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Abstract In biomechanics, recent mathematical force-fatigue models allow to predict the muscular response to functional electric stimulations. This opens the road to compute optimized electrical pulses trains. Such pulses are modeled as Dirac pulses and this leads to a sampled-data control problem. The aim of this article is to establish the corresponding Pontryagin-type first-order necessary optimality conditions and to compute optimized pulses trains using numerical shooting techniques.

Keywords Muscle mechanics · Functional electrical stimulation · Sampled-data control problems · Pontryagin-type necessary optimality conditions · Geometric control

Mathematics Subject Classification (2000) 49J15 · 93C57 · 92B05

Introduction

Predicted muscular force response to functional electrical stimulation (FES) is utilized in biomechanics to muscular reeducation and in case of paralysis.

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The first part of the model (see [1–4]) concerns non fatigued model and comes from the Hill equation in the context of biochemistry and pharmacology [5]. More complete models (see [6, 7]) taking into account the muscle fatigue were obtained recently in the framework of model identification to produce a dynamics described by a set of five differential equations and in this article we shall use the so-called Ding et al. force-fatigue control model [7]. Functional electrical stimulation is modeled by Dirac pulses and are integrated using a linear dynamics to produce a sampled-data control problem where the control parameters are the interpulses and the amplitudes of the Dirac pulses. A cost function taking into account the force response and the fatigue variables will lead to a Mayer problem. The problem fits in the frame of optimal problems with sampled-data controls. Adaptation of the techniques of [8] allows to derive Pontryagin-type first-order necessary optimality conditions. This led to the computation of the optimized pulses train using a numerical shooting algorithm. It can be compared to a preliminary study where suboptimal controls are computed using a Model Predictive Control (MPC) [9].

The organization of this article is the following. In Section 1, the Ding et al. model is briefly recalled and the problem is set in the sampled-control data frame using a proper integration of the Dirac pulses. The problem is analyzed using geometric control methods [10, 11] to obtain a reduced problem where only one fatigue parameter is varying. The first part of Section 2 is to recall necessary optimality conditions obtained in [8] to reach a self-contained presentation and the Pontryagin necessary conditions are compared in the permanent case (where the set of controls is the set of bounding measurable mappings) versus the sampled-data control case. The second part of Section 2 contains the main contribution of this article that is Pontryagin-type optimality conditions applicable to the optimal muscular force-fatigue problem. In Section 3, they are implemented using a shooting algorithm.

1 Ding et al. model and geometric properties

The first part of the model concerns the *non fatigued* model describing the force response to a pulses train modeled as Dirac impulses at times $0 = t_1 < t_2 < \ldots < t_n$:

$$v(t) = \sum_{i=1}^{n} \delta(t - t_i)$$

where $I_i = t_i - t_{i-1}$ is the interpulse and convexifying leads to apply the input:

$$v(t) = \sum_{i=1}^{n} \eta_i \, \delta(t - t_i)$$

with the amplitudes $\eta_i \in [0, 1]$. Such pulses train feeds a first-order model linear dynamics to produce the response $q(t)$ to pulses train according to

$$\dot{q}(t) + \frac{q(t)}{\tau_c} = \sum_{i=1}^{n} \eta_i \delta(t - t_i). \tag{1}$$

Integrating (1) with $q(0) = 0$ gives for $t \geq t_n$

$$q(t) = e^{\frac{-t}{\tau_c}} \sum_{i=1}^{n} \int_0^t \eta_i \, e^{\frac{s}{\tau_c}} \delta(s - t_i) \, \mathrm{d}s = e^{\frac{-t}{\tau_c}} \sum_{i=1}^{n} \eta_i \, e^{\frac{t_i}{\tau_c}}.$$

