,t) \in \mathbb{T}^2_+$.

For this, let t_π (resp. r_π) be the greatest (resp. smallest) point of π below (resp. above) t (resp. r). Then with the definition of ϕ^π (Notation 2),

$$\phi^\pi_{t,r} - \phi_{t,r} = \phi_{t,t_\pi} \circ \phi^\pi_{t_\pi,r} - \phi_{t,t_\pi} \circ \phi_{t_\pi,r} + \phi_{t,t_\pi,r}.$$

With (11) and (6) of Definition 2 with $\gamma = 1$,

$$\|\phi^\pi_{t,r} - \phi_{t,r}\|_{\text{Lip}} \leqslant (1+\delta_T)\|\phi^\pi_{t_\pi,r} - \phi_{t_\pi,r}\|_{\text{Lip}} + C\varpi(\omega_{r,t}). \tag{17}$$

Similarly,

$$\phi^\pi_{t_\pi,r} - \phi_{t_\pi,r} = \phi^\pi_{t_\pi,r_\pi} \circ \phi^\pi_{r_\pi,r} - \phi_{t_\pi,r_\pi} \circ \phi_{r_\pi,r} + \phi_{t_\pi,r_\pi,r}.$$

Using (9) and (16),

$$\begin{aligned} \|\phi^\pi_{t_\pi,r} - \phi_{t_\pi,r}\|_{\text{Lip}} &\leqslant \|\phi^\pi_{t_\pi,r_\pi} - \phi_{t_\pi,r_\pi}\|_{\text{Lip}} \cdot \|\phi_{r_\pi,r}\|_{\text{Lip}} + C\varpi(\omega_{r,t}) \\ &\leqslant (1+\delta_T)K\varpi(\omega_{r,t}) + C\varpi(\omega_{r,t}). \end{aligned} \tag{18}$$

Inequality (12), which is (16) applied for any $(r,t) \in \mathbb{T}^2_+$, stems from (16), (17) and (18). □

Corollary 1. *If $\phi \in \mathcal{SA}_{\delta_T,\varpi}(\mathrm{V})$ is a stable almost flow then there exists a unique Lipschitz flow ψ in the galaxy containing ϕ. Moreover, there is a constant $L \geqslant 0$ such that for all $(s,t) \in \mathrm{V}$,*

$$\|\psi_{t,s} - \phi_{t,s}\|_{\text{Lip}} \leqslant L\varpi(\omega_{s,t}). \tag{19}$$

Proof. According to Theorem 2, ϕ satisfies the UL condition of Proposition 4. Hence, it converges in the sup-norm to a Lipschitz flow $\psi \sim \phi$. According to Proposition 2, ψ is the only flow in the galaxy of ϕ.

Passing to the limit in (12) leads to (19). □

5 Perturbations

In [6], we have introduced the notion of perturbation of almost flow. This notion still gives an almost flow.

We recall that η, δ_T and γ are defined in Section 2.1.

Definition 8 (Perturbation)**.** A *perturbation* is an element $\epsilon \in \mathscr{F}(\mathrm{V})$ such that for any $(s,t) \in \mathbb{T}_+^2$ and $a, b \in \mathrm{V}$,

$$\epsilon_{t,t} = 0, \tag{20}$$

$$\|\epsilon_{t,s}\|_\infty \leqslant C\varpi(\omega_{s,t}), \tag{21}$$

$$|\epsilon_{t,s}(b) - \epsilon_{t,s}(a)| \leqslant \delta_T|b-a| + \eta(\omega_{s,t})|b-a|^\gamma, \tag{22}$$

where η is defined by (3) and $C \geqslant 0$ is a constant.

Proposition 5 ([6, Proposition 1])**.** *If $\phi \in \mathscr{F}(\mathrm{V})$ is an almost flow and $\epsilon \in \mathscr{F}(\mathrm{V})$ is a perturbation, then $\psi := \phi + \epsilon$ is an almost flow in the same galaxy as ϕ.*

We introduce now the notion of Lipschitz perturbation, which is a perturbation on which a control stronger than (22) holds.

Definition 9 (Lipschitz perturbation)**.** A *Lipschitz perturbation* is a perturbation $\epsilon \in \mathscr{F}^+(\mathrm{V})$ with satisfies for a constant $C \geqslant 0$

$$\|\epsilon_{t,s}\|_{\mathrm{Lip}} \leqslant C\varpi(\omega_{s,t}),\ \forall (s,t) \in \mathbb{T}_+^2. \tag{23}$$

Stable almost flows remain stable almost flows under Lipschitz perturbations.

Proposition 6 (Stability of stable almost flow under Lipschitz perturbation)**.** *If $\phi \in \mathcal{SA}_{\delta_T,\varpi}$ is a stable almost flow (see Definition 7) and ϵ is a Lipschitz perturbation, then $\psi := \phi + \epsilon$ is also a stable almost flow.*

Proof. It is proved in Proposition 5 that $\phi + \epsilon$ is an almost flow. Here we show that $\phi + \epsilon$ is a stable almost flow.

First, for any $a, b, c, d \in V$,

$$|\epsilon_{t,s}(a) - \epsilon_{t,s}(b) - \epsilon_{t,s}(c) + \epsilon_{t,s}(d)| \leqslant 2C\varpi(\omega_{s,t})|a-c| \vee |b-d|,$$

so that $\epsilon_{t,s}$ satisfies a ϖ-compatible 4-points control (see Definition 6) with $\widehat{\epsilon}_{t,s} := 2C\varpi(\omega_{s,t})$ and $\widecheck{\epsilon}_{t,s} := 0$. Thus, $\phi + \epsilon$ satisfies a ϖ-compatible 4-points control with $\widehat{\phi + \epsilon} = \widehat{\phi} + 2C\varpi(\omega_{s,t})$ and $\widecheck{\phi_{t,s} + \epsilon_{t,s}} = \widecheck{\phi_{t,s}} \leqslant 1 + \delta_T$.

It remains to show that for any $(r,s,t) \in \mathbb{T}_+^3$, $\|\psi_{t,s,r}\|_{\mathrm{Lip}} \leqslant C\varpi(\omega_{r,t})$, with $\psi_{t,s,r} := \psi_{t,s} \circ \psi_{s,r} - \psi_{t,r}$. For any $a \in V$, we write

$$\psi_{t,s,r}(a) = \underbrace{[\phi_{t,s} \circ (\phi_{s,r} + \epsilon_{s,r})(a) - \phi_{t,s} \circ \phi_{s,r}(a)]}_{\mathrm{I}_{r,s,t}(a)} + \underbrace{[\epsilon_{t,s} \circ (\phi_{s,r} + \epsilon_{s,r})(a) - \epsilon_{t,s} \circ \epsilon_{s,r}(a)]}_{\mathrm{II}_{r,s,t}(a)} + \underbrace{\phi_{t,s,r}(a) + \epsilon_{t,s,r}(a)}_{\mathrm{III}_{r,s,t}(a)}.$$

On the one hand, using the ϖ-compatible 4-points control of $\phi_{t,s}$, (13), (21) and (23) we write,

$$\begin{aligned}\|\mathrm{I}_{r,s,t}\|_{\mathrm{Lip}} &\leqslant \widehat{\phi_{t,s}}(\|\epsilon_{s,r}\|_\infty)(\|\phi_{s,r}\|_{\mathrm{Lip}} + \|\epsilon_{s,r}\|_{\mathrm{Lip}}) + \widecheck{\phi_{t,s}}\|\epsilon_{s,r}\|_{\mathrm{Lip}} \\ &\leqslant \widehat{\phi_{t,s}}(C\varpi(\omega_{r,t}))(1+\delta_T + C\varpi(\omega_{0,T})) + (1+\delta_T)C\varpi(\omega_{r,t}) \\ &\leqslant (C'(1+\delta_T + C\varpi(\omega_{0,T})) + 1 + \delta_T)\varpi(\omega_{r,t}),\end{aligned}$$

where C' is a constant.

On the other hand, with (6), (23)

$$\begin{aligned}\|\mathrm{II}_{r,s,t}\|_{\mathrm{Lip}} &\leqslant \|\epsilon_{t,s}\|_{\mathrm{Lip}}(\|\phi_{s,r}\|_{\mathrm{Lip}} + \|\epsilon_{s,r}\|_{\mathrm{Lip}}) + \|\epsilon_{t,s}\|_{\mathrm{Lip}}\|\epsilon_{s,r}\|_{\mathrm{Lip}} \\ &\leqslant C\varpi(\omega_{r,t})(1+\delta_T + C\varpi(\omega_{0,T})) + C^2\varpi(\omega_{0,T})\varpi(\omega_{r,t}) \leqslant K_T\varpi(\omega_{r,t}),\end{aligned}$$

where $K_T \to 1$ when $T \to 0$.

Finally, with (11) and (23),

$$\|\mathrm{III}_{r,s,t}\|_{\mathrm{Lip}} \leqslant \|\phi_{t,s,r}\|_{\mathrm{Lip}} + \|\epsilon_{t,r}\|_{\mathrm{Lip}} + \|\epsilon_{t,s}\|_{\mathrm{Lip}}\|\epsilon_{s,r}\|_{\mathrm{Lip}} \leqslant (2C + C^2\varpi(\omega_{0,T}))\varpi(\omega_{r,t}).$$

This concludes the proof. □

Now, we prove another perturbation formula which is useful in Subsection 8.3. We recall the δ_T, η and γ are defined in Section 2.1.

Proposition 7. *Let ψ be a flow which may be decomposed as*

$$\psi_{t,s}(a) = \phi_{t,s}(a) + \epsilon_{s,t}(a),\ a \in \mathrm{V},\ (s,t) \in \mathbb{T}^2$$

with for any $(s,t) \in \mathbb{T}_+^2$ and $a, b \in \mathrm{V}$,

$$\phi_{t,t} = \mathfrak{i},\ \epsilon_{t,t} = 0, \tag{24}$$

$$|\phi_{t,s}(a) - \phi_{t,s}(b)| \leqslant (1+\delta_T)|a-b| + \eta(\omega_{s,t})|a-b|^\gamma,\ (a,b) \in \mathrm{V}, \tag{25}$$

$$\|\phi_{t,s} - \mathfrak{i}\|_\infty \leqslant \delta_T, \tag{26}$$

$$\|\epsilon_{s,t}\|_\infty \leqslant M\varpi(\omega_{s,t}). \tag{27}$$

Then ϕ is an almost flow in the same galaxy as ψ. Besides, for any partition π of $\mathbb{T}$,

$$\|\phi^{\pi}_{t,s} - \phi_{t,s}\|_\infty \leqslant L\varpi(\omega_{s,t}) \tag{28}$$

where $L \leqslant 2\left[(3+\delta_T)M + \delta_T M^\gamma\right]/(1-(1+\delta_T)\kappa - \delta_T)$.

