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# A REFINED MEAN FIELD APPROXIMATION OF SYNCHRONOUS DISCRETE-TIME POPULATION MODELS

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ABSTRACT. Mean field approximation is a popular method to study the behaviour of stochastic models composed of a large number of interacting objects. When the objects are asynchronous, the mean field approximation of a population model can be expressed as an ordinary differential equation. When the objects are (clock-) synchronous the mean field approximation is a discrete time dynamical system. We focus on the latter.

We study the accuracy of mean field approximation when this approximation is a discrete-time dynamical system. We extend a result that was shown for the continuous time case and we prove that expected performance indicators estimated by mean field approximation are  $O(1/N)$ -accurate. We provide simple expressions to effectively compute the asymptotic error of mean field approximation, for finite time-horizon and steady-state, and we use this computed error to propose what we call a *refined* mean field approximation. We show, by using a few numerical examples, that this technique improves the quality of approximation compared to the classical mean field approximation, especially for relatively small population sizes.

## 1. INTRODUCTION

Stochastic models are often used to model and analyse the performance of computer (and many other) systems. A particularly rich and popular class of models is given by stochastic population models. These have been used, for instance, to model biological systems [26], epidemic spreading [1] or queuing networks [25]. These systems are composed of a set of homogeneous objects interacting with one another. These models have a high expressive power, but an exact analysis of any such a model is often computationally prohibitive when the number of objects of the system grows. This results in the need for approximation techniques.

A popular technique is to use mean field approximation. The idea behind mean field approximation is to replace the study of the original stochastic system by the one of a, much simpler, deterministic dynamical system. The success of mean field approximation can be explained by multiple factors : (a) it is fast – many models can be solved in closed form [25, 19, 23, 18] or easily solved numerically [17, 10, 24] – (b) it is proven to be asymptotically optimal as the number of objects in the system goes to infinity [15, 5, 2, 12, 4]; and (c) it is often very accurate also for systems of moderate size, composed of  $N \approx 100$  objects.

The mean field approximation of a given model is constructed by considering the limit of the original stochastic model as the number of objects  $N$  goes to infinity. There can be two types of limits. The first type arises when the dynamics of

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This paper and the simulations it contains are fully reproducible : [https://github.com/ngast/RefinedMeanField\\_SynchronousPopulation](https://github.com/ngast/RefinedMeanField_SynchronousPopulation).

the objects are asynchronous. In this case the mean field approximation is given by a continuous time dynamical system (often a system of ordinary differential equations) – this is the most studied case *e.g.* [15, 2, 4]. The second type arises when the objects are synchronous. In this case the mean field approximation is a discrete time dynamical system [5, 11, 22]. We focus on the latter.

**Contributions.** Our main contribution is an extension to (synchronous) DTMC population models of the results proposed in [13] for (asynchronous) CTMC population models, thus providing a new approximation technique that is significantly more accurate than classical mean field approximation, especially for relatively small systems. Our results apply to the classical model of [5, 11, 16]. We prove our result for the transient and the steady state dynamics. Moreover, it retains an interesting feature of mean field approximation by being computationally non-intensive.

More precisely, if  $M_i^{(N)}(t)$  denotes the proportion of objects in a state  $i$  at time  $t$ , then the classical result of [5] states that, as  $N$  grows large, if the vector  $M^{(N)}(0)$  converges almost surely to  $m$ , for some vector  $m$ , then the vector  $M^{(N)}(t)$  converges almost surely to a deterministic quantity  $\mu(t)$  that satisfies a recurrence equation of the form  $\mu(t+1) = \mu(t)\mathbf{K}(\mu(t))$  with  $\mu(0) = m$ . We show that, for any twice differentiable function  $h$ , there exists a constant  $V_{t,h}$  such that

$$(1) \quad \lim_{N \rightarrow \infty} N(\mathbb{E} [h(M_t^{(N)})] - h(\mu(t))) = V_{t,h}.$$

We provide an algorithm to compute the constant  $V_{t,h}$  by a linear dynamical system that involves the first and second derivative of the functions  $m \mapsto m\mathbf{K}(m)$  and  $h$ . We also show that if the function  $m \mapsto m\mathbf{K}(m)$  has a unique fixed point  $\mu(\infty)$  that is globally exponentially stable, then the same result holds for the steady-state : in this case,  $V_{\infty,h} = \lim_{t \rightarrow \infty} V_{t,h}$  exists and can be expressed as the solution of a discrete-time Lyapunov equation that involves the first and second derivative of  $m \mapsto m\mathbf{K}(m)$  and  $h$  evaluated at the point  $\mu(\infty)$ .

By using these results, we define a quantity  $h(\mu(t)) + V_{t,h}/N$  that we call the *refined mean field* approximation. As opposed to the classical mean field approximation, this approximation depends on the system size  $N$ . We illustrate our theoretical results with four different examples. While these examples all show that our refined model is clearly more accurate than the classical approximations, they illustrate different characteristics. The first two examples are cases where the dynamical system has a unique exponentially stable attractor. In these examples, refined mean field provides performance estimates that are extremely accurate (the typical error between  $M^{(N)}$  and  $\mu$  is less than 1% for  $N = 10$ ). The third example is different as it is a case when the stochastic system has two absorbing states. In this case, the refined mean field is still more accurate than the classical mean field approximation but remains far from the exact values for  $N = 10$ . It is only for larger values of  $N$  that the refined mean field provides a very accurate estimate. Finally, the fourth example is a case where the mean field approximation has a unique attractor that is not exponentially stable. We observe that in this case the refined approximation provides an accurate approximation of  $\mathbb{E}[M^{(N)}(t)]$  for small values of  $t$  but fails to predict correctly what happens when  $t$  is large compared to  $N$ . In fact in this case, one cannot refine the steady-state expectation by a term in  $O(1/N)$  because the convergence is only in  $O(1/\sqrt{N})$  in this case.

This suggests that, when using a mean field or refined mean field approximation, one has to be careful : the approximations of a system with more than one stable

equilibrium or a unique but non-exponentially stable equilibrium is likely to be inaccurate for small values of  $N$ , even when one focuses on the transient behaviour. Related work. Our results extend the recent results of [13]. The authors of [13] study the steady-state of stochastic models that have a continuous-time mean field approximation. They show that Equation (1) is true in this case and provide a numerical algorithm to compute the constant. Our paper has two theoretical contributions with respect to [13] : First we show that the results also hold for models that have a discrete-time mean field approximation, and second we show how to derive these equations for the transient and steady state regimes of such systems. This means that our results remain in the realm of discrete-time models whereas in [13] it is shown how some discrete-time models can be transformed into density-dependent (continuous time) population models by replacing time steps with steps that last for a random time that is exponentially distributed with mean  $1/N$ , where  $N$  is the population size. The resulting continuous time model can then be analysed using the approximation techniques for CTMC population models discussed in [13].

The results of [13] and the one of the current paper follow from a series of recent results concerning the rate of convergence of stochastic models to their mean field approximation [9, 27, 28, 14]. The key idea behind these works is to study the convergence of the generator of the stochastic processes to the one of its mean field approximation and to use this convergence rate to obtain a bound on Equation (1). For the steady-state regime, this is made possible by using Stein's methods [21, 7, 6]. Note that the approach taken in the current paper is fundamentally different from the one that is usually used to obtain convergence rates, like [12, 3, 11] in which the authors focus on sample path convergence and obtain bounds on the convergence of the expected distance  $\mathbb{E}[\|M^{(N)} - \mu\|]$  between the stochastic system and its mean field approximation. When focusing on sample path convergence, the refinement of the mean field approximation would be to consider an additive term of  $1/\sqrt{N}$  times a Gaussian noise as for example in [12] and not a  $1/N$  term as in this paper. Outline. The rest of the paper is organised as follows. In Section 2, we introduce the model that we study. In Section 3 we provide the main results and in particular Theorem 1. In Sections 4, 5, 6 and 7, we provide a few numerical examples that demonstrate the accuracy of the refined mean field approximation and its limits. Finally, we conclude in Section 8.

## 2. PRELIMINARIES

In this section we introduce some terminology and notation as well as some preliminary definitions, setting the context for the rest of the paper.

**2.1. Notations.** We let  $\mathbb{N}$  denote the set of natural numbers and  $\mathbb{R}_{\geq 0}^n$  the set of  $n$ -tuples of non-negative real numbers; we conventionally see any such an  $n$ -tuple  $m = (m_1, \dots, m_n)$  as a row-vector, i.e. a  $1 \times n$  matrix. We let  $\mathcal{U}^n \subset \mathbb{R}_{\geq 0}^n$  be the unit simplex of  $\mathbb{R}_{\geq 0}^n$ , that is  $\mathcal{U}^n = \{m \in [0, 1]^n \mid m_1 + \dots + m_n = 1\}$ .

For function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$  continuous and twice differentiable, with  $f(m) = (f_1(m), \dots, f_p(m))$ , we denote by  $Df$  and  $D^2f$  its first and second derivatives, respectively.  $Df(m)$  is the  $p \times n$  (function) matrix such that  $(Df(m))_{ij} = \frac{\partial f_i(m)}{\partial m_j}$ .  $D^2f(m)$  is the  $p \times n \times n$  tensor such that  $(D^2f(m))_{ijk} = \frac{\partial^2 f_i(m)}{\partial m_j \partial m_k}$ .

Moreover, for a  $p \times n \times n$  tensor  $P$  and a  $n \times n$  matrix  $Q$ , we let  $P \cdot Q$  be the (row) vector in  $\mathbb{R}^p$  such that  $(P \cdot Q)_i = \sum_{j,k=1}^n (P)_{ijk}(Q)_{jk}$ , for  $i = 1, \dots, p$ . In addition, for  $p \times n$  matrix  $A$  and  $n \times m$  matrix  $B$  we use the notation  $AB$  for standard matrix product:  $(AB)_{ij} = \sum_{k=1}^n (A)_{ik}(B)_{kj}$ ; obviously this includes the case of vector inner product  $u^T v$  for  $u$  and  $v$  being  $n \times 1$  (column) vectors, where  $A^T$  is the transpose of  $A$ , i.e.  $(A^T)_{ij} = (A)_{ji}$ ; the vector outer product  $u \otimes v$ , sometimes also denoted by  $u v^T$ , is the matrix such that  $(u \otimes v)_{i,j} = u_i v_j$ .

Finally, for any vector  $v$ , we let  $\|v\|$  denote the norm of  $v$  and  $\|v\|^2$  denote the square of  $\|v\|$ . Note that the choice of the specific norm is left unspecified because the results presented in the present paper hold for any norm<sup>1</sup>. Note that in the proofs, what we denote by  $E, o(\|E\|^2), \mathbb{E}[o(\|E\|^2)]$  are  $p$ -dimensional vectors while their norm  $\|E\|, \|E\|^2, \|\mathbb{E}[o(\|E\|^2)]\| \in \mathbb{R}$  are real non-negative numbers.