Introduce the FES-input

$$E_s(t) = \frac{1}{\tau_c} \sum_{i=1}^{n} R_i \, \eta_i H(t - t_i) \exp\left(-\frac{t - t_i}{\tau_c}\right) \tag{2}$$

where R_i is a scaling factor associated to the *phenomenon of tetania* that is the memory effect of successive pulses and given by

$$R_i = \begin{cases} 1 \text{ for } i = 1 \\ 1 + (R_0 - 1) \exp\left(-\frac{t_i - t_{i-1}}{\tau_c}\right) \text{ for } i > 1 \end{cases}$$

and H denotes the Heaviside step function.
The FES-signal drives the evolution of Ca^{2+} concentration denoted by $C_N(t)$ according to

$$\frac{\mathrm{d}C_N}{\mathrm{d}t} + \frac{C_N}{\tau_c} = E_s(t). \tag{3}$$

Write E_s as

$$E_s(t) = \frac{1}{\tau_c} e^{\frac{-t}{\tau_c}} \sum_{i=1}^{n} v_i$$

and integrating the (resonant) linear system (3) leads for $t \geq t_n$ to

$$C_N(t) = \frac{1}{\tau_c} \sum_{i=1}^{n} R_i \eta_i H(t - t_i) \frac{t - t_i}{\tau_c} \exp\left(-\frac{t - t_i}{\tau_c}\right). \tag{4}$$

Definition 1.1 A sampled-data control system is a dynamics given by

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = f(t, x(t), u(t)), \, t \in [0, T]$$

such that there exists sampling times $0 = t_1 < t_2 < \ldots < t_n < T$ where u is constant on each interpulse I_i.

Clearly the control representation $E_s(t)$ will lead to a sampled-data control problem where the constant controls are: v_1, $v_1 + v_2$, $\ldots$, $v_1 + \ldots + v_n$. This gives a first geometric framework at our analysis.

Introducing

$$a = \frac{C_N}{K_m + C_N}, \quad b = \frac{1}{\tau_1 + \tau_2 \, a}, \tag{5}$$

the force response is described by the dynamic

$$\frac{\mathrm{d}F}{\mathrm{d}t} = A \, a - F \, b \tag{6}$$

and this defines the unfatigued model.
The complete *force-fatigue model* is obtained by describing the evolution of the parameters (A, K_m, τ_1) associated to the fatigue and corresponds to the linear dynamics

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{A - A_{rest}}{\tau_{fat}} + \alpha_A F \tag{7}$$

$$\frac{\mathrm{d}K_m}{\mathrm{d}t} = -\frac{K_m - K_{m,rest}}{\tau_{fat}} + \alpha_{K_m} F \tag{8}$$

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}t} = -\frac{\tau_1 - \tau_{1,rest}}{\tau_{fat}} + \alpha_{\tau_1} F. \tag{9}$$

Table 1 contains the definitions and details of the model.

Denoting $x = (x_1, x_2, x_3, x_4, x_5) = (C_N, F, A, K_m, \tau_1)$, the model can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = F_0(x(t)) + u(t)\, F_1(t) \tag{10}$$

where the sampled physical control data on $[0, t]$ is defined by:

$$(I_1, I_2, \ldots, I_n, \eta_1, \eta_2, \ldots, \eta_n)$$

with constraints:

$$I_{min} \leq I \leq I_{max}, 0 \leq \eta_i \leq 1$$

corresponding to interpulses bounds and Dirac amplitude convexification.
For the purpose of Lie algebraic analysis, the system (10) can be written as a time independent control system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \tilde{F}_0(x) + u(t)\tilde{F}_1(x) \tag{11}$$

adjoining the equation:

$$\begin{cases} \dot{x}_0 + 1/\tau_c\, x_0 = 0 \\ x_0(0) = 1. \end{cases}$$

Fixing the initial condition $x(0)$, such a control system defined on *input-output* mapping:

$$E^{x(0),t} : (\eta, I) \longmapsto x(t, x(0), \eta, I). \tag{12}$$

A first property concerns the dynamics of the force model.