Proof. To show that ϕ is an almost flow, it is sufficient to consider (24)-(27) as well as controlling $\phi_{t,s,r}$. For $(r,s,t) \in \mathbb{T}^3_+$,

$$\psi_{t,s}\circ\psi_{s,r}(a) = \overbrace{\phi_{t,s}(\phi_{s,r}(a)+\epsilon_{s,r}(a)) - \phi_{t,s}(\phi_{s,r}(a))}^{\mathrm{I}_{r,s,t}} + \phi_{t,s}(\phi_{s,r}(a)) + \epsilon_{t,s}(\psi_{s,r}(a)).$$

Since ψ is a flow, $\psi_{r,s,t} = 0$ and then

$$\phi_{t,s,r}(a) = \mathrm{I}_{r,s,t} + \epsilon_{t,s}(\psi_{s,r}(a)) - \epsilon_{t,r}(a).$$

With (25),

$$|\mathrm{I}_{r,s,t}| \leqslant (1+\delta_T)M\varpi(\omega_{r,s}) + \eta(\omega_{s,t})M^\gamma\varpi(\omega_{r,s})^\gamma.$$

With (3),

$$|\mathrm{I}_{r,s,t}| \leqslant A\varpi(\omega_{r,t}) \text{ with } A := (1+\delta_T)M + \delta_T M^\gamma.$$

It follows that $\|\phi_{t,s,r}\|_\infty \leqslant (2M + A)\varpi(\omega_{r,t})$. This proves that ϕ is an almost flow following Definition 3. The control (28) follows from Theorem 1. □

Corollary 2. *Assume that* V *is a finite-dimensional Banach space. Let $\{\psi^m\}_{m\in\mathbb{N}}$ be a family of flows with decomposition $\psi^m = \phi^m + \epsilon^m$ where (ϕ^m, ϵ^m) satisfy* (24)-(27) *uniformly in m. Assume moreover that*

$$\|\phi^m_{s,t}\|_\infty \leqslant \delta_{t-s},\ \forall (s,t) \in \mathbb{T}^2_+. \tag{29}$$

Then any possible limit ϕ of $\phi^m_{t,s}$ (at least one exists) satisfies (24)-(27) *as well as* (29) *with the same constants.*

Proof. With (29), Lemma 1 in [6] can be applied uniformly. As (25) is also uniform in m, this proves that for any $R > 0$, $\{\phi^m(a)_{s,t}\}_{(s,t)\in\mathbb{T}^2_+,\ a\in\overline{B(0,R)}}$ is equi-continuous where $\overline{B(0,R)}$ is the closed ball of center 0 and some radius $R > 0$. The Ascoli-Arzelà shows that at least one limit of ϕ^m exists. Clearly, this limit satisfies the same properties as ϕ^m. □

6 Inversion of the flow

In this section, we prove that our definition of stable almost (Definition 7) flow is stable respect to inversion.

Proposition 8. *Let $\phi \in \mathcal{ST}_{\delta_T,\varpi}$ be a stable almost flow and ψ the unique flow in the same galaxy as ϕ (Corollary 1). We assume that $\chi := \phi - \mathfrak{i}$ satisfies a 4-point control such that*

$$\forall (s,t) \in \mathbb{T}_+^2,\ \widecheck{\chi_{t,s}} = \widecheck{\phi_{t,s}} - 1,\ \widehat{\chi_{t,s}} = \widehat{\phi_{t,s}},\ \text{and } \|\chi_{t,s}\|_{\mathrm{Lip}} \leqslant \delta_T.$$

Then, ϕ is invertible and $(\zeta_{s,t})_{(s,t)\in\mathbb{T}_+^2} := (\phi_{t,s}^{-1})_{(s,t)\mathbb{T}_+^2}$ is a stable reverse almost flow which galaxy contains a unique flow which equal to ψ^{-1}.

Proof. According to item (iv) of Lemma 2 and because $\|\chi_{t,s}\|_{\mathrm{Lip}} \leqslant \delta_T$ we know that for $T > 0$ such that $\delta_T < 1$, $\phi_{t,s}$ is invertible and that $\phi_{t,s}^{-1}$ satisfies a 4-points control with $\widecheck{\phi_{t,s}^{-1}} = 1/(1-\widecheck{\chi_{t,s}})$ and $\widehat{\phi_{t,s}^{-1}}(x) = \widehat{\chi_{t,s}}(\|\phi_{t,s}\|_{\mathrm{Lip}}x)\|\phi_{t,s}\|_{\mathrm{Lip}}$ for any $x \in \mathbb{R}_+$. It follows that $\widecheck{\phi_{t,s}^{-1}} \leqslant 1 + \delta'_T$, with $\delta'_T := \delta_T/(1-\delta_T)$ and that $\phi_{t,s}$ satisfies a ϖ-compatible 4-points control.

Moreover, $\|\phi_{t,s}^{-1}\|_{\mathrm{Lip}} \leqslant 1/(1-\|\chi_{t,s}\|_{\mathrm{Lip}})$ and we assume $\|\chi_{t,s}\|_{\mathrm{Lip}} \leqslant \delta_T$. It follows that $\|\phi_{t,s}^{-1}\|_{\mathrm{Lip}} \leqslant 1 + \delta'_T$ which proves that (6) holds for ϕ^{-1}. In substituting a by $\phi_{t,s}^{-1}(a)$ in (5) we show that (5) holds for ϕ^{-1}.

To prove that $(\zeta_{s,t})_{(s,t)\in\mathbb{T}_+^2} := (\phi_{t,s}^{-1})_{(s,t)\in\mathbb{T}_+^2}$ is a reverse stable almost flow, it remains to show that the conditions (7) and (11) hold for any $(r,s,t) \in \mathbb{T}_+^3$. Firstly, we compute with (7), since $\phi_{t,s} \circ \phi_{s,r}$ is one-to-one,

$$\begin{aligned}\|\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1} \circ \phi_{t,s} \circ \phi_{s,r} - \phi_{t,r}^{-1} \circ \phi_{t,s} \circ \phi_{s,r}\|_\infty &= \|\phi_{t,r}^{-1} \circ \phi_{t,r} - \phi_{t,r}^{-1} \circ \phi_{t,s} \circ \phi_{s,r}\|_\infty \\ &\leqslant (1+\delta'_T)\|\phi_{t,r} - \phi_{t,s} \circ \phi_{s,r}\|_\infty \leqslant M\varpi(\omega_{r,t}),\end{aligned}$$

which yields with to $\|\zeta_{r,s} \circ \zeta_{s,t} - \zeta_{r,t}\|_\infty \leqslant M\varpi(\omega_{r,t})$.

Secondly, for any $a, b \in \mathrm{V}$ and $(r,s,t) \in \mathbb{T}_+^3$, we set $a' := \phi_{s,r}^{-1} \circ \phi_{t,s}^{-1}(a)$, $b' := \phi_{s,r}^{-1} \circ \phi_{t,s}^{-1}(b)$, and

$$\begin{aligned}\Phi_{r,s,t} &:= (\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1} - \phi_{t,r}^{-1}) \circ \phi_{t,s} \circ \phi_{s,r}(b') - (\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1} - \phi_{t,r}^{-1}) \circ \phi_{t,s} \circ \phi_{s,r}(a') \\ &= \phi_{t,r}^{-1} \circ \phi_{t,r}(b') - \phi_{t,r}^{-1} \circ \phi_{t,s} \circ \phi_{s,r}(b') - \phi_{t,r}^{-1} \circ \phi_{t,r}(a') + \phi_{t,r}^{-1} \circ \phi_{t,s} \circ \phi_{s,r}(a').\end{aligned}$$

We know that $\phi_{t,r}^{-1}$ satisfies a ϖ-compatible 4-points control and we use (11),

$$\begin{aligned}|\Phi_{r,s,t}| &\leqslant \widehat{\phi_{t,r}^{-1}}(\|\phi_{t,s,r}\|_\infty)\,[\|\phi_{t,r}\|_{\mathrm{Lip}} \vee \|\phi_{t,s} \circ \phi_{s,r}\|_{\mathrm{Lip}}]\,|b'-a'| + \widecheck{\phi_{t,r}^{-1}}\|\phi_{t,s,r}\|_{\mathrm{Lip}}|b'-a'| \\ &\leqslant \phi^{-1,\circledast}(M)\varpi(\omega_{r,t})(1+\delta_T)^2|b'-a'| + (1+\delta_T)C\varpi(\omega_{r,t})|b'-a'|.\end{aligned}$$

Then substituting a' and b' by $\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1}(a)$ and $\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1}(a)$,

$$\begin{aligned}\|\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1} - \phi_{t,r}^{-1}\|_{\mathrm{Lip}} &\leqslant [\phi^{-1,\circledast}(M)\varpi(\omega_{r,t})(1+\delta_T)^2 + (1+\delta_T)C\varpi(\omega_{r,t})]\|\phi_{s,r}^{-1} \circ \phi_{t,s}^{-1}\|_{\mathrm{Lip}} \\ &\leqslant \left[\phi^{-1,\circledast}(M))(1+\delta_T)^2 + (1+\delta_T)C\right](1+\delta'_T)^2\varpi(\omega_{r,t}).\end{aligned}$$

Hence ζ is a stable reverse almost flow. According to Corollary 1, ζ^π converges to a unique Lipschitz flow ζ^∞ in $\mathscr{F}(\mathrm{V})$. But, $\zeta_{s,t}^\pi = (\phi_{t,s}^\pi)^{-1}$, which yields to $\zeta_{s,t}^\pi \circ \phi_{t,s}^\pi = \mathfrak{i}$ and passing to limit $\zeta_{s,t}^\infty \circ \psi_{t,s} = \mathfrak{i}$. This concludes the proof. ☐

7 Generalized solution to rough differential equations

Almost flows approximates of flows, similarly to numerical algorithms. In classical analysis, flows are strongly related to solutions of ordinary differential equations (ODE). Rough differential equations (RDE) were solve first using fixed point theorems on paths [25]. The technical difficulty with this approach is that the solution itself should be a rough path.

Later, A.M. Davie introduced in [14] another notion of solution of RDE which no longer involves solutions as rough paths, but only as paths. We abstract here this approach in order to relate almost flows and paths.

Definition 10 (Generalized solution in the sense of Davie)**.** Let ϕ be an almost flow. Let $a \in \mathrm{V}$ and $r \in \mathbb{T}$. A V-valued path $\{y_{r\rightsquigarrow t}\}_{(r,t)\in\mathbb{T}^2}$ is said to be a *solution in the sense of Davie* if $y_{r\rightsquigarrow r} = a$ and there exists a constant C such that

$$|y_{r\rightsquigarrow t} - \phi_{t,s}(y_{r\rightsquigarrow s})| \leqslant C\varpi(\omega_{s,t}),\ \forall r \leqslant s \leqslant t \leqslant T. \tag{30}$$

Definition 11 (Manifold of solutions)**.** A family $\{y_{r\rightsquigarrow\cdot}(a)\}_{r\in\mathbb{T},\ a\in\mathrm{V}}$ of solutions satisfying (30) and $y_{r\rightsquigarrow r}(a) = a$ is called a *manifold of solutions*. We write $\dot{y} = \phi'(y)$ to denote the whole family of solutions.

Definition 12 (Lipschitz manifold of solutions)**.** If $a \mapsto y_{r\rightsquigarrow\cdot}(a)$ is uniformly Lipschitz continuous from (V, d) to $(\mathscr{C}([r,T],\mathrm{V}), \|\cdot\|_\infty)$, then we say that the manifold of solutions is *Lipschitz*.

Remark 3. When $\phi_{t,s} = \mathfrak{i} + \chi_{t,s}$, then (30) may be written $|y_{s,t} - \chi_{t,s}(y_s)| \leqslant C\varpi(\omega_{s,t})$ with $y_{s,t} := y_{r\rightsquigarrow t} - y_{r\rightsquigarrow s}$. for $(r,s,t) \in \mathbb{T}_2^+$ This is the form used by A.M. Davie in [14].

Flows and manifold of solutions are of closely related. Besides, a manifold of solutions is in relation with a whole galaxy. The proof of the next lemma is immediate so that we skip it.

Lemma 3. *A flow ψ generates a manifold of solutions to $\dot{y} = \psi'(y)$ through $y_{r\rightharpoondown t}(a) := \psi_{t,r}(a)$, $(r,t) \in \mathbb{T}_2^+$, $a \in \mathrm{V}$. Besides, y is also solution to $\dot{y} = \phi'(y)$ for any almost flow ϕ in the galaxy containing ψ.*

7.1 Existence of a flow from a family of solutions

In a first time, we show how to construct a flow from a suitable family of paths.

