# A Note on the Ei Function and a Useful Sum-Inequality

Lilian Besson

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**Lilian Besson** LILIAN.BESSON@CENTRALESUPELEC.FR
*CentraleSupélec (campus of Rennes), IETR, SCEE Team,*
*Avenue de la Boulaie – CS 47601, F-35576 Cesson-Sévigné, France*

## Abstract

This short note defines formally the Ei function, and gives some interesting inequalities and integration using it. I illustrate the inequalities and detail what is still to be proved.
**Keywords:** Analysis; Inequalities; Primitive.

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## 1. Introduction and Motivation

Take $a > 1$, $b > 1$ and $\gamma > 0$, and for an integer $L > 0$ consider the sum $\sum_{i=1}^{L} (a^{b^i})^\gamma$. We want to bound it, and the goal is to show that it is bounded by a constant times its last term. A first naive bound is $\sum_{i=1}^{L} (a^{b^i})^\gamma \leq (L+1)(a^{b^L})^\gamma$ which is too brutal as soon as $L \to \infty$.

I first remind and prove two useful elementary results, and then we define and study the Ei function, to finally prove the desired inequality.

## 2. Lemma and Proof

**Lemma 1** *For any $n \in \mathbb{N}^*$, $a > 1$, $b > 1$ and $\gamma > 0$, we have*

$$\sum_{i=0}^{n} (a^{b^i})^\gamma \leq a^\gamma + \left(1 + \frac{1}{(\log(a))(\log(b^\gamma))}\right)(a^{b^n})^\gamma = \mathcal{O}\Big((a^{b^n})^\gamma\Big). \tag{1}$$

**Proof** We first isolate both the first and last term in the sum and focus on the from $i = 1$ sum up to $i = n - 1$. As the function $t \mapsto (a^{b^t})^\gamma$ is increasing for $t \geq 1$, we use a sum-integral inequality, and then the change of variable $u := \gamma b^t$, of Jacobian $\mathrm{d}t = \frac{1}{\log b} \frac{\mathrm{d}u}{u}$, gives

$$\sum_{i=1}^{n-1} (a^{b^i})^\gamma \leq \int_1^n a^{\gamma b^t} \,\mathrm{d}t \leq \frac{1}{\log(b^\gamma)} \int_{\gamma b}^{\gamma b^n} \frac{a^u}{u} \,\mathrm{d}u$$

Now for $u \geq 1$, observe that $\frac{a^u}{u} \leq a^u$, and as $\gamma b > 1$, we have

$$\leq \frac{1}{\log(b^\gamma)} \int_{\gamma b}^{\gamma b^n} a^u \,\mathrm{d}u \leq \frac{1}{\log(b^\gamma)} \frac{1}{\log(a)} a^{\gamma b^n} = \frac{1}{(\log(a))(\log(b^\gamma))} (a^{b^n})^\gamma.$$

Finally, we obtain as desired, $\sum\limits_{i=0}^{n} (a^{b^i})^\gamma \leq a^\gamma + (a^{b^n})^\gamma + \frac{1}{(\log(a))(\log(b^\gamma))} (a^{b^n})^\gamma$. ∎

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## 3. Elementary Results

### 3.1. Integration by Part

The Integration by Part is a basic but useful result to establish inequalities, *e.g.*, for Lemma 10 using Lemma 3, and to prove the existence of finite integrals, *e.g.*, for Lemma 5 using two chained IP.

**Lemma 2 (Integration by Part (IP))** *Let $x, y \in \mathbb{R}$, $x \leq y$, and $u, v$ two functions of class*[1] $\mathcal{C}^1$*, and with this notation $\left[uv\right]_x^y := u(y)v(y) - u(x)v(x)$, then*

$$\int_x^y u(t)v'(t)\ \mathrm{d}t = \left[uv\right]_x^y - \int_x^y u'(t)v(t)\ \mathrm{d}t. \tag{2}$$

**Proof** The two integrals and the two evaluations are well defined by the $\mathcal{C}^1$ hypothesis on both $u$ and $v$ ($u$ and $v$ are continuous at $x$ and $y$ and $u'v$ is continuous so integrable on the interval $[x, y]$).