Lemma 1.1 *Using a time reparameterization, the force model is integrable by quadratures.*

Proof One sets:

$$\mathrm{d}s = b(t)\,\mathrm{d}t,\ c = A\frac{a}{b}.$$

So that (6) becomes:

$$\frac{\mathrm{d}F(s)}{\mathrm{d}s} = c(s) - F(s),$$

and the dynamics can be integrated using Lagrange formulae.

Lemma 1.2 *Let for the fatigue model the force response mapping:* $(t, \eta, I) \mapsto F(t, \eta, I)$. *This mapping is smooth with respect to* I *and* η *and piecewise smooth with respect to* t.

Table 1 Margin settings

Symbol	Unit	Value	description
C_N	—	—	Normalized amount of Ca^{2+}-troponin complex
F	N	—	Force generated by muscle
t_i	ms	—	Time of the i^{th} pulse
n	—	—	Total number of the pulses before time t
i	—	—	Stimulation pulse index
τ_c	ms	20	Time constant that commands the rise and the decay of C_N
R_0	—	1.143	Term of the enhancement in C_N from successive stimuli
A	$\frac{N}{ms}$	—	Scaling factor for the force and the shortening velocity of muscle
τ_1	ms	—	Force decline time constant when strongly bound cross-bridges absent
τ_2	ms	124.4	Force decline time constant due to friction between actin and myosin
K_m	—	—	Sensitivity of strongly bound cross-bridges to C_N
A_{rest}	$\frac{N}{ms}$	3.009	Value of the parameter A when muscle is not fatigued
$K_{m,rest}$	—	0.103	Value of the parameter K_m when muscle is not fatigued
$\tau_{1,rest}$	ms	50.95	The value of the parameter τ_1 when muscle is not fatigued
α_A	$\frac{1}{ms^2}$	$-4.0\ 10^{-7}$	Coefficient for the force-model parameter A in the fatigue model
α_{K_m}	$\frac{1}{msN}$	$1.9\ 10^{-8}$	Coefficient for the force-model parameter K_m in the fatigue model
α_{τ_1}	$\frac{1}{N}$	$2.1\ 10^{-5}$	Coefficient for force-model parameter τ_1 in the fatigue model
τ_{fat}	s	127	Time constant controlling the recovery of $(A,\ K_m,\ \tau_1)$

1.1 **Geometric properties**

Consider a smooth single input control system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x) + u\,Y(x) \tag{13}$$

where $x \in \mathcal{O} \subset \mathbb{R}^n$, $u \in \Omega$ (control domain).
We denote $[U, V]$ the Lie bracket of two smooth vector fields on $\mathbb{R}^n$ computed with the convention:

$$[U,V](x) = \frac{\partial U}{\partial x}(x)V(x) - \frac{\partial V}{\partial x}(x)U(x). \tag{14}$$

A (local) frame on an open subset $\mathcal{O}'$ of $\mathcal{O}$ is a set of vector fields $\{X_i,\, i = 1,\ldots,n\}$ of the polysystem $D = \{X + uY;\, u \in \Omega :$ constant$\}$ such that $\text{span}\{X_i,\, i = 1,\ldots,n\} = \mathbb{R}^n$. The system is written as:

$$\begin{cases} \dot{x}_0 = \lambda_1 x_0 \\ \dot{x}_1 = \lambda_1 x_1 + x_0 u \\ \dot{x}_2 = x_3\, a(x_1,x_4) - x_2\, b(x_1,x_4,x_5) \\ \dot{x}_3 = (x_3 - \bar{x}_3)\lambda_2 + \alpha_A x_2 \\ \dot{x}_4 = (x_4 - \bar{x}_4)\lambda_2 + \alpha_{K_m} x_2 \\ \dot{x}_5 = (x_5 - \bar{x}_5)\lambda_2 + \alpha_{\tau_1} x_2 \end{cases} \tag{15}$$