**2.2. Synchronous Mean Field Model.** We consider a system of  $0 < N \in \mathbb{N}$  identical<sup>2</sup> interacting objects;  $N$  is called the *size* of the system. We assume that the number of local states of each object is finite<sup>3</sup>, say  $n$ ; for the sake of simplicity, in the sequel we let the set of states of local objects be  $\{1, \dots, n\}$ , when not specified otherwise. Time is *discrete* and the behaviour of the system is characterised by a (time homogeneous) discrete time Markov chain (DTMC)  $X^{(N)}(t) = (X_1^{(N)}(t), \dots, X_N^{(N)}(t))$ , where  $X_i^{(N)}(t)$  is the state of object  $i$  at time  $t$ , for  $i = 1, \dots, N$ .

We define the occupancy measure at time  $t$  as the row-vector  $M^{(N)}(t) = (M_1^{(N)}(t), \dots, M_n^{(N)}(t))$  where, for  $j = 1, \dots, n$ ,  $M_j^{(N)}(t)$  is the *fraction* of objects in state  $j$  at time  $t$ , over the total population of  $N$  objects:

$$M_j^{(N)}(t) = \frac{1}{N} \sum_{i=1}^N 1_{\{X_i^{(N)}(t)=j\}}$$

where  $1_{\{x=j\}}$  is equal to 1 if  $x = j$  and 0 otherwise.

At each time step  $t \in \mathbb{N}$  each object performs a local transition, that may also be a self-loop to its current state. The transition probabilities of an object state depend on the current local state of the object and may depend also on  $M(t)$ .

We let  $\mathbf{K}(m)$  denote the one-step transition probability  $n \times n$  matrix of an object in the system:  $\mathbf{K}_{ij}(m)$  is the probability for the object to jump from state  $i$  to state  $j$  in the system when the occupancy measure vector is  $m$ . We assume that, given the occupancy measure, the transitions made by two objects are independent. Our model is identical to the one of [5] up to the fact that the authors of [5] add a continuous resource to the model and allow object transition matrix  $\mathbf{K}$  to depend also on the size  $N$  of the system (in which case they assume that the sequence of transition matrices  $\mathbf{K}^N$  converges to a function  $\mathbf{K}$  as  $N$  goes to infinity). To

<sup>1</sup>Of course, technical details in the proofs may depend on the specific norm (see e.g. footnote 4 in the proof of Lemma 4)

<sup>2</sup>It is worth pointing out here that the requirement that the  $N$  objects be identical can be relaxed, since a system with *different* classes of *identical* objects can easily be modelled by considering an equivalent system with instances of an object whose set of states is the union of those of the original objects and similarly for the set of its transitions, as shown in the example in Section 5.

<sup>3</sup>In fact, the same theoretical results could be derived for infinite dimensional models with two additional assumptions to cope with the fact that the simplex  $\mathcal{U}^\infty$  is not compact : (i) imposing all functions to be *uniformly* continuous and (ii) Imposing tightness assumptions.

simplify the exposition, in this paper we consider a case without resource and we assume  $\mathbf{K}(m)$  is a continuous function of  $m$  that does not depend on  $N$ . The results presented in this paper could be extended to the more general case where  $\mathbf{K}(m)$  is also a function of  $N$  and could be modified to incorporate a resource, essentially by replacing our equation for the variance  $\Gamma$  by the one presented in [11, Equation (7)].

Below we recall Theorem 4.1 of [5] on classical mean field approximation, under the simplifying assumptions mentioned above:

**Theorem 4.1 of [5] (Convergence to Mean Field)** *Assume that the initial occupancy measure  $M^{(N)}(0)$  converges almost surely to the deterministic limit  $\mu(0)$ . Define  $\mu(t)$  iteratively by (for  $t \geq 0$ ):*

$$(2) \quad \mu(t+1) = \mu(t) \mathbf{K}(\mu(t)).$$

*Then for any fixed time  $t$ , almost surely:*

$$\lim_{N \rightarrow \infty} M^{(N)}(t) = \mu(t).$$

In the sequel, we will write  $M(t)$  or simply  $M$  instead of  $M^{(N)}(t)$ , leaving  $N$  and  $t$  implicit, when this does not cause confusion.

### 3. REFINED DETERMINISTIC APPROXIMATION THEOREM

In this section, we present our main results (Theorem 1 and Theorem 2) and provide their proofs.

**3.1. First Main Result : Transient Behaviour.** The iterative procedure of Theorem 4.1 of [5] can be formalised as a  $t$ -indexed family of functions  $\Phi_t$  from  $\mathcal{U}^n$  to  $\mathcal{U}^n$ , where the functions  $\Phi_t : \mathcal{U}^n \rightarrow \mathcal{U}^n$  are defined as follows:

$$\Phi_0(m) = m; \quad (\Phi_1(m))_j = \sum_{i=1}^n m_i \mathbf{K}_{ij}(m); \quad \Phi_{t+1}(m) = \Phi_1(\Phi_t(m))$$

This implies that for all  $m \in \mathcal{U}^n$  and  $t \in \mathbb{N}$ , we have  $\Phi_1(\Phi_t(m)) = \Phi_t(\Phi_1(m))$ .

In the following we assume that  $\Phi_1$  is continuous and twice differentiable with respect to  $m$  and that its second derivative is continuous (note that as  $\mathcal{U}^n$  is compact, this implies that  $\Phi_1$  and its first two derivative are uniformly continuous). Moreover, in what follows, we assume that  $M^{(N)}(0)$  converges to  $\mu(0)$  (a deterministic value) as  $N$  goes to infinity and we let  $\mu(t)$  be defined as in Equation (2), or, equivalently,  $\mu(t+1) = \Phi_1(\mu(t)) = \Phi_{t+1}(\mu(0))$ .

Our main theorem can be stated as follows.

**Theorem 1.** *Assume that the function  $\Phi_1$  is twice differentiable with continuous second derivative and that  $M^{(N)}(0)$  converges weakly to  $\mu(0)$ . Let  $A_t$  and  $B_t$  be respectively the  $n \times n$  matrix  $A_t = (D\Phi_1)(\mu(t))$  and the  $n \times n \times n$  tensor  $B_t = (D^2\Phi_1)(\mu(t))$ . Then for any continuous and twice differentiable function with continuous second derivative  $h : \mathcal{U}^n \rightarrow \mathbb{R}_{\geq 0}^p$  we have:*

$$\lim_{N \rightarrow \infty} N \mathbb{E} \left[ h(M^{(N)}(t)) - h(\Phi_t(M^{(N)}(0))) \right] = Dh(\mu(t))V_t + \frac{1}{2}D^2h(\mu(t)) \cdot W_t,$$

where  $V_t$  is an  $n \times 1$  vector and  $W_t$  is an  $n \times n$  matrix, defined as follows:

$$\begin{aligned} V_{t+1} &= A_t V_t + \frac{1}{2} B_t \cdot W_t \\ W_{t+1} &= \Gamma(\mu(t)) + A_t W_t A_t^T, \end{aligned}$$

with  $V_0 = 0$ ,  $W_0 = 0$  and  $\Gamma(m)$  is the following  $n \times n$  matrix:

$$\begin{aligned} \Gamma_{jj}(m) &= \sum_{i=1}^n m_i \mathbf{K}_{ij}(m) (1 - \mathbf{K}_{ij}(m)) \\ \Gamma_{jk}(m) &= - \sum_{i=1}^n m_i \mathbf{K}_{ij}(m) \mathbf{K}_{ik}(m) \end{aligned}$$

The key idea of the proof is to use a Taylor expansion of  $\mathbb{E}[h(\Phi_1(m))]$  around  $\Phi_1(m)$ . We postpone the proof to Section 3.3.1.

One of the main consequences of Theorem 1 is that it allows us to compute precisely a development in  $O(1/N)$  of the mean and the covariance of the vector  $M^{(N)}(t)$ . This first order development is what we call the *refined mean field approximation*. In our numerical simulations, we will show that this refined approximation can greatly improve the accuracy of the original mean field approximation when the number of entities  $N$  is relatively small.

**Corollary 1.** *Let  $t \in \mathbb{N}$ . Under the assumptions of Theorem 1, and denoting  $\mu(t) = \Phi_t(m)$ , it holds that*

(i) *For any coordinate  $i$  and any time-step*

$$\mathbb{E} \left[ M_i^{(N)}(t) \right] = \mu_i(t) + \frac{(V_t)_i}{N} + o\left(\frac{1}{N}\right).$$

(ii) *For any pair of coordinates  $i, j$ , the co-variance satisfies*

$$\text{cov} \left( M_i^{(N)}(t), M_j^{(N)}(t) \right) = \frac{1}{N} (W_t)_{i,j} + o\left(\frac{1}{N}\right).$$

**3.2. Second Main Result : Steady-State.** Mean field approximation can also be used to characterise the steady-state behaviour of a population model. It has been shown that, in the case of continuous time or discrete-time mean field approximation, if this approximation has a unique attractor, then the stationary distribution of the system of size  $N$  concentrates on this attractor. In this section, we refine this result by computing the rate of convergence and by defining the refined approximation in this case.

We say that a point  $\mu(\infty)$  is an exponentially stable attractor if

- For any  $m \in \mathcal{U}^n$  :  $\lim_{t \rightarrow \infty} \Phi_t(m) = \mu(\infty)$  (i.e. it is a global attractor).
- There exists an open neighbourhood  $V$  of  $\mu(\infty)$  and two constants  $a, b$  such that all  $m \in V$  :  $\|\Phi_t(m) - \mu(\infty)\| \leq a e^{-bt} \|m - \mu(\infty)\|$  (i.e. it is exponentially stable).

**Theorem 2.** *Assume that  $M^{(N)}$  has a unique stationary distribution (for each  $N$ ), that the function  $\Phi_1$  is twice differentiable and that the flow has a unique exponentially stable attractor  $\mu(\infty)$ . Then there exists a  $n \times 1$  vector  $V_\infty$  and a  $n \times n$  matrix  $W_\infty$  such that the constants  $V_t$  and  $W_t$  defined in Theorem 1 satisfy:*

$$\lim_{t \rightarrow \infty} V_t = V_\infty \quad \text{and} \quad \lim_{t \rightarrow \infty} W_t = W_\infty$$

Moreover

(i)  $W_\infty$  is the unique solution of the discrete-time Lyapunov equation:

$$A_\infty W A_\infty^T - W + \Gamma(\mu(\infty)) = 0$$

and  $V_\infty$  is uniquely determined by

$$V_\infty = \frac{1}{2}(I - A_\infty)^{-1} B_\infty W_\infty,$$

where  $A_\infty = D\Phi_1(\mu(\infty))$ ,  $B_\infty = D^2\Phi_1(\mu(\infty))$  and  $I$  is the identity matrix.

(ii) for any twice differentiable function  $h$ , we can exchange the limits :

$$\begin{aligned} & \lim_{N \rightarrow \infty} \lim_{t \rightarrow \infty} N(\mathbb{E}[h(M(t))] - h(\Phi_t(M^{(N)}(0)))) \\ &= \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} N(\mathbb{E}[h(M(t))] - h(\Phi_t(M^{(N)}(0)))) \\ &= Dh(\mu(\infty))V_\infty + \frac{1}{2}D^2h(\mu(\infty)) \cdot W_\infty, \end{aligned}$$

This result is interesting for at least two reasons. First, it generalises Theorem 1 to the case of stationary distribution. Second, it is also the first result that provides a rate of convergence for the steady-state distribution of a model that has a discrete-time mean field approximation (to the best of our knowledge, the rate of convergence had only been obtained for the finite-time horizon in [11]). We postpone the proof to Section 3.3.2.