The product function $uv$ is differentiable, and $(uv)' = u'v + uv'$, so $\left[uv\right]_x^y = \int_x^y (uv)'(t)\ \mathrm{d}t = \int_x^y u(t)v'(t)\ \mathrm{d}t + \int_x^y u'(t)v(t)\ \mathrm{d}t$ as wanted, by the linearity of the integral. ■

**Lemma 3 (IP Inequality)** *If both $u, v$ are non-negative, and non-decreasing, then*

$$\int_x^y u(t)v'(t)\ \mathrm{d}t \leq u(y)v(y). \tag{3}$$

**Proof** The non-negativeness gives that $-u(x)v(x) \leq 0$ and the monotony hypothesis gives that $u(t)v'(t)$ is non-negative on the interval $[x, y]$, and so $-\int_x^y u(t)v'(t)\ \mathrm{d}t \leq 0$, so an Integration by Part gives the desired inequality. ■

### 3.2. Sum-Integral Inequality

A well known result is the following, which bound a discrete sum $\sum_{i=x}^{y} f(i)$ by two integrals for non-decreasing functions, and it is used for Lemma 10.

**Lemma 4** *For any $x, y \in \mathbb{N}^*$, $x \leq y$, and $f$ a non-decreasing function on $[0, +\infty)$, then*

$$\int_{x-1}^{y} f(t)\ \mathrm{d}t \leq \sum_{i=x}^{y} f(i) \leq \int_x^{y+1} f(t)\ \mathrm{d}t, \tag{4}$$

*and*

$$f(x) + \int_x^y f(t)\ \mathrm{d}t \leq \sum_{i=x}^{y} f(i) \leq f(y) + \int_x^y f(t)\ \mathrm{d}t. \tag{5}$$

**Proof** For the first inequality, both parts comes from the monotony of $f$ and monotony and additivity of the integral. On any interval $[i, i+1]$, $f(i) = \min_{t\in[i,i+1]} f(t) \leq \int_i^{i+1} f(t)\ \mathrm{d}t$, and $f(i) = \max_{t\in[i-1,i]} f(t) \geq \int_{i-1}^{i} f(t)\ \mathrm{d}t$, And so, if we sum these terms from $i = x$ to $y$, we get

$$\sum_{i=x}^{y} f(i) \leq \sum_{i=x}^{y} \int_i^{i+1} f(t)\ \mathrm{d}t = \int_x^{y+1} f(t)\ \mathrm{d}t.$$

1. A function of class $\mathcal{C}^1$ is continuous, differentiable and of continuous derivative on its interval of definition.

as well as

$$\sum_{i=x}^{y} f(i) \geq \sum_{i=x}^{y} \int_{i-1}^{i} f(t)\,\mathrm{d}t = \int_{x-1}^{y} f(t)\,\mathrm{d}t.$$

The two sides of second inequality are immediate by isolating the first (or last) term of the sum $f(y)$ (or $f(x)$), and applying the first inequality to $x-1$ instead of $x$ (or $y-1$ instead of $y$). ■

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## 4. The Exponential Integral Ei Function

This last Subsection is rather long, and actually not required to obtain the Lemma 1. But I find this Ei function to be quite interesting, so I wanted to write down these proofs. We define the Ei function (Weisstein, 2017; Collective, 2017), by carefully justifying its existence, and then we give two results using it, to obtain the non-trivial Lemma 10.

