Stability Analysis of Switched Systems with Time-Varying Discontinuous Delays *

Frédéric Mazenc, Michael Malisoff, H Ozbay

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Abstract

A new technique is proposed for the stability analysis of nonlinear switched time-varying systems with time-varying discontinuous delays. It is based on an adaptation of Halanay's inequality to switched systems, combined with a recent trajectory based stability analysis technique. The result is applied to a family of the linear time-varying systems with time-varying delays. In particular, it is shown that this approach can be used for stabilization of LTV systems with time varying delays by a set of properly designed switched controllers.

Key Words: switched system, time delay, time-varying system, asymptotic stability.

1 Introduction

Switched systems in continuous-time are systems with discrete switching events. They use a switching signal that indicates which subsystem operates at each instant; see, e.g., [13] and [26], for more details. They are fundamental systems which are encountered in a wide range of applications, including communication networks; see [5, 12, 23, 29, 33]. In addition, delays are frequently present in models describing engineering processes. They also appear in the control inputs (actuator delays) or the outputs (measurement delays) in the feedback loops. Moreover, in some applications time delays can be time-varying and discontinuous: prime example is control over a network, where congestion and failures in certain links of the network leads to sudden changes in the routing, that causes the return-trip-time to change abruptly.

Even for systems without switches, the problem of analyzing the stability of systems with time-varying delays is difficult in general, especially when the delay is discontinuous. For systems with discontinuous delays, a strategy for stability analysis consists of representing the systems as switched systems, and then applying techniques from the theory of the switched systems; this is explained in [24] and [32]. These facts motivate the recent works [26] and [30], where switched nonlinear systems having lumped delays are studied, and many other contributions, such as [4] and [25]. See also [28], for stability results for non-delayed switched systems, based on assuming the existence of a stable convex combination of their subsystems.

In the present paper, we will study switched systems whose switching signal only depends on time. Our main result is in Section 2, and is a new result that is based on combining Halanay's inequality [9] with the main result of [17]. The key advantage of the technique is that it applies to broad classes of systems, including time-varying systems with switchings and discontinuous delays. This contrasts with much of the literature,

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†Inria EPI DISCO, L2S-CNRS-CentraleSupélec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France, (frederic.mazenc@l2s.centralesupelec.fr).

‡Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA (malisoff@lsu.edu).

§Department of Electrical and Electronics Engineering, Bilkent University, Ankara 06800, Turkey (hitay@bilkent.edu.tr).

since in many cases, the time-varying delays are assumed to be continuously differentiable [2, 3, 6, 27] or the discontinuous part of the delay is assumed to be small in a suitable sense [18]. Our key assumption here is a condition on the switching signal. This is basically a limitation on the number of switchings on all intervals of a certain length, but a key feature of our work is that we do not impose an upper bound on the size of the variation of the delay. Hence, our work contrasts with other works that involve delays and switching such as [32] (which constructs a switching rule that yields stability properties, instead of proving stability properties for large classes of switching rules as we do here) and [26], which introduces upper bound on the delays that makes it possible to use Lyapunov-Krasovskii functionals (or LKFs) to prove stability properties.

We do not assume that all of the subsystems of the switched system are stable, nor we assume that a common Lypunov function for the subsystems is available. This is a valuable feature because there are several techniques for systems with strong or even weak common Lyapunov functions (such as those in [10] and [16]), but, in many cases, common Lyapunov functions do not exist, and the presence of a time-varying delay precludes the application of standard invariance principles. In many cases, the existence of a common strict Lyapunov function implies stability for any switching signal, and we aim to establish stability results under restricted switching signals because only this type of result makes it possible to solve problems for time-varying systems with time-varying delays; see Section 2.1. In Section 3, we apply the nonlinear result from Section 2 to a crucial family of systems, namely, linear time-varying systems with time-varying pointwise delays. We show that their stability can be analyzed under assumptions that can be verified by finding LKFs for time-invariant systems with constant pointwise delays. All our results rely on an assumption on the dwell time. We present illustrative examples in Section 4. Finally, concluding remarks are made in Section 5.

Notation. We will use the following notation and conventions. We omit arguments of functions when the arguments are clear from the context. We set $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{Z}_{\geq 0} = \mathbb{N} \cup \{0\}$. For any dimensions k and n in $\mathbb{N}$, the $k \times n$ matrix all of whose entries are 0 will also be denoted by 0. The usual Euclidean norm of vectors, and the induced norm of matrices, of any dimensions are denoted by $|\cdot|$, and I is the identity matrix in the dimension under consideration. The floor function $E : [0, +\infty) \to \mathbb{Z}_{\geq 0}$ is defined by $E(x) = k$ when $k \in \mathbb{Z}_{\geq 0}$ is such that $x \in [k, k+1)$. We let $f(t^-)$ denote the limit from the left of functions f at points t in their domain where the left limit is defined. Given any constant $\tau_b > 0$, we let C_{in} denote the set of all continuous functions $\phi : [-\tau_b, 0] \to \mathbb{R}^n$, which we call the set of all *initial functions*. We define $\Xi_t \in C_{\text{in}}$ by $\Xi_t(s) = \Xi(t+s)$ for all choices of Ξ, s, and t for which the equality is defined. Let $\mathcal{K}_\infty$ be the set of all continuous functions $g : [0, +\infty) \to [0, +\infty)$ such that $g(0) = 0$, g is strictly increasing, and $\lim_{s \to +\infty} g(s) = +\infty$.

2 A Fundamental Result for Nonlinear Switched Systems with Time Delay

2.1 Problem Definition and Preliminary Remarks

In this section we consider a nonlinear time-varying switched system with a delay, $\tau(t) \geq 0$, represented by

$$\dot{x}(t) = f_{\sigma(t)}(t, x(t - \tau(t)) \tag{1}$$

where x is valued in $\mathbb{R}^n$, $\sigma : [0, +\infty) \to \{1, ..., k\}$ is called the *switching signal*, $k \in \mathbb{N}$ and $n \in \mathbb{N}$ are arbitrary, and each f_i is locally Lipschitz with respect to its last argument and piecewise continuous with respect to its first argument t for all $i \in \{1, ..., k\}$. We assume that the delay is bounded by a constant $\tau_b \geq \tau(t)$ for all $t \geq 0$ and the initial condition ϕ is in C_{in}. The system (1) includes the important special case of dynamics with a time-varying piecewise continuous delay $\tau : [0, +\infty) \to [0, +\infty)$.

Let $\{t_i\}$ be the switching sequence of non-negative real numbers associated with σ, i.e., the times σ changes to a new value, with $t_0 = 0$ and σ is such that $\sigma(t) = \sigma(t_i)$ for all $t \in [t_i, t_{i+1})$. Moreover, we assume that there are two constants $\mathcal{T}_1$ and $\mathcal{T}_2$ such that

$$0 < \mathcal{T}_1 < t_{i+1} - t_i \leq \mathcal{T}_2 \text{ for all } i \in \mathbb{Z}_{\geq 0}. \tag{2}$$

The constant $\mathcal{T}_1$ is usually called the *dwell time*, and the sequence $\{t_i\}$ is called a partition of $[0, +\infty)$. See [21], and [11] for design methods minimizing dwell times to ensure stability of delay systems. These papers develop stability results in the form "if $\mathcal{T}_1 < t_{i+1} - t_i$ for all i, then the system is stable" under certain technical assumptions (usually requiring all subsystems to be stable), [4]. In recent years there has been a considerable effort in trying to find the smallest possible $\mathcal{T}_1$ for a given switched system, see e.g. [11] and its references. In this paper, the stability condition to be derived depends not only on τ_b and $\mathcal{T}_2$ (see Assumption 3 below) but also on some other parameters to be defined below.

We next state our technical assumptions; see the end of this subsection for their motivations.

Assumption 1. *There are k absolutely continuous functions $V_j : [0, +\infty) \times C_{\text{in}} \to [0, +\infty)$ for $j = 1, 2, \ldots, k$, real numbers $\alpha_1, \ldots, \alpha_k$, nonnegative constants $\beta_1, \ldots, \beta_k$, a continuous function $W : \mathbb{R}^n \to [0, +\infty)$, and class $\mathcal{K}_\infty$ functions χ_1 and χ_2 such that*

$$\chi_1(|\phi(0)|) \leq V_j(t, \phi) \leq \chi_2(|\phi|_\infty) \tag{3}$$

hold for all $\phi \in C_{\text{in}}$, $t \in [0, +\infty)$, and $j \in \{1, 2, \ldots, k\}$ and such that

$$\dot{V}_{\sigma(t_i)}(t) \ \leq \ \alpha_{\sigma(t_i)} V_{\sigma(t_i)}(t, x_t) + \beta_{\sigma(t_i)} \sup_{\ell \in [t - \tau_b, t]} W(x(\ell)) \tag{4}$$

holds almost everywhere along all trajectories of $\dot{x}(t) = f_{\sigma(t_i)}(t, x_t)$ for almost all $t \in [t_i, t_{i+1})$ and all $i \in \mathbb{Z}_{\geq 0}$.

Assumption 2. *The functions V_j and W from Assumption 1 admit a constant $\mu > 1$ such that*

$$W(\phi(0)) \leq V_1(t, \phi) \tag{5}$$

and

$$V_i(t, \phi) \leq \mu V_j(t, \phi) \tag{6}$$

hold for all $\phi \in C_{\text{in}}$, all i and j in $\{1, ..., k\}$, and all $t \in [0, +\infty)$.

Assumption 3. *There are constants $T \geq \tau_b + \mathcal{T}_2$ and $\lambda(T) > 0$ such that the inequality*

$$\int_{t-T}^{t} \alpha_{\sigma(\ell)} \mathrm{d}\ell \leq -\lambda(T) \tag{7}$$

holds for all $t \geq 0$.

With the above notation we can now find a constant $\nu > 0$ such that

$$\int_{t-T}^{t} \beta_{\sigma(s)} e^{\int_s^t \alpha_{\sigma(s)} \mathrm{d}\ell} \mathrm{d}s \leq \nu \tag{8}$$

holds for all $t \geq 0$. Note that ν exists because $\alpha_{\sigma(s)}$ and $\beta_{\sigma(s)}$ take only finitely many possible values on bounded intervals of time. Then, we let $N(r, t)$ denote the number of switching instants t_i on $[t - r, t)$ for all $r > 0$ and $t \geq 0$, and define $L(r) := \sup_{t \geq 0} N(r, t)$ (which is finite, because of (2)). The next assumption is the stability condition of the main result to be stated in the next section.