with $\lambda_1 = \frac{-1}{\tau_c}$ and $\lambda_2 = \frac{-1}{\tau_{fat}}$. Hence:

$$X = \begin{pmatrix} \lambda_1 x_0 \\ \lambda_1 x_1 \\ x_3 a(x) - x_2 b(x) \\ (x_3 - \bar{x}_3)\lambda_2 + \alpha_A x_2 \\ (x_4 - \bar{x}_4)\lambda_2 + \alpha_{K_m} x_2 \\ (x_5 - \bar{x}_5)\lambda_2 + \alpha_{\tau_1} x_2 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ x_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Geometric properties are related to Lie brackets computations.

Proposition 1.1 *We have the following:*

(i) $[Y,[X,Y]](x) \in \mathit{span}\{Y,[X,Y]\}(x)$.
(ii) *Denoting* $adX \cdot Y = [X,Y]$, *one has:* $\mathit{span}\{ad^k X \cdot Y(\bar{x}),\, k = 0,\ldots,+\infty\} = \mathbb{R}^3$ *at* $\bar{x} = (x_0, 0, 0, \bar{x}_3, \bar{x}_4, \bar{x}_5)$, $x_0 \neq 0$ *and the linearized system at* $\bar{x}$ *is not controllable.*

Definition 1.2 The reduced force-fatigue model is obtained assuming K_m, τ_1 fixed and the dynamics is given by (3)-(6)-(7).

2 Pontryagin Maximum Principle with sampled-data control and necessary optimality conditions

One makes a brief introduction to the main techniques coming from the standard calculus of variations, the Maximum Principle in the permanent case and the standard computations in the sampled-data control case.

2.1 **Classical calculus of variations: Erdmann-Weierstrass condition [12]**

In the classical calculus of variations, one considers the problem:

$$\min_{x(\cdot)} \int_0^T L(x, \dot{x}, t)\,\mathrm{d}t \tag{16}$$

where L is a smooth Lagrangian. We denote by C the cost associated to (16) and if $x^*(\cdot)$ is an optimal curve with extremities $(t_0, x_0), (t_1, x_1)$ and $x(\cdot)$ be any curve with extremities $(t_0 + \delta t_0, x_0 + \delta x_0), (t_1 + \delta t_1, x_1 + \delta x_1)$ and denoting $h(t) = \bar{x}(t) - x^*(t)$ and $\Delta C = C(x) - C(x^*)$, one has

$$\Delta C = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}} \right)_{|x^*} h(t)\,\mathrm{d}t + \left[\frac{\partial L}{\partial \dot{x}}_{|x^*} \delta x \right]_{t_0}^{t_1} + \left[\left(L - \frac{\partial L}{\partial \dot{x}}\,\dot{x} \right)_{|x^*} \delta t \right]_{t_0}^{t_1} + o(\delta t, \delta x, h, \dot{h}).$$

Introducing $p = \frac{\partial L}{\partial \dot{x}}$ (adjoint vector), $H = p\,\dot{x} - L$ (Hamiltonian), if $x^*(\cdot)$ is an optimal curve, one has

$$\left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}} \right)_{|x^*} = 0$$

and at a corner (t_1, x_1) of a broken extremal one has

- *Variation* $\delta t = 0$:
$$\frac{\partial L}{\partial \dot{x}}(t_1^-) = \frac{\partial L}{\partial \dot{x}}(t_1^+),$$
that is $p(t_1^-) = p(t_1^+)$.
- *Variation* $\delta x = 0$:
$$\left(L - \frac{\partial L}{\partial \dot{x}}\,\dot{x} \right)(t_1^-) = \left(L - \frac{\partial L}{\partial \dot{x}}\,\dot{x} \right)(t_1^+),$$
that is $H(t_1^-) = H(t_1^+)$.