Proposition 9. *Consider an almost flow ϕ. Assume that there exists a family $\{y_{0\rightharpoondown t}(a)\}_{t\in\mathbb{T},a\in\mathrm{V}}$ of V-valued paths, continuous in time and Lipschitz continuous in space such that such that*

$$y_{0\rightharpoondown 0} = \mathfrak{i},\ \|y_{0\rightharpoondown t} - \phi_{t,s}(y_{0\rightharpoondown s})\|_\infty \leqslant C\varpi(\omega_{s,t}),\ \forall(s,t) \in \mathbb{T}_+^2,$$
$$\sup_{t\in\mathbb{T}}\{\|y_{0\rightharpoondown t} - \mathfrak{i}\|_{\mathrm{Lip}} + \|y_{0\rightharpoondown t} - \mathfrak{i}\|_\infty\} \leqslant K_T \text{ where } K_T \xrightarrow[T\to 0]{} 0.$$

Then the family $\{y_{0\rightharpoondown t}(a)\}_{t\in\mathbb{T},a\in\mathrm{V}}$ may be extended to a manifold of solutions to $\dot{y} = \phi'(y)$. Besides, if $\psi_{s,t}(a) := y_{s\rightharpoondown t}(a)$, then ψ is an invertible, Lipschitz flow in the same galaxy as ϕ.

Proof. The Lipschitz inverse mapping shows that $y_{0\rightharpoondown t}$ is invertible with a Lipschitz continuous inverse $y_{0\rightharpoondown t}^{-1}$ when $K_T < 1$ ([1] p. 124).

Assuming that T is small enough, we define $\psi_{t,s}(a) := y_{0\rightharpoondown t} \circ y_{0\rightharpoondown s}^{-1}(a)$ for any $(s,t) \in \mathbb{T}^2$ and $a \in \mathrm{V}$. Clearly, ψ is a flow which is invertible. Besides, for any $(s,t) \in \mathbb{T}_+^2$, $\psi_{s,t}$ is Lipschitz continuous since both $y_{0\rightharpoondown t}$ and $y_{0\rightharpoondown s}^{-1}$ are Lipschitz continuous.

It remains to prove that $\phi \sim \psi$. For $a \in \mathrm{V}$, let us set $b := y_{0\rightharpoondown s}^{-1}(a)$. Thus,

$$\begin{aligned}|\psi_{t,s}(a) - \phi_{t,s}(a)| &= |\psi_{t,s}(y_{0\rightharpoondown s}(b)) - \phi_{t,s}(y_{0\rightharpoondown s}(b))| \\ &\leqslant |\phi_{t,s}(y_{0\rightharpoondown s}(b)) - y_{0\rightharpoondown t}(b)| \leqslant C\varpi(\omega_{s,t}).\end{aligned}$$

Thus, $\psi \sim \phi$. ☐

7.2 Uniqueness and continuity of a solution in the sense of Davie

A stable almost flow ϕ satisfies the condition UL (see Theorem 2), so that there exists a unique flow ψ in the same galaxy as ϕ. Furthermore, ψ is Lipschitz.

The flow ψ generates a manifold of solutions. We show that there exists only one such manifold with a Lipschitz continuity result. Note that in the following proposition, ζ is not assumed to be stable.

Proposition 10. *Let ϕ be a stable almost flow and ζ be an almost flow. Let y and z be two paths from $[0,T]$ to* V *such that*

$$|y_t - \phi_{t,s}(y_s)| \leqslant K\varpi(\omega_{s,t}) \text{ and } |z_t - \zeta_{t,s}(z_s)| \leqslant K\varpi(\omega_{s,t}),\ \forall (s,t) \in \mathbb{T}_+^2.$$

Let us write $\alpha_{t,s} := \zeta_{t,s} - \phi_{t,s}$ and $\alpha_{t,s,r} := \zeta_{t,s,r} - \phi_{t,s,r}$. Let $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ be such that for any $(r,s,t) \in \mathbb{T}_+^3$,

$$|\alpha_{t,s,r}(z_r)| \leqslant \epsilon_1 \varpi(\omega_{r,t}),\ \|\alpha_{s,t}\|_{\mathrm{Lip}} \leqslant \epsilon_2 \text{ and } |\alpha_{s,t}(z_s)| \leqslant \epsilon_3.$$

Then there exists a time T small enough and a constant C that depends only on ϕ, K, $T \mapsto \delta_T$, $\varkappa$ and $\sup_{(r,s,t)\in\mathbb{T}^3} \|\phi_{t,s,r}\|_{\mathrm{Lip}}/\varpi(\omega_{r,t})$ such that

$$\begin{aligned} |y_t - z_t| &\leqslant C(\epsilon_1 + \epsilon_2 + \epsilon_3 + |y_0 - z_0|) \\ \text{and } |y_t - \phi_{t,s}(y_s) - z_t + \zeta_{t,s}(z_s)| &\leqslant C(\epsilon_1 + \epsilon_2 + \epsilon_3 + |y_0 - z_0|)\varpi(\omega_{s,t}) \end{aligned}$$

for all $(s,t) \in \mathbb{T}^2$.

The following corollary is then immediate by applying $\phi = \zeta$ to Proposition 10.

Corollary 3. *If ϕ is a stable almost flow, there exists one and only one manifold of solutions to $\dot{y} = \phi'(y)$. Besides, this manifold of solutions is Lipschitz.*

Remark 4. As seen in Lemma 3, the notion of manifold of solution is associated to a galaxy. Hence, a galaxy with a stable almost flow is associated to a unique manifold of solutions (actually, we have not proved that if ϕ is a stable almost flow, then the associated flow is also stable).

Proof of Proposition 10. We define

$$V_{s,t} = |z_t - \zeta_{t,s}(z_s) - y_t + \phi_{t,s}(y_s)|,\ \forall (s,t) \in \mathbb{T}^2.$$

Clearly, $V_{s,t} \leqslant 2K\varpi(\omega_{s,t})$.

For any $(r,s,t) \in \mathbb{T}^3$,

$$\begin{aligned} z_t - \zeta_{t,r}(z_r) - y_t + \phi_{t,r}(y_r) \\ &= z_t - \zeta_{t,s}(z_s) - y_t + \phi_{t,s}(y_s) + \zeta_{t,s}(z_s) - \zeta_{t,s}(\zeta_{s,r}(z_r)) \\ &\quad - \phi_{t,s}(y_s) + \phi_{t,s}(\phi_{s,r}(y_r)) + \zeta_{t,s,r}(z_r) - \phi_{t,s,r}(y_r) \\ &= z_t - \zeta_{t,s}(z_s) - y_t + \phi_{t,s}(y_s) + \alpha_{t,s}(z_s) - \alpha_{t,s}(\zeta_{s,r}(z_r)) \\ &\quad + \phi_{t,s}(z_s) - \phi_{t,s}(\zeta_{s,r}(z_r)) - \phi_{t,s}(y_s) + \phi_{t,s}(\phi_{s,r}(y_r)) \\ &\qquad + \alpha_{t,s,r}(z_r) + \phi_{t,s,r}(z_r) - \phi_{t,s,r}(y_r). \end{aligned}$$

Set $L := \sup_{(r,s,t)\in\mathbb{T}^3}\|\phi_{t,s,r}\|_{\mathrm{Lip}}$. With the 4-points control of $\phi_{s,t}$,

$$\begin{aligned}V_{t,r} \leqslant V_{t,s} + \check{\phi}_{t,s}V_{r,t} + L|z_r - y_r|\varpi(\omega_{r,t})\\ + \widehat{\phi}_{t,s}(|z_s - \zeta_{s,r}(z_r)| \vee |y_s - \phi_{s,r}(y_r)|)\cdot(|z_s - y_s| \vee |\zeta_{s,r}(z_r) - \phi_{s,r}(y_r)|)\\ + |\alpha_{t,s,r}(z_r)| + \|\alpha_{t,s}\|_{\mathrm{Lip}}|z_s - \zeta_{s,r}(z_r)|.\end{aligned}$$

Since $\check{\phi}_{t,s} \leqslant 1 + \delta_T$ and $\|\phi_{t,s}\|_{\mathrm{Lip}} \leqslant 1 + \delta_T$,

$$\begin{aligned}V_{t,r} \leqslant V_{t,s} + (1+\delta_T)V_{r,t} + L\|z - y\|_\infty\varpi(\omega_{r,t})\\ + \widehat{\phi}_{t,s}(K\varpi(\omega_{s,r}))\big((1+\delta_T)\|z-y\|_\infty + \epsilon_3\big) + (\epsilon_1 + \epsilon_2 K)\varpi(\omega_{r,t}).\end{aligned}$$

Since $\widehat{\phi}_{t,s}$ is ϖ-compatible (see Defintion 6), $\widehat{\phi}_{t,s}(K\varpi(\omega_{r,s})) \leqslant \Phi(K)\varpi(\omega_{r,t})$ so that

$$V_{r,t} \leqslant V_{s,t} + (1+\delta_T)V_{r,t} + B\varpi(\omega_{r,t}) \tag{31}$$

$$\text{with } B := (L + (1+\delta_T)\Phi(K))\|y - z\|_\infty + \epsilon_1 + \epsilon_2 K + \epsilon_3\Phi(K)). \tag{32}$$

Owing to (31)-(32), from the continuous time version of the Davie's Lemma 10, for T small enough (depending only on $\varkappa$ and $T \mapsto \delta_T$), for all $(r,t) \in \mathbb{T}^2$,

$$V_{r,t} \leqslant BC\varpi(\omega_{r,t}) \text{ with } C := \frac{2+\delta_T}{1 - (\varkappa(1+\delta_T)^2 + \delta_T)}.$$

For any $t \in [0,T]$, since $\|\phi_{t,0}\|_{\mathrm{Lip}} \leqslant 1 + \delta_T$,

$$\begin{aligned}|y_t - z_t| &\leqslant |y_t - z_t - \phi_{t,0}(y_0) + \zeta_{t,0}(z_0)| + |\phi_{t,0}(z_0) - \zeta_{t,0}(z_0)| + |\phi_{t,0}(z_0) - \phi_{t,0}(y_0)|\\ &\leqslant V_{0,t} + \epsilon_3 + (1+\delta_T)|y_0 - z_0|.\end{aligned}$$

With the expression of B in (32), for any $t \in [0,T]$, we see that there exists constants A and A' that depend only on $\varkappa$, L, δ_T, K and $\Phi(K)$ such that

$$|y_t - z_t| \leqslant A\|y - z\|_\infty\varpi(\omega_{0,T}) + A'(\epsilon_1 + \epsilon_2 + \epsilon_3)\varpi(\omega_{0,T}) + \epsilon_3 + (1+\delta_T)|y_0 - z_0|.$$

Choosing T small enough so that $A\varpi(\omega_{0,T}) \leqslant 1/2$ implies that

$$\|y - z\|_\infty \leqslant 2A'(\epsilon_1 + \epsilon_2 + \epsilon_3)\varpi(\omega_{0,T}) + 2\epsilon_3 + 2(1+\delta_T)|y_0 - z_0|.$$

This concludes the proof. □

8 Application to Rough differential equation

In this section, we show how our framework allows us to link the different flow based approaches. The key is to show that Friz-Victoir's and Bailleul's almost flows are different perturbation of the Davie's almost flow.

We start by giving some notations. We did not recall notions of the rough path theory. The reader can find a clear introduction in [23] and in [18].

In this section, the remainder introduced in Section 2.1 is of the type

$$\varpi(\delta) = \delta^{(2+\gamma)/p},\ \forall \delta > 0.$$

with $\gamma \in (0,1]$ and a real number $p > 0$ satisfying $2+\gamma > p$.

8.1 Rough path notations

Before showing the link between the different based flow approaches, we set notations of classical objects of the rough path theory.

Given another Banach space $(\mathrm{U}, |\cdot|)$ and a real number $p \geqslant 1$, let us denote by $\mathscr{C}^{p-\omega}(\mathbb{T}, \mathrm{U})$ the space of $1/p$-Hölder paths controlled by ω, which we equip we the semi-norm

$$\|x\|_p := \sup_{(s,t)\in\mathbb{T}_+^2, s\neq t} \frac{|x_{s,t}|}{\omega_{s,t}^{1/p}},$$

this quantity being bounded by definition.

We define also $\mathscr{C}^{p-\mathrm{var}}([s,t], \mathrm{U})$ the space of bounded p-variation paths from $[s,t]$ to U which we equip with the p-variation semi-norm on $[s,t]$ denoted by $\|x\|_{[s,t],p}$.