Assumption 4. *The inequality*

$$\mu^{L(T)+1} e^{-\lambda(T)} + \left[\frac{\mu^{L(T)+2} - 1}{\mu - 1} - L(T) - 1 \right] \nu\mu < 1 \tag{9}$$

is satisfied.

The following remarks summarize the notable features of the preceding assumptions.

Remark 2.1. The constants α_i in Assumption 1 can be positive or negative. In other words, some systems $\dot{z}(t) = f_i(t, z_t)$ for some $i \in \{1, ..., k\}$ may be unstable. In this case, to have stability for (1), their instability should be compensated in a certain sense by the stability of other systems $\dot{z}(t) = f_i(t, z_t)$, which is why we cannot extend our result to the case of switchings without restriction. Assumption 1 does not make it possible to apply Razumikhin's theorem or its recent extensions in [19] and [34] or Halanay's inequality to prove global asymptotic stability, because several Lyapunov functions are involved.

Remark 2.2. When a common Lyapunov functional is available, i.e., $V_1 = V_j$ for all $j \in \{2, ..., k\}$, the problem solved below can be solved using [19] and [34]. Recall that globally uniformly exponential stability of a switched system does not imply that its subsystems have a common Lyapunov function; see [13, Section 2.1.5]. Also, if a switched system is input-to-state stable (or ISS) under arbitrary switching, then uniform (with respect to the switching signals) ISS is equivalent to the existence of a common ISS Lyapunov function; see [15]. Using [20, Lemma A.1], we can extend our result to cases where the α_i's and β_i's are time varying. For simplicity, we do not present this extension. Setting $U(t, \phi) = V_{\sigma(t)}(t, \phi)$, Assumption 1 gives

$$\dot{U}(t) \ \leq \ a(t)U(t, x_t) + b(t) \sup_{\ell \in [t-\tau_b, t]} W(x(\ell)) \tag{10}$$

almost everywhere, where $a(t) = \alpha_{\sigma(t)}$ and $b(t) = \beta_{\sigma(t)}$. This inequality does not make it possible to conclude as in [19], because U is discontinuous.

Remark 2.3. The inequalities (6) in Assumption 2 are a standard assumption that is imposed in switching contexts; see [13]. The inequality (7) is the main assumption on the constants α_i and the switching signal σ which ensures the stability of the system (1). Assumption 4 pertains to σ as well, because L depends on the dwell time $\mathcal{T}_1$. In fact, Assumption 4 and (7) are the fundamental constraints that σ has to satisfy for ensuring asymptotic stability.

Notice that the inequalities (2) ensure that the functions N and L are well-defined and that they imply that for all $r \geq 0$, the inequality

$$L(r) \leq E\left(\frac{r}{\mathcal{T}_1}\right) + 1 \tag{11}$$

is satisfied. Indeed, if we consider g successive switching instants t_i, t_{i+1}, ... t_{i+g-1}, then the length of the interval $[t_i, t_{i+g-1}]$ is equal to the sum of the lengths of the intervals $[t_{i+s}, t_{i+s+1}]$, $s = 0$ to $g-2$ which is strictly larger than $(g-1)\mathcal{T}_1$. Any interval $[t-r, t)$ contains $N(r, t)$ switching instants, so it follows that $r > [N(r, t) - 1]\mathcal{T}_1$. Since $N(r, t)$ is an integer, we obtain $N(r, t) - 1 < E(r/\mathcal{T}_1)$, which is equivalent to (11).

2.2 Main Result

With the notation introduced above we are now ready to state our main result.

Theorem 1. *Let the system (1) satisfy Assumptions 1-4. Then its origin is a globally uniformly asymptotically stable equilibrium point.*

Proof. For the sake of brevity, we continue to use the notation $a(t) = \alpha_{\sigma(t)}$, $b(t) = \beta_{\sigma(t)}$, and

$$U(t, \phi) = V_{\sigma(t)}(t, \phi) \tag{12}$$

from the previous subsection. One can prove that Assumptions 1-2 ensure that the system is forward complete. This can be proved by induction, because if the solution is defined over $[0, t_j)$, then by integrating (4), we can prove that the solution is defined over $[0, t_{j+1})$. In fact, forward completeness on $[0, t_1]$ follows by combining Assumption 1 with the bounds $W(x(\ell)) \leq \mu V_j(\ell, x(\ell))$ from Assumption 2, and applying Gronwall's inequality to the function $\sup_{\ell \in [0,t]} V_j(\ell, x(\ell))$ for all $t \in [0, t_1]$, and then arguing inductively.

We next analyze the behavior of U along the trajectories of (1). Observe that Assumption 1 implies that if $t \in [t_m, t_{m+1})$, $t_* \in [t_m, t]$, and $m \in \mathbb{Z}_{\geq 0}$ are such that $\sigma(t_m) = j$, then the inequality

$$V_j(t, x_t) \leq e^{\alpha_j (t-t_*)} V_j(t_*, x_{t_*}) + \beta_j \int_{t_*}^{t} e^{\alpha_j (t-s)} \sup_{\ell \in [s-\tau_b, s]} W(x(\ell)) \mathrm{d}s \tag{13}$$

is satisfied. It follows that for all $j \in \mathbb{Z}_{\geq 0}$, all $t \in [t_j, t_{j+1})$, and all $t_* \in [t_j, t]$, the inequality

$$U(t, x_t) \leq e^{\int_{t_*}^t a(s)\mathrm{d}s} U(t_*, x_{t_*}) + \int_{t_*}^t b(s) e^{\int_s^t a(r)\mathrm{d}r} \sup_{\ell \in [s-\tau_b, s]} W(x(\ell))\mathrm{d}s \tag{14}$$

is satisfied.

Next, let $t \geq T$, $j \in \mathbb{Z}_{\geq 0}$, and $q \geq 2$ be such that $t \in [t_j, t_{j+1})$, $j - 1 \geq q$ and $t - T \in [t_{j-q-1}, t_{j-q})$. Observe for later use that $q \leq L(T)$. We introduce the simplifying notation

$$\mathcal{E}(\ell, t) = e^{\int_\ell^t a(s)\mathrm{d}s}, \;\; \mathcal{G}_i = \mu \mathcal{E}(t_{i-1}, t_i), \;\; \text{and} \;\; w(t) = \sup_{\ell \in [t-\tau_b, t]} W(x(\ell)) \; . \tag{15}$$

From (14) with the choice $t_* = t_j$, and using the simplifying notation $U(t)$ to mean $U(t, x_t)$, we deduce that

$$U(t) \;\; \leq \;\; \mathcal{E}(t_j, t) U(t_j) + b(t) \int_{t_j}^t \mathcal{E}(\ell, t) w(\ell)\mathrm{d}\ell \;\; \leq \;\; \mu \mathcal{E}(t_j, t) U(t_j^-) + \int_{t_j}^t b(\ell) \mathcal{E}(\ell, t) w(\ell)\mathrm{d}\ell \; , \tag{16}$$

where the last inequality is a consequence of (6) in Assumption 2. From the first inequality in (16) (with the choice $j = l - 1$, and by taking the limit $t \to t_l^-$ from the left in (16)), we deduce that for all integers $l \geq 1$, the inequality

$$U(t_l^-) \leq \mathcal{E}(t_{l-1}, t_l) U(t_{l-1}) + \int_{t_{l-1}}^{t_l} b(\ell) \mathcal{E}(\ell, t_l) w(\ell)\mathrm{d}\ell \tag{17}$$

is satisfied. Consequently, using (6) in Assumption 2 and the nonnegativity of the $\mathcal{G}_j$'s, we obtain

$$\begin{array}{rcl} U(t_j^-) & \leq & \mathcal{G}_j U(t_{j-1}^-) + \displaystyle\int_{t_{j-1}}^{t_j} b(\ell)\mathcal{E}(\ell, t_j) w(\ell)\mathrm{d}\ell \\ \mathcal{G}_j U(t_{j-1}^-) & \leq & \mathcal{G}_j \mathcal{G}_{j-1} U(t_{j-2}^-) + \mathcal{G}_j \displaystyle\int_{t_{j-2}}^{t_{j-1}} b(\ell)\mathcal{E}(\ell, t_{j-1}) w(\ell)\mathrm{d}\ell \\ & \vdots & \\ \displaystyle\prod_{r=j-q+2}^{j} \mathcal{G}_r U(t_{j-q+1}^-) & \leq & \displaystyle\prod_{r=j-q+1}^{j} \mathcal{G}_r U(t_{j-q}^-) + \prod_{r=j-q+2}^{j} \mathcal{G}_r \int_{t_{j-q}}^{t_{j-q+1}} b(\ell)\mathcal{E}(\ell, t_{j-q+1}) w(\ell)\mathrm{d}\ell, \end{array} \tag{18}$$

where we used the fact that $q \geq 2$.

By summing all of the inequalities in (18), we obtain

$$\begin{array}{rcl} U(t_j^-) & \leq & \displaystyle\prod_{r=j-q+1}^{j} \mathcal{G}_r U(t_{j-q}^-) + \int_{t_{j-1}}^{t_j} b(\ell)\mathcal{E}(\ell, t_j) w(\ell)\mathrm{d}\ell + \mathcal{G}_j \int_{t_{j-2}}^{t_{j-1}} b(\ell)\mathcal{E}(\ell, t_{j-1}) w(\ell)\mathrm{d}\ell \\ & & + ... + \displaystyle\prod_{r=j-q+2}^{j} \mathcal{G}_r \int_{t_{j-q}}^{t_{j-q+1}} b(\ell)\mathcal{E}(\ell, t_{j-q+1}) w(\ell)\mathrm{d}\ell \; . \end{array} \tag{19}$$

Next note that for all $h \in \mathbb{Z}_{\geq 0}$ and $j \in \mathbb{Z}_{\geq 0}$ such that $j \geq h$, we have

$$\prod_{r=h}^{j} \mathcal{G}_r = \mu^{j-h+1} \mathcal{E}(t_{h-1}, t_j) \;\text{ and }\; \mathcal{E}(\ell, t_{j-1}) = \mathcal{E}(\ell, t_j)(\mathcal{E}(t_{j-1}, t_j))^{-1} \tag{20}$$

for all $\ell \in [0, t_{j-1}]$. We deduce that

$$\begin{array}{rcl} U(t_j^-) & \leq & \mu^q \mathcal{E}(t_{j-q}, t_j) U(t_{j-q}^-) + \displaystyle\int_{t_{j-1}}^{t_j} b(\ell)\mathcal{E}(\ell, t_j) w(\ell)\mathrm{d}\ell + \mu \int_{t_{j-2}}^{t_{j-1}} b(\ell)\mathcal{E}(\ell, t_j) w(\ell)\mathrm{d}\ell \\ & & + ... + \mu^{q-1} \displaystyle\int_{t_{j-q}}^{t_{j-q+1}} b(\ell)\mathcal{E}(\ell, t_j) w(\ell)\mathrm{d}\ell \; . \end{array} \tag{21}$$