2.2 **A brief presentation of the Maximum Principle in the permanent case [13]**

Let $x^*(\cdot)$ be a reference optimal curve on $[0, T]$ associated to the control $u^*(\cdot)$ on $[0, T]$. Consider a needle variation (L^1-perturbation) defined by $u_\varepsilon = u_1$ on $t_1 - \varepsilon \leq t \leq t_1$ where $\varepsilon > 0$ and t_1 is a Lebesgue point and $u_\varepsilon(\cdot) = u^*(\cdot)$ elsewhere. The associated tangent vector is

$$w(t_1) = f(x^*(t_1), u_1) - f(x^*(t_1), u^*(t_1)) \tag{17}$$

which is transported for $t > t_1$ using the variational equation

$$\dot{w}(t) = \frac{\partial f}{\partial x}(x^*(t), u^*(t))\, w(t) \tag{18}$$

and $w(t_1)$ is given by the above formulae (17).

Define the adjoint equation

$$\dot{p}(t) = -\frac{\partial f}{\partial x}^{\intercal}(x^*(t), u^*(t))\, p(t) \tag{19}$$

and consider the Mayer problem

$$\min_{u(\cdot)} \varphi(x(T)).$$

One has

$$\frac{\varphi(x^*(T, u^*)) - \varphi(x(T, u_\varepsilon))}{\varepsilon} \geq 0$$

and taking the limit one gets

$$\langle \nabla\varphi(x^*(T, u^*)), w(T)\rangle \geq 0.$$

Integrating (18) we obtain

$$\langle \nabla\varphi(x^*(T, u^*)), \Phi(T, s)\, w(s)\rangle \geq 0$$

where $\Phi(T, s)$ is the transition matrix of (18).

Denote $p_T = -\nabla\varphi(x^*(T, u^*))$ and let $p(t)$ be the solution of (19) with $p(T) = p_T$, one gets

$$\langle p_T, w_T\rangle = \langle p(t), w(t)\rangle \leq 0, \quad \forall t \leq T.$$

Taking $t = t_1$ as a Lebesgue time, we obtain:

$$\langle p(t_1), f(x^*(t_1, u_1) - f(x^*(t_1), u^*(t_1))\rangle \leq 0 \tag{20}$$

for each constant u_1.

This is the standard maximization condition of Pontryagin maximum principle. Introducing $H(x, p, u) = \langle p, f(x, u)\rangle$, one gets

$$H(x^*, p, u^*) = \max_{u_1} H(x^*, p, u_1). \tag{21}$$

2.3 **Sampled-data control problem (classical case) [8]**

Let u^* be the reference control, $u^*+h = v$. Take the L^∞-variation $u_\varepsilon = u+\varepsilon\, h$. For such a L^∞-variation, the tangent vector satisfies the equation

$$\dot w(t) = A(t)\, w + B(t)\, h, \qquad w(0) = 0, \tag{22}$$

where $A(t) = \frac{\partial f}{\partial x}(x^*, u^*)$, $B(t) = \frac{\partial f}{\partial u}(x^*, u^*)$. For the Mayer problem, if u^* is optimal one has:

$$\frac{\varphi\,(x^*(T, u^*) - \varphi(x(T, u_\varepsilon))}{\varepsilon} \geq 0.$$

Taking the limit, one gets: $\dfrac{\partial\varphi}{\partial x}(x^*(T), w(T)) \geq 0$ and integrating (22) one obtains:

$$\langle \frac{\partial\varphi}{\partial x}(x^*(T)), \int_0^T \varPhi(T,s)B(s)(v-u^*)\,\mathrm{d}s\rangle \geq 0$$

where $\varPhi(T,s)$ is the transition matrix associated to $\dot x = A(t)x$. This gives, using the adjoint equation:

$$\langle \int_0^T p(s)\,\mathrm{d}s, \frac{\partial f}{\partial u}(x^*(T), u^*(s))\,(v(s)-u^*(s))\,\mathrm{d}s\rangle \leq 0.$$

Decomposing on each subinterval $[t_k, t_{k+1}]$, we get

$$\langle \int_{t_k}^{t_{k+1}} \frac{\partial H}{\partial u}(x^*(s), u_k^*(s), p^*(s))\,\mathrm{d}s\,, (v-u_k^*)\rangle \leq 0 \tag{23}$$

for each admissible constant v.