Moreover, if $x \in \mathscr{C}^{p-\omega}([0,T], \mathrm{U})$, then $x \in \mathscr{C}^{p-\mathrm{var}}([s,t], \mathrm{U})$ and

$$\|x\|_{[s,t],p} \leqslant \|x\|_p \omega_{s,t}^{1/p}.$$

We denote by $\lfloor\cdot\rfloor$ the floor function.

For an integer $N \geqslant \lfloor p \rfloor$, let $\mathscr{T}^{p,N}(\mathrm{U})$ be the space of $1/p$-Hölder rough path controlled by ω of order N. If $\mathbf{x} \in \mathscr{T}^{p,N}(\mathrm{U})$ we denote by $\mathbf{x}^{(k)}$ the component of $\mathbf{x}$ in $\mathrm{U}^{\otimes k}$ with $0 \leqslant k \leqslant N$ an integer and $S_j(\mathbf{x}) := \sum_{j=0}^{k} \mathbf{x}^{(j)}$. Obviously, $\mathbf{x} = S_N(\mathbf{x})$. We denote the homogeneous semi-norm

$$\|\mathbf{x}\|_p := \sup_{k\leqslant N} \sup_{[s,t]\in\mathbb{T}^2, s\neq t} \frac{|\mathbf{x}_{s,t}^{(k)}|}{\omega_{s,t}^{k/p}},$$

which is finite by definition. Moreover we set $\mathscr{T}^p(U) := \mathscr{T}^{p,\lfloor p\rfloor}(U)$.

For $N \geqslant 0$, we denote $G^N(\mathrm{U})$ the free nilpotent group (Chapter 7 in [19]).

Let $\mathscr{G}^p(\mathrm{U}) := \mathscr{C}^{p-\omega}([0,T], G^{\lfloor p\rfloor}(\mathrm{U}))$ be the set of weak-geometric rough paths of finite $1/p$-Hölder rough path controlled by ω with values in U.

When $\mathrm{U} = \mathbb{R}^\ell$ ($\ell \geqslant 0$ an integer). For any multi-indice $I := (i_1, \dots, i_k) \in \{1, \dots, \ell\}^k$ we set $|I| := k$ and $e_I := e_{i_1} \otimes \dots \otimes e_{i_k}$ where $\{e_1, \dots, e_\ell\}$ is the canonical basis of $\mathbb{R}^\ell$. If $\mathbf{x} \in \mathscr{T}^p(\mathbb{R}^\ell)$. If $\mathbf{x} \in \mathscr{T}^p(\mathbb{R}^\ell)$, then $\mathbf{x}^I$ denote the coordinates of $\mathbf{x}^{(k)}$ in the basis $(e_I)_{|I|=k}$. It follows that $S_k(\mathbf{x}) = \sum_{|I|\leqslant k} \mathbf{x}^I e_I$. If $x \in \mathscr{C}^{1-\mathrm{var}}(\mathbb{T}, \mathbb{R}^l)$ then for any integer $N \geqslant 0$,

$$S_N(x) = \sum_{|I|\leqslant N} x^I e_I,$$

where $x^I_{s,t} := \int_{s\leqslant t_k \leqslant \dots \leqslant t_1 \leqslant t} \mathrm{d}x^{i_k}_{t_k} \dots \mathrm{d}x^{i_1}_{t_1}$.

8.2 The Davie's approach

Let us consider now a p-rough path $\mathbf{x} \in \mathscr{T}^p(\mathrm{U})$ with $2 \leqslant p < 3$ for a Banach space U. A Rough Differential Equation (RDE) is a solution y taking its values in another Banach space V to

$$y_t = a + \int_s^t f(y_u)\,\mathrm{d}\mathbf{x}_u,\ \forall (s,t) \in \mathbb{T}^2_+, \tag{33}$$

provided that $f : \mathrm{V} \to L(\mathrm{U}, \mathrm{V})$ is regular enough.

Existence of solution to (33) was proved first by T. Lyons using a Picard fixed point theorem [25]. In [14], A.M. Davie provided an alternative approach based on Euler-type schemes. Over the approach of T. Lyons, it has the advantage that the solution is sought as a path with values in V and not in the tensor space $\mathrm{T}_2(\mathrm{U}\oplus\mathrm{V})$.

For $f \in \mathscr{C}^1_b(\mathrm{V}, L(\mathrm{U}, \mathrm{V}))$, we define

$$\phi_{t,s}(a) = a + f(a)\mathbf{x}^{(1)}_{s,t} + \mathrm{d}f(a)\cdot f(a)\mathbf{x}^{(2)}_{s,t}, \tag{34}$$

where $\mathrm{d}f(a)$ is the differential of f in a. Definition 10 coincides with the one defined by A. M. Davie in [14] for the notion of solution to (33).

The following lemma is a generalization in an infinite dimensional setting of Theorems 3.2 and 3.3 in [14] with a bounded function f.

Lemma 4. *Let $p \in [2,3)$ and $\gamma > p-2$.*
(i) *If $f \in \mathscr{C}^{1+\gamma}_b$, then the family ϕ defined by (34) is an almost flow.*

(ii) *If f is of class $\mathscr{C}_b^{2+\gamma}$, then ϕ is a stable almost flow.*

Proof. According to Proposition 5 in [6], ϕ is an almost flow as soon as $f \in \mathscr{C}_b^{1+\gamma}$ with $2+\gamma > p$.

We now assume that $f \in \mathscr{C}_b^{2+\gamma}$. We show that ϕ verifies a ϖ-compatible 4-points control (see Definition 6). For any $a, b, c, d \in \mathrm{V}$, $(s,t) \in \mathbb{T}^2$, we compute

$$\phi_{t,s}(a) - \phi_{t,s}(b) - \phi_{t,s}(c) + \phi_{t,s}(d) = a - b - c + d + \underbrace{[f(a) - f(b) - f(c) + f(d)]\mathbf{x}^{(1)}_{s,t}}_{\mathrm{I}'_{s,t}} \\ + \underbrace{[\,\mathrm{d}f \cdot f(a) - \,\mathrm{d}f \cdot f(b) - \,\mathrm{d}f \cdot f(c) + \,\mathrm{d}f \cdot f(d)]\mathbf{x}^{(2)}_{s,t}}_{\mathrm{II}'_{s,t}}. \quad (35)$$

Then, we apply Lemma 1 to $f \in \mathscr{C}_b^2$ and to $\mathrm{d}f \cdot f \in \mathscr{C}_b^{1+\gamma}$ to obtain

$$|\mathrm{I}'_{s,t}| \leqslant \|\mathbf{x}^{(1)}\|_p \omega_{s,t}^{1/p} 4\|\,\mathrm{d}^2 f\|_\infty (|a-c| \vee |b-d|)^2 |a-b| \\ + \|\mathbf{x}^{(1)}\|_p \omega_{0,T}^{1/p} \|\,\mathrm{d}f\|_\infty |a-b-c+d|, \quad (36)$$

and

$$|\mathrm{II}'_{s,t}| \leqslant \|\mathbf{x}^{(2)}\|_{\frac{p}{2}} \omega_{s,t}^{2/p} \Big[2\|\,\mathrm{d}(\,\mathrm{d}f \cdot f)\|_\gamma \left(|a-c| \vee |b-d|\right)^\gamma |a-b| \\ + \|\,\mathrm{d}(\,\mathrm{d}f \cdot f)\|_\infty |a-b-c+d| \Big]. \quad (37)$$

Combining (35), (36) and (37) we set for all $y \in \mathbb{R}_+$, $\widehat{\phi_{t,s}}(y) := c_1' \left[\omega_{s,t}^{1/p} y^2 + \omega_{s,t}^{2/p} y^\gamma\right]$ where $c_1' := \|\mathbf{x}^{(1)}\|_p 4\|\,\mathrm{d}^2 f\|_\infty + \|\mathbf{x}^{(2)}\|_{\frac{p}{2}} 2\|\,\mathrm{d}(\,\mathrm{d}f \cdot f)\|_\gamma$ and

$$\widetilde{\phi_{t,s}} := 1 + \|\mathbf{x}^{(1)}\|_p \omega_{0,T}^{1/p} + \|\,\mathrm{d}(\,\mathrm{d}f \cdot f)\|_\infty \|\mathbf{x}^{(2)}\|_{\frac{p}{2}} \omega_{0,T}^{2/p}.$$

It follows that $\widetilde{\phi_{t,s}} \leqslant 1 + \delta_T$ and that $\phi_{t,s}$ is ϖ-compatible. Indeed, for $\alpha \in \mathbb{R}_+$,

$$\widehat{\phi_{t,s}}\left(\alpha \omega_{s,t}^{(2+\gamma)/p}\right) \leqslant c_1'(\alpha^2 \vee \alpha^\gamma)\left(\omega_{s,t}^{(5+2\gamma)/p} + \omega_{s,t}^{(2+2\gamma+\gamma^2)/p}\right) \\ \leqslant c_1'(\alpha^2 \vee \alpha^\gamma)\left(\omega_{0,T}^{(3+\gamma)/p} + \omega_{0,T}^{(\gamma+\gamma^2)/p}\right)\omega_{s,t}^{(2+\gamma)/p} \leqslant \delta_T \varpi(\omega_{s,t}).$$

It remains to show that (11) holds. As $\phi_{t,s} \in \mathscr{C}^{1+\gamma}$, thus the two semi-norms $\|\phi_{t,s}\|_{\mathrm{Lip}}$ and $\|\,\mathrm{d}\phi_{t,s}\|_\infty$ are equivalent. We recall that $\phi_{t,s,r} = \phi_{t,s} \circ \phi_{s,r} - \phi_{t,r}$. For any $a \in \mathrm{V}$ and $(r,s,t) \in \mathbb{T}^3_+$,

$$\begin{aligned}
\mathrm{d}\phi_{t,s,r}(a) =& (\mathrm{d}\phi_{s,r}(a)\,\mathrm{d}f \circ \phi_{s,r}(a) - \mathrm{d}f(a))\mathbf{x}^{(1)}_{s,t} - \mathrm{d}(\mathrm{d}f\cdot f)(a)(\mathbf{x}^{(2)}_{s,t} - \mathbf{x}^{(1)}_{r,s}\otimes \mathbf{x}^{(1)}_{s,t})\\
&+ \mathrm{d}\phi_{s,r}(a)\,\mathrm{d}(\mathrm{d}f\cdot f)\circ\phi_{s,r}(a)\mathbf{x}^{(2)}_{s,t}\\
=& \underbrace{\left(-\mathrm{d}f(a) + \mathrm{d}\phi_{s,r}(a)\,\mathrm{d}f\circ\phi_{s,r}(a) - \mathrm{d}(\mathrm{d}f\cdot f)(a)\mathbf{x}^1_{r,s}\right)\mathbf{x}^{(1)}_{s,t}}_{\mathrm{I}''_{r,s,t}}\\
&- \underbrace{\left(\mathrm{d}(\mathrm{d}f\cdot f)(a) - \mathrm{d}\phi_{s,r}(a)\,\mathrm{d}(\mathrm{d}f\cdot f)\circ\phi_{s,r}(a)\right)\mathbf{x}^2_{s,t}}_{\mathrm{II}''_{r,s,t}}. \qquad (38)
\end{aligned}$$

Each term is estimated separately. For the first one,

$$\begin{aligned}
|\mathrm{I}''_{r,s,t}| \leqslant& |\,\mathrm{d}f\circ\phi_{s,r}(a) - \mathrm{d}f(a) - \mathrm{d}^2 f(a)(\phi_{s,r}(a)-a)||\mathbf{x}^{(1)}_{s,t}|\\
&+ |\,\mathrm{d}f(a)\,\mathrm{d}f(a)\mathbf{x}^{(2)}_{r,s}\mathbf{x}^1_{s,t}| + |\,\mathrm{d}(\mathrm{d}f\cdot f)(a)\,\mathrm{d}f\circ\phi_{s,r}(a)\mathbf{x}^{(2)}_{r,s}\mathbf{x}^{(1)}_{s,t}|\\
&+ |\,\mathrm{d}f(a)[\,\mathrm{d}f\circ\phi_{s,r}(a) - \mathrm{d}f(a)]\mathbf{x}^1_{r,s}\otimes\mathbf{x}^1_{s,t}\\
\leqslant& \|\,\mathrm{d}^2 f\|_\gamma|\phi_{s,r}(a)-a|^{1+\gamma}|\mathbf{x}^{(1)}_{s,t}| + (\|\,\mathrm{d}f\|^2_\infty + \|\,\mathrm{d}(\mathrm{d}f\cdot f)\,\mathrm{d}f\|_\infty)|\mathbf{x}^{(2)}_{r,s}||\mathbf{x}^{(1)}_{s,t}|\\
&+ \|\,\mathrm{d}f\|_\infty\|\,\mathrm{d}^2 f\|_\infty|\phi_{s,r}(a)-a||\mathbf{x}^1_{r,s}||\mathbf{x}^1_{s,t}|\\
\leqslant& c''_1(\omega^{(2+\gamma)/p}_{r,t} + 2\omega^{3/p}_{r,t}) \leqslant c''_1(1+2\omega^{(1-\gamma)/p}_{0,T})\omega^{(2+\gamma)/p}_{r,t}, \qquad (39)
\end{aligned}$$

where c''_1 is a constant which depends on f, $\mathbf{x}$, γ.