**Lemma 5** *For any $\varepsilon > 0$, $I(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{e}^u}{u}\,\mathrm{d}u$ exists and is finite, it satisfies this identity*

$$I(\varepsilon) = (\mathrm{e}^{\varepsilon} - \mathrm{e}^{-\varepsilon}) \log \varepsilon - (\mathrm{e}^{\varepsilon} + \mathrm{e}^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) + \int_{0}^{\varepsilon} (\mathrm{e}^{u} - \mathrm{e}^{-u})(u \log u - u)\,\mathrm{d}u. \tag{6}$$

*Additionally, it stays finite when $\varepsilon \to 0$, and $\lim_{\varepsilon\to 0} I(\varepsilon) = \lim_{\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{e}^u}{u}\,\mathrm{d}u = 0$.*

**Proof** Fix $\varepsilon > 0$, and let $I(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{e}^u}{u}\,\mathrm{d}u$.

Roughly, one just needs to observe[2] that for $u$ close to 0, $\frac{\mathrm{e}^u}{u} \sim \frac{1}{u}$ as $\mathrm{e}^u \sim 1$, and $\frac{1}{u}$ can be integrated on $[-\varepsilon, \varepsilon]$, even if it is not defined at 0, because it is odd: $\int_{-\varepsilon}^{\varepsilon} \frac{1}{u}\,\mathrm{d}u = \lim_{t\to 0}(\int_{-\varepsilon}^{t} \frac{1}{u}\,\mathrm{d}u + \int_{t}^{\varepsilon} \frac{1}{u}\,\mathrm{d}u)$ (as Cauchy's principal values), and $\int_{-\varepsilon}^{t} \frac{1}{u}\,\mathrm{d}u = -\int_{t}^{\varepsilon} \frac{1}{v}\,\mathrm{d}v$ with the change of variable $v = -u$. So $\int_{-\varepsilon}^{\varepsilon} \frac{1}{u}\,\mathrm{d}u = 0$ for any $\varepsilon \geq 0$.

But we have to justify more properly that $I(\varepsilon)$ exists for any $\varepsilon > 0$ and that $I(\varepsilon) \to 0$ for $\varepsilon \to 0$. A first Integration by Part (Lemma 2) with $a(u) = \mathrm{e}^u$ and $b'(u) = \frac{1}{u}$, that is $a'(u) = \mathrm{e}^u$ and by choosing $b(u) = \log|u|$, gives

$$\begin{aligned} I(\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{e}^u}{u}\,\mathrm{d}u \\ &= \left[\mathrm{e}^u \log|u|\right]_{-\varepsilon}^{\varepsilon} - \int_{-\varepsilon}^{\varepsilon} \mathrm{e}^u \log|u|\,\mathrm{d}u \\ &= (\mathrm{e}^{\varepsilon} - \mathrm{e}^{-\varepsilon}) \log \varepsilon - \int_{0}^{\varepsilon} (\mathrm{e}^u + \mathrm{e}^{-u}) \log|u|\,\mathrm{d}u. \end{aligned}$$

Let $I_2(\varepsilon) := \int_0^{\varepsilon} (\mathrm{e}^u + \mathrm{e}^{-u}) \log u\,\mathrm{d}u$. A second Integration by Part (Lemma 2) with $a(u) = \mathrm{e}^u + \mathrm{e}^{-u}$ and $b'(u) = \log u$, that is $a'(u) = \mathrm{e}^u - \mathrm{e}^{-u}$ and $b(u) = u \log u - u$ ($\mathcal{C}^1$ on $(0, \varepsilon]$), gives

$$\begin{aligned} I_2(\varepsilon) &= \int_{0}^{\varepsilon} (\mathrm{e}^u + \mathrm{e}^{-u}) \log u\,\mathrm{d}u \\ &= \left[(\mathrm{e}^u + \mathrm{e}^{-u})(u \log u - u)\right]_0^{\varepsilon} - \int_{0}^{\varepsilon} (\mathrm{e}^u - \mathrm{e}^{-u})(u \log u - u)\,\mathrm{d}u. \end{aligned}$$