Also, using (14) with the choice $t_* = t - T$ (and by letting $t \to t_{j-q}^-$ in (14)), we obtain

$$U(t_{j-q}^-) \le \mathcal{E}(t-T, t_{j-q}) U(t-T) + \int_{t-T}^{t_{j-q}} b(\ell)\mathcal{E}(\ell, t_{j-q}) w(\ell) \mathrm{d}\ell \,, \tag{22}$$

since $t - T \in [t_{j-q-1}, t_{j-q})$. Consequently,

$$\begin{aligned} U(t_j^-) \;&\le\; \mu^q \mathcal{E}(t_{j-q}, t_j)\mathcal{E}(t-T, t_{j-q})U(t-T) + \mu^q \mathcal{E}(t_{j-q}, t_j) \int_{t-T}^{t_{j-q}} b(\ell)\mathcal{E}(\ell, t_{j-q}) w(\ell)\mathrm{d}\ell \\ &\quad + \int_{t_{j-1}}^{t_j} b(\ell)\mathcal{E}(\ell, t_j)w(\ell)\mathrm{d}\ell + \mu \int_{t_{j-2}}^{t_{j-1}} b(\ell)\mathcal{E}(\ell, t_j)w(\ell)\mathrm{d}\ell \\ &\quad + ... + \mu^{q-1} \int_{t_{j-q}}^{t_{j-q+1}} b(\ell)\mathcal{E}(\ell, t_j)w(\ell)\mathrm{d}\ell \\ &= \; \mu^q \mathcal{E}(t-T, t_j)U(t-T) + \int_{t_{j-1}}^{t_j} b(\ell)\mathcal{E}(\ell, t_j)w(\ell)\mathrm{d}\ell + \mu \int_{t_{j-2}}^{t_{j-1}} b(\ell)\mathcal{E}(\ell, t_j)w(\ell)\mathrm{d}\ell \\ &\quad + ... + \mu^{q-1} \int_{t_{j-q}}^{t_{j-q+1}} b(\ell)\mathcal{E}(\ell, t_j)w(\ell)\mathrm{d}\ell + \mu^q \int_{t-T}^{t_{j-q}} b(\ell)\mathcal{E}(\ell, t_j)w(\ell)\mathrm{d}\ell \,. \end{aligned} \tag{23}$$

From the second inequality in (16), it follows that

$$\begin{aligned} U(t) \;&\le\; \mu^{q+1}\mathcal{E}(t-T, t)U(t-T) + \int_{t_j}^{t} b(\ell)\mathcal{E}(\ell, t)w(\ell)\mathrm{d}\ell + \mu \int_{t_{j-1}}^{t_j} b(\ell)\mathcal{E}(\ell, t)w(\ell)\mathrm{d}\ell \\ &\quad + \mu^2 \int_{t_{j-2}}^{t_{j-1}} b(\ell)\mathcal{E}(\ell, t)w(\ell)\mathrm{d}\ell + ... + \mu^q \int_{t_{j-q}}^{t_{j-q+1}} b(\ell)\mathcal{E}(\ell, t)w(\ell)\mathrm{d}\ell \\ &\quad + \mu^{q+1} \int_{t-T}^{t_{j-q}} b(\ell)\mathcal{E}(\ell, t)w(\ell)\mathrm{d}\ell \\ &\le\; \mu^{q+1} e^{-\lambda(T)} U(t-T) + \Lambda(t) \sup_{\ell \in [t-\tau_b - T, t]} W(x(\ell)) \,, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \Lambda(t) \;&=\; \int_{t_j}^{t} b(\ell)\mathcal{E}(\ell, t)\mathrm{d}\ell + \mu \int_{t_{j-1}}^{t_j} b(\ell)\mathcal{E}(\ell, t)\mathrm{d}\ell + \mu^2 \int_{t_{j-2}}^{t_{j-1}} b(\ell)\mathcal{E}(\ell, t)\mathrm{d}\ell + ... \\ &\quad + \mu^q \int_{t_{j-q}}^{t_{j-q+1}} b(\ell)\mathcal{E}(\ell, t)\mathrm{d}\ell + \mu^{q+1} \int_{t-T}^{t_{j-q}} b(\ell)\mathcal{E}(\ell, t)\mathrm{d}\ell \,, \end{aligned} \tag{25}$$

and where the last inequality is a consequence of (7) in Assumption 3. Since $\mu > 1$, we have

$$\begin{aligned} \Lambda(t) \;&=\; \int_{t-T}^{t} b(\ell) e^{\int_\ell^t a(s)\mathrm{d}s} \mathrm{d}\ell + (\mu - 1) \int_{t_{j-1}}^{t_j} b(\ell)\mathcal{E}(\ell, t)\mathrm{d}\ell + (\mu^2 - 1) \int_{t_{j-2}}^{t_{j-1}} b(\ell)\mathcal{E}(\ell, t)\mathrm{d}\ell + ... \\ &\quad + (\mu^q - 1) \int_{t_{j-q}}^{t_{j-q+1}} b(\ell)\mathcal{E}(\ell, t)\mathrm{d}\ell + (\mu^{q+1} - 1) \int_{t-T}^{t_{j-q}} b(\ell)\mathcal{E}(\ell, t)\mathrm{d}\ell \\ &\le\; \int_{t-T}^{t} b(\ell) e^{\int_\ell^t a(s)\mathrm{d}s} \mathrm{d}\ell + \left[\mu - 1 + \mu^2 - 1 + ... + \mu^{L(T)+1} - 1\right] \int_{t-T}^{t} b(\ell) e^{\int_\ell^t a(s)\mathrm{d}s} \mathrm{d}\ell \\ &\le\; \left[\frac{\mu^{L(T)+2} - 1}{\mu - 1} - L(T) - 1\right] \nu \end{aligned} \tag{26}$$

where the last inequality is a consequence of our choice of the bound ν from (8) and the formula for the geometric sum.

Using (26) to upper bound $\Lambda(t)$ in (24), recalling that $\mu > 1$ and $q \leq L(T)$, and using Assumption 2 to upper bound $\sup_{\ell\in[t-\tau_b-T,t]} W(x(\ell))$, we obtain

$$U(t,x_t) \leq \mu^{L(T)+1} e^{-\lambda(T)} U(t-T, x_{t-T}) + \left[\frac{\mu^{L(T)+2}-1}{\mu-1} - L(T) - 1\right] \nu\mu \sup_{\ell\in[t-\tau_b-T,t]} U(\ell, x_\ell) \tag{27}$$

for all $t \geq T + \tau_b$. We can therefore use Assumption 4 and [17, Lemma 1] (with the choices $T^* = T + \tau_b$ and $w(\ell) = U(\ell + T + \tau_b)$ in the lemma) to obtain an exponential decay estimate on $U(t, x_t)$ that is valid for all $t \geq T + \bar{\tau}$; see Appendix A below for a statement of [17, Lemma 1]. Also, we can use Gronwall's inequality (as we did to prove the forward completeness property at the beginning of the proof) to obtain an exponential decay estimate on $U(t, x_t)$ that is valid for all $t \in [0, T + \tau_b]$ (by applying Gronwall's inequality on successive intervals of the form $[t_j, t_{j+1})$). The desired global asymptotic stability estimate now follows from the upper and lower bounds for V in (3) in Assumption 1, so this concludes the proof. □

Remark 2.4. If the χ_i's from Assumption 1 are quadratic functions, then we can also deduce that $x(t)$ converges exponentially to zero.

3 Application to Linear Time Varying Systems

This section is devoted to application of the main result to a fundamental family of systems, namely, linear time-varying systems with time-varying lumped delays. For the sake of simplicity, we do not assume that these systems have switches, but we illustrate connections with switched systems. Throughout this section, we adopt the notation of Section 2.

3.1 Problem Definition and Preliminary Remarks

In this section we consider the system

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau(t)) \tag{28}$$

with x valued in $\mathbb{R}^n$ for any $n \in \mathbb{N}$ and $t \geq 0$. The following assumptions are in effect.

Assumption 5. *The functions A, B and τ are piecewise C^1, and there are constants a_b, b_b, and τ_b such that*

$$|A(t)| \leq a_b, \quad |B(t)| \leq b_b, \quad \text{and} \quad \tau(t) \leq \tau_b \tag{29}$$

hold for all $t \geq 0$.

Assumption 6. *There exist an integer $k \in \mathbb{N}$, a switching signal $\sigma : [0, +\infty) \to \{1, 2, \ldots, k\}$ whose switching instants $\{t_i\}$ admit positive constants $\mathcal{T}_1$ and $\mathcal{T}_2$ that satisfy (2) for all $i \in \mathbb{Z}_{\geq 0}$, a finite set $\{(A_i, B_i, \tau_i) : i \in \{1, 2, \ldots, k\}\}$ in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times [0, +\infty)$, and nonnegative constants $\bar{a}$, $\bar{b}$, and $\bar{\tau}$ such that*

$$|A(t) - A_{\sigma(t)}| \leq \bar{a}\,, \quad |B(t) - B_{\sigma(t)}| \leq \bar{b} \quad \text{and} \quad |\tau(t) - \tau_{\sigma(t)}| \leq \bar{\tau} \tag{30}$$

hold for all $t \geq 0$.

Assumption 7. *For each $i \in \{1, \ldots, k\}$, there exist an absolutely continuous function $V_i : C_{\text{in}} \to [0, +\infty)$, a real constant α_i, and a positive constant γ_i such that the time derivative of V_i along all trajectories of*

$$\dot{z}(t) = A_i z(t) + B_i z(t-\tau_i) + \delta_i(t) \tag{31}$$

for all continuous functions $\delta_i : [0, +\infty) \to \mathbb{R}^n$ satisfies

$$\dot{V}_i \leq \alpha_i V_i(z_t) + \gamma_i |\delta_i(t)|^2 \tag{32}$$

for almost all t, where the triples (A_i, B_i, τ_i) are from Assumption 6. Moreover, there are constants $\Psi_1 > 0$ and $\Psi_2 > 0$ such that for all $\phi \in C_{\rm in}$ and all $i \in \{1, ..., k\}$, we have

$$\Psi_1|\phi(0)|^2 \leq V_i(\phi) \leq \Psi_2 \sup_{\ell\in[-\tau_b,0]} |\phi(\ell)|^2 \, . \tag{33}$$

Also, the α_i's satisfy Assumption 3 with σ as defined in Assumption 6. Finally, there is a constant $\mu > 1$ such that $V_i(\phi) \leq \mu V_j(\phi)$ holds for all $(i, j) \in \{1, ..., m\}^2$ and for all $\phi \in C_{\rm in}$.