**3.3. Proofs.** To ease notation, in all the proofs we denote  $M^N(0)$  by  $m$ , unless specified otherwise. In particular, when we write  $\mathbb{E}[h(M(t))]$  we formally mean  $\mathbb{E}[h(M^{(N)}(t)) | M^{(N)}(0) = m]$ .

**3.3.1. Proof of Theorem 1.** One of the key ingredients to prove our result is to study what happens for  $t = 1$ . This is what we do in Lemma 1. Then Theorem 1 will follow by using an induction on  $t$ . Note that one of the main technicalities in all these lemmas is to prove that the convergence is uniform in the initial condition. This is why we make use of various functions  $\varepsilon(N)$  or  $\varepsilon_{t,g}(N)$  that control the uniform convergence to 0.

**Lemma 1.** *Let  $h : \mathcal{U}^n \rightarrow \mathbb{R}_{\geq 0}^p$  (for  $p \geq 1$ ), be a twice differentiable function such that  $D^2h$  is continuous. Then, there exists a function  $\varepsilon(N)$  such that  $\lim_{N \rightarrow \infty} \varepsilon(N) = 0$  and for all  $M^{(N)}(0) = m \in \mathcal{U}^n$ , the following holds:*

$$\left| \left| N(\mathbb{E}[h(M(1))] - h(\Phi_1(m))) - \frac{1}{2}(D^2h)(\Phi_1(m)) \cdot \Gamma(m) \right| \right| \leq \varepsilon(N).$$

where  $\Gamma(m)$  is defined in Theorem 1.

*Proof.* The key idea of this proof is to consider the Taylor expansion of  $h$  for  $M(1)$  in the neighbourhood of  $\Phi_1(m)$ . Let  $E = M(1) - \Phi_1(m)$ . We get the following :

$$h(M(1)) - h(\Phi_1(m)) = (Dh)(\Phi_1(m))E + \frac{1}{2}(D^2h)(\Phi_1(m)) \cdot (E \otimes E) + o(\|E\|^2)$$

Taking the expectation on both sides, we get :

$$\begin{aligned} & \mathbb{E}[h(M(1))] - h(\Phi_1(m)) \\ &= (Dh)(\Phi_1(m))\mathbb{E}[E] + \frac{1}{2}(D^2h)(\Phi_1(m)) \cdot \mathbb{E}[(E \otimes E)] + \mathbb{E}[o(\|E\|^2)] \end{aligned}$$

The result follows by using Lemma 3, that establishes that  $\mathbb{E}[E] = 0$  and  $N\mathbb{E}[(E \otimes E)] = \Gamma(m)$  and Lemma 4 that shows  $\|N\mathbb{E}[o(\|E\|^2)]\| \leq \varepsilon(N)$ .  $\square$



The following lemma is a direct generalisation of Lemma 1 and it will be used in the proof of Theorem 1. The proof is essentially the same as that of Lemma 1 and exploits the time-homogeneity of the Markov chain.

**Lemma 2.** *Let  $h : \mathcal{U}^n \rightarrow \mathbb{R}_{\geq 0}^p$  be a twice differentiable function whose second derivative is continuous. Then there exists a function  $\varepsilon(N)$  such that  $\lim_{N \rightarrow \infty} \varepsilon(N) = 0$  and for all  $t \in \mathbb{N}$ ,  $N \in \mathbb{N}_{>0}$  and  $M^{(N)}(t) = m' \in \mathcal{U}^n$ , the following holds:*

$$\left| N(\mathbb{E}[h(M(t+1)) | M(t) = m'] - h(\Phi_1(m'))) - \frac{1}{2}(D^2h)(\Phi_1(m')) \cdot \Gamma(m') \right| \leq \varepsilon(N)$$

**Lemma 3.** *Under the assumptions of Lemma 1, we obtain that*

$E = \mathbb{E}[M^{(N)}(1) - \Phi_1(m) | M^{(N)}(0) = m]$  *satisfies  $\mathbb{E}[E] = 0$  and  $\mathbb{E}[E \otimes E] = \Gamma(m)/N$ .*

*Proof.* We observe that by definition of our model,  $M_j(1)$  is the following random variable:

$$(3) \quad M_j(1) = \frac{1}{N} \sum_{i=1}^n \widehat{B}_{ij}$$

where  $(\widehat{B}_{i,\cdot})$  is a random vector with multinomial distribution, with parameters  $Nm_i$  and  $(\mathbf{K}_{i,\cdot}(m))$ . The variables are independent for different values of  $i$  (in particular, if  $i \neq i'$  we have  $\text{cov}(\widehat{B}_{ij}, \widehat{B}_{i'k}) = 0$ ). Moreover, for all  $i$  and all  $j \neq k$ :

$$\begin{aligned} \mathbb{E}[\widehat{B}_{ij}] &= Nm_i \mathbf{K}_{ij}(m) \\ \text{var}(\widehat{B}_{ij}) &= Nm_i \mathbf{K}_{ij}(m)(1 - \mathbf{K}_{ij}(m)) \\ \text{cov}(\widehat{B}_{ij}, \widehat{B}_{ik}) &= Nm_i \mathbf{K}_{ij}(m) \mathbf{K}_{ik}(m). \end{aligned}$$

This implies that

$$(4) \quad \mathbb{E}[M_j(1)] = \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^n \widehat{B}_{ij}\right] = \frac{1}{N} \sum_{i=1}^n \mathbb{E}[\widehat{B}_{ij}] = \Phi_1(m)_j.$$

The case of  $E \otimes E$  makes again use of Equation (3). Note that by (4),  $E_j = M_j(1) - \Phi_1(m)_j = M_j(1) - \mathbb{E}[M_j(1)]$ . This shows that  $\mathbb{E}[(E \otimes E)_{jk}] = \text{cov}(M_j(1), M_k(1))$ , *i.e.* the covariance of  $M_j(1)$  and  $M_k(1)$ . We consider the case  $k = j$  and the case  $k \neq j$  separately.

**Case  $k = j$ .**

$$\begin{aligned} N\mathbb{E}[(E \otimes E)_{jj}] &= N\text{var}(M_j(1)) \\ &= N\text{var}\left(\frac{1}{N} \sum_{i=1}^n \widehat{B}_{ij}\right) \\ &= \frac{1}{N} \sum_{i=1}^n \text{var}(\widehat{B}_{ij}) \\ &= \sum_{i=1}^n m_i \mathbf{K}_{ij}(m)(1 - \mathbf{K}_{ij}(m)), \end{aligned}$$

where the one but last equality comes from the independence of the variables  $\widehat{B}_{ij}$  for  $i \in \{1 \dots n\}$ .

**Case  $k \neq j$ .** This case is similar.

$$\begin{aligned} N\mathbb{E}[(E \otimes E)_{jk}] &= N\text{cov}(M_j(1), M_k(1)) \\ &= N \sum_{i=1}^n \sum_{i'=1}^n \frac{1}{N^2} \text{cov}(\widehat{B}_{ij}, \widehat{B}_{i'k}) \\ &= - \sum_{i=1}^n m_i \mathbf{K}_{ij}(m) \mathbf{K}_{ik}(m), \end{aligned}$$

where in the double sum, only the terms  $i = i'$  are non-zero because  $\widehat{B}_{ij}$  and  $\widehat{B}_{i'k}$  are independent when  $i \neq i'$ .  $\square$

**Lemma 4.** *Under the assumptions of Lemma 1 and using the notations of Lemma 1, there exists a function  $\varepsilon(N)$  such that  $\lim_{N \rightarrow \infty} \varepsilon(N) = 0$  and that  $\|N\mathbb{E}[o(\|E\|^2)]\| \leq \varepsilon(N)$ .*

*Proof.* First of all we note that, as  $D^2h$  is continuous, it is uniformly continuous (because  $\mathcal{U}$  is compact). Hence, the term  $o(\|E\|^2) \in \mathbb{R}^p$  is uniform in  $m$  and  $\|E\|^2$ , *i.e.*, there exist a function  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}$  and a constant  $\gamma > 0$  such that  $\lim_{\|e\| \rightarrow 0} \delta(e) = 0$ ,  $\delta(e) \leq \gamma$  for all  $e \in \mathbb{R}^n$ , and  $\|o(\|E\|^2)\| \leq \|E\|^2 \delta(E)$ . This implies  $\mathbb{E}[\|o(\|E\|^2)\|] \leq \mathbb{E}[\|E\|^2 \delta(E)]$ . The proof proceeds with the following derivation:

First, note that  $\lim_{\|e\| \rightarrow 0} \delta(e) = 0$  implies that for all  $\epsilon > 0$ , there exists  $a_\epsilon > 0$  such that  $\delta(e) \leq \epsilon$  for all  $e$  such that  $\|e\| \leq a_\epsilon$ . Therefore

$$\begin{aligned} \mathbb{E}[\|E\|^2 \delta(E)] &\leq \mathbb{E}[\|E\|^2 \delta(E) \mathbf{1}_{\|E\| \geq a_\epsilon}] + \mathbb{E}[\|E\|^2 \epsilon \mathbf{1}_{\|E\| < a_\epsilon}] \\ &\leq \eta \gamma \mathbb{E}[\mathbf{1}_{\|E\| \geq a_\epsilon}] + \epsilon \mathbb{E}[\|E\|^2], \end{aligned}$$

for some constant<sup>4</sup>  $\eta < \infty$ .

As indicated by Equation (3),  $E_j = \sum_i (\widehat{B}_{ij}/N - m_i K_{ij}(m))$  is the sum of the  $n$  independent random variables  $(\widehat{B}_{ij}/N - m_i K_{ij}(m))$  and  $\widehat{B}_{ij}$  has a binomial distribution of parameters  $(Nm_i, K_{ij}(m))$ . Hence,  $E_j$  can be expressed as the sum of  $N$  independent Bernoulli random variables. By Hoeffding's inequality,

$$\mathbf{P}[\|E_j\| \geq t] \leq e^{-2Nt^2}.$$

This implies that  $\mathbf{P}[\|E\| \geq a_\epsilon] \leq \sum_{j=1}^n \mathbf{P}[\|E_j\| \geq a_\epsilon/n] \leq ne^{-2Na_\epsilon/n^2}$ .