The second one is more simple,

$$\begin{aligned}
|\mathrm{II}''_{r,s,t}| \leqslant& |\,\mathrm{d}(\mathrm{d}f\cdot f)\circ\phi_{s,r}(a) - \mathrm{d}(\mathrm{d}f\cdot f)(a)||\mathbf{x}^{(2)}_{s,t}| + |\,\mathrm{d}\phi_{s,r}(a) - 1|\,\mathrm{d}(\mathrm{d}f\cdot f)\circ\phi_{s,r}(a)\mathbf{x}^2_{s,t}\\
\leqslant& \|\,\mathrm{d}(\mathrm{d}f\cdot f)\|_\infty|\phi_{s,r}(a)-a|^\gamma|\mathbf{x}^{(2)}_{s,t}| + \|\,\mathrm{d}(\mathrm{d}f\cdot f)\|_\infty|\,\mathrm{d}\phi_{s,r}(a)-1||\mathbf{x}^{(2)}_{s,t}|\\
\leqslant& c''_2(\omega^{(2+\gamma)/p}_{r,t} + \omega^{3/p}_{r,t}) \leqslant c''_2(1+\omega^{(1-\gamma)/p}_{0,T})\omega^{(2+\gamma)/p}_{r,t}. \qquad (40)
\end{aligned}$$

Finally, combining (38), (39) and (40), $\|\,\mathrm{d}\phi_{t,s,r}\|_\infty \leqslant (c''_1 + c''_2)(1+\omega^{(1-\gamma)/p}_{0,T})\varpi(\omega_{r,t})$ with $\varpi(\omega_{r,t}) = \omega^{(2+\gamma)/p}_{r,t}$. This proves (11) and that ϕ is a stable almost flow. □

Combining Lemma 4, Proposition 2, Corollaries 1 and 3 leads to the following result.

Corollary 4. *If f is of class $\mathscr{C}^{2+\gamma}_b$, then ϕ^π converges to a unique Lipschitz flow ψ. Moreover, if χ is an almost flow in a galaxy containing ϕ, then, χ^π converges to ψ. Besides, there exists a unique manifold of solutions to $\dot{y} = \phi'(y)$ which is Lipschitz.*

8.3 Almost flows constructed from sub-Riemannian geodesics, as in P. Friz and N. Victoir

In [19], P. Friz and N. Victoir proposed an approach based on the use of geodesics. The following proposition is one of the fundamental result of their framework.

Now, we assume that $\mathrm{U} = \mathbb{R}^\ell$.

Proposition 11 (Remark 10.10, [19, p. 216]). *Let $p \geqslant 1$ a real number and an integer $N \geqslant \lfloor p \rfloor$. For any $\mathbf{x} \in \mathscr{C}^{p-\omega}([0,T], G^N(\mathbb{R}^\ell))$ and any $(s,t) \in \mathbb{T}^2$, there exists a path $x^{s,t} \in \mathscr{C}^{1-\mathrm{var}}(\mathbb{R}^\ell)$ defined on $[s,t]$ such that*

$$S_N(x^{s,t})_{s,t} = \mathbf{x}_{s,t} \text{ and } \|x^{s,t}\|_{[s,t],1} \leqslant K\|\mathbf{x}\|_p \omega_{s,t}^{1/p} \leqslant K'\|S_{\lfloor p \rfloor}(\mathbf{x})\|_p \omega_{s,t}^{1/p}.$$

for some universal constant K (resp. K') that depends only on p (p and N). We say that $x^{s,t}$ is a geodesic path *associated to $\mathbf{x}$.*

Remark 5. If $x \in \mathscr{C}^{1-\mathrm{var}}([s,t], \mathbb{R}^\ell)$, then $\|x\|_{[s,t],1} = \int_s^t |\,\mathrm{d}x_r|$.

For notational convenience, we prefer now to express differential equations with respect to vector fields, that is a family of functions $\overrightarrow{f} := (\overrightarrow{f}_1, \dots, \overrightarrow{f}_\ell)$ that acts on $\mathscr{C}^1_{\mathrm{b}}(\mathrm{V}, L(\mathrm{U},\mathrm{V}))$. Therefore for $x \in \mathscr{C}^{1-\mathrm{var}}(\mathbb{R}_+, \mathbb{R}^\ell)$, the equation $z_t = a + \int_0^t \overrightarrow{f}\mathfrak{i}(z_s)\,\mathrm{d}x_s$ is equivalent to $y_t = a + \int_0^t f(z_s)\,\mathrm{d}x_s$ with $f = \overrightarrow{f}\mathfrak{i}$. For a multi-indice $I := (i_1, \dots, i_k) \in \{1, \dots, \ell\}^k$ and $(s,t) \in \mathbb{T}$ we denote $\overrightarrow{f}_I := \overrightarrow{f}_{i_1} \dots \overrightarrow{f}_{i_n}$. By convention, $\overrightarrow{f}_\varnothing \mathfrak{i} := \mathfrak{i}$. We employ the Einstein convention of summation.

Let us fix $n \geqslant 2$ in $\overrightarrow{f}\mathfrak{i} \in \mathscr{C}^{\lambda-1}_{\mathrm{b}}$ for $\lambda \geqslant n$. Let $\mathbf{x}$ be a rough path with values in $\mathrm{T}_n(\mathrm{U})$ and V be a vector field such that $\overrightarrow{f}\mathfrak{i} \in \mathscr{C}^{n-1}_{\mathrm{b}}$. Let us define

$$\phi^{(n)}_{t,s}[\mathbf{x}, \overrightarrow{f}](a) := \sum_{k=0}^{n} \sum_{|I|=k} \overrightarrow{f}_I \mathfrak{i}(a) \mathbf{x}^I_{s,t},\ \forall (s,t) \in \mathbb{T}^2_+. \tag{41}$$

The next proposition summarizes various results on RDEs (Theorem 10.26 in [19, p. 233], Theorem 10.30 in [19, p. 238]). When $\overrightarrow{f}\mathfrak{i} \in \mathscr{C}^{1+\gamma}_{\mathrm{b}}$ but no in $\mathscr{C}^{2+\gamma}_{\mathrm{b}}$ with $2 + \gamma > p$, several solutions to the RDE (33) may exist (See Example 2 in [14]).

Theorem 3. *Assume that* U *and* V *are finite dimensional Banach spaces. Choose $\lambda > 2$ as well as an integer n with $2 \leqslant n \leqslant \lfloor \lambda \rfloor$. Let $\mathbf{x}$ be a p-rough paths with values in $\mathrm{T}_n(\mathrm{V})$. Let us assume that $\overrightarrow{f}\mathfrak{i} \in \mathscr{C}^{\lambda-1}$ for a vector field $\overrightarrow{f} : \mathrm{V} \to L(\mathrm{U},\mathrm{V})$. It holds that*

(i) *When $\lambda > p$, there exists a flow $\psi[\mathbf{x}, \overrightarrow{f}]$ in the same galaxy as $\phi^{(n)}[\mathbf{x}, \overrightarrow{f}]$,*

(ii) *When $1+\lambda > p$, then there exists a unique flow as well as a unique Lipschitz manifold of solutions to $\dot{y} = {\phi^{(n)}}'(y)$ ($\phi^{(n)}$ is defined in (41)).*
In addition, for any partition $\pi = \{t_i\}_{i=0}^k$,

$$|\psi_{t,s}[\mathbf{x}, \overrightarrow{f}](a) - {\phi^{(n)}}^{\pi}_{t,s}[\mathbf{x}, \overrightarrow{f}]| \leqslant C\|\mathbf{x}\|_p \sup_{i=0,\dots,k-1} \omega_{t_i,t_{i+1}}^{\frac{n+\gamma}{p}-1}, \tag{42}$$

with $\gamma := \min\{\lambda - n, 1\}$ and a constant C that depends on $\omega_{0,T}$, $\|\mathbf{x}\|_p$ and V.

From the next lemma, we obtain that of $\overrightarrow{f}\mathfrak{i} \in \mathscr{C}_{\mathrm{b}}^{\lambda}$ with $\lambda > p$, then there exists a unique flow associated to $\phi^{(n)}[\mathbf{x}, \overrightarrow{f}]$.

Lemma 5. *For any $n \geqslant 2$, $\phi^{(n)}[\mathbf{x}, \overrightarrow{f}]$ belongs to the same galaxy as $\phi^{(2)}[\mathbf{x}, \overrightarrow{f}]$, which the Davie expansion given by (34) with $f = \overrightarrow{f}\mathfrak{i}$.*

Proof. Let us write for $n \geqslant 2$,

$$R^{(n)}_{s,t}[\mathbf{x}, \overrightarrow{f}] := \phi^{(n)}_{s,t}[\mathbf{x}, \overrightarrow{f}] - \phi^{(2)}_{s,t}[\mathbf{x}, \overrightarrow{f}] = \sum_{k=3}^{n} \sum_{|I|\leqslant k} \overrightarrow{f}_I\mathfrak{i}(a)\mathbf{x}^I_{s,t},\ \forall (s,t) \in \mathbb{T}^2_+.$$

Clearly, $R^{(n)}[\mathbf{x}, \overrightarrow{f}]$ is a perturbation with $\varpi(\delta) = \delta^{3/p}$. Moreover, as $\overrightarrow{f}\mathfrak{i} \in \mathscr{C}_{\mathrm{b}}^{\lambda-1}$, $\overrightarrow{f}_I\mathfrak{i} \in \mathscr{C}_{\mathrm{b}}^{\lambda-k}$ for any word I with $|I| = k$ so that when $\lambda - n \geqslant 1$, $R^{(n)}[\mathbf{x}, \overrightarrow{f}]$ is a Lipschitz perturbation (Definition 5). ☐

This is however not sufficient to obtain the rate in (42).
For a path $x \in \mathscr{C}^{1-\mathrm{var}}$, we denote for $a \in \mathrm{V}$ by $\psi_{\cdot,s}[x, \overrightarrow{f}](a)$ the unique solution to

$$\psi_{t,s}[x, \overrightarrow{f}](a) = a + \int_s^t \overrightarrow{f}\mathfrak{i}(\psi_{r,s}[x, \overrightarrow{f}](a))\,\mathrm{d}x_r,\ t \geqslant s. \tag{43}$$

The family $\psi[x, \overrightarrow{f}]$ satisfies the flow property.
We set for any $(s,t) \in \mathbb{T}^2_+$,

$$\phi^{(n)}_{t,s}[x, \overrightarrow{f}](a) := \phi^{(n)}_{t,s}[S_n(x), \overrightarrow{f}](a),$$
$$\epsilon^{(n)}_{t,s}[x, \overrightarrow{f}](a) = \sum_{|I|=n} \int_{s\leqslant t_{k-1}\leqslant\cdots\leqslant t_1\leqslant t} \left(\overrightarrow{f}_I\mathfrak{i}(\psi_{t_k,s}[x, \overrightarrow{f}](a)) - \overrightarrow{f}_I\mathfrak{i}(a)\right) \mathrm{d}x^I_{t_k},$$

where $\mathrm{d}x^I_{t_k} = \mathrm{d}x^{i_k}_{t_k} \dots \mathrm{d}x^{i_1}_{t_1}$. Using iteratively the Newton formula on (43),

$$\psi_{t,s}[x, \overrightarrow{f}](a) = \phi^{(n)}_{t,s}[x, \overrightarrow{f}](a) + \epsilon^{(n)}_{t,s}[x, \overrightarrow{f}](a).$$

We denote by $\|f\|_\gamma$ the γ-Hölder semi-norm of a function f. For $\gamma = 1$, this is the Lipschitz semi-norm. Thus, $\gamma := \min\{\lambda - n, 1\}$ is the Hölder indice of $\overrightarrow{f}_I\mathrm{i}$ with $|I| = n$.