Assumption 8. *Assumption 4 is satisfied, where ν is a constant that satisfies (8) as before, and σ, μ, and the α_i's are defined as in Assumptions 6-7, and the β_i's defined by*

$$\beta_i = \frac{\gamma_i\left[\overline{a} + \overline{b} + b_b\overline{\tau}\left(a_b + b_b\right)\right]^2}{\Psi_1} \tag{34}$$

for $i = 1, 2, \ldots, k$.

Before stating our main result of this section, we point out several notable features of Assumptions 5-8.

Remark 3.1. In Assumption 6, we can always choose $\overline{a} = a_b + \max_i |A_i|$, $\overline{b} = b_b + \max_i |B_i|$ and $\overline{\tau} = \tau_b + \max_i \tau_i$. However, these choices will not in general get the best (non-conservative) values for (9) in Assumption 4. It is also possible to see the triplets (A_i, B_i, τ_i) as nominal LTI systems; time varying perturbations around these systems (along with switchings) determine the LTV system (28). Therefore, stability of the LTI switched systems together with certain conditions on the bounds defined in Assumption 6 will guarantee stability of the LTV system. For similar arguments see [32].

Remark 3.2. Assumption 6 covers the important special case where A, B, and τ all have some period $\mathcal{P} > 0$, by choosing $t_i = i\mathcal{P}/k$ and $(A_i, B_i, \tau_i) = (A(i\mathcal{P}/k), B(i\mathcal{P}/k), \tau(i\mathcal{P}/k))$ for all $i \in \{1, 2, \ldots, k\}$, in which case the system (31) is replaced by

$$\dot{z}(t) = A\left(\frac{i\mathcal{P}}{k}\right)z(t) + B\left(\frac{i\mathcal{P}}{k}\right)z\left(t - \tau\left(\frac{i\mathcal{P}}{k}\right)\right) + \delta_i(t), \tag{35}$$

and we take the switching signal σ defined by $\sigma(t) = i$ for all $t \in [i\mathcal{P}/k, (i+1)\mathcal{P}/k)$ and $i \in \{1, \ldots, k\}$. However, Assumption 6 is much more general, e.g., because we do not require periodicity of the coefficient matrices or periodicity of the delay, nor do we require the t_i's to be evenly spaced. Besides, even when A, B and τ are periodic, it is worth trying to find a switching sequence with less switching instants than $i\mathcal{P}/k$. All these comments motivate Appendix B.

Remark 3.3. For systems (31) classical results, notably using LMIs [8], make it possible to construct time-invariant quadratic LKFs. In Appendix C, we explain how LKFs are designed such that (32) and (33) are satisfied can be constructed. Also, we do not require the constants α_i to be non-positive. Thus, we can allow some of the systems (31) to be unstable.

Remark 3.4. Since we only require $\tau(t)$ to be piecewise C^1 and bounded (instead of being C^1), it follows that the case of sampling can be handled. Indeed, sampling can be represented as sawtooth shaped delay functions, whose values are 0 at the sampling times, see e.g. [7]. In fact, the present set-up allows non-uniform sampling and drifts in the sampling clocks. This further motivates our work, because controllers are typically implemented using sampled observations of their state arguments.

Remark 3.5. Even if the matrix $A(t)$ is Hurwitz for all $t \geq 0$, it does not follow that $\dot{y} = A(t)y$ is globally exponentially stable; see [22, Exercise 5.7.21]. The systems (31) do not in general admit a common Lyapunov functional. As noted in [14], the fact that a system $\dot{y} = Ay + By(t - h)$ is globally exponentially stable to 0 for any constant $h \in [0, \overline{h}]$ does not imply that any system $\dot{z} = Az + Bz(t - h(t))$ is globally exponentially stable to 0 when h is a function such that $h(t) \in [0, \overline{h}]$ for all $t \geq 0$. This motivates Assumption 8.

3.2 Stability of the LTV System

The main result of this section is the stability of the LTV system (28).

Theorem 2. *If the system (28) satisfies Assumptions 5-8, then it is globally exponentially stable to* 0.

Proof. For each $j \in \{1, 2, \dots, k\}$, we can rewrite (28) as

$$\dot{x}(t) = A_j x(t) + B_j x(t - \tau_j) + [A(t) - A_j]x(t) + [B(t) - B_j]x(t - \tau_j) + B(t)[x(t - \tau(t)) - x(t - \tau_j)] \ . \quad (36)$$

Consequently, the system (28) can be represented as the following system with switches:

$$\begin{array}{l} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}x(t - \tau_{\sigma(t)}) + \Lambda(t, \sigma(t), x_t), \text{ where} \\ \Lambda(t, j, x_t) = [A(t) - A_j]x(t) + [B(t) - B_j]x(t - \tau_j) + B(t) \int_{t-\tau_j}^{t-\tau(t)} [A(\ell)x(\ell) + B(\ell)x(\ell - \tau(\ell))] \, \mathrm{d}\ell \end{array} \quad (37)$$

for all $t \in [t_j, t_{j+1})$ and $j \in \mathbb{Z}_{\geq 0}$.

Assumption 6 implies that for all $t \geq 2\tau_b + \max_i \tau_i$, we have

$$\begin{array}{rcl} |\Lambda(t, \sigma(t), x_t)| & \leq & \overline{a}|x(t)| + \overline{b}|x(t - \tau_{\sigma(t)})| + b_b \left| \int_{t-\tau_{\sigma(t)}}^{t-\tau(t)} [a_b|x(\ell)| + b_b|x(\ell - \tau(\ell))|] \, \mathrm{d}\ell \right| \\ & \leq & \overline{a}|x(t)| + \overline{b}|x(t - \tau_{\sigma(t)})| + b_b\overline{\tau}\,(a_b + b_b) \sup_{\ell \in [t-2\tau_b - \max_i \tau_i, t]} |x(\ell)| \\ & \leq & \left[\overline{a} + \overline{b} + b_b\overline{\tau}\,(a_b + b_b)\right] \sup_{\ell \in [t-2\tau_b - \max_i \tau_i, t]} |x(\ell)| \ . \end{array} \quad (38)$$

We deduce from Assumption 7 that

$$\dot{V}_{\sigma(t)}(t) \leq \alpha_{\sigma(t)} V_{\sigma(t)}(x_t) + \frac{\gamma_{\sigma(t)} \left[\overline{a} + \overline{b} + b_b\overline{\tau}\,(a_b + b_b)\right]^2}{\Psi_1} \sup_{\ell \in [t-2\tau_b - \max_i \tau_i, t]} \Psi_1 |x(\ell)|^2 \quad (39)$$

for all $t \geq 2\tau_b + \max_i \tau_i$. Hence, Assumptions 1-4 are satisfied by (37), with $W(x) = \Psi_1|x|^2$. Then Theorem 1 allows us to conclude that this system is uniformly globally asymptotically stable. In fact, the proof of Theorem 1 provides an exponential decay estimate on $U(t, x_t) = V_{\sigma(t)}(t, x_t)$. By using the features of the V_i's, we deduce that (28) is also uniformly globally exponentially stable to 0. This completes the proof. □

3.3 State Feedback Design for LTV Systems with Input Delay

In this section, in the spirit of what is done in [32], we show how a control law with switchings (and a piecewise constant gain) can be fruitfully used to stabilize a time-varying system, even when they have no switchings.

Consider the system

$$\dot{x}(t) = A(t)x(t) + B_0(t)u(t - \tau(t)) \quad (40)$$

with x valued in $\mathbb{R}^n$ for any $n \in \mathbb{N}$, u valued in $\mathbb{R}^p$ for any $p \in \mathbb{N}$ and $t \geq 0$.

Similar to the previous section, the following assumptions are made.

Assumption 9. *The functions A, B_0 and τ are piecewise C^1, and there are constants a_b, b_b, and τ_b such that*

$$|A(t)| \leq a_b, \ \ |B_0(t)| \leq b_{0,b}, \ \ \textit{and} \ \ \tau(t) \leq \tau_b \quad (41)$$

hold for all $t \geq 0$.

Assumption 10. *There exist an integer $k \in \mathbb{N}$, a switching signal $\sigma : [0, +\infty) \to \{1, 2, \dots, k\}$ whose switching instants $\{t_i\}$ admit positive constants $\mathcal{T}_1$ and $\mathcal{T}_2$ that satisfy (2) for all $i \in \mathbb{Z}_{\geq 0}$, a finite set $\{(A_i, B_{0,i}, \tau_i) : i \in \{1, 2, \dots, k\}\}$ in $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times [0, +\infty)$, and nonnegative constants $\bar{a}$, $\bar{b}_0$, and $\bar{\tau}$ such that*

$$|A(t) - A_{\sigma(t)}| \leq \overline{a} \ , \quad |B_0(t) - B_{0,\sigma(t)}| \leq \overline{b}_0 \quad \textit{and} \quad |\tau(t) - \tau_{\sigma(t)}| \leq \overline{\tau} \quad (42)$$

hold for all $t \geq 0$.

Assumption 11. *For each $i \in \{1,\ldots,k\}$, there exist an absolutely continuous function $V_i : C_{\text{in}} \to [0,+\infty)$, a matrix K_i a real constant α_i, and a positive constant γ_i such that the time derivative of V_i along all trajectories of*

$$\dot{z}(t) = A_i z(t) + B_{0,i} K_i z(t-\tau_i) + \delta_i(t) \tag{43}$$

for all continuous functions $\delta_i : [0,+\infty) \to \mathbb{R}^n$ satisfies

$$\dot{V}_i \le \alpha_i V_i(z_t) + \gamma_i |\delta_i(t)|^2 \tag{44}$$

for almost all t, where the triples $(A_i, B_{0,i}, \tau_i)$ are from Assumption 10. Moreover, there are constants $\Psi_1 > 0$ and $\Psi_2 > 0$ such that for all $\phi \in C_{\text{in}}$ and all $i \in \{1,...,k\}$, we have

$$\Psi_1 |\phi(0)|^2 \le V_i(\phi) \le \Psi_2 \sup_{\ell \in [-\tau_b, 0]} |\phi(\ell)|^2 \,. \tag{45}$$

Also, the α_i's satisfy Assumption 3 with σ as defined in Assumption 10. Finally, there is a constant $\mu > 1$ such that $V_i(\phi) \le \mu V_j(\phi)$ holds for all $(i,j) \in \{1,...,m\}^2$ and for all $\phi \in C_{\text{in}}$.