2. The notation $f(u) \sim g(u)$ for $u \to u_0$ means that $g(u) \neq 0$ and $f(u)/g(u) \to 1$ for $u \to u_0$.

Indeed, $b(u) = u \log u - u$ is well defined for $u \to 0$, as $u \log u \to 0$, so we can define $b(0) = 0$ to have $b$ of class $\mathcal{C}^1$ on $[0, \varepsilon]$. Therefore, $I_2(\varepsilon)$ exists, and we have, as wanted, the following identity

$$I(\varepsilon) = (\mathrm{e}^{\varepsilon} - \mathrm{e}^{-\varepsilon}) \log \varepsilon - (\mathrm{e}^{\varepsilon} + \mathrm{e}^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) + \int_0^{\varepsilon} (\mathrm{e}^u - \mathrm{e}^{-u})(u \log u - u) \, \mathrm{d}u.$$

The last integral is well defined and finite, as the integrated function is continuous and finite for all $u$, even at 0. So this proves that $I(\varepsilon)$ is finite for any $\varepsilon > 0$.

Now, taking $\varepsilon \to 0$ gives, for each of the three terms in $I(\varepsilon)$,

$$\begin{cases} (\mathrm{e}^{\varepsilon} - \mathrm{e}^{-\varepsilon}) \log \varepsilon \sim ((1+\varepsilon) - (1-\varepsilon)) \log \varepsilon = 2\varepsilon \log \varepsilon \to 0 \\ (\mathrm{e}^{\varepsilon} + \mathrm{e}^{-\varepsilon})(\varepsilon \log \varepsilon - \varepsilon) \sim 2b(\varepsilon) \to 0 \\ \int_0^{\varepsilon} (\mathrm{e}^u - \mathrm{e}^{-u})(u \log u - u) \, \mathrm{d}u \to 0, \end{cases}$$

so $I(\varepsilon) \to 0$, as wanted. ■

**Lemma 6** *For any $0 < \varepsilon \leq 1$, $I(\varepsilon)$ satisfies $I(\varepsilon) \leq \mathrm{e}^{\varepsilon} - \mathrm{e}^{-\varepsilon}$. In particular, $I(1) \leq \mathrm{e} - \mathrm{e}^{-1}$.*

**Proof** For $0 < \varepsilon \leq 1$, $b(1) \geq -1$, and $(\mathrm{e}^{\varepsilon} + \mathrm{e}^{-\varepsilon}) \log(\varepsilon) \leq 0$, and so the identity (6) gives $I(\varepsilon) \leq (\mathrm{e}^{\varepsilon} - \mathrm{e}^{-\varepsilon}) + \int_0^{\varepsilon} (\mathrm{e}^u - \mathrm{e}^{-u})(u \log u - u) \, \mathrm{d}u$, but $(\mathrm{e}^u - \mathrm{e}^{-u})(u \log u - u) \leq 0$ for all $u \in [0, 1]$, so $I(\varepsilon) \leq (\mathrm{e}^{\varepsilon} - \mathrm{e}^{-\varepsilon})$ as wanted. In particular, $I(1) \leq (\mathrm{e} - \mathrm{e}^{-1})$. ■

**Definition 7** *The* Exponential Integral Ei *function is defined for $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ by*

$$\mathrm{Ei}(x) := \int_{-\infty}^{x} \frac{\mathrm{e}^u}{u} \, \mathrm{d}u, \tag{7}$$

*where the Cauchy's principal value of the integral is taken.*

**Proof** This integral exists and is finite for $x < 0$, as the function $u \mapsto \frac{\mathrm{e}^u}{u}$ is of class $\mathcal{C}^1$ on $(-\infty, 0)$.

For $x \to 0$ (from above or from below), $\mathrm{Ei}(x) \to -\infty$.