Remark 2.1 Compare the maximization conditions (21) in the permanent case and (23) in the sampled-data control case.

2.4 **Pontryagin type first-order necessary conditions for muscular force-fatigue model**

Our system obtained by integrating Dirac pulses and subject to the phenomenon of tetania doesn't fit exactly in the frame introduced in [8]. Hence derivative of the input-output mapping with respect to the parameters (η, I) has to be determined. Note it can be also computed in the frame of geometric control described in [14], calculations performed using Baker-Campbell-Hausdorff formulae and Lie brackets.

Let $T > 0$, $\tau_c > 0$ and $R_0 > 1$ be fixed. We introduce the three functions

$$\begin{aligned}
b : [0,T] &\to \mathbb{R}_+^* \\
t &\mapsto \tfrac{1}{\tau_c} e^{-\frac{t}{\tau_c}},\\
G :\ [0,T]^2 &\to \mathbb{R}_+^* \\
(t_1,t_2) &\mapsto (R_0-1)e^{t_1/\tau_c} + e^{t_2/\tau_c},\\
B : [0,T] &\to \mathbb{R}^q \\
t &\mapsto (\, b(t)\ 0\ \ldots\ 0\,)^\intercal
\end{aligned}$$

where $q \in \mathbb{N}^*$. In the sequel we fix $n \in \mathbb{N}^*$ and $I_{\min} > 0$.
We denote by $t_0 := -\infty$, $t_1 = 0$ and $t_{n+1} := T$. We consider the optimal control problem of Mayer form given by:

$$\begin{aligned}
&\min \varphi(x_{i_0}(T)) \\
&\quad \dot{x}(t) = f(x(t)) + B(t) \sum_{i=1}^{n} G(t_{i-1}, t_i) H(t - t_i) \eta_i, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in [0, T] \\
&\quad x(0) = \tilde{x}_0 \in \mathbb{R}^q, \\
&\quad (\eta_1, \ldots, \eta_n, t_2, \ldots, t_n) \in \mathbb{R}^{2n-1}, \\
&\quad \eta_i \in [0, 1], \quad \forall i = 1, \ldots, n, \\
&\quad 0 = t_1 < t_2 < \ldots < t_n < T, \\
&\quad t_i - t_{i-1} \geq I_{\min}, \quad \forall i = 2, \ldots, n,
\end{aligned}$$

where $\tilde{x}_0 \in \mathbb{R}^q$ is fixed (its first coordinate is zero), $x = (x_1, \ldots, x_q) \in \mathbb{R}^q$, $2 \leq i_0 \leq q$, $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable, and $f : \mathbb{R}^q \to \mathbb{R}^q$ is of class C^1.

2.4.1 Sensitivity analysis and variation vectors

For all $\delta = (\eta_1, \ldots, \eta_n, t_2, \ldots, t_n) \in \mathbb{R}^{2n-1}$, we denote by x_δ the unique solution to the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(x(t)) + B(t) \displaystyle\sum_{i=1}^{n} G(t_{i-1}, t_i) H(t - t_i) \eta_i \\ x(0) = x_0. \qquad\qquad\qquad\qquad \text{a.e. } t \in [0, T]. \end{cases}$$

In particular note that

$$x_{\delta,1}(t) = b(t) \sum_{i=1}^{n} G(t_{i-1}, t_i) H(t - t_i)(t - t_i) \eta_i,$$

for all $t \in [0, T]$.