Assumption 12. *Assumption 4 is satisfied, where ν is a constant that satisfies (8) as before, and σ, μ, and the α_i's are defined as in Assumptions 9-11, and the β_i's defined by*

$$\beta_i = \frac{\gamma_i \left[\overline{a} + \overline{b} + b_b \overline{\tau} \left(a_b + b_b\right)\right]^2}{\Psi_1} \tag{46}$$

with $\overline{b} = \overline{b}_0 \max_{i \in \{1,\ldots,k\}} \{|K_i|\}$, $b_b = b_{0,b} \max_{i \in \{1,\ldots,k\}} \{|K_i|\}$ for $i = 1, 2, \ldots, k$.

We are ready to state the following result:

Theorem 3. *If the system (28) satisfies Assumptions 9-12, then it is globally exponentially stabilized to 0 by the control law*

$$u(t-\tau(t)) = K_{\sigma(t)} x(t-\tau(t)). \tag{47}$$

Proof. The closed-loop system is

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau(t)) \tag{48}$$

with $B(t) = B_0(t) K_{\sigma(t)}$. Let $B_i = B_{0,i} K_i$

Then

$$|B(t) - B_{\sigma(t)}| = |B_0(t) K_{\sigma(t)} - B_{0,\sigma(t)} K_{\sigma(t)}| \le |B_0(t) - B_{0,\sigma(t)}| |K_{\sigma(t)}| \tag{49}$$

From Assumption 9, it follows that

$$|B(t) - B_{\sigma(t)}| \le \overline{b}_0 \max_{i \in \{1,\ldots,k\}} \{|K_i|\} \tag{50}$$

Also, using Assumption 9, we can prove that

$$|B(t)| \le b_{0,b} \max_{i \in \{1,\ldots,k\}} \{|K_i|\} \tag{51}$$

for all $t \ge 0$. Now, one can check that Assumptions 9-12 imply that the system (48) satisfies Assumptions 5-8. Then Theorem 2 allows us to conclude. □

Obviously, the key which makes the above result useful is the design of the feedback gains K_i, once $(A_i, B_{0,i}, \tau_i)$ are fixed. This can be done by taking into account stability robustness, considering the bounds defined in (42), and an optimality criterion as in [31].

4 Examples

In this section two examples are presented to illustrate the above results.

4.1 A Periodic System Example

Consider the switched linear system

$$\dot{x}(t) = A_{\sigma(t)}x(t) + Bx(t - \tau(t)) \tag{52}$$

with x valued in $\mathbb{R}^2$, $\tau(t) \in [0, \tau_b]$ for any bound $\tau_b > 0$,

$$A_1 = -I, \quad A_2 = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}, \text{ and } B = \begin{bmatrix} b & b \\ b & b \end{bmatrix} \tag{53}$$

with $b \geq 0$ being a constant, and $\sigma : [0, +\infty) \to \{1, 2\}$ being periodic of some period $\mathcal{P} > 0$ and defined by $\sigma(t) = 1$ when $t \in \left[0, \frac{\mathcal{P}}{2}\right)$ and $\sigma(t) = 2$ when $t \in \left[\frac{\mathcal{P}}{2}, \mathcal{P}\right)$. We provide conditions on the constants b and $\mathcal{P}$ ensuring that the origin of (52) is globally exponentially stable.

We use the positive definite quadratic functions $V_1(x) = \sqrt{3}|x|^2/2$ and $V_2(x) = x_1^2 + x_1x_2 + x_2^2$. Then for all $t \in \left[i\mathcal{P}, \frac{\mathcal{P}}{2} + i\mathcal{P}\right)$ and integers $i \in \mathbb{Z}_{\geq 0}$, we have

$$\begin{array}{rcl} \dot{V}_1(t) & = & -\sqrt{3}|x(t)|^2 + \sqrt{3}x(t)^\top Bx(t - \tau(t)) \\ & = & -2V_1(x(t)) + \sqrt{3}b[x_1(t) + x_2(t)][x_1(t - \tau(t)) + x_2(t - \tau(t))] \\ & \leq & -2V_1(x(t)) + \sqrt{3}b(|x(t)|^2 + |x(t - \tau(t))|^2) \\ & \leq & -2V_1(x(t)) + 2b(V_1(x(t)) + V_1(x(t - \tau(t)))) \\ & \leq & -2V_1(x(t)) + 4b \sup_{\ell \in [t - \overline{\tau}, t]} V_1(x(\ell)) \end{array} \tag{54}$$

and when $t \in \left[i\mathcal{P} + \frac{\mathcal{P}}{2}, i\mathcal{P} + \mathcal{P}\right)$, we instead have

$$\begin{array}{rcl} \dot{V}_2(t) & = & -(2x_1 + x_2)^2 - (x_1 + 2x_2)^2 + [2x_1 + x_2 \quad x_1 + 2x_2]Bx(t - \tau(t)) \\ & = & -(2x_1 + x_2)^2 - (x_1 + 2x_2)^2 + 3b[x_1 + x_2][x_1(t - \tau(t)) + x_2(t - \tau(t))] \\ & \leq & -2V_2(x(t)) + 4\sqrt{3}b \sup_{\ell \in [t - \overline{\tau}, t]} V_1(x(\ell)) , \end{array} \tag{55}$$

where we used the fact that $(c_1 + c_2)(c_3 + c_4) \leq c_1^2 + c_2^2 + c_3^2 + c_4^2$ holds for all nonnegative values c_1, c_2, c_3, and c_4. Also, $V_2(x) \leq \sqrt{3}V_1(x)$ and $V_1(x) \leq \sqrt{3}V_2(x)$ hold for all $x \in \mathbb{R}^2$. Then, with our general notation from Assumptions 1-2, we can choose $\mu = \sqrt{3}$, $\alpha_1 = -2$, $\alpha_2 = -2$, $\beta_1 = 4b$, and $\beta_2 = 4\sqrt{3}b$.

Also, choosing $T = 2\mathcal{P}$, we obtain

$$\int_{t-T}^{t} a(\ell)\mathrm{d}\ell = 2\int_0^{\mathcal{P}} a(\ell)\mathrm{d}\ell = -4\mathcal{P}$$

for all $t \geq T$. Thus, Assumption 3 also holds, with $\lambda(T) = 4\mathcal{P}$, so Assumptions 1-3 are satisfied with $W = V_1$. Also, we can choose $\nu = 2\sqrt{3}b(1 - e^{-4\mathcal{P}})$. Moreover, $L(2\mathcal{P}) = 4$.

From (9) in Assumption 4 and Theorem 1, we deduce that if

$$\mu^{L(T)+1}e^{-\lambda(T)} + \left[\frac{\mu^{L(T)+2} - 1}{\mu - 1} - L(T) - 1\right]\nu\mu = 9\sqrt{3}e^{-4\mathcal{P}} + \left[\frac{27\sqrt{3} - 1}{\sqrt{3} - 1} - 5\right]6b\left(1 - e^{-4\mathcal{P}}\right) < 1 \tag{56}$$

then the system (52) is globally asymptotically stable to 0. Thus, roughly speaking, this system is globally asymptotically stable to 0 if $\mathcal{P}$ is sufficiently large and b is sufficiently small. Notice that $\mathcal{P}$ being large means that the system is slowly varying, and that the condition (56) is independent of τ_b.

4.2 Switched State Feedback Design for a Periodic System

In this section, we illustrate Theorem 3 with the linear system with a pointwise time-varying delay

$$\dot{x}(t) = A(t)x(t) + B_0 u(t - \tau(t)) \tag{57}$$

where x is valued in $\mathbb{R}^2$,

$$B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{58}$$

with A and τ periodic of period $\mathcal{P} > 0$ and defined by $\tau(t) = \tau_s$ when $t \in [0, q\mathcal{P})$ and $\tau(t) = \tau_b > 0$ when $t \in [q\mathcal{P}, \mathcal{P})$, and where $\tau_s \in \left[0, \frac{1}{12}\right]$, $q \in (0,1)$ and $\tau_b > \frac{1}{12}$ are constants and

$$A(t) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{when} \quad t \in [0, q\mathcal{P}) \quad \text{and} \quad A(t) = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \quad \text{when} \quad t \in [q\mathcal{P}, \mathcal{P}) \tag{59}$$

.

Step 1. We start our control design by choosing the following switching sequence: $t_{2j} = j\mathcal{P}$, $t_{2j+1} = q\mathcal{P} + j\mathcal{P}$ and we define the switching signal σ by $\sigma(t) = 1$ when $t \in [t_{2j}, t_{2j+1})$ and $\sigma(t) = 2$ when $t \in [t_{2j+1}, t_{2(j+1)})$.

We select

$$u(t - \tau(t)) = K_{\sigma(t)} x(t - \tau(t)). \tag{60}$$

with

$$K_1 = \begin{bmatrix} 0 & -e^{-\tau_s} \end{bmatrix} \quad \text{and} \quad K_2 = 0 \tag{61}$$

Notice that we have $A(t) = A_{\sigma(t)}$ for all $t \geq 0$. Moreover the eigenvalues of $(A_1 + B_0 K_1)$ are $\{-1, -e^{-\tau_s}\}$, and those of $(A_2 + B_0 K_2) = A_2$ are $\pm j\sqrt{3}$. So, the feedback system in mode $\sigma(t) = 2$ is not asymptotically stable, yet with proper switching scheme the system can be stabilized. Below computations derive a condition on admissible switching scheme

Step 2. We study the stability properties of the system

$$\left\{ \begin{array}{rcl} \dot{z}_1(t) & = & -z_1 + \delta_{11}(t) \\ \dot{z}_2(t) & = & -e^{-\tau_1} z_2(t - \tau_s) + \delta_{12}(t) \end{array} \right. \tag{62}$$

with $\delta_1 = (\delta_{11}, \delta_{12})$. Let us introduce

$$S(z_{2,t}) = \int_{t-\tau_s}^{t} e^{2(m-t)} z_2(m)^2 dm + 6 \left[z_2(t) - \int_{t-\tau_s}^{t} e^{m-t} z_2(m) dm \right]^2 \tag{63}$$