From Proposition 10.3 in [19, p. 213], for a constant C that depends on λ and on

$$\|V\|_* := \max_{|I|\leqslant n}\|\overrightarrow{f}_I\mathrm{i}\|_\infty + \max_{|I|\leqslant n-1}\|\overrightarrow{f}_I\mathrm{i}\|_{\mathrm{Lip}} + \max_{|I|=n}\|\overrightarrow{f}_I\mathrm{i}\|_\gamma,$$

it holds that

$$|\epsilon^{(n)}_{t,s}[x, \overrightarrow{f}](a)| \leqslant C\|x\|^{n+\gamma}_{[s,t],1}. \tag{44}$$

Lemma 6. *Let x be a path of finite 1-variation with $\|x\|_{[s,t],1} \leqslant A\omega_{s,t}$ for any $(s, t) \in \mathbb{T}^2_+$. Let $x^{s,t}$ be the geodesic path given by Proposition 11. Assume that there exists a constant K such that $\|x^{s,t}\|_{[s,t],1} \leqslant K\omega^{1/p}_{s,t}$. Then there exists a time $T > 0$ small enough and a constant D that depend only on ω, K and $\|\overrightarrow{f}\mathrm{i}\|_\star$ such that*

$$\|\psi_{t,s}[x^{s,t}, \overrightarrow{f}] - \psi_{t,s}[x, \overrightarrow{f}]\|_\infty \leqslant D\omega^{\frac{n+\gamma}{p}}_{s,t},\ \forall(s, t) \in \mathbb{T}^2_+.$$

In particular, the choice of T and D does not depend on A.

Proof. Let us assume that $\|x\|_{[s,t],1} \leqslant A\omega_{s,t}$ and $\|x^{s,t}\|_{[s,t],1} \leqslant K\omega^{1/p}_{s,t}$.

Let $x^{r,s,t}$ be the concatenation of $x^{r,s}$ and $x^{s,t}$. Then

$$\psi_{t,r}[x, \overrightarrow{f}](a) - \psi_{t,r}[x^{r,t}, \overrightarrow{f}](a) = \underbrace{\psi_{t,s}[x, \overrightarrow{f}](\psi_{s,r}[x, \overrightarrow{f}](a)) - \psi_{t,r}[x^{r,s,t}, \overrightarrow{f}](a)}_{\mathrm{I}_{r,s,t}}$$
$$\underbrace{+\psi_{t,r}[x^{r,s,t}, \overrightarrow{f}](a) - \psi_{t,r}[x^{r,t}, \overrightarrow{f}](a)}_{\mathrm{II}_{r,s,t}}.$$

Since $x^{r,s,t}$ is the concatenation between two paths,

$$\psi_{t,r}[x^{r,s,t}, \overrightarrow{f}](a) = \psi_{t,s}[x^{s,t}, \overrightarrow{f}](\psi_{s,r}[x^{r,s}, \overrightarrow{f}](a)).$$

Thus,

$$|\mathrm{I}_{r,s,t}| \leqslant |\psi_{t,s}[x, \overrightarrow{f}](\psi_{s,r}[x, \overrightarrow{f}](a)) - \psi_{t,s}[x^{s,t}, \overrightarrow{f}](\psi_{s,r}[x, \overrightarrow{f}](a))|$$
$$+ |\psi_{t,s}[x^{s,t}, \overrightarrow{f}](\psi_{s,r}[x, \overrightarrow{f}](a)) - \psi_{t,s}[x^{s,t}, \overrightarrow{f}](\psi_{s,r}[x^{r,s}, \overrightarrow{f}](a))|.$$

Writing $U_{r,t} := \|\psi_{t,r}[x, \overrightarrow{f}] - \psi_{t,r}[x^{r,t}, \overrightarrow{f}]\|_\infty$, it holds that

$$|\mathrm{I}_{s,r,t}| \leqslant U_{t,s} + \|\psi_{t,s}[x^{s,t}, \overrightarrow{f}]\|_{\mathrm{Lip}}U_{r,s}.$$

From (43), we derive that for $t \geqslant s$,

$$\begin{aligned}\|\psi_{t,s}[x^{s,t}, \overrightarrow{f}]\|_{\mathrm{Lip}} &\leqslant 1 + \|\overrightarrow{f}\mathfrak{i}\|_{\mathrm{Lip}} \int_s^t \|\psi_{r,s}[x^{s,t}, \overrightarrow{f}]\|_{\mathrm{Lip}} |\,\mathrm{d}x_r^{s,t}| \\ &\leqslant 1 + \|\overrightarrow{f}\mathfrak{i}\|_{\mathrm{Lip}} \int_s^t \|\psi_{r,s}[x^{s,t}, \overrightarrow{f}]\|_{\mathrm{Lip}} |\dot{x}_r^{s,t}|\,\mathrm{d}r, \qquad (45)\end{aligned}$$

where the derivative $\dot{x}^{s,t}$ is almost everywhere defined because $x^{s,t} \in \mathscr{C}^{1-\mathrm{var}}(\mathbb{R}^\ell)$.

Then, using the Grönwall's inequality with (45) and Proposition 11, there is constant C that depends only on K (defined in Proposition 11), $\|\mathbf{x}\|_p$ and $\|\overrightarrow{f}\mathfrak{i}\|_{\mathrm{Lip}}$ such that

$$\|\psi_{t,s}[x^{s,t}, \overrightarrow{f}]\|_{\mathrm{Lip}} \leqslant \exp(C\omega_{0,T}^{1/p}).$$

Besides,

$$S_n(x^{r,s,t})_{r,t} = S_n(x^{r,s})_{r,s} \otimes S_n(x^{s,t})_{s,t} = \mathbf{x}_{r,s} \otimes \mathbf{x}_{s,t} = \mathbf{x}_{r,t} = S_n(x^{r,t})_{r,t}.$$

It follows that $\phi^{(n)}[x^{r,s,t}, \overrightarrow{f}] = \phi^{(n)}[x^{r,t}, \overrightarrow{f}]$. Thus,

$$\begin{aligned}|\mathrm{II}_{r,s,t}| &= |\epsilon_{t,r}^{(n)}[x^{r,s,t}, \overrightarrow{f}](a) - \epsilon_{t,r}^{(n)}[x^{r,t}, \overrightarrow{f}](a)| \\ &\leqslant \|\epsilon_{t,r}^{(n)}[x^{r,s,t}, \overrightarrow{f}]\|_\infty + \|\epsilon_{t,r}^{(n)}[x^{r,t}, \overrightarrow{f}]\|_\infty \leqslant C'\omega_{s,t}^{\frac{n+\gamma}{p}},\end{aligned}$$

where $C' \geqslant 0$ is a new constant and using (44) and Proposition 11 for the last estimation.

Thus,

$$U_{r,t} \leqslant U_{s,t} + \exp(C\omega_{0,T}^{1/p})U_{r,s} + C'\omega_{s,t}^{\frac{n+\gamma}{p}}.$$

On the other hand, when $\omega_{s,t} \leqslant 1$,

$$\begin{aligned}U_{s,t} &= \|\psi_{t,s}[x, \overrightarrow{f}] - \psi_{t,s}[x^{s,t}, \overrightarrow{f}]\|_\infty \\ &\leqslant \|\psi_{t,s}[x, \overrightarrow{f}] - \phi_{t,s}^{(n)}[x, \overrightarrow{f}]\|_\infty + \|\psi_{t,s}[x^{s,t}, \overrightarrow{f}] - \phi_{t,s}^{(n)}[x^{s,t}, \overrightarrow{f}]\|_\infty \\ &\leqslant C'' \max\{A^{n+\gamma}\omega_{s,t}^{n+\gamma}, K^{\frac{n+\gamma}{p}}\omega_{s,t}^{\frac{n+\gamma}{p}}\} \\ &\leqslant B\omega_{s,t}^{\frac{n+\gamma}{p}} \text{ with } B := C'' \max\{A^{n+\gamma}, K^{\frac{n+\gamma}{p}}\}.\end{aligned}$$

From the continuous time Davie lemma (Lemma 10 in Appendix), there exists a constant D that does not depend on B (hence on A) such that $U_{s,t} \leqslant D\omega_{s,t}^{\frac{n+\gamma}{p}}$. □

Lemma 7. *Let $x \in \mathscr{C}^{1-\mathrm{var}}$ be as in Lemma 7. Then*

$$\|\epsilon_{t,s}^{(n)}[x, \overrightarrow{f}]\|_\infty = \|\psi_{t,s}[x, \overrightarrow{f}] - \phi_{t,s}^{(n)}[x, \overrightarrow{f}]\|_\infty \leqslant E\omega_{s,t}^{\frac{n+\gamma}{p}},$$

where E depends only on ω, T, and K.

Proof. Since $\phi^{(n)}_{t,s}[x, \overrightarrow{f}] = \phi^{(n)}_{t,s}[x^{s,t}, \overrightarrow{f}]$,

$$\epsilon^{(n)}_{t,s}[x, \overrightarrow{f}] := \psi_{t,s}[x, \overrightarrow{f}] - \phi^{(n)}_{t,s}[x, \overrightarrow{f}] = \psi_{t,s}[x, \overrightarrow{f}] - \psi^{(n)}_{t,s}[x, \overrightarrow{f}] + \epsilon^{(n)}_{t,s}[x, \overrightarrow{f}].$$

The results is then an immediate consequence of (44) and Lemma 6. □

Proof of Theorem 3. We recall that we assume that U and V are finite dimensional Banach spaces. For any $(a, b) \in \mathrm{V}$, any $(s, t) \in \mathbb{T}^2_+$,

$$\begin{aligned}|\phi^{(n)}_{t,s}[x, \overrightarrow{f}](a) - \phi^{(n)}_{t,s}[x, \overrightarrow{f}](b)| \\ \leqslant \sum_{k=0}^{n-1} \sum_{|I|=k} \|\overrightarrow{f}_I \mathbf{i}\|_{\mathrm{Lip}} |x^I_{s,t}| \cdot |a-b| + \sum_{|I|=n} \|\overrightarrow{f}_I \mathbf{i}\|_\gamma \cdot |x^I_{s,t}||a-b|^\gamma \\ \leqslant \sum_{k=0}^{n-1} \|\overrightarrow{f}_k \mathbf{i}\|_{\mathrm{Lip}} \|x\|^k_{[s,t],1} \cdot |a-b| + \|\overrightarrow{f}_n \mathbf{i}\|_\gamma \|x\|^n_{[s,t],1} |a-b|^\gamma.\end{aligned}$$

It then follows from Proposition 7 that $\phi^{(n)}[x, \overrightarrow{f}]$ is an almost flow with

$$\left\|\phi^{(n)}_{t,s}[x, \overrightarrow{f}](\phi^{(n)}_{s,r}[x, \overrightarrow{f}](a)) - \phi^{(n)}_{t,r}[x, \overrightarrow{f}](a)\right\|_\infty \leqslant L\omega_{s,t}^{\frac{n+\gamma}{p}}, \forall (r, s, t) \in \mathbb{T}^3_+,$$

for a constant L that depends only on $\|x\|_{[s,t],1}$ and of $\|V\|_*$.