And for $x > 0$, let $\varepsilon > 0$, and observe that we can write the integral from $-\infty$ to $x$ as three terms, $\mathrm{Ei}(x) = \mathrm{Ei}(-\varepsilon) + \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{e}^u}{u} \, \mathrm{d}u + \int_{\varepsilon}^{x} \frac{\mathrm{e}^u}{u} \, \mathrm{d}u$. $\mathrm{Ei}(-\varepsilon)$ and the last integral both exists and are finite, thanks to the first case of $x < 0$ and as the function $u \mapsto \frac{\mathrm{e}^u}{u}$ is of class $\mathcal{C}^1$ on $(\varepsilon, +\infty)$. And thanks to Lemma 5, $\int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{e}^u}{u} \, \mathrm{d}u$ is finite. So all the three terms in the decomposition of $\mathrm{Ei}(x)$ exist and are finite, therefore $\mathrm{Ei}(x)$ is well defined. ■

A few properties of Ei worth noting include the following: it has a unique zero (located at $x_0 \simeq 0.327$), it is negative for $x < x_0$ and in particular for $x < 0$, it is positive for $x > x_0$, and it is decreasing on $(-\infty, 0)$ and increasing on $(0, +\infty)$. Ei is also concave on $(-\infty, 0)$ and $(0, 1)$, and convex on $(1, +\infty)$.

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**Illustration** We can plot this function[3]:

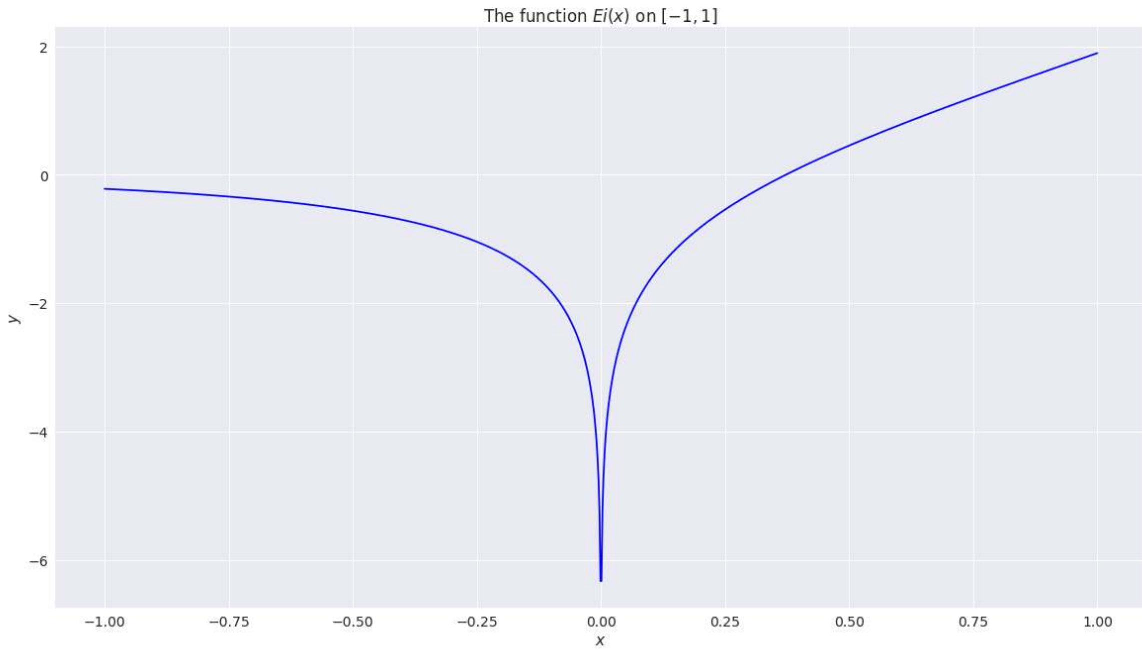


Figure 1: The Ei function on $[-1, 1]$.