Elementary calculations give that the derivative of $S(z_{2,t})$ along (62) satisfies:

$$\begin{array}{rcl} \dot{S}(t) & = & -2 \int_{t-\tau_s}^{t} e^{2(m-t)} z_2(m)^2 dm + z_2(t)^2 - e^{-2\tau_s} z_2(t-\tau_s)^2 - 12 \left[\int_{t-\tau_s}^{t} e^{m-t} z_2(m) dm - z_2(t) \right]^2 \\ & & +12 \left[z_2(t) - \int_{t-\tau_s}^{t} e^{m-t} z_2(m) dm \right] \left[-e^{-\tau_s} z_2(t-\tau_s) + \delta_{12}(t) + e^{-\tau_s} z_2(t-\tau_s) \right] \\ & \leq & -2 \int_{t-\tau_s}^{t} e^{2(m-t)} z_2(m)^2 dm + z_2(t)^2 - 12 \left[\int_{t-\tau_s}^{t} e^{m-t} z_2(m) dm - z_2(t) \right]^2 \\ & & +12 \left[z_2(t) - \int_{t-\tau_s}^{t} e^{m-t} z_2(m) dm \right] \delta_{12}(t) \end{array} \tag{64}$$

Now, observe that

$$\begin{array}{rcl} z_2(t)^2 & = & \left[\int_{t-\tau_s}^{t} e^{m-t} z_2(m) dm - z_2(t) - \int_{t-\tau_s}^{t} e^{m-t} z_2(m) dm \right]^2 \\ & \leq & \frac{9}{8} \left(\int_{t-\tau_s}^{t} e^{m-t} z_2(m) dm \right)^2 + 3 \left[z_2(t) - \int_{t-\tau_s}^{t} e^{m-t} z_2(m) dm \right]^2 \\ & \leq & \frac{9\tau_s}{8} \int_{t-\tau_s}^{t} e^{2(m-t)} z_2(m)^2 dm + 3 \left[z_2(t) - \int_{t-\tau_s}^{t} e^{m-t} z_2(m) dm \right]^2 \end{array} \tag{65}$$

where the last inequality is a consequence of Cauchy-Schwarz's inequality. An an immediate consequence, we have

$$\begin{array}{rcl}\dot{S}(t) & \leq & \left(-2+\frac{9\tau_s}{8}\right)\int_{t-\tau_s}^{t} e^{2(m-t)}z_2(m)^2dm - 9\left[\int_{t-\tau_s}^{t} e^{m-t}z_2(m)dm - z_2(t)\right]^2 \\ & & +12\left[z_2(t) - \int_{t-\tau_s}^{t} e^{m-t}z_2(m)dm\right]\delta_{12}(t)\end{array} \tag{66}$$

Since $\tau_s \in \left[0, \frac{1}{12}\right]$, it follows that

$$\begin{array}{rcl}\dot{S}(t) & \leq & -\int_{t-\tau_s}^{t} e^{2(m-t)}z_2(m)^2dm - 9\left[\int_{t-\tau_s}^{t} e^{m-t}z_2(m)dm - z_2(t)\right]^2 \\ & & +12\left[z_2(t) - \int_{t-\tau_s}^{t} e^{m-t}z_2(m)dm\right]\delta_{12}(t)\end{array} \tag{67}$$

Thus

$$\dot{S}(t) \quad \leq \quad -\int_{t-\tau_s}^{t} e^{2(m-t)}z_2(m)^2dm - 3\left[\int_{t-\tau_s}^{t} e^{m-t}z_2(m)dm - z_2(t)\right]^2 + 6\delta_{12}(t)^2 \tag{68}$$

Now, let

$$V_1(z_t) = S(z_{2,t}) + 3z_1(t)^2 \tag{69}$$

Then

$$\dot{V}_1(t) \quad \leq \quad -3z_1^2 - \int_{t-\tau_s}^{t} e^{2(m-t)}z_2(m)^2dm - 3\left[\int_{t-\tau_s}^{t} e^{m-t}z_2(m)dm - z_2(t)\right]^2 + 6|\delta_1(t)|^2 \tag{70}$$

It follows that

$$\dot{V}_1(t) \leq -\frac{1}{2}V_1(z_t) + 6|\delta_1(t)|^2 \tag{71}$$

Step 3. We study the instability property of the system

$$\left\{\begin{array}{rcl}\dot{z}_1(t) & = & z_1 + 2z_2 + \delta_{21}(t) \\ \dot{z}_2(t) & = & -2z_1 - z_2 + \delta_{22}(t)\end{array}\right. \tag{72}$$

Let us introduce the positive definite quadratic function

$$U_1(z) = z_1^2 + z_2^2 + z_1z_2 \tag{73}$$

Elementary calculations give that the derivative of $U_1(z)$ along (72) satisfies

$$\dot{U}_1(t) = (2z_1(t) + z_2(t))\delta_{21}(t) + (2z_2(t) + z_1(t))\delta_{22}(t) \tag{74}$$

Let

$$U_2(z_t) = U_1(z(t)) + \frac{1}{2}\int_{t-\tau_s}^{t} z_2(m)^2dm \tag{75}$$

Then

$$\dot{U}_2(t) \leq \frac{1}{2}z_2(t)^2 + (2z_1(t) + z_2(t))\delta_{21}(t) + (2z_2(t) + z_1(t))\delta_{22}(t) \tag{76}$$

Consequently,

$$\begin{array}{rcl}\dot{U}_2(t) & \leq & \frac{1}{2}z_2(t)^2 + \frac{1}{4}(2z_1(t) + z_2(t))^2 + \frac{1}{4}(2z_2(t) + z_1(t))^2 + |\delta_2(t)|^2 \\ & = & \frac{1}{4}[5z_1(t)^2 + 7z_2(t)^2 + 8z_1(t)z_2(t)] + |\delta_2(t)|^2 \\ & \leq & \frac{1}{4}[7U_1(z(t)) + z_1(t)z_2(t)] + |\delta_2(t)|^2 \\ & \leq & 2U_1(z(t)) + |\delta_2(t)|^2 \\ & \leq & 2U_2(z_t) + |\delta_2(t)|^2\end{array} \tag{77}$$

Now, observe that

$$V_1(z_t) = \int_{t-\tau_s}^{t} e^{2(m-t)}z_2(m)^2dm + 6\left[z_2(t) - \int_{t-\tau_s}^{t} e^{m-t}z_2(m)dm\right]^2 + 3z_1(t)^2 \tag{78}$$

Therefore

$$\begin{array}{rcl} V_1(z_t) & \le & \int_{t-\tau_s}^t e^{2(m-t)} z_2(m)^2 dm + 12\left[z_2(t)^2 + \left(\int_{t-\tau_s}^t e^{m-t} z_2(m) dm\right)^2\right] + 3z_1(t)^2 \\ & \le & \int_{t-\tau_s}^t z_2(m)^2 dm + 12\int_{t-\tau_s}^t e^{2(m-t)} dm \int_{t-\tau_s}^t z_2(m)^2 dm + 3z_1(t)^2 + 12 z_2(t)^2 \\ & \le & 6\int_{t-\tau_s}^t z_2(m)^2 dm + 3z_1(t)^2 + 12z_2(t)^2 \end{array} \tag{79}$$

Also,

$$\begin{array}{rcl} V_1(z_t) & = & \int_{t-\tau_s}^t e^{2(m-t)} z_2(m)^2 dm + 6\left[z_2(t)^2 + \left(\int_{t-\tau_s}^t e^{m-t} z_2(m) dm\right)^2 - 2z_2(t)\int_{t-\tau_s}^t e^{m-t} z_2(m) dm\right] \\ & & +3z_1(t)^2 \end{array} \tag{80}$$

Using $\left|2z_2(t)\int_{t-\tau_s}^t e^{m-t} z_2(m) dm\right| \le c z_2(t)^2 + \frac{1}{c}\left(\int_{t-\tau_s}^t e^{m-t} z_2(m) dm\right)^2$ for any $c \in (0,1)$, we deduce that

$$\begin{array}{rcl} V_1(z_t) & \ge & \int_{t-\tau_s}^t e^{2(m-t)} z_2(m)^2 dm + 6\left[(1-c)z_2(t)^2 + \left(1-\frac{1}{c}\right)\left(\int_{t-\tau_s}^t e^{m-t} z_2(m) dm\right)^2\right] + 3z_1(t)^2 \\ & = & \int_{t-\tau_s}^t e^{2(m-t)} z_2(m)^2 dm + \left(6-\frac{6}{c}\right)\left(\int_{t-\tau_s}^t e^{m-t} z_2(m) dm\right)^2 + 3z_1(t)^2 + 6(1-c)z_2(t)^2 \end{array} \tag{81}$$

Since $c \in (0,1)$,

$$\begin{array}{rcl} V_1(z_t) & \ge & \int_{t-\tau_s}^t e^{2(m-t)} z_2(m)^2 dm + \left(6-\frac{6}{c}\right)\int_{t-\tau_s}^t e^{2(m-t)} z_2(m)^2 dm + 3z_1(t)^2 + 6(1-c)z_2(t)^2 \\ & = & \left(7-\frac{6}{c}\right)\int_{t-\tau_s}^t e^{2(m-t)} z_2(m)^2 dm + 3z_1(t)^2 + 6(1-c)z_2(t)^2 \end{array} \tag{82}$$

Choosing $c = \frac{13}{14}$, we obtain

$$\begin{array}{rcl} V_1(z_t) & \ge & \frac{7}{13}\int_{t-\tau_s}^t e^{2(m-t)} z_2(m)^2 dm + 3z_1(t)^2 + \frac{3}{7}z_2(t)^2 \\ & \ge & \frac{7e^{-2\tau_s}}{13}\int_{t-\tau_s}^t z_2(m)^2 dm + \frac{3}{7}(z_1(t)^2 + z_2(t)^2) \\ & \ge & \frac{7e^{-\frac{1}{6}}}{13}\int_{t-\tau_s}^t z_2(m)^2 dm + \frac{3}{7}(z_1(t)^2 + z_2(t)^2) \end{array} \tag{83}$$

We deduce that

$$\begin{array}{rcl} U_2(z_t) & \le & \frac{3}{2}(z_1^2 + z_2^2) + \frac{1}{2}\int_{t-\tau_s}^t z_2(m)^2 dm \\ & \le & \frac{7}{2}V_1(z_t) \end{array} \tag{84}$$

Now, from (79), we deduce that

$$\begin{array}{rcl} V_1(z_t) & \le & 6\int_{t-\tau_s}^t z_2(m)^2 dm + 12z_1^2 + 12z_2(t)^2 \\ & \le & 6\int_{t-\tau_s}^t z_2(m)^2 dm + 24[z_1^2 + z_2^2 + z_1 z_2] \\ & \le & 24U_2(z_t) \end{array} \tag{85}$$

Let $V_2 = gU_2$, where $g > 0$ is to be selected. Then

$$\begin{array}{rcl} V_1(z_t) & \le & \frac{24}{g}V_2(z_t) \\ V_2(z_t) & \le & \frac{7}{2}gV_1(z_t) \end{array} \tag{86}$$

So we choose $g = \sqrt{\frac{48}{7}}$ and obtain

$$\begin{array}{rcl} V_1(z_t) & \le & \mu V_2(z_t) \\ V_2(z_t) & \le & \mu V_1(z_t) \end{array} \tag{87}$$

with $\mu = 2\sqrt{21}$.