Let $(x^m)_{m\in\mathbb{N}}$ be a sequence of bounded variation paths such that $S_n(x^m) \xrightarrow[m\to\infty]{} \mathbf{x}$ uniformly on $[0, T]$ and such that $\sup_{m\in\mathbb{N}} \|S_n(x^m)\|_p \leqslant c\|\mathbf{x}\|_p$ for a uniform constant c in m. Such a sequence exists according to Remark 10.32 in [19]. It consists in concatenating the geodesic approximations given by Proposition 11. From this, $|S_n(x^m)_{s,t}| \leqslant K\omega^{1/p}_{s,t}$ with $K = c\|\mathbf{x}\|_p$.

Clearly, $\phi^{(n)}_{t,s}[x^m, \overrightarrow{f}](a)$ converges to $\phi^{(n)}_{t,s}[\mathbf{x}, \overrightarrow{f}]$ for any $(s, t) \in \mathbb{T}^2_+$ and any $a \in \mathrm{V}$. The result follows from Corollary 2 and Lemma 7. □

8.4 Bailleul's approach

8.4.1 Classical control of ODE solutions

For $a \in \mathrm{V}$, the solution to the ordinary differential equation

$$y_t(a) = a + \int_0^t \overrightarrow{f}\mathbf{i}(y_s(a))\,\mathrm{d}s$$

is a path from $[0, T]$ to V such that

$$\phi(y_t(a)) = \phi(a) + \int_0^t V\phi(y_s(a))\,\mathrm{d}s \tag{46}$$

for any $\phi \in \mathscr{C}^1(\mathrm{V},\mathrm{V})$. Assuming enough regularity on both ϕ and $\overrightarrow{f}$, we iterate (46) so that

$$\phi(y_t(a)) = \phi(a) + tV\phi(a) + \cdots + \frac{t^k}{k!}V^k\phi(a) + R(V^k\phi, a; t)$$

with $\overrightarrow{f}^0\phi = \phi$, $\overrightarrow{f}^{k+1}\phi = \overrightarrow{f}(\overrightarrow{f}^k\phi)$, $k = 0, 1, \dots$ and

$$R_k(\psi, a; t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} \big(\psi(y_{t_k}(a)) - \psi(a)\big)\,\mathrm{d}t_k \cdots \mathrm{d}t_1.$$

for a function $\psi : \mathrm{V} \to \mathrm{V}$.

Lemma 8. *If $\overrightarrow{f}\mathfrak{i}$ is uniformly Lipschitz, then for any $a \in \mathrm{V}$ and any $t \geqslant 0$.*

$$|y_t(a) - a| \leqslant t|\overrightarrow{f}\mathfrak{i}(a)|\exp(\|\overrightarrow{f}\mathfrak{i}\|_{\mathrm{Lip}}t). \tag{47}$$

Moreover, if $\overrightarrow{f}\mathfrak{i}$ satisfies a 4-points control, then for any $a, b \in \mathrm{V}$ and any $t \geqslant 0$,

$$\begin{aligned}\Delta_t(a,b) &:= |y_s(a) - a - y_s(b) + b| \\ &\leqslant t\,\widehat{\overrightarrow{f}\mathfrak{i}}(\alpha_t(a,b))\exp\left((\widehat{\overrightarrow{f}\mathfrak{i}}(\alpha_t(a,b)) + \widecheck{\overrightarrow{f}\mathfrak{i}})t\right)|a-b| \end{aligned}\tag{48}$$

with

$$\alpha_t(a,b) := \sup_{s\in[0,t]} |y_s(a) - a| \vee |y_s(b) - b| \leqslant t\big((|\overrightarrow{f}\mathfrak{i}(a)| \vee |\overrightarrow{f}\mathfrak{i}(b)|)\exp(\|\overrightarrow{f}\mathfrak{i}\|_{\mathrm{Lip}}t)\big). \tag{49}$$

In particular, if $\overrightarrow{f}\mathfrak{i}$ satisfies a 4-points control and is bounded, then $a \mapsto y(a)$ is Lipschitz from V to $(\mathscr{C}([0,T],\mathrm{V}), \|\cdot\|_\infty)$.

Proof. Let us write $v := \overrightarrow{f}\mathfrak{i} \in \mathscr{C}^1(\mathrm{V},\mathrm{V})$. Since

$$y_t(a) - a = \int_0^t (v(y_s(a)) - v(a))\,\mathrm{d}s + tv(a), \tag{50}$$

an immediate application of the Gronwall lemma gives (47).

Since v satisfies a 4-points control, for $a, b \in \mathrm{V}$, with $\Delta_s(a,b) := |y_s(a) - a - y_s(b) + b|$,

$$\begin{aligned}&|v(y_s(a)) - v(a) - v(y_s(b)) + v(b)| \\ &\qquad \leqslant \widehat{v}(|y_s(a) - a| \vee |y_s(b) - b|)|y_s(a) - y_s(b)| \vee |a-b| + \widecheck{v}\Delta_s(a,b).\end{aligned} \tag{51}$$

Besides,

$$|y_s(a) - y_s(b)| \leqslant |a - b| + \Delta_s(a,b).$$

Injecting (51) into (50) shows that

$$\Delta_t(a,b) \leqslant \alpha_t(a,b)t\widehat{v}(\alpha_t(a,b))|a-b| + (\widehat{v}(\alpha_t(a,b)) + \widecheck{v})\int_0^t \Delta_s(a,b)\,\mathrm{d}s$$

with $\alpha_t(a,b)$ given by (49). Again, the Gronwall lemma yields (48). □

8.4.2 Bailleul's approach by truncated logarithmic series

Here $\mathrm{U} = \mathbb{R}^\ell$ for a dimension $\ell \geqslant 1$.

Let $V := (\overrightarrow{f}_1, \dots, \overrightarrow{f}_\ell)$ be a family of vector fields which acts on $C^1(\mathrm{V}, \mathrm{V})$ and $\mathbf{x} \in \mathscr{G}^p(\mathbb{R}^\ell))$ be a weak-geometric p-rough path with $2 \leqslant p < 3$. By definition of the weak geometric rough paths, $\mathbf{x}^{i,j} + \mathbf{x}^{j,i} = \mathbf{x}^i \mathbf{x}^j$ for any $i, j \in \{1, \dots, \ell\}$. We denote by $[\overrightarrow{f}_i, \overrightarrow{f}_j] := \overrightarrow{f}_i \overrightarrow{f}_j - \overrightarrow{f}_j \overrightarrow{f}_i$, the Lie bracket of vector fields $\overrightarrow{f}_i$ and $\overrightarrow{f}_j$. The Lie bracket is itself a vector field.

Assuming that V is smooth, we define for any $(s, t) \in \mathbb{T}^2$, $\alpha \in \mathbb{T}$ and $a \in \mathrm{V}$, the solution $(\alpha, a) \mapsto y_{s,t}(\alpha, a)$ of the ODE,

$$y_{s,t}(\alpha, a) = a + \int_0^\alpha \overrightarrow{f}_i \mathfrak{i}(y_{s,t}(\beta, a)) \mathbf{x}^i_{s,t} \,\mathrm{d}\beta + \frac{1}{2} \int_0^\alpha [\overrightarrow{f}_i, \overrightarrow{f}_j] \mathfrak{i}(y_{s,t}(\beta, a)) \mathbf{x}^{i,j}_{s,t} \,\mathrm{d}\beta, \tag{52}$$

where we omit the summation over all indice $i, j \in \{1, \dots, \ell\}$. We write

$$\chi_{t,s}(a) := y_{s,t}(1, a) = \phi_{s,t} + \epsilon_{t,s}, \tag{53}$$

where, by iterating (52),

$$\begin{aligned}
\phi_{t,s}(a) &:= a + \overrightarrow{f}_i \mathfrak{i}(a) \mathbf{x}^i_{s,t} + \frac{1}{2} \overrightarrow{f}_i \overrightarrow{f}_j \mathfrak{i}(a) \mathbf{x}^i_{s,t} \mathbf{x}^j_{s,t} + \frac{1}{2} [\overrightarrow{f}_i, \overrightarrow{f}_j] \mathfrak{i}(a) \mathbf{x}^{i,j}_{s,t}, \\
\epsilon_{t,s}(a) &:= \int_0^1 \int_0^\beta \overrightarrow{f}_i \overrightarrow{f}_j (\mathfrak{i}(y_{s,t}(\gamma, a)) - \mathfrak{i}(a)] \mathbf{x}^i_{s,t} \mathbf{x}^j_{s,t} \,\mathrm{d}\gamma \,\mathrm{d}\beta \\
&\quad + \frac{1}{2} \int_0^1 \int_0^\beta [\overrightarrow{f}_i, \overrightarrow{f}_j][\mathfrak{i}(y_{s,t}(\gamma, a)) - \mathfrak{i}(a)] \mathbf{x}^{i,j}_{s,t} \,\mathrm{d}\gamma \,\mathrm{d}\beta \\
&\quad + \frac{1}{2} \int_0^1 \int_0^\beta \overrightarrow{f}_i [\overrightarrow{f}_j, \overrightarrow{f}_k] \mathfrak{i}(y_{s,t}(\gamma, a)) \mathbf{x}^i_{s,t} \mathbf{x}^{j,k}_{s,t} \,\mathrm{d}\gamma \,\mathrm{d}\beta.
\end{aligned}$$

With the weak geometric property of $\mathbf{x}$: $\mathbf{x}^{i,j}_{s,t} + \mathbf{x}^{j,i}_{s,t} = x^i_{s,t} x^j_{s,t}$, we simplify the expression of ϕ such that

$$\begin{aligned}
\phi_{t,s}(a) &= a + \overrightarrow{f}_i \mathfrak{i}(a) \mathbf{x}^i_{s,t} + \frac{1}{2} \overrightarrow{f}_i \overrightarrow{f}_j \mathfrak{i}(a) \left(\mathbf{x}^{i,j}_{s,t} + \mathbf{x}^{j,i}_{s,t} \right) + \frac{1}{2} [\overrightarrow{f}_i, \overrightarrow{f}_j] \mathfrak{i}(a) \mathbf{x}^{i,j}_{s,t} \\
&= a + \overrightarrow{f}_i \mathfrak{i}(a) \mathbf{x}^i_{s,t} + \overrightarrow{f}_i \overrightarrow{f}_j \mathfrak{i}(a) \mathbf{x}^{i,j}_{s,t}.
\end{aligned}$$

So ϕ corresponds to the Davie's almost flow defined in (34).

Proposition 12. *Assume that $V \in \mathscr{C}^{2+\gamma}_b$ with $2 + \gamma > p$. Then, χ defined by (53) is an almost flow which generates a Lipschitz manifold of solutions. Moreover χ^π converges to the Davie's flow ψ of the Corollary 4.*

Proof. We proved in Lemma 4 that ϕ is a stable almost flow. We shall show that $\epsilon_{t,s}$ is a perturbation in the sense of Definition 8 and then we use Proposition 5 to conclude that χ is an almost flow which is in the galaxy of ϕ. We use Corollary 4 and Remark 4 to conclude the proof.