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## 5. Using Ei to compute primitives

**Lemma 8** *For any $a, b, \gamma \in \mathbb{R}$ and $L \in \mathbb{N}$ such that $a, b > 1$, $\gamma > 0$ and $L > 0$,*

$$\int_0^L (a^{b^t})^\gamma \, \mathrm{d}t = \frac{1}{\log b} \left( \mathrm{Ei}\left( \gamma \log(a^{b^L}) \right) - \mathrm{Ei}\left( \gamma \log(a) \right) \right). \tag{8}$$

**Proof** A first change of variable with $u := b^t$ gives $\mathrm{d}t = \frac{1}{\log b} \frac{1}{u} \, \mathrm{d}u$ ($\log b > 0$ as $b > 1$), and so

$$\int_0^L (a^{b^t})^\gamma \, \mathrm{d}t = \frac{1}{\log b} \int_1^{b^L} \frac{1}{u} (a^u)^\gamma \, \mathrm{d}u = \frac{1}{\log b} \int_1^{b^L} \frac{1}{u} (a^\gamma)^u \, \mathrm{d}u$$

And a second change of variable with $v := \log(a^\gamma)u = \gamma \log(a)u$ gives $\frac{1}{u} \, \mathrm{d}u = \frac{1}{v} \, \mathrm{d}v$ (and no change in the order of the integral's bounds, as $\log a > 0$ as $a > 1$), and so

$$= \frac{1}{\log b} \int_{\gamma \log(a)}^{\gamma \log(a) b^L} \frac{\mathrm{e}^v}{v} \, \mathrm{d}v = \frac{1}{\log b} \Big[ \mathrm{Ei}(v) \Big]_{\gamma \log(a)}^{\gamma \log\left(a^{b^L}\right)}$$
$$= \frac{1}{\log b} \left( \mathrm{Ei}\left( \gamma \log\left( a^{b^L} \right) \right) - \mathrm{Ei}\left( \gamma \log(a) \right) \right).$$

3. See for instance, the `scipy.special.expi` function, on https://docs.scipy.org/doc/scipy/reference/generated/scipy.special.expi.html, if you use Python and SciPy (Foundation, 2017; Jones et al., 2001–).

■

## 6. Inequalities for Ei

**Lemma 9 (First Inequalities Using** $\mathrm{Ei}$**)** *For any* $x \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$, $\mathrm{Ei}(x) \leq \mathrm{e}^x$.

*Moreover, for* $x \geq 1$*, we also have* $\mathrm{Ei}(x) \geq \mathrm{Ei}(1) + \frac{\mathrm{e}^x - \mathrm{e}}{x} \geq -1 + \frac{\mathrm{e}^x}{x}$.

*A useful consequence is that for any* $y \geq 1$ *and* $0 \leq \gamma \leq 1$*,*

$$\mathrm{Ei}(\log(y^\gamma)) = \mathrm{Ei}(\gamma \log(y)) \leq y^\gamma. \tag{9}$$

**Proof** Let $x \in \mathbb{R}$. First, if $x < 0$, then clearly $\mathrm{Ei}(x) \leq 0 < \mathrm{e}^x$.

If $0 < x < 1$, we can split the integral defining $\mathrm{Ei}(x)$ in two terms, and as $I(x) = \int_{-x}^{x} \frac{\mathrm{e}^u}{u} \,\mathrm{d}u \leq \mathrm{e}^x - \mathrm{e}^{-x}$ (see Lemma 6),

$$\begin{aligned} \mathrm{Ei}(x) = \underbrace{\int_{-\infty}^{-x} \frac{\mathrm{e}^u}{u} \,\mathrm{d}u}_{=\mathrm{Ei}(-x) \leq 0} + \underbrace{\int_{-x}^{x} \frac{\mathrm{e}^u}{u} \,\mathrm{d}u}_{=I(x)} \leq I(x) \\ \leq \mathrm{e}^x - \mathrm{e}^{-x} \leq \mathrm{e}^x. \end{aligned}$$