Step 4. Now, let us choose $T = \mathcal{P}$. Now, observe that with the notation of Section 3.3, we have $\overline{a} = 0$, $\overline{b}_0 = 0$, $\overline{\tau} = 0$. It follows that $\beta_1 = \beta_2 = 0$ and $\nu = 0$. We have $\alpha_1 = -\frac{1}{2}$ (see (71)), $\alpha_2 = 2$ (see (77)). We deduce that we can take $\lambda(T) = \frac{1}{2}qT - 2(1-q)T$. In addition, notice that $L(\mathcal{P}) = 2$. Clearly, V_1 and V_2 satisfy (33).

From Assumption 4, we deduce from Theorem 3 that the following is a sufficient condition for stability:

$$(2\sqrt{21})^3 < e^{[\frac{5}{2}q-2]T}, \tag{88}$$

which is equivalent to

$$\frac{6\ln(2) + 3\ln(21)}{\mathcal{P}} + \frac{4}{5} < q \tag{89}$$

This inequality is in accordance with what the intuition suggests: the interval where the system is unstable should be sufficiently small and the switchings should be not be too frequent.

5 Conclusion

We presented a new technique making it possible to establish the globally asymptotic stability of switched time-varying systems with time-varying delays. The result is used for switched controller design for the stability of a class of LTV systems. This technique relies on a trajectory based approach. It applies to a wide family of systems for which no other technique of stability analysis was available in the literature. We applied it to linear systems and have shown that it provides a result which only requires the construction of simple Lyapunov Krasovskii functionals. Much remains to be done: in particular, control designs under certain optimality and robustness conditions and ISS properties based on the results of the present paper will be subjects of future works.

Appendices

A Key Lemma

We used the following key lemma:

Lemma A.1. Let $T^* > 0$ be a constant. Let a piecewise continuous function $\mathfrak{w} : [-T^*, +\infty) \to [0, +\infty)$ admit a sequence of real numbers v_i and positive constants $\overline{v}_a$ and $\overline{v}_b$ such that $v_0 = 0$, $v_{i+i} - v_i \in [\overline{v}_a, \overline{v}_b]$ for all $i \geq 0$, $\mathfrak{w}$ is continuous on each interval $[v_i, v_{i+1})$ for all $i \geq 0$, and $\mathfrak{w}(v_i^-)$ exists and is finite for each $i \in \mathbb{Z}_{\geq 0}$. Let $d : [0, +\infty) \to [0, +\infty)$ be any piecewise continuous function, and assume that there is a constant $\rho \in (0, 1)$ such that

$$\mathfrak{w}(t) \leq \rho |\mathfrak{w}|_{[t-T^*,t]} + d(t) \tag{90}$$

holds for all $t \geq 0$. Then

$$\mathfrak{w}(t) \leq |\mathfrak{w}|_{[-T^*,0]} e^{\frac{\ln(\rho)}{T^*}t} + \tfrac{1}{(1-\rho)^2} |d|_{[0,t]} \tag{91}$$

holds for all $t \geq 0$. □

The preceding lemma is [17, Lemma 1]. For its proof, see [17].

B Construction of a special switching signal

When we adopt the technique of stability analysis exposed in Section 3, it is useful to minimize the number of switching instants associated with the function σ to diminish the conservatism of the stability conditions. This motivates the construction of the switching signal σ that we propose in this section. For the sake of simplicity, we consider the case where only A is time-varying, but the generalization to the case where A, B

and τ are time-varying is straightforward.

Let $A : [0, +\infty) \to \mathbb{R}^{n\times n}$ be a continuous function such that $|A(t)| \leq a_b$ for all $t \geq 0$. Let $\Delta > 0$ be any constant.

The boundedness of A implies that one can establish the following fact:

There are $k_a \in \mathbb{N}$ constants matrices $A_i \in \mathbb{R}^{n\times n}$ such that for all $t \in [0, +\infty)$, there is $l \in \mathfrak{E}_{k_a}$ with $\mathfrak{E}_{k_a} = \{1, ..., k_a\}$ such that the inequality

$$|A(t) - A_l| < \Delta \tag{92}$$

holds.

Now, we assume that there is no $j \in \mathfrak{E}_{k_a}$ such that there is an instant $t_c \geq 0$ such that for all $t \geq t_c$, $|A(t) - A_j| < 2\Delta$. Indeed, this case is not interesting and trivial.

Now, let us define an increasing sequence of nonnegative real numbers s_i as follows.

1) Let $s_0 = 0$. Notice that the fact above implies that there is $i_0 \in \mathfrak{E}_{k_a}$ such that $|A(s_0) - A_{i_0}| < \Delta$.

2) Let $s_1 > 0$ be such that, for all $t \in [s_0, s_1)$, $|A(t) - A_{i_0}| < 2\Delta$ and $|A(s_1) - A_{i_0}| = 2\Delta$. Then, the fact above implies that there is $i_1 \in \mathfrak{E}_{k_a}$ such that $|A(s_1) - A_{i_1}| < \Delta$.

3) Now, we proceed by induction. Induction assumption: we assume we have s_0, s_1, ..., s_l, $l > 0$ with $0 = s_0 < s_1 < ... < s_l$ and $i_0 \in \mathfrak{E}_{k_a}$, ..., $i_l \in \mathfrak{E}_{k_a}$ such that for all $m \in \{1, ..., l\}$, $|A(s_m) - A_{i_{m-1}}| = 2\Delta$, $|A(s_m) - A_{i_m}| < \Delta$ and for all $t \in [s_{m-1}, s_m)$, $|A(t) - A_{i_{m-1}}| < 2\Delta$.

4) According to our induction assumption, $|A(s_l) - A_{i_l}| < \Delta$. Therefore there is $s_{l+1} > s_l$ such that, for all $t \in [s_l, s_{l+1})$, $|A(t) - A_{i_l}| < 2\Delta$ and $|A(s_{l+1}) - A_{i_l}| = 2\Delta$. Now, we let $i_{l+1} \in \mathfrak{E}_{k_a}$ be such that $|A(s_{l+1}) - A_{i_{l+1}}| < \Delta$. The induction assumption is satisfied at the step $i + 1$.

Now, we let σ be defined by $\sigma(t) = i_m$ when $t \in [s_m, s_{m+1})$. Then for all $t \geq 0$, the inequality

$$|A(t) - A_{\sigma(t)}| \leq 2\Delta \tag{93}$$

is satisfied.

Now, a question arises: does a constant $\mathcal{T}_1 > 0$ such that

$$\mathcal{T}_1 \leq s_{i+1} - s_i \tag{94}$$

for all $i \in \mathbb{N}$ exist ? We present a case for which the answer is yes. Let us assume that A is of class C^1 and there is a constant $d_a > 0$ such that for all $t \in [0, +\infty)$,

$$|\dot{A}(t)| \leq d_a \tag{95}$$

We have, for all $m > 1$,

$$\begin{array}{rcl} |A(s_{m-1}) - A(s_m)| & = & |A(s_{m-1}) - A_{i_{m-1}} + A_{i_{m-1}} - A(s_m)| \\ & \geq & -|A(s_{m-1}) - A_{i_{m-1}}| + |A_{i_{m-1}} - A(s_m)| \end{array} \tag{96}$$

Also, for all $q > 0$, $|A(s_q) - A_{i_{q-1}}| = 2\Delta$ and $|A(s_q) - A_{i_q}| < \Delta$. We deduce that

$$\begin{array}{rcl} -|A(s_{m-1}) - A_{i_{m-1}}| + |A_{i_{m-1}} - A(s_m)| & = & 2\Delta - |A(s_{m-1}) - A_{i_{m-1}}| \\ & \geq & 2\Delta - \Delta \end{array} \tag{97}$$

Combining (97) and (96), we obtain

$$|A(s_{m-1}) - A(s_m)| \geq \Delta \tag{98}$$

Now, from (95), we deduce that

$$|A(s_{m-1}) - A(s_m)| \leq d_a(s_m - s_{m-1}) \tag{99}$$

By combining (100) and (99), we deduce that

$$\Delta \leq d_a(s_m - s_{m-1}) \tag{100}$$

Thus, we can take $\mathcal{T}_1 = \frac{\Delta}{d_a}$.

If there is no constant $\mathcal{T}_2 > 0$ such that $s_{i+1} - s_i \leq \mathcal{T}_2$ for all $i \in \mathbb{N}$, then we can always replace s_i by a new sequence t_j which is composed of s_i and additional switchings that are introduced to get a sufficiently large constant $\mathcal{T}_2 > 0$ such that $t_{i+1} - t_i \leq \mathcal{T}_2$ holds for all $i \geq 0$. We can construct the t_i's as follows. We choose $t_0 = 0$ and $t_1 = s_1$, and we set $\mathcal{L}_i = \text{Floor}((s_i - s_{i-1})/s_1)$ for all integers $i \geq 2$. If $\mathcal{L}_2 \in \{0, 1\}$, then we choose $t_2 = s_2$, and so we do not introduce any new switching instants on $[0, s_2]$. On the other hand, if $\mathcal{L}_2 \geq 2$, then we introduce the switching times $t_{j+1} = s_1 + js_1$ for $j = 1, 2, \ldots, \mathcal{L}_2 - 1$, and we set $t_{\mathcal{L}_2+1} = s_2$. By the definition of the floor function, we have $t_j \in (s_1, s_2)$ for all $j \in \{2, 3, \ldots, \mathcal{L}_2\}$ and $s_2 - \mathcal{L}_2 s_1 \in [s_1, 2s_1]$, so all of the switching times t_i on $[0, s_2]$ will satisfy $\mathcal{T}_1 \leq t_i - t_{i+1} \leq 2s_1$. We continue in the same way, by introducing the switching times $s_{p-1} + js_1$ for $j = 1, 2, \ldots, \mathcal{L}_p - 1$ on the interval (s_{p-1}, s_p) if $p \geq 2$ is such that $\mathcal{L}_p \geq 2$, but not introducing any switching times in the interval (s_{p-1}, s_p) for integers $p \geq 2$ for which $\mathcal{L}_p \in \{0, 1\}$. This produces a sequence t_i such that $\mathcal{T}_1 \leq t_i - t_{i-1} \leq 2s_1$ for all integers $i \geq 1$.