Moreover, by Lemma 3,  $\mathbb{E}[\|E\|^2] \leq 1/N$ . This shows that

$$\begin{aligned} \mathbb{E}[\|E\|^2 \delta(E)] &\leq \eta \gamma \mathbb{P}(\|E\| \geq a_\epsilon) + \epsilon \mathbb{E}[\|E\|^2] \\ &\leq \eta \gamma ne^{-2Na_\epsilon/n^2} + \frac{\epsilon}{N} \end{aligned}$$

The assert follows by using  $\varepsilon(N) = \inf_{\epsilon > 0} (N \eta \gamma ne^{-2Na_\epsilon/n^2} + \epsilon)$ .  $\square$

**Lemma 5.** *Under the assumptions of Lemma 1 and that  $M^{(N)}(0)$  converges weakly to  $\mu(0)$  as  $N$  goes to infinity, for any continuous function  $g : \mathcal{U}^n \rightarrow \mathbb{R}_{\geq 0}^p$  and all  $t$ , there exists a function  $\varepsilon_{t,g}(N)$  such that  $\lim_{N \rightarrow \infty} \varepsilon_{t,g}(N) = 0$  and*

$$\|\mathbb{E}[g(M(t))] - g(\mu(t))\| \leq \varepsilon_{t,g}(N).$$

<sup>4</sup> The specific value of  $\eta$  depends on the norm used; for instance  $\eta = 1$  for the infinity norm.

*Proof.* We proceed by induction on  $t$ . The lemma holds for  $t = 0$  because  $M^{(N)}(0)$  converges weakly to  $\mu(0)$ . As  $g$  is continuous, there exists a function  $\delta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that  $\|g(m) - g(m')\| \leq \delta(\|m - m'\|)$  and  $\lim_{r \rightarrow 0} \delta(r) = 0$ . Moreover, as  $g$  and  $\Phi_t$  are continuous,  $g \circ \Phi_t$  is also uniformly continuous (and the continuity is uniform since  $\mathcal{U}^n$  is compact). Hence

$$\begin{aligned} & \mathbb{E} [\|g(M(t+1)) - g(\mu(t+1))\|] \\ &= \mathbb{E} [\|g(M(t+1)) - g(\Phi_1(M(t))) + g(\Phi_1(M(t))) - g(\mu(t+1))\|] \\ &\leq \mathbb{E} [\delta(\|M(t+1) - \Phi_1(M(t))\|)] + \mathbb{E} [g \circ \Phi_1(M(t)) - g \circ \Phi_1(\mu(t))] \\ &\leq \mathbb{E} [\mathbb{E} [\delta(\|E\|) \mid M(t)]] + \varepsilon_{t, g \circ \Phi_1}, \end{aligned}$$

where  $E = M(t+1) - \Phi_1(M(t))$  converges to 0 (uniformly in  $M(t)$ ) by Lemma 3.  $\square$

*Of the main theorem.* We proceed by induction on  $t$ . The theorem clearly holds for  $t = 0$  because  $\Phi_0(M^{(N)}(0)) = M^{(N)}(0)$  by definition of  $\Phi_0$ . Assume that the theorem now holds for some  $t \geq 0$ . We have :

$$\begin{aligned} N(\mathbb{E}[h(M(t+1))] - h(\mu(t+1))) &= N\mathbb{E}[h(M(t+1)) - h(\Phi_1(M(t)))] \\ &\quad + N(\mathbb{E}[h(\Phi_1(M(t)))] - h(\mu(t+1))). \end{aligned}$$

We will analyse the two lines separately. For the first line, the idea is to use Lemma 1. Indeed this line is equal to

$$\begin{aligned} & \mathbb{E} [N\mathbb{E} [h(M(t+1)) - h(\Phi_1(M(t))) \mid M(t)]] \\ &= \mathbb{E} \left[ \frac{1}{2} (D^2h)(\Phi_1(M(t))) \cdot \Gamma(M(t)) \right] + \Theta(N), \end{aligned}$$

where by Lemma 1,  $\Theta(N)$  is such that  $\|\Theta(N)\| \leq \varepsilon(N)$ . By Lemma 5 with  $g = (D^2h)(\Phi_1) \cdot \Gamma$ , as  $N$  goes to infinity, this quantity converges to

$$(5) \quad \frac{1}{2} (D^2h)(\Phi_1(\mu(t))) \cdot \Gamma(\mu(t)) = \frac{1}{2} (D^2h)(\mu(t+1)) \cdot \Gamma(\mu(t)).$$

For the second line, the idea is to apply the induction hypothesis to  $h \circ \Phi_1$  which can be done because the  $h \circ \Phi_1$  is twice differentiable (because both  $h$  and  $\Phi_1$  are). This shows that

$$\begin{aligned} & N(\mathbb{E}[h(\Phi_1(M(t)))] - h(\mu(t+1))) \\ &= N(\mathbb{E}[h(\Phi_1(M(t)))] - h(\Phi_1(\mu(t)))) \\ &= D(h \circ \Phi_1)(\mu(t))V_t + \frac{1}{2} D^2(h \circ \Phi_1)(\mu(t)) \cdot W_t + \varepsilon_{t, h \circ \Phi_1}(N) \end{aligned}$$

The first term can be dealt with by applying the chain rule  $D(h \circ \Phi_1) = (Dh)(\Phi_1)(D\Phi_1)$  which shows that:

$$\begin{aligned} (6) \quad D(h \circ \Phi_1)(\mu(t))V_t &= (Dh)(\Phi_1(\mu(t)))(D\Phi_1)(\mu(t))V_t \\ &= (Dh)(\mu(t+1))(D\Phi_1)(\mu(t))V_t \\ &= (Dh)(\mu(t+1))A_tV_t. \end{aligned}$$

For the second term, we apply the product rule and again the chain rule:

$$\begin{aligned} \frac{1}{2}D^2(h \circ \Phi_1) \cdot W_t &= \frac{1}{2}D((Dh)(\Phi_1)(D\Phi_1)) \cdot W_t \\ &= \left( D((Dh)(\Phi_1)) \cdot (D\Phi_1) + (Dh)(\Phi_1) \cdot D(D\Phi_1) \right) \cdot \frac{1}{2}W_t \\ &= \left( (D^2h)(\Phi_1) \cdot (D\Phi_1)(D\Phi_1)^T + (Dh)(\Phi_1)(D^2\Phi_1) \right) \cdot \frac{1}{2}W_t \end{aligned}$$

By applying the last function at the point  $\mu(t)$  we get that:

$$\begin{aligned} \frac{1}{2}D^2(h \circ \Phi_1)(\mu(t)) \cdot W_t &= (D^2h)(\Phi_1(\mu(t))) \cdot (D\Phi_1)(\mu(t)) \frac{1}{2}W_t (D\Phi_1)^T(\mu(t)) \\ &\quad + (Dh)(\Phi_1(\mu(t)))(D^2\Phi_1(\mu(t))) \cdot \frac{1}{2}W_t, \end{aligned}$$

which, using the definition of  $\mu(t+1) = \Phi_1(\mu(t))$  and the assumptions  $A_t = (D\Phi_1)(\mu(t))$  and  $B_t = (D^2\Phi_1(\mu(t)))$ , is the same as:

$$(7) \quad \frac{1}{2}(D^2h)(\mu(t+1)) \cdot A_t W_t A_t^T + \frac{1}{2}(Dh)(\mu(t+1))(B_t \cdot W_t).$$

The theorem holds by combining Equations (5), (6) and (7).  $\square$

### 3.3.2. Proof of Theorem 2.

*Proof.* The proof is inspired by the proof of [13, Theorem 3.1] and uses ideas of Stein's method. Because many details are similar to the proof of Theorem 1, we skip some details of computation in this proof.

Let  $h$  be a twice-differentiable function and let  $G_h$  be the function defined for all  $m$  by:

$$G_h(m) = \sum_{t=0}^{\infty} [h(\Phi_t(m)) - h(\mu(\infty))].$$

$G_h(m)$  is well defined because  $\mu(\infty)$  is exponentially stable attractor.

By construction, for any  $m$  we have

$$\begin{aligned} G_h(m) &= h(m) - h(\mu(\infty)) + \sum_{t=1}^{\infty} [h(\Phi_t(m)) - h(\mu(\infty))] \\ &= h(m) - h(\mu(\infty)) + G_h(\Phi_1(m)) \end{aligned}$$

The above equation is a discrete time Poisson equation and implies that for any  $m$ :

$$(8) \quad h(m) - h(\mu(\infty)) = G_h(m) - G_h(\Phi_1(m))$$

Assume that at time 0, the initial state  $M(0)$  is distributed according to the stationary distribution of the system of size  $N$ . By the definition of stationarity, at time 1,  $M(1)$  is also distributed according to the same stationary distribution and we have  $\mathbb{E}[G_h(M(0))] = \mathbb{E}[G_h(M(1))]$ .

By using (8) and then the above equation, we get:

$$\begin{aligned} \mathbb{E}[M(0)] - h(\mu(\infty)) &= \mathbb{E}[M(0) - h(\mu(\infty))] \\ &= \mathbb{E}[G_h(M(0)) - G_h(\Phi_1(M(0)))] \\ &= \mathbb{E}[G_h(M(1)) - G_h(\Phi_1(M(0)))] \end{aligned}$$

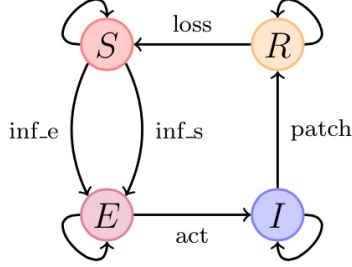


FIGURE 1. SEIR model of individual object

By Lemma 1, this shows that :

$$\begin{aligned}
 N(\mathbb{E}[M(\infty)] - h(\mu(\infty))) &= \mathbb{E}[G_h(M(1)) - G_h(\Phi_1(M(0)))] \\
 &= \mathbb{E}\left[\frac{1}{2}D^2(G_h)(\Phi_1(M(0))) \cdot \Gamma(M(0))\right] + o(1) \\
 &= \frac{1}{2}D^2(G_h)(\Phi_1(\mu(\infty))) \cdot \Gamma(\mu(\infty)) + o(1),
 \end{aligned}$$

where the last equality comes from the fact that the stationary distribution of the system of size  $N$  converges weakly to a Dirac measure in  $\mu(\infty)$  as  $N$  goes to infinity (see [11, Corollary 14]).

To conclude the proof, the only remaining step is to compute the second differential of  $G_h$  which can be expressed as the infinite sum:

$$D^2(G_h)(\mu(\infty)) = \sum_{t=0}^{\infty} D^2(h \circ \Phi_t)(\mu(\infty)).$$

The expressions for  $V_\infty$  and  $W_\infty$  come from plugging the above equations into Equation (5), (6) and (7). The uniqueness of the solution of the Lyapunov equation is due to the fact that the fixed point  $\mu(\infty)$  is exponentially stable and therefore also linearly stable.  $\square$

#### 4. REFINED MEAN FIELD MODEL FOR SEIR

In this section we provide a simple example that illustrates the results for the refined mean field model of the simple computer epidemic SEIR example presented in [4]. Each object in the model consists of four local states: Susceptible (S), Exposed (E), Infected (I) (and active) and Recovered (R). The four-state SEIR model of an individual object is shown in Figure 1.