If $x > 1$, we do the same with three terms, and by using $\mathrm{Ei}(-1) = \int_{-\infty}^{-1} \frac{\mathrm{e}^u}{u} \,\mathrm{d}u \leq 0$, and $I(1) = \int_{-1}^{1} \frac{\mathrm{e}^u}{u} \,\mathrm{d}u \leq \mathrm{e} - \mathrm{e}^{-1}$ (see Lemma 6), we have

$$\begin{aligned} \mathrm{Ei}(x) = \underbrace{\int_{-\infty}^{-1} \frac{\mathrm{e}^u}{u} \,\mathrm{d}u}_{=\mathrm{Ei}(-1) \leq 0} + \underbrace{\int_{-1}^{1} \frac{\mathrm{e}^u}{u} \,\mathrm{d}u}_{=I(1)} + \underbrace{\int_{1}^{x} \frac{\mathrm{e}^u}{u} \,\mathrm{d}u}_{\leq \mathrm{e}^x - \mathrm{e}^1} \leq I(1) + \mathrm{e}^x - \mathrm{e} \\ \leq \mathrm{e} - \mathrm{e}^{-1} + \mathrm{e}^x - \mathrm{e} = \mathrm{e}^x - \mathrm{e}^{-1} \leq \mathrm{e}^x. \end{aligned}$$

Now for the lower bound, let $x \geq 1$, and we use the same splitting. For $I(1)$, we use conversely that $I(1) \geq 0$ (see Lemma 6), and for the integral we have $\int_1^x \frac{\mathrm{e}^u}{u} \,\mathrm{d}u \geq \frac{1}{x}(\mathrm{e}^x - \mathrm{e})$. So $\mathrm{Ei}(x) \geq \mathrm{Ei}(1) + \frac{\mathrm{e}^x - \mathrm{e}}{x}$. We also have $-\frac{\mathrm{e}}{x} \geq -\mathrm{e}$ and numerically, $\mathrm{Ei}(1) - \mathrm{e} \geq -1$ (as $\mathrm{Ei}(1) \simeq 1.895$), so $\mathrm{Ei}(x) \geq -1 + \frac{\mathrm{e}^x}{x}$.

Finally, if $x = \log(y^\gamma)$ and $y \geq 0$, then $\mathrm{e}^x = y^\gamma$, so $\mathrm{Ei}(x) = \mathrm{Ei}(\gamma \log(y)) \leq \mathrm{e}^x = y^\gamma$. ■

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**Illustration** We can check this inequality $\mathrm{Ei}(x) \leq \mathrm{e}^x$ graphically, as well as a tighter inequality $\mathrm{Ei}(x) \leq \mathrm{Ei}(-1) - \frac{1}{\mathrm{e}} + \mathrm{e}^x$.