It is straightforward that $\epsilon_{t,t} = 0$. We start by computing an *a priori* estimate of $(\alpha, a) \mapsto y_{s,t}(\alpha, a)$, for any $(s,t) \in \mathbb{T}^2$, $a \in \mathrm{V}$ and $\alpha \in [0,1]$,

$$\begin{aligned} |y_{s,t}(\alpha, a) - a| &\leqslant \alpha \|\overrightarrow{f}_i \mathfrak{i}\|_\infty \|\mathbf{x}^i\|_p \omega_{s,t}^{1/p} + \frac{\alpha}{2} \|[\overrightarrow{f}_i, \overrightarrow{f}_j]\mathfrak{i}\|_\infty \|\mathbf{x}^{i,j}\|_{\frac{p}{2}} \omega_{s,t}^{2/p} \\ &\leqslant \left(\|\overrightarrow{f}_i \mathfrak{i}\|_\infty + \|[\overrightarrow{f}_i, \overrightarrow{f}_j]\mathfrak{i}\|_\infty \omega_{0,T}^{1/p} \right) \|\mathbf{x}\|_p \omega_{s,t}^{1/p} \qquad (54) \\ &\leqslant C_\infty \|\mathbf{x}\|_p \omega_{s,t}^{1/p}, \qquad (55) \end{aligned}$$

where $C_\infty := \|\overrightarrow{f}_i \mathfrak{i}\|_\infty + \|[\overrightarrow{f}_i, \overrightarrow{f}_j]\mathfrak{i}\|_\infty \omega_{0,T}^{1/p}$. With (55), we control the remainder $\epsilon_{t,s}$,

$$\begin{aligned} \|\epsilon_{t,s}\|_\infty \leqslant &\left[\|\mathbf{x}^i\|_p \|\mathbf{x}^j\|_p \|\overrightarrow{f}_i \overrightarrow{f}_j \mathfrak{i}\|_{\mathrm{Lip}} + \|\mathbf{x}^{i,j}\|_{2p} \|[\overrightarrow{f}_i, \overrightarrow{f}_j]\mathfrak{i}\|_{\mathrm{Lip}} \right] C_\infty \|\mathbf{x}\|_p \omega_{s,t}^{3/p} \\ &+ \|\overrightarrow{f}_i [\overrightarrow{f}_j, \overrightarrow{f}_k]\mathfrak{i}\|_\infty \|\mathbf{x}^i\|_p \|\mathbf{x}^{j,k}\|_{2p} \omega_{s,t}^{3/p}, \end{aligned}$$

which proves (21).

To show the last estimation (22), we compute for any $(s,t) \in \mathbb{T}_+^2$ and any $a, b \in \mathrm{V}$,

$$\begin{aligned} \epsilon_{t,s}(b) - \epsilon_{t,s}(a) = &\underbrace{\int_0^1 \int_0^\beta \overrightarrow{f}_i \overrightarrow{f}_j (\mathfrak{i}(y_{s,t}(\gamma, b)) - \mathfrak{i}(b) - \mathfrak{i}(y_{s,t}(\gamma, a)) + \mathfrak{i}(a)] \mathbf{x}_{s,t}^i \mathbf{x}_{s,t}^j \,\mathrm{d}\gamma \,\mathrm{d}\beta}_{\mathrm{I}} \\ &+ \underbrace{\frac{1}{2} \int_0^1 \int_0^\beta [\overrightarrow{f}_i, \overrightarrow{f}_j][\mathfrak{i}(y_{s,t}(\gamma, b)) - \mathfrak{i}(b) - \mathfrak{i}(y_{s,t}(\gamma, a)) + \mathfrak{i}(a)] \mathbf{x}_{s,t}^{i,j} \,\mathrm{d}\gamma \,\mathrm{d}\beta}_{\mathrm{II}} \\ &+ \underbrace{\frac{1}{2} \int_0^1 \int_0^\beta \overrightarrow{f}_i [\overrightarrow{f}_j, \overrightarrow{f}_k][\mathfrak{i}(y_{t,s}(\gamma, b)) - \mathfrak{i}(y_{t,s}(\gamma, a))] \mathbf{x}_{s,t}^i \mathbf{x}_{s,t}^{j,k} \,\mathrm{d}\gamma \,\mathrm{d}\beta}_{\mathrm{III}}. \end{aligned}$$

We assume that $\overrightarrow{f}\mathfrak{i} \in \mathscr{C}_b^{2+\gamma}$, so $\overrightarrow{f}_i \overrightarrow{f}_j \mathfrak{i} \in \mathscr{C}_b^{1+\gamma}$. It follows from Lemma 1 that $\overrightarrow{f}_i \overrightarrow{f}_j$ satisfies a 4-points control such that $\widehat{\overrightarrow{f}_i \overrightarrow{f}_j \mathfrak{i}}(x) = C|x|^\gamma$ where C a positive constant with depends on the γ-Hölder norm of the derivative of $\overrightarrow{f}_i \overrightarrow{f}_j$. It follows that,

$$\begin{aligned} |\mathrm{I}| \leqslant C &\sup_{\gamma \in [0,1], a \in \mathrm{V}} |y_{s,t}(\gamma, a) - a|^\gamma \left[\sup_{\gamma \in [0,1]} |y_{s,t}(\gamma, b) - y_{s,t}(\gamma, a)| + |b - a| \right] \|\mathbf{x}^{(1)}\|_p^2 \omega_{s,t}^{2/p} \\ &+ \|\overrightarrow{f}_i \overrightarrow{f}_j\|_{\mathrm{Lip}} \sup_{\gamma \in [0,1]} |y_{s,t}(\gamma, b) - b - y_{s,t}(\gamma, a) + a| \|\mathbf{x}^{(1)}\|_p^2 \omega_{s,t}^{2/p}, \end{aligned}$$

which yields combining with (48), (49) and (55) to

$$|\mathrm{I}| \leqslant CC^{\gamma}_{\infty,T}\|\mathbf{x}\|^{\gamma}_{p}\omega^{\gamma/p}_{s,t}(1+C_T)|b-a|\|\mathbf{x}^{(1)}\|^2_p\omega^{2/p}_{s,t} + \|\overrightarrow{f}_i\overrightarrow{f}_j\|_{\mathrm{Lip}}C_T|b-a|\|\mathbf{x}^{(1)}\|^2_p\omega^{2/p}_{s,t},$$

where C_T is a constant which is compute in (48). And finally, $|\mathrm{I}| \leqslant \delta_T|b-a|$ where δ_T is a constant depending on the norms of V, $\mathbf{x}$ which decreases to 0 when $T \to 0$. Similarly, we obtain the same estimation for II. To estimate III, we note that $\overrightarrow{f}_i[\overrightarrow{f}_j, \overrightarrow{f}_k]\mathfrak{i} \in \mathscr{C}^{\gamma}_b$. Then with (48) it follows that $|\mathrm{III}| \leqslant C''_T\omega^{2/p}_{s,t}|b-a|^{\gamma}$, where C''_T is another constant which has the same dependencies as C'_T. Thus $|\epsilon(b)-\epsilon(a)| \leqslant \delta|b-a| + C''_T\omega^{2/p}_{s,t}|b-a|^{\gamma}$. This concludes the proof. □

Remark 6. This results can be extend to the case U is a Banach case. It is an advantage compare to the Friz-Victoir's approach of Subsection 8.3.

A The Davie lemma

In this section, we introduce our main tool to control the iterated products (Notation 2) on a partition.

Definition 13 (Successive points)**.** Let π be a partition of $[0,T]$. Two points s and t of π are said to be *at distance* k if there are $k-1$ points between them in π. Points at distance 1 are then *successive points* in π.

We now state the Davie lemma[1].

Lemma 9 (The Davie lemma, discrete time version)**.** *Let us consider a family* $U := \{U_{s,t}\}_{s,t\in\pi,s\leqslant t}$ *with values in* $\mathbb{R}_+$ *satisfying for any* $(r,s,t) \in \pi \bigcap \mathbb{T}^3$, $U_{r,r} = 0$,

$$\begin{aligned} U_{r,s} &\leqslant D\varpi(\omega_{r,s}) \text{ when } r \text{ and } s \text{ are successive points}, \\ U_{r,t} &\leqslant (1+\alpha_T)U_{r,s} + (1+\alpha_T)U_{s,t} + B\varpi(\omega_{r,t}), \end{aligned} \tag{56}$$

for some constants $D \geqslant 1$, $B \geqslant 0$ *and* $\alpha_T \geqslant 0$ *that decreases to* 0 *as* $T \to 0$.

Then for all $T > 0$ *such that* $\varkappa(1+\alpha_T)^2 + \alpha_T < 1$,

$$U_{r,t} \leqslant A\varpi(\omega_{r,t}),\ \forall (r,t) \in [0,T] \cap \pi^2, \tag{57}$$

$$\text{with } A := \frac{D(1+\alpha_T)(1+\alpha_T)^2 + B(2+\alpha_T)}{1-(\varkappa(1+\alpha_T)^2+\alpha_T)}. \tag{58}$$

In particular, A does not depend on the choice of the partition.

[1]What we call here the Davie lemma differs from the Davie lemma A and B in [19], and also from the one in [14]. However, they all share the same key idea which is due to A.M. Davie.

Proof. We perform an induction on the distance m between points in π.

If $m = 0$ or $m = 1$, (57) is true since $A \geqslant D$.

Let us assume that this is true for any two points at distance m. Fix two points r and t at distance $m+1$ in π. Hence, there exists two successive points s and s' in π such that

$$\omega_{r,s} \leqslant \frac{\omega_{r,t}}{2} \text{ and } \omega_{s',t} \leqslant \frac{\omega_{r,t}}{2}$$

with possibly $r = s$ or $s' = t$.

Applying (56) twice with (r, s, t) and (s, s', t),

$$\begin{aligned} U_{r,t} &\leqslant (1+\alpha_T)U_{r,s} + (1+\alpha_T)U_{s,t} + B\varpi(\omega_{r,t}) \\ &\leqslant (1+\alpha_T)U_{r,s} + (1+\alpha_T)^2(U_{s,s'} + U_{s',t}) + (2+\alpha_T)B\varpi(\omega_{r,t}). \end{aligned}$$

Both $U_{r,s}$ and $U_{s,t}$ satisfy the induction property. With (2),

$$\begin{aligned} U_{r,t} &\leqslant 2A(1+\alpha_T)^2\varpi\left(\frac{\omega_{r,t}}{2}\right) + (1+\alpha_T)^2 D\varpi(\omega_{s,s'}) + (2+\alpha_T)B\varpi(\omega_{r,t}) \\ &\leqslant \left[A\left(\varkappa(1+\alpha_T)^2 + \alpha_T\right) + D(1+\alpha_T)^2(1+\alpha_T) + (2+\alpha_T)B\right]\varpi(\omega_{r,t}). \end{aligned}$$

Our choice of $A \geqslant D$ in (58) ensures the results at level $m+1$. The control (57) is then true whatever the partition. ☐

We could now state a continuous time version of the Davie lemma.

Lemma 10 (The Davie lemma, continuous time version)**.** *Let us consider a family* $U := \{U_{s,t}\}_{s,t\in,\mathbb{T}^2}$ *with values in* $\mathbb{R}_+$ *satisfying for any* $(r, s, t) \in \mathbb{T}^3$,

$$\begin{aligned} &U_{r,s} \leqslant E\varpi(\omega_{r,s}), \\ &U_{r,t} \leqslant (1+\alpha_T)U_{r,s} + (1+\alpha_T)U_{s,t} + B\varpi(\omega_{r,t}), \end{aligned} \tag{59}$$

for some constants $E \geqslant 1$*,* $B \geqslant 0$ *and* $\alpha_T \geqslant 0$ *that decreases to* 0 *as* $T \to 0$*. Then for any* T *such that* $\varkappa(1+\alpha_T)^2 + \alpha_T < 1$*,*

$$\begin{aligned} &U_{r,t} \leqslant A\varpi(\omega_{r,t}),\ \forall (r,t) \in \mathbb{T}^2, \\ &\text{with } A := B\frac{(2+\alpha_T)}{1 - (\varkappa(1+\alpha_T)^2 + \alpha_T)}. \end{aligned} \tag{60}$$

In particular, the choice of A *in* (60) *does not depend on the bound* E *in* (59)*.*

Proof. The proof is similar as the one of Lemma 7 in [6] from Eq. (31). ☐

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