Then when a switching sequence σ_1 is associated with the sequence s_i, we can choose the switching sequence σ_2 associated with t_i by setting $\sigma_1(t) = \sigma_2(t)$ for all $t \in [0, +\infty)$.

C Technical Result

Finding LKFs such that (32) and (33) hold is an easy task. We illustrate the key steps on an example involving two systems, this can be directly extended to any arbitrary number of systems.

Lemma C.1. Consider the systems

$$\dot{X}(t) = A_a X(t) + A_b X(t - T_a) \text{ and } \dot{Z}(t) = B_a Z(t) + B_b Z(t - T_b) \tag{101}$$

where X and Z are valued in $\mathbb{R}^n$ for any dimension n, and where T_b and T_a are any nonnegative values. Assume that the systems admit LKFs of the form

$$\begin{aligned} Q_a(X_t) &= X(t)^\top P_a X(t) + \int_{t-T_a}^{t} \theta_a(t-\ell) X(\ell)^\top R_a X(\ell) \mathrm{d}\ell \text{ and} \\ Q_b(Z_t) &= Z(t)^\top P_b Z(t) + \int_{t-T_b}^{t} \theta_b(t-\ell) Z(\ell)^\top R_b Z(\ell) \mathrm{d}\ell \end{aligned} \tag{102}$$

respectively, where $P_a \in \mathbb{R}^{n\times n}$, $P_b \in \mathbb{R}^{n\times n}$, $R_a \in \mathbb{R}^{n\times n}$ and $R_b \in \mathbb{R}^{n\times n}$ are symmetric positive definite matrices and θ_a and θ_b are bounded, continuous and nonnegative values, such that their time derivatives along the corresponding systems in (101) satisfy

$$\dot{Q}_a(t) \leq -|X(t)|^2 \quad \text{and} \quad \dot{Q}_b(t) \leq -|Z(t)|^2, \tag{103}$$

respectively. Then the functionals

$$Q_a^*(X_t) = Q_a(X_t) + \frac{1}{2}\int_{t-T_b}^{t} e^{-t+\ell}|X(\ell)|^2 \mathrm{d}\ell \quad \text{and} \quad Q_b^*(Z_t) = Q_a(Z_t) + \frac{1}{2}\int_{t-T_b}^{t} e^{-t+\ell}|Z(\ell)|^2 \mathrm{d}\ell \tag{104}$$

admit positive constants $\varrho_i > 0$ for $i = 1, 2, \ldots, 7$ such that

$$\dot{Q}_a^*(t) \leq -\varrho_1 Q_a^*(X_t) \text{ and } \dot{Q}_b^*(t) \leq -\varrho_2 Q_b^*(Z_t) \tag{105}$$

hold along all solutions of the corresponding systems. Moreover, we have that

$$\varrho_3|\phi(0)|^2 \leq Q_a^*(\phi) \leq \varrho_4 \sup_{\ell\in[-T_b,0]} |\phi(\ell)|^2 \qquad \varrho_5|\phi(0)|^2 \leq Q_b^*(\phi) \leq \varrho_6 \sup_{\ell\in[-T_b,0]} |\phi(\ell)|^2 \tag{106}$$

and

$$Q_a^*(\phi) \leq \varrho_7 Q_b^*(\phi) \text{ and } Q_b^*(\phi) \leq \varrho_7 Q_a^*(\phi) \tag{107}$$

hold for all $\phi \in C_{\text{in}}$. Thus ϱ_7 plays the role of μ appearing in Assumption 2.

Proof. The inequalities (106) are consequences of the definitions of Q_a and Q_b. Simple calculations give

$$\dot{Q}_a^*(t) \leq -\frac{1}{2}|X(t)|^2 - \frac{1}{2}\int_{t-T_b}^{t} e^{-t+\ell}|X(\ell)|^2 \mathrm{d}\ell \text{ and } \dot{Q}_b^*(t) \leq -\frac{1}{2}|Z(t)|^2 - \frac{1}{2}\int_{t-T_b}^{t} e^{-t+\ell}|Z(\ell)|^2 \mathrm{d}\ell\,. \tag{108}$$

We easily deduce that (105) (107) are satisfied. □

References

[1] Y. Ariba, F. Gouaisbaut, and K. Johansson. Stability interval for time-varying delay systems. In *Proceedings of the 49th IEEE Conference on Decision and Control*, Atlanta, GA, 2010, pp. 1017-1022.

[2] C. Briat, J-F. Lafay, and O. Sename, Delay-scheduled state-feedback design for time-delay systems with time-varying delays - a LPV approach. *Systems and Control Letters*, 58(9):664-671, 2009.

[3] C. Briat, O. Sename, and J-F. Lafay. Memory-resilient gain-scheduled state-feedback control of uncertain LTI/LPV systems with time-varying delays. *Systems and Control Letters*, 59(8):451-459, 2010

[4] S. Caliskan, H. Ozbay, and S-I. Niculescu. Dwell-time computation for stability of switched systems with time delays. *IET Control Theory and Applications*, 7(10):1422-1428, 2013.

[5] D. Cheng, J. Wang, and X. Hu. An extension of LaSalle's Invariance Principle and its application to multi-agent consensus. *IEEE Transactions on Automatic Control*, 53(7):1765-1770, 2008.

[6] E. Fridman. On robust stability of linear neutral systems with time-varying delays. *IMA Journal of Mathematical Control and Information*, 25(4):393-407, 2008.

[7] E. Fridman. *Introduction to Time-Delay Systems: Analysis and Control*. Birkhauser, 2014.

[8] K. Gu, V. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhauser, Boston, 2003.

[9] A. Halanay. *Differential Equations, Stability, Oscillations, Time Lags*. Academic Press, New York, 1966.

[10] J. Hespanha. Uniform stability of switched linear systems: extensions of La Salle's invariance principle. *IEEE Transactions on Automatic Control*, 49(4):470-482, 2004.

[11] A. Koru, A. Delibasi, and H. Ozbay. On dwell time minimization for switched delay systems: Free-weighting matrices method. In *Proceedings of the 53rd IEEE Conference on Decision and Control*, Los Angeles, CA, 2014, pp. 1978-1982.

[12] T-C. Lee and Z-P. Jiang. Uniform asymptotic stability of nonlinear switched systems with an application to mobile robots. *IEEE Transactions on Automatic Control*, 53(5):1235-1252, 2008.

[13] D. Liberzon. *Switching in Systems and Control*. Birkhauser, Berlin, Germany, 2003.

[14] J. Louisell. Delay differential systems with time-varying delay: new directions for stability theory. *Kybernetika*, 31(3):239-251, 2001.

[15] J. Mancilla-Aguilar and R. Garcia. On converse Lyapunov theorems for ISS and iISS switched nonlinear systems. *Systems and Control Letters*, 42(1):47-53, 2001.

[16] J. Mancilla-Aguilar and R. Garcia. An extension of LaSalle's invariance principle for switched systems. *Systems and Control Letters*, 55(5):376-384, 2006.

[17] F. Mazenc and M. Malisoff. Trajectory based approach for the stability analysis of nonlinear systems with time delays. *IEEE Transactions on Automatic Control*, 60(6):1716-1721, 2015.

[18] F. Mazenc and M. Malisoff. Reduction model approach for systems with a time-varying delay. In *Proceedings of the 54th IEEE Conference on Decision and Control*, Osaka, Japan, 2015, pp. 7723-7727.

[19] F. Mazenc and M. Malisoff. Extension of Razumikhin's theorem for time-varying systems with delay. In *Proceedings of the 2016 American Control Conference*, Boston, MA, 2016, pp. 84-88.

[20] F. Mazenc, M. Malisoff, and S-I. Niculescu, Stability analysis for systems with time-varying delay: trajectory based approach. In *Proceedings of the 54th IEEE Conference on Decision and Control*, Osaka, Japan, 2015, pp. 1811-1816.

[21] P. Pepe. Stabilization in the sample-and-hold sense of nonlinear retarded systems. *SIAM Journal on Control and Optimization*, 52(5):3053-3077, 2014.

[22] E. Sontag. *Mathematical Control Theory, Second Edition*. Springer, New York, 1998.

[23] Z. Sun and S. Ge. *Stability Theory of Switched Dynamical Systems*. Communications and Control Engineering Series. Springer, New York, 2011.

[24] X-M. Sun, X-F. Wang, F. Mazenc, and Y-E Wang. Delay-hybrid-dependent stability for systems with time-varying delay. *Submitted*.

[25] L. Vu and K. Morgansen. Stability of time-delay feedback switched linear systems. *IEEE Transactions on Automatic Control*, 55(10):2385-2390, 2010.

[26] Y-E. Wang, X-L. Sun, and F. Mazenc. Stability of switched nonlinear systems with delay and disturbance. *Automatica*, 69:78-86, 2016.

[27] M. Wu, Y. He, J-H. She, and G-P. Liu. Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica*, 40(8):1435-1439, 2004.

[28] G. Xie and L. Wang. Periodical stabilization of switched linear systems. *Journal of Computational and Applied Mathematics*, 181:176-187, 2005.

[29] D. Xie and Y. Wu. Stabilisation of time-delay switched systems with constrained switching signals and its applications in networked control systems. *IET Control Theory and Applications*, 4(10):2120-2128, 2010.

[30] P. Yan and H. Ozbay. Stability analysis of switched time-delay systems. *SIAM Journal on Control and Optimization*, 47(2):936-949, 2008.

[31] P. Yan, H. Özbay, and M. Şansal. Dwell Time Optimization in Switching Control of Parameter Varying Time Delay Systems. *Proc. of the 50th IEEE Conference on Decision and Control*, Orlando, Florida, USA, December 2011, pp. 4909–4914.

[32] P. Yan, H. Özbay, and M. Şansal. Robust Stabilization of Parameter Varying Time Delay Systems by Switched Controllers. *Applied and Computational Mathematics*, 13(1):31–45, 2014.

[33] H. Yang, V. Cocquempot, and B. Jiang. On stabilization of switched nonlinear systems with unstable modes. *Systems and Control Letters*, 58(10-11):703-708, 2009.

[34] B. Zhou and A. Egorov. Razumikhin and Krasovskii stability theorems for time-varying delay systems. *Automatica*, 71:281-291, 2016.