Its discrete time evolution is given by the following probability transition matrix  $\mathbf{K}$  in which  $m_S$ ,  $m_E$ ,  $m_I$  and  $m_R$  denote the fraction of objects in the system that are in local state S, E, I and R, respectively:

$$\mathbf{K}(m_S, m_E, m_I, m_R) = \begin{pmatrix} 1 - (\alpha_e + \alpha_i m_I) & \alpha_e + \alpha_i m_I & 0 & 0 \\ 0 & 1 - \alpha_a & \alpha_a & 0 \\ 0 & 0 & 1 - \alpha_r & \alpha_r \\ \alpha_l & 0 & 0 & 1 - \alpha_l \end{pmatrix}$$

In other words, a susceptible becomes exposed with probability  $(\alpha_e + \alpha_i m_I)$  – *i.e.*,  $\alpha_e$  denotes the external and  $\alpha_i$  the internal infection probability –; An exposed node activates his infection with probability  $\alpha_a$ ; An infected recovers with probability  $\alpha_r$ ; and  $\alpha_l$  is the probability to loose the protection against infection.

**4.1. Computation of  $A$ ,  $B$  and  $\Gamma$ .** We illustrate how to apply Theorem 1 in its simplified form, when  $h$  is the identity function, as in Corollary 1–(i). The first step is to compute the Jacobian and the Hessian of the function  $\Phi_1$  for a generic occupancy measure vector  $m$  at time step  $t$ . Written as a column vector, the function  $\Phi_1(m) = mK(m)$  is given by

$$\Phi_1(m) = \begin{pmatrix} m_S(1 - \alpha_e - \alpha_i m_I) + \alpha_l m_R \\ m_S(\alpha_e + \alpha_i m_I) + (1 - \alpha_a) m_E \\ m_E \alpha_a + (1 - \alpha_r) m_I \\ \alpha_r m_I + (1 - \alpha_l) m_R \end{pmatrix}$$

Hence, the Jacobian is the following  $4 \times 4$  matrix:

$$D(\Phi_1)(m_S, m_E, m_I, m_R) = \begin{pmatrix} 1 - (\alpha_e + \alpha_i m_I) & 0 & -\alpha_i m_S & \alpha_l \\ \alpha_e + \alpha_i m_I & 1 - \alpha_a & \alpha_i m_S & 0 \\ 0 & \alpha_a & 1 - \alpha_r & 0 \\ 0 & 0 & \alpha_r & 1 - \alpha_l \end{pmatrix}$$

The Hessian is a  $4 \times 4 \times 4$  tensor. We provide them as 4 matrices of  $4 \times 4$ , one for each function  $(\Phi_1)_j$ , where  $j \in \{S, E, I, R\}$ :

$$D^2((\Phi_1)_S)(m_S, m_E, m_I, m_R) = \begin{pmatrix} 0 & 0 & -\alpha_i & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D^2((\Phi_1)_E)(m_S, m_E, m_I, m_R) = \begin{pmatrix} 0 & 0 & \alpha_i & 0 \\ 0 & 0 & 0 & 0 \\ \alpha_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrices for  $I$  and  $R$  are two  $4 \times 4$  zero-matrices. The  $4 \times 4$  matrix  $\Gamma$  depends on  $K$  and the occupancy measure  $m$  as defined in Lemma 1. The refined mean field approximation of the occupancy measure vector is thus given by  $\mathbb{E}[M^{(N)}(t)] \approx \mu(t) + V_t/N$ , where  $V_t$  is computed recursively, according to Theorem 1.

**4.2. Dynamics of the SEIR Model and its Approximations.** We consider a model with the following parameter values for the local transition probabilities:  $\alpha_e = 0.01$ ,  $\alpha_i = 0.08$ ,  $\alpha_r = 0.02$ ,  $\alpha_l = 0.01$  and  $\alpha_a = 0.04$ . Initially,  $M(0) = (0.2, 0.2, 0.2, 0.4)$ . Figure 2 shows the results for the classical mean field approximation, the refined mean field approximation and the average of 100,000 runs of a stochastic simulation of the model obtained. The results are given for population size  $N = 10$ ,  $N = 20$ ,  $N = 50$  and  $N = 100$ , respectively; time  $t$  ranges from 0 to 500 time units.

We observe that, as expected, the gap between the classical mean field approximation and the simulation is relatively small and decreases with  $N$ . Still, for  $N = 10$ , we observe a clear difference between the classical mean field approximation and the simulation, whereas the refined mean field provides a much closer

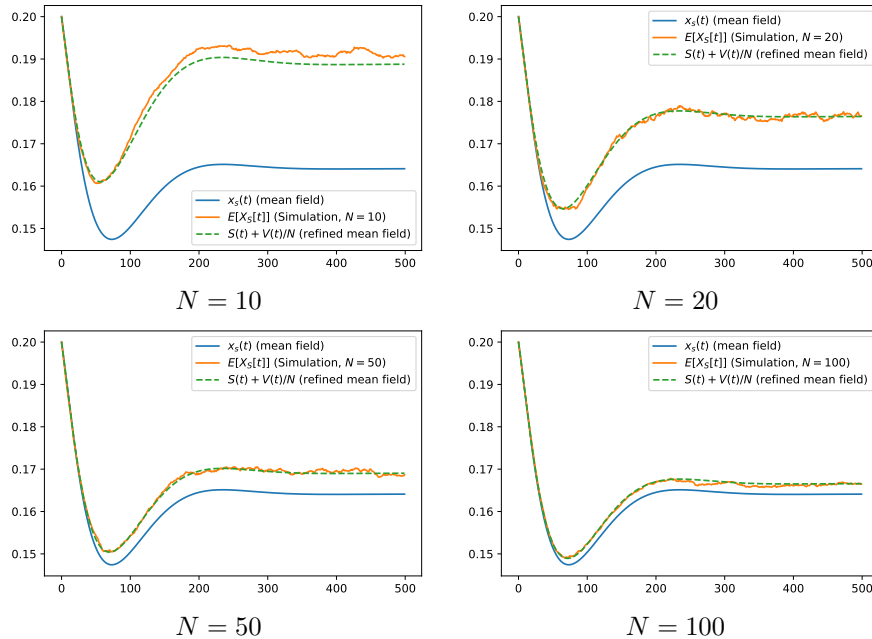
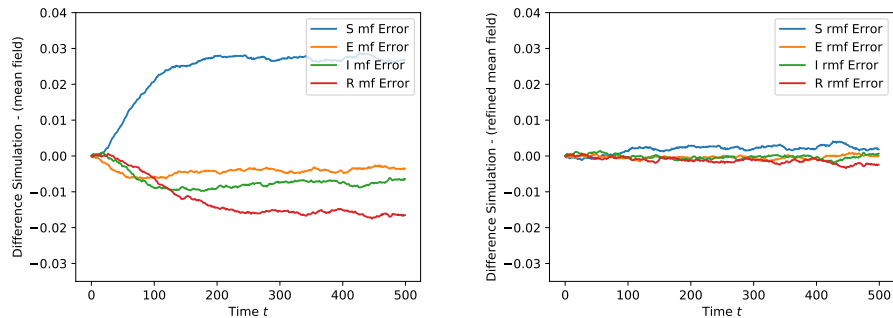


FIGURE 2. Evolution of the fraction of objects in state  $S$  for population sizes  $N = 10$ ,  $N = 20$ ,  $N = 50$  and  $N = 100$ . The figures compare the classical mean field approximation (obtained with `python-numpy`) with the refined one and with the average of 10,000 simulation runs of the system.

approximation (in this case, the graphs overlap almost everywhere). With the increase of the population size  $N$  both approximations converge to the same value, as well as the value obtained by simulation: for  $N \geq 50$ , the curves are almost indistinguishable.

To highlight the differences, we plot in Figure 3 the difference between the two approximations with respect to the simulation: On the left panel, we plot a function of time for the quantities  $\mathbb{E}[M(t)] - \mu(t)$ ; On the right we plot  $\mathbb{E}[M(t)] + V_i/N - \mu(t)$  (in both cases for  $N = 10$ ). We observe that the refined mean field approximation (right panel) is an order of magnitude closer to the value obtained by simulation: while the error of the classical mean field approximation can be larger than 0.05, the error of the refined mean field approximation remains always smaller than 0.01.

These two figures illustrate that Theorem 1 is not just valid asymptotically, but actually it refines the classical mean field approximation for relatively small values of  $N$ . To go further, we study the steady-state distribution in Table 1 in which we display the average proportion of objects in states  $S$ ,  $E$ ,  $I$  or  $R$  estimated by simulation, refined mean field approximation and classical mean field approximation. This illustrates the approximation accuracy for the steady state of the SEIR example for each local state of an object. As for the two previous figures, this table illustrates that the refined mean field approximation provides very accurate estimates of the true stationary distribution even for very small values of  $N$ , which shows that the asymptotic results presented in Theorem 2 are also useful for small values of  $N$ .



Error of the mean field approximation    Error of the refined mean field approx.

FIGURE 3. SEIR model: Quantification of the difference (error) between the simulation results and the classical mean field approximation (left) and the refined mean field result (right), respectively, for  $N = 10$ . The results show simulation value minus mean field value.

These results are in line with the results presented in [13] for continuous time mean field models.

State	$S$	$E$	$I$	$R$
Simulation ( $N = 10$ )	0.191	0.115	0.231	0.462
Refined mean field ( $N = 10$ )	0.189	0.116	0.232	0.464
Mean field ( $N = 10$ )	0.164	0.119	0.239	0.478

TABLE 1. SEIR model: Comparison of the accuracy of the mean field and refined mean field approximation. The columns show the average proportion of objects in the states susceptible (S), exposed (E), infected (I) and recovered (R), respectively. Each item in the table was computed by measuring the occupancy measure for times  $t = 1000$ , i.e. when the systems' occupancy measure has reached a sufficiently stable value. Simulation values are averages over 100,000 simulations.

## 5. REFINED MEAN FIELD MODEL FOR WSN

The next example concerns a simple model of a wireless sensor network [3]. Such networks are composed of wireless sensor nodes and gateways. This example serves two purposes. First it shows that the assumption of homogeneous objects is not restrictive: In this example, there are two classes of objects, which is represented by having a block-diagonal matrix  $K$ . Second, we use it to consider a function  $h$  that is not just the projection on one coordinate.

Wireless sensor nodes have three local states. In the initial state,  $e$ , a sensor node waits for detecting an event of interest and collects data for that event. After that, the node moves to state  $c$  to communicate its data to an available gateway. The communication attempt may timeout if no gateway is available. In that case



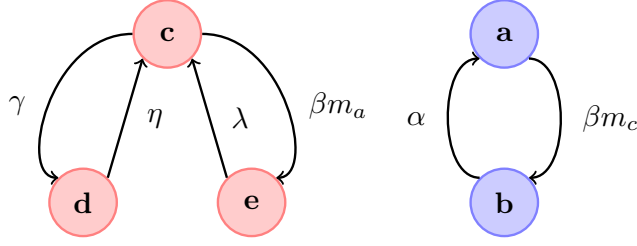


FIGURE 4. WSN model of individual objects: Sensor Node (left) and Gateway (right)

the sensor node moves to state  $d$ , introducing some delay before moving back to state  $c$  for a further communication attempt.