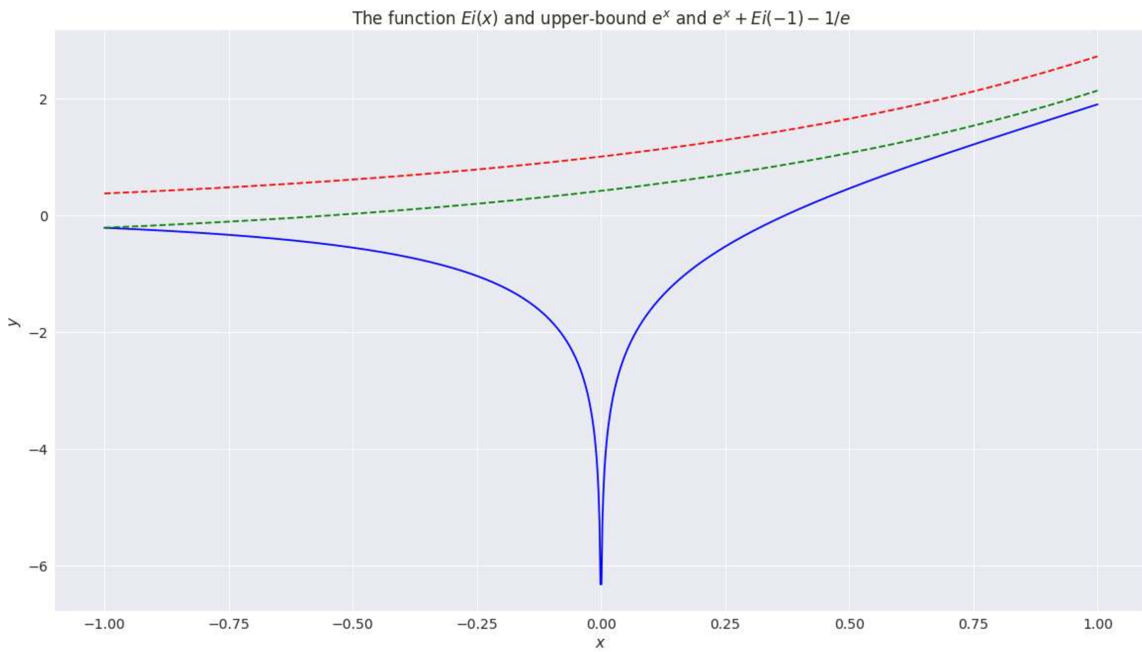


Figure 2: The $\mathrm{Ei}$ function and two upper-bounds valid respectively on $\mathbb{R}$ and $[1, +\infty)$.

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This last sum-inequality is the result we were looking for.

**Lemma 10 (Sum Inequality Using $\mathrm{Ei}$)** *For any $a, b, \gamma \in \mathbb{R}$ and $L \in \mathbb{N}$ such that $a, b > 1$, $\gamma > 0$ and $L > 0$, and if $\mathrm{Ei}\left(\gamma \log(a)\right) \geq 0$, then*

$$\sum_{i=0}^{L-1} (a^{b^i})^\gamma \leq \frac{1}{\log b} (a^{b^L})^\gamma. \tag{10}$$

*And by isolating the last term, we also have*

$$\sum_{i=0}^{L} (a^{b^i})^\gamma \leq \left(1 + \frac{1}{\log b}\right) (a^{b^L})^\gamma \tag{11}$$

**Proof** Using the sum-integral inequality (Lemma 4) and then Lemma 8, we have directly that

$$\begin{aligned} \sum_{i=0}^{L-1} (a^{b^i})^\gamma &\leq \int_0^L (a^{b^t})^\gamma \,\mathrm{d}t \\ &\leq \frac{1}{\log b} \mathrm{Ei}\left(\gamma \log(a^{b^L})\right) \leq \frac{1}{\log b} (a^{b^L})^\gamma. \end{aligned}$$

■

In particular, this inequality (11) holds as soon as $a \geq \mathrm{e}^{0.373/\gamma}$, as $\gamma \log(a) \geq 0.373 > x_0 \implies \mathrm{Ei}\left(\gamma \log(a)\right) \geq \mathrm{Ei}(x_0) > 0$ and $x_0 \simeq 0.372507 \geq 0.373$. For instance, $\gamma = 1/2$ gives $a \geq \mathrm{e}^{0.373/\gamma} = \mathrm{e}^{0.746} \simeq 2.107$, close to the simplest value $a = 2$.

And if $\gamma = 0$, then $\mathrm{Ei}(\gamma \log(a))$ cannot be $\geq 0$, but the sum in (10) is constant and equals to $L$.

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## 7. Conclusion

This small note defines and studies a useful non-canonical function called the "exponential integral" function, Ei, and we use it to find a bound on any sum of the form $\sum_{i=0}^{L}(a^{b^i})^{\gamma}$.

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*Note*: the simulation code used for the experiments is using Python 3, (Foundation, 2017), and Matplotlib (Hunter, 2007) for plotting, as well as SciPy (Jones et al., 2001–). It is open-sourced at `github.com/Naereen/notebooks/blob/master/Exponential_Integral_Python.ipynb`.  and is available online at `perso.crans.org/besson/publis/A_note_on_the_Ei_function.pdf`.

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