Gateway nodes have two states. Initially they are in state  $a$  and available to receive data from a sensor node. Upon connection to a sensor node they move to state  $b$  during which they are busy processing the data. When in state  $b$  they are temporarily unavailable for communication with other sensor nodes. After processing the batch of data they move back to state  $a$ .

We consider a model where objects have five local states  $\{a, b, c, d, e\}$ , where  $a$  and  $b$  are states of a gateway node and  $c, d$  and  $e$  are states of a sensor node, i.e. each object in the model can behave either as a gateway or as a sensor, but it cannot change its behaviour from that of a gateway to that of a sensor, or vice-versa. A system is then composed of  $N$  (syntactical) homogeneous objects with a fixed fraction  $m_G = m_a + m_b$  of gateway nodes and a fraction  $m_W = m_c + m_d + m_e$  of wireless sensor nodes, such that  $m_W = 1 - m_G$ . To keep the model simple for the purpose of illustrating the refined mean field approach, we do not consider interference due to collision in the communication between nodes and gateways.

The probability transition matrix is given below:

$$\mathbf{K}(m) = \begin{pmatrix} 1 - \beta m_c & \beta m_c & 0 & 0 & 0 \\ \alpha & 1 - \alpha & 0 & 0 & 0 \\ 0 & 0 & 1 - \gamma - \beta m_a & \gamma & \beta m_a \\ 0 & 0 & \eta & 1 - \eta & 0 \\ 0 & 0 & \lambda & 0 & 1 - \lambda \end{pmatrix},$$

where  $\alpha$  denotes the probability of the gateway to get again available,  $\beta$  the probability of data communication between the gateway and a sensor node,  $\lambda$  the probability that a sensor node is ready to send data,  $\gamma$  the probability that a sensor node performs a time-out and  $\eta$  the probability that a delayed sensor node tries to communicate again.

In the example we will use the following values for the above parameters:  $\alpha = 0.09$ ,  $\beta = 0.9$ ,  $\lambda = 0.09$ ,  $\gamma = 0.01$ , and  $\eta = 0.01$ , and let  $M^{(N)}(t)$  denote, as usual, the occupancy measure process of the WSN model (leaving  $N$  and  $t$  implicit for the sake of notation simplicity). We are interested in the average response time of a sensor node, i.e. the time a sensor node needs to wait to be able to communicate its data to the gateway. This expected response time can be defined as the fraction between the sensor nodes that are already waiting to communicate their data, i.e. the sensor nodes in state  $c$  and state  $d$ , and the new sensor nodes that became

ready in the current time step, i.e.  $\lambda$  times the nodes in local state  $e$ :

$$\mathbb{E}[R] = \mathbb{E} \left[ \frac{(M_c + M_d)}{\lambda M_e} \right],$$

With reference to Theorem 1, we define  $h(x_1, x_2, x_3, x_4, x_5) = \frac{(x_3 + x_4)}{\lambda x_5}$ .

**5.1. Computation of  $A$ ,  $B$  and  $\Gamma$  for the WSN model.** In the sequel, we make reference to  $m = (m_a, m_b, m_c, m_d, m_e) \in \mathcal{U}^5$ . The Jacobian of function  $\Phi_1$ :

$$D(\Phi_1)(m) = \begin{pmatrix} 1 - \beta m_c & \alpha & -\beta m_a & 0 & 0 \\ \beta m_c & 1 - \alpha & \beta m_a & 0 & 0 \\ -\beta m_c & 0 & 1 - \gamma - \beta m_a & \eta & \lambda \\ 0 & 0 & \gamma & 1 - \eta & 0 \\ \beta m_c & 0 & \beta m_a & 0 & 1 - \lambda \end{pmatrix}$$

The Hessian of the function  $\Phi_1$  satisfies  $D^2((\Phi_1)_d)(m) = 0$ :

$$D^2((\Phi_1)_a)(m) = D^2((\Phi_1)_c)(m) = \begin{pmatrix} 0 & 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D^2((\Phi_1)_b)(m) = D^2((\Phi_1)_e)(m) = \begin{pmatrix} 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Jacobian of function  $h$  is  $D(h)(m) = (0, 0, \frac{1}{\lambda m_e}, \frac{1}{\lambda m_e}, -\frac{m_c + m_d}{\lambda m_e^2})$  and its Hessian is

$$D^2(h)(m) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\lambda m_e^2} \\ 0 & 0 & 0 & 0 & -\frac{1}{\lambda m_e^2} \\ 0 & 0 & -\frac{1}{\lambda m_e^2} & -\frac{1}{\lambda m_e^2} & \frac{2*(m_c + m_d)}{\lambda * m_e^3} \end{pmatrix}.$$

The  $5 \times 1$  vector  $V_t$  and the  $5 \times 5$  matrix  $W_t$  are computed recursively according to Theorem 1, using the new Jacobian and Hessian for function  $\Phi$ ; thus the refined mean field approximation of the measure of interest is given by  $\mathbb{E}[h(M^{(N)}(t))] \approx h(\Phi_t(m)) + (D(h) \cdot V_t + \frac{1}{2} * (D^2(h) \cdot W_t))/N$ .

**5.2. Results.** In Figure 5 various approximations of the expected response time for the WSN model are shown, for time values  $t$  ranging from 0 to 400 time units. We consider a relatively small system with 15 nodes (10 sensor and 5 gateway nodes). Recall that the function  $h$  is defined by :  $h(x_1, x_2, x_3, x_4, x_5) = \frac{(x_3 + x_4)}{\lambda x_5}$ . We compare five curves :

- (1) The blue curve labelled Classic Mean Field (1) (obtained with Octave) shows the expected response time when this is approximated by defining  $\mathbb{E}[R]$  as in [3]:

$$\mathbb{E}[R] = \frac{(m_c + m_d)}{\lambda m_e} = h(m),$$

where  $m_c$ ,  $m_d$  and  $m_e$  denote the classical mean field approximation values for the fractions of sensor nodes being in state  $c$ ,  $d$  and  $e$ , respectively.

- (2) The red curve (2) is the expectation of  $h(M)$ , computed by stochastic simulation:

$$\mathbb{E}[h(M)] = \min\left(\mathbb{E}\left[\frac{(M_c + M_d)}{\lambda M_e}\right], 100\right)$$

In the case of individual simulation runs it may of course happen that  $M_e = 0$  occasionally. This is why we defined  $\mathbb{E}[h(M)]$  as the minimum between the actual value and 100. The latter is the value one obtains when one but all nodes are waiting and the last node is getting ready for communication too:  $M_c + M_d = 9$  and  $M_e = 1$ , i.e.  $9/(0.09*1) = 100$ .

- (3) The orange curve (3) shows the expected response time approximated using the refined mean field approximation of Theorem 1 with the function  $h$ .

For comparison, we also compute two other quantities :

4. The purple curve (4) shows the response time approximated as follows, using the refined mean field approximation for the fraction of sensor nodes in each state (i.e. we use Theorem 1 with the identity function and then apply  $h$ ):

$$h(rmf) = \frac{(rmf_c + rmf_d)}{\lambda rmf_e}$$

where  $rmf_c = \mu_c + V_c/N$  denotes the refined mean field approximation of the fraction of sensor nodes in state  $c$ , and similarly for  $rmf_d$  and  $rmf_e$ .

5. Finally, the green curve (5) shows the expected response time defined as:

$$h(\mathbb{E}[M]) = \frac{(\mathbb{E}[M_c] + \mathbb{E}[M_d])}{\lambda \mathbb{E}[M_e]}$$

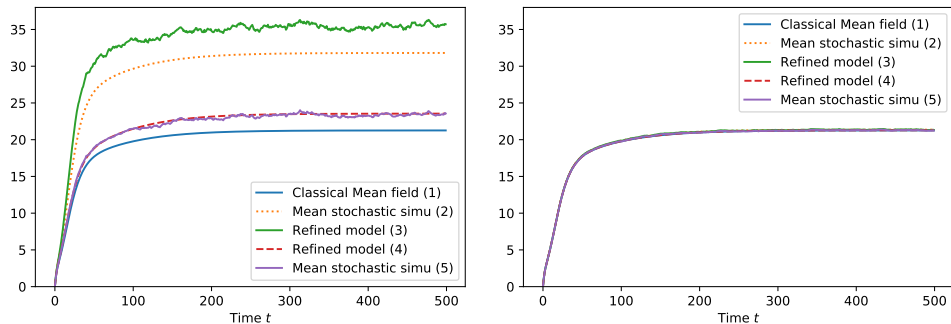
where  $\mathbb{E}[M_c]$  is the average fraction of sensor nodes in state  $c$  obtained via the average of 100,000 individual simulation runs of the model.  $\mathbb{E}[M_d]$  and  $\mathbb{E}[M_e]$  are obtained in a similar way.

In Figure 5(a), we plot these various curves for 15 nodes in total. We make two observations. First, in this case the value obtained by simulation (red curve (2)) is almost 50% larger than the classic mean field approximation (1) whereas the refined approximation (3) is much closer. Second, the purple curve (4) is close to the green curve (5) but quite far away from the red (2) and orange (3) curves. This shows that when applying Theorem 1, computing a refined model for  $\mathbb{E}[h(M)]$  and for  $h(\mathbb{E}[M])$  might lead to very different results.

Of course, the larger  $N$  gets (in an otherwise equal model), the closer the orange (3) and red (2) curves will get to the blue curve (1), i.e. the classic mean field approximation, as illustrated in Figure 5(b). In both cases, all curves collapse into to a single curve.

## 6. REFINED MEAN FIELD MODEL FOR MAJORITY RULE DECISION-MAKING

The example in this section concerns a model for collective decision-making. The model is inspired by the work of Montes de Oca et al. (see [8, 20] and references therein). Collective decision-making is a process whereby the members of a group decide on a course of action by consensus. Such collective decision-making processes have also been applied in swarm robotics. In particular, in that context the robots



(a)  $N = 15$  : 10 sensors; 5 gateways (b)  $N = 1500$  : 1000 sensors; 500 gateways

FIGURE 5. Expected response time  $E[R]$  for a sensor node to communicate its data to a GW for a WSN model with  $N$  nodes of which  $2N/3$  are sensor nodes and  $N/3$  are gateway nodes. The red line (2) is an average over 20,000 simulations for  $N = 15$  and 1000 for  $N = 1500$ .

where asked to choose between two actions that have the same effect but differ in their execution times [8].

One strategy of collective decision making is the use of the majority rule. In this strategy the agents in a population are initially divided into two groups. One in which all members have opinion A and one where all members have opinion B. In every step three agents are selected randomly from the total population to form a temporary team. The team applies the majority rule such that all its members adopt the opinion held by the majority (i.e. at least two) of the members, after which they return to the total population until the population has reached a consensus on one of the two opinions.

In the majority rule strategy extended with differential latency the agents in the population are not all the time available for team formation. Both types of agents are assumed to perform an action with a certain duration during which they cannot participate in team formation. For example, agents with opinion B perform such actions taking (on average) relatively more time than those with opinion A. In [8] such latency periods for agents with opinion A and B are modelled by random variables with exponential distributions with rate  $\lambda_A$  and  $\lambda_B$  respectively. For simplicity, it can also be assumed that the A-type actions take 1 time unit on average (i.e.  $\lambda_a = 1$ ) and that B-type actions take  $1/\lambda$  time units on average, where  $\lambda$  takes a value in  $(0, 1]$ . This variant of self-organised collective decision-making is known as majority rule with differential latency (MRDL).

In the following we develop a probabilistic, discrete time variant of the MRDL strategy which we call MRDL-DT. In this variant agents can have either opinion A or opinion B, and in both cases they can be either be latent or not, leading to a partition of the population into exactly four classes:  $LA$  (latent A),  $NA$  (non-latent A),  $LB$  (latent B),  $NB$  (non-latent B). It is assumed that if an agent is latent it cannot be selected for team formation. The four state MRDL-DT model of an individual object of the population is shown in Figure 6. The name of the states indicate in which class the object is.

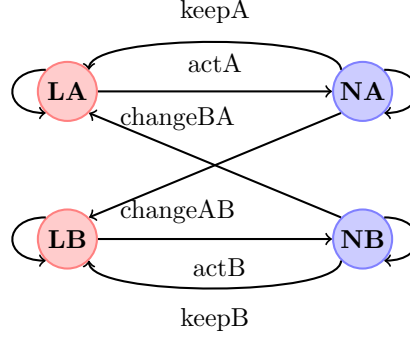


FIGURE 6. Majority rule differential latency model of an individual object. Latent states are red, non-latent ones blue.

The behaviour of an individual object is as follows. Initially the object is latent, and, assuming it has opinion A (state  $LA$ ), it finishes its job and becomes available for team-formation (transition  $actA$ ) moving to state  $NA$  with probability  $1/q$ , for appropriate  $q$ . When in  $NA$  it gets selected in a team with two other members. If the two other members have opinion B it changes its opinion into B and moves to state  $LB$ . This can happen with a probability  $3m_{NB}^2$  where factor 3 models the fact that we abstract from the exact order in which the members of the team are selected, which can happen in 3 different ways. In alternative the two other members can have both opinion A, or one opinion A and the other opinion B. In that case the opinion of the object does not change and the object moves back to  $LA$  with probability  $\frac{3}{q}(m_{NA}^2 + m_{NA}m_{NB})$ .

If the object is in state  $LB$  it becomes available for team formation, moving to state  $NB$  with a probability  $\lambda/q$ , where  $\lambda$  is a value in  $(0, 1]$ . The latter models the relative longer duration of activity B with respect to A. The behaviour in state  $NB$  is similar to that in state  $NA$ , except that now the opinion may change from B to A.

The discrete time evolution of the model is given by probability transition matrix  $\mathbf{K}$  in which  $m_{LA}$ ,  $m_{NA}$ ,  $m_{LB}$  and  $m_{NB}$  denote the fraction of objects in the system that are in local state  $LA$ ,  $NA$ ,  $LB$  and  $NB$ , respectively and  $m = (m_{LA}, m_{NA}, m_{LB}, m_{NB})$ :

$$\mathbf{K}(m) = \begin{pmatrix} 1 - \frac{1}{q} & \frac{1}{q} & 0 & 0 \\ \frac{3}{q}(m_{NA}^2 + m_{NA}m_{NB}) & NA(m) & \frac{3}{q}m_{NB}^2 & 0 \\ 0 & 0 & 1 - \frac{\lambda}{q} & \frac{\lambda}{q} \\ \frac{3}{q}m_{NA}^2 & 0 & \frac{3}{q}(m_{NB}^2 + m_{NA}m_{NB}) & NB(m) \end{pmatrix}$$

where

$$NA(m) = 1 - \frac{3}{q}(m_{NB}^2 + m_{NA}^2 + m_{NA}m_{NB})$$

and

$$NB(m) = 1 - \frac{3}{q}(m_{NA}^2 + m_{NB}^2 + m_{NA}m_{NB}).$$

Since we are dealing with clock-synchronous discrete systems, we also introduced the discretisation factor  $q = 10$  so that only a fraction of the population is moving from the latent to the non-latent state at any time.

In the example we use the values  $q = 10$  and  $\lambda$  taking values 1.0, 0.5 and 0.25 in the various analyses, modelling that task B takes the same time as task A, or twice as much time or four times as much time as task A, *on average*, respectively. We are interested in the evolution of the consensus,  $C_A$ , on opinion A as a function of the initial values and the differential latency  $\lambda$ . Let

$$\mathbb{E}[C_A] = \mathbb{E}[M_{LA} + M_{NA}] = \mathbb{E}[h(M_{LA}, M_{NA}, M_{LB}, M_{NB})]$$

where, with reference to Theorem 1,  $h(x_1, x_2, x_3, x_4) = x_1 + x_2$ .

**6.1. Computation of  $A$ ,  $B$  and  $\Gamma$  for the MRDL-DT model.** As before, we first need to compute the Jacobian and the Hessian of the function  $\Phi_1$  for a generic occupancy measure vector  $m$  at time step  $t$ . The Jacobian of function  $\Phi_1$  is:

$$D(\Phi_1)(m) = \begin{pmatrix} 1 - 1/q & \frac{9}{q}m_{NA}^2 + \frac{12}{q}m_{NA}m_{NB} & 0 & \frac{6}{q}m_{NA}^2 \\ 1/q & JNA(m) & 0 & -\frac{3}{q}m_{NA}^2 - \frac{6}{q}m_{NA}m_{NB} \\ 0 & \frac{6}{q}m_{NB}^2 & 1 - \frac{\lambda}{q} & \frac{12}{q}m_{NB}m_{NA} + \frac{9}{q}m_{NB}^2 \\ 0 & -\frac{3}{q}m_{NB}^2 - \frac{6}{q}m_{NA}m_{NB} & \frac{\lambda}{q} & JNB(m) \end{pmatrix}$$

where

$$JNA(m) = 1 - \left( \frac{9}{q}m_{NA}^2 + \frac{6}{q}m_{NA}m_{NB} + \frac{3}{q}m_{NB}^2 \right)$$

and

$$JNB(m) = 1 - \left( \frac{3}{q}m_{NA}^2 + \frac{9}{q}m_{NB}^2 + \frac{6}{q}m_{NA}m_{NB} \right).$$

The Hessian of function  $\Phi_1$  is:

$$D^2((\Phi_1)_{LA})(m) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{18}{q}m_{NA} + \frac{12}{q}m_{NB} & 0 & \frac{12}{q}m_{NA} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{12}{q}m_{NA} & 0 & 0 \end{pmatrix}$$

$$D^2((\Phi_1)_{NA})(m) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{18}{q}m_{NA} - \frac{6}{q}m_{NB} & 0 & -\frac{6}{q}m_{NA} - \frac{6}{q}m_{NB} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{q}m_{NA} - \frac{6}{q}m_{NB} & 0 & -\frac{6}{q}m_{NA} \end{pmatrix}$$

$$D^2((\Phi_1)_{LB})(m) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{12}{q}m_{NB} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{12}{q}m_{NB} & 0 & \frac{18}{q}m_{NB} + \frac{12}{q}m_{NA} \end{pmatrix}$$

$$D^2((\Phi_1)_{NB})(m) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{q}m_{NB} & 0 & -\frac{6}{q}m_{NA} - \frac{6}{q}m_{NB} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{6}{q}m_{NA} - \frac{6}{q}m_{NB} & 0 & -\frac{18}{q}m_{NB} - \frac{6}{q}m_{NA} \end{pmatrix}$$

**6.2. Results for the MRDL-DT example.** We first show some results for a medium size population of  $N = 160$ . Figure 7 shows the dynamics of the fractions of the population having opinion A and B, for both the latent and non-latent objects, for the first 200 time units. Similarly to the previous examples, a good correspondence can be observed between the results of the mean of 1000 simulation runs and the mean field approximations. Also in this case the refined mean field provides a better approximation than the classical mean field approximation for the period ranging approximately from 50 to 150 time units.

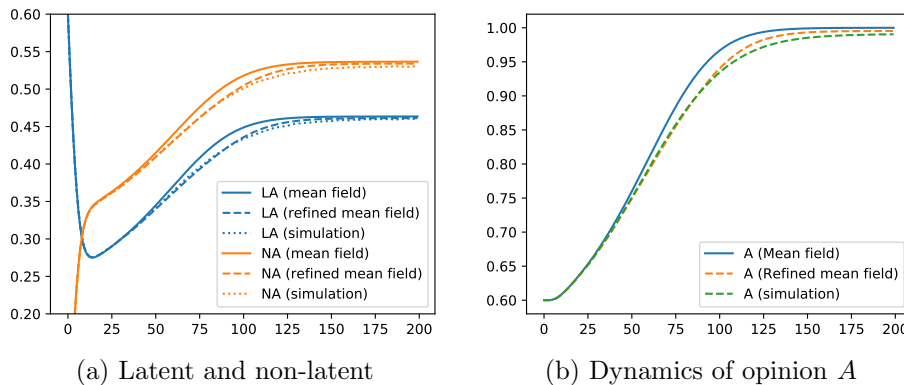


FIGURE 7. Dynamics of opinion A, latent and non-latent. Classical mean field (plain lines), refined mean field (dashed lines) and simulation results (dotted lines) of MRDL model with 160 objects,  $\lambda = 1.0$ ,  $q=10$  and initially  $0.6 * 160 = 96$  have opinion A and the population is initially latent.

A similar improved correspondence can be observed when considering the aggregated populations with opinion A or opinion B, respectively, as shown in Figure 7(b).

However, if a much smaller population is considered, e.g.  $N = 32$ , the mean field approximation differs considerably from the simulation results and the refined approximation does not really improve the accuracy of the approximation. This is what can be observed in Figure 8, where the results for  $N = 32$  are shown for a model without differential latency (i.e.  $\lambda = 1.0$ ), for  $t \in [0, 500]$ . This can be explained as follows : For a large population, a system that is initially biased towards one opinion will reach consensus on this opinion. For a small population, however, a system that is initially biased towards one opinion still can reach consensus on the other opinion due to intrinsic stochastic fluctuations. This cannot be taken into account in a mean field population model. However, both analytical models, derived from a master equation approach, and simulation show that the probability to reach consensus on a given opinion rapidly converges to a step function for a growing population size  $N$  [20], where the critical density is given by  $m_A = \frac{\lambda}{(1+\lambda)}$ , where  $m_A$  is the initial fraction of the population with opinion A. For large  $N$ , if  $m_A > \frac{\lambda}{(1+\lambda)}$  the system almost surely reaches consensus on A, whereas for  $m_A < \frac{\lambda}{(1+\lambda)}$  it almost surely reaches consensus on B. This explains why for

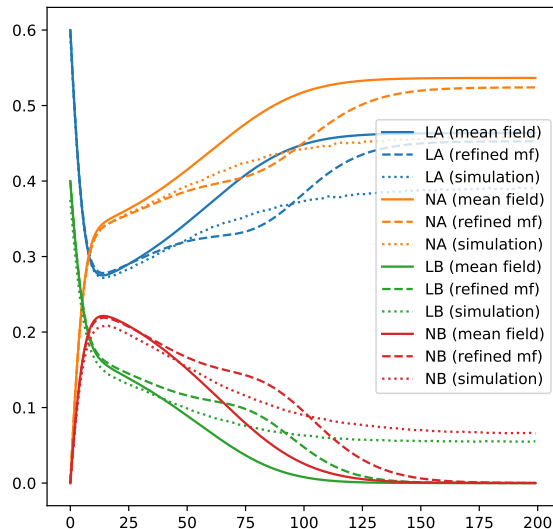


FIGURE 8. Classical mean field (plain lines), refined mean field (dashed lines) and simulation results (dotted lines) of MRDL model with 32 objects,  $\lambda = 1.0$ ,  $q=10$  and initially  $\lfloor 0.6 * 32 \rfloor = 9$  have opinion A and the population is initially latent.

larger populations the mean field approximations become increasingly accurate as shown in Figure 7.

This example shows a limit of the refined mean field approximation: when a system has multiple equilibrium point (and in particular when there are multiple absorbing states as in this example), the dynamics of the mean field approximation depends on the initial state of the system: For a given initial state the mean field will always follow the same trajectory (it is a deterministic system). When the system is large, the random fluctuations will remain small and the corresponding stochastic system will stay in the same basin of attraction. In the case of a small population, however, the dynamics will be greatly affected by the random fluctuations. These fluctuations can lead the system to another basin of attraction than the original one.

## 7. NON-EXPONENTIALLY STABLE EQUILIBRIUM : ACCURACY VERSUS TIME

In the previous sections, the dynamical system  $m = \Phi_1(m)$  has either one exponentially stable attractor (Section 4) or multiple exponentially stable attractors (Section 5). When the attractor is unique and exponentially stable, the accuracy of the mean field or refined mean field approximation is uniform in time (Theorem 2(ii)). In this section, we study the case of a system that has a unique attractor but that is not exponentially stable. We show that, in this case, the accuracy of the (refined) mean field approximation is no longer uniform in time.

We consider a system with  $N$  objects in which each object is in state 0 or 1. An object in state 1 goes to state 0 with probability 1 and an object in state 0 goes to 1 with probability  $\alpha m_0$ , where  $\alpha \in (0, 1)$  is a parameter. The transition matrix  $K$



is

$$K(m) = \begin{bmatrix} 1 - \alpha m_0 & \alpha m_0 \\ 1 & 0 \end{bmatrix}$$

The function  $\Phi_1 : m \mapsto mK(m)$  has a unique fixed point whose first component is  $\mu_0(\infty) = (\sqrt{1 + 4\alpha} - 1)/(2\alpha)$ . This fixed point is exponentially stable if and only if  $\alpha < 0.75$ .

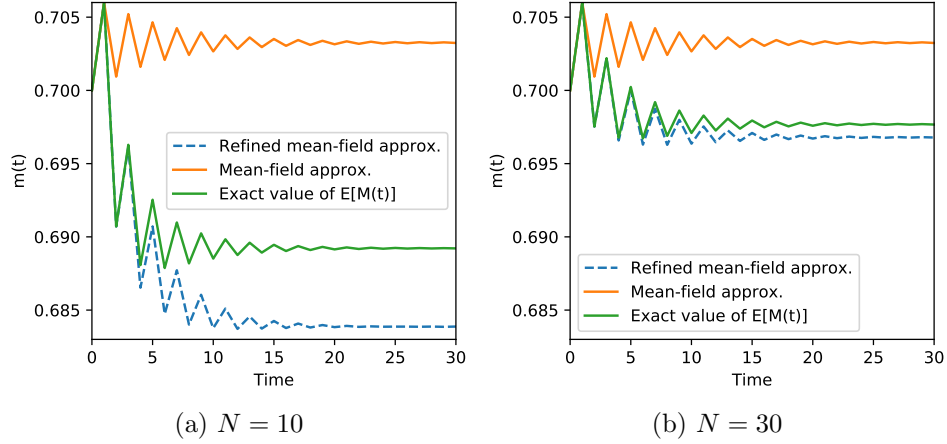


FIGURE 9. Exponentially stable case ( $\alpha = 0.6$ ).

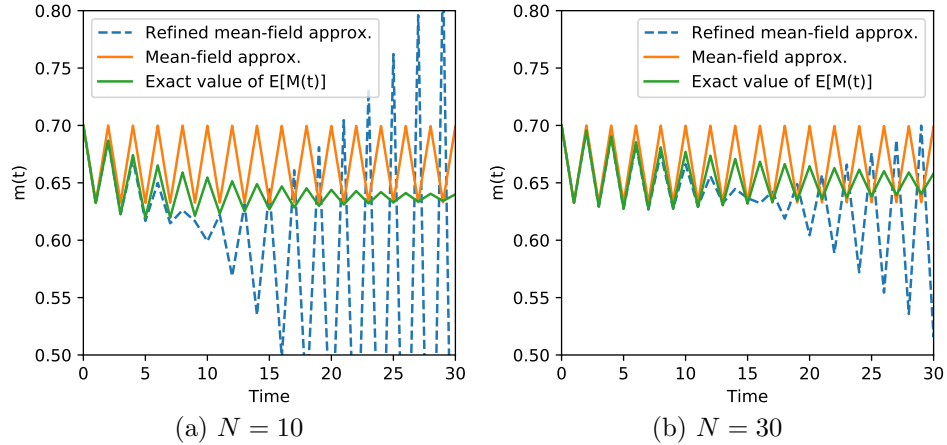


FIGURE 10. Non-exponentially stable case ( $\alpha = 0.75$ ).

**7.1. Transient regime and accuracy for large  $t$ .** In Figure 9 and Figure 10, we plot the first component of the mean field  $\mu(t)$  and refined mean field approximation  $\mu(t) + V(t)/N$  as well as an exact value of  $\mathbb{E}[M(t)]$  for  $N = 10$  and  $N = 30$ . The initial value is  $m = (0.7, 0.3)$ . The exact value of  $\mathbb{E}[M(t)]$  was computed by a numerical method that uses the fact that the system with  $N$  objects can be described by a Markov chain with  $N + 1$  states.

These figures show that the refined approximation always improves the accuracy compared to the classical mean field approximation for small values of  $t$ , both for  $\alpha = 0.6$  and  $\alpha = 0.75$ . The situation for large values of  $t$  is quite different. On the one hand, when the fixed point is exponentially stable ( $\alpha = 0.6$ , Figure 9), the refined approximation is very accurate for all values of  $t$ . On the other hand, when the fixed point is not exponentially stable ( $\alpha = 0.75$ , Figure 10), the refined approximation seems to be unstable and is not a good approximation of  $\mathbb{E}[M(t)]$  for values of  $t$  that are too large compared to  $N$  ( $t > 7$  for  $N = 10$  or  $t > 12$  for  $N = 30$ ).

**7.2. Steady-state convergence.** To explore how the non-exponentially stable case affects the accuracy of mean field approximation, we now study in more details the steady-state convergence when  $\alpha = 0.75$ . It is known (see for example [11, Corollary 14]) that when the dynamical system  $m = \Phi_1(m)$  has a unique attractor  $\mu(\infty)$ , then the steady-state expectation  $\mathbb{E}[M^{(N)}]$  converges to  $\mu(\infty)$  as  $N$  goes to infinity. Theorem 2 shows that if in addition the attractor is exponentially stable then  $\mathbb{E}[M^{(N)}] \approx \mu(\infty) + V/N$ .

In Figure 11, we show that the latter no longer holds when the mean field system has a unique attractor that is *not* exponentially stable. We consider the same model with  $\alpha = 0.75$  for which  $\mu(\infty) = 2/3$ . We plot in Figure 11  $\sqrt{N}(\mathbb{E}[M^{(N)}] - \mu(\infty))$ , where we computed  $\mathbb{E}[M^{(N)}]$  by inverting the transition matrix of the system of size  $N$ . This figure shows that  $\sqrt{N}(\mathbb{E}[M^{(N)}] - \mu(\infty))$  does not converge to 0 as  $N$  goes to infinity but seems to converge to approximately  $-0.0975$  (as indicated by the fitted line in orange). This suggests that for this model, one has in steady-state :

$$(9) \quad \mathbb{E}[M^{(N)}] \approx \mu(\infty) - \frac{0.0975}{\sqrt{N}} + \frac{0.14}{N}.$$

Note that the constants  $-0.0975$  and  $0.14$  were obtained by a purely numerical method that consist in finding the best curve of the form  $a + b/\sqrt{N}$  that fits  $\sqrt{N}(\mathbb{E}[M^{(N)}] - \mu(\infty))$ . For now, we do not know if there exists a systematic way to obtain these values for another model that would also have a non-exponentially equilibrium point. We left this question for future work.

We remark that the convergence of  $\mathbb{E}[M^{(N)}]$  to  $\mu(\infty)$  observed Equation (9) is in  $O(1/\sqrt{N})$  and not in  $O(1/N)$ . This model satisfies all the assumptions of Theorem 2 but one : The attractor  $\mu(\infty)$  is not exponentially stable. This suggests that having an exponentially stable attractor is needed to obtain a convergence in  $1/N$ .

## 8. CONCLUSION

In this paper we studied population models composed of (clock-)synchronous objects. A classical method to study such systems is to consider the mean field approximation. By studying the accuracy of this deterministic approximation, we developed a new approximation, that we call the *refined* mean field approximation. We illustrated on a few examples that this approximation can greatly improve the accuracy of the classical mean field limit, also for systems with a relatively small size (10 – 20 objects). Yet, this refined approximation has some limitations when the deterministic approximation has multiple basins of attraction or has a unique attractor that is not exponentially stable.

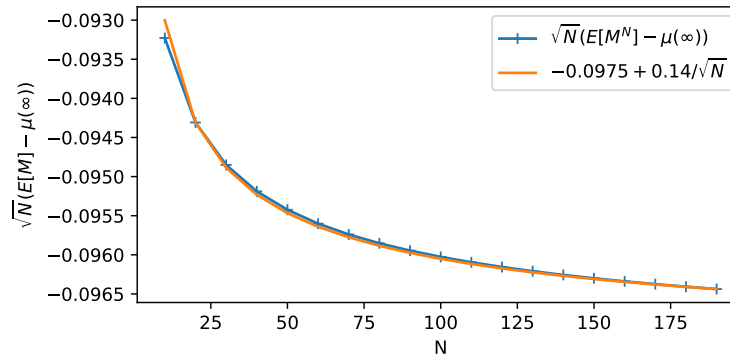


FIGURE 11. Non-exponentially stable case ( $\alpha = 0.75$ ) : convergence of  $\mathbb{E}[M(t)]$  to  $\mu(\infty)$ .

The proposed refined approximation is given by a set of linear equations that scales as the square of the dimension of the model (but does not depend on the system size). For now, we limited our study to relatively small models, for which the Jacobian and Hessian can be computed in closed form. We are currently investing means to make this computation automatic which will allow us to study large-scale examples.

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