Perturbation of η_j. Let $\delta = (\eta_1, \ldots, \eta_n, t_2, \ldots, t_n) \in \mathbb{R}^{2n-1}$ and let $j \in \{1, \ldots, n\}$. Let $\tilde{\eta_j} \in \mathbb{R}$. We denote by

$$\tilde{\delta} := (0, \ldots, 0, \tilde{\eta_j}, 0, \ldots, 0, 0, \ldots, 0) \in \mathbb{R}^{2n-1}$$

and we denote by

$$w(t) = \lim_{\alpha \to 0} \frac{x_{\delta + \alpha \tilde{\delta}}(t) - x_\delta(t)}{\alpha} \tag{24}$$

for all $t \in [t_j, T]$.
It can be proved that $w : [t_j, T] \to \mathbb{R}^q$ is the unique solution to the linear Cauchy problem given by

$$\begin{cases} \dot{w}(t) = \nabla f(x_\delta(t)) \times w(t) + B(t)\, G(t_{j-1}, t_j) \tilde{\eta_j}, \\ w(t_j) = 0_{\mathbb{R}^q}. \qquad\qquad\qquad\qquad \text{a.e. } t \in [t_j, T]. \end{cases}$$

Perturbation of t_j. Let $\delta = (\eta_1, \dots, \eta_n, t_2, \dots, t_n) \in \mathbb{R}^{2n-1}$ and let $j \in \{2, \dots, n\}$. Let $\tilde{t_j} \in \mathbb{R}$. We denote by

$$\tilde{\delta} := (0, \dots, 0, 0, \dots, 0, \tilde{t_j}, 0, \dots, 0) \in \mathbb{R}^{2n-1}$$

and we denote by

$$w(t) = \lim_{\alpha \to 0} \frac{x_{\delta+\alpha\tilde{\delta}}(t) - x_\delta(t)}{\alpha} \tag{25}$$

for all $t \in [t_j, T]$.
It can be proved that $w : [t_j, T] \to \mathbb{R}^q$ is the unique solution to the linear Cauchy problem given by

$$\begin{cases} \dot{w}(t) = \nabla f(x_\delta(t)) \times w(t) + B(t)B(-t_j)\Big(\eta_j + \\ \quad (R_0 - 1)H(t - t_{j+1})\eta_{j+1}\Big)\tilde{t_j}, \text{ a.e. } t \in [t_j, T], \\ w(t_j) = -B(t_j)G(t_{j-1}, t_j)\eta_j\tilde{t_j}. \end{cases}$$

In the above Cauchy problem, the product $B(t)B(-t_j)$ has to be understood component by component.

2.4.2 Main result

Let $\delta^* = (\eta_1^*, \dots, \eta_n^*, t_2^*, \dots, t_n^*) \in \mathbb{R}^{2n-1}$ be an optimal solution and let x^* stand for the corresponding optimal state. In particular note that $x^* = x_{\delta^*}$. Let us denote by

$$p_T := -\big(0, \dots, 0, \varphi'(x_{i_0}^*(T)), 0, \dots, 0\big)^\intercal \in \mathbb{R}^q,$$

and let $p : [0, T] \to \mathbb{R}^q$ be the adjoint vector defined as the unique solution to the backward linear Cauchy problem given by

$$\begin{cases} \dot{p}(t) = -\nabla f(x^*(t))^\top \times p(t), \quad \text{a.e. } t \in [0, T], \\ p(T) = p_T. \end{cases}$$

Denoting by $\Phi(\cdot, \cdot)$ the standard state transition matrix associated to the matrix function $t \mapsto \nabla f(x^*(t))$, recall that

$$p(t) = \Phi(T, t)^\top \times p_T,$$

for all $t \in [0, T]$.

Some terminology. Let us introduce the notion of *admissible perturbations.*

Definition 2.1 Let $j \in \{1, \dots, n\}$. We say that $\tilde{\eta_j} \in \mathbb{R}$ is an admissible perturbation of η_j^* if there exists $\alpha_0 > 0$ such that

$$\eta_j^* + \alpha\tilde{\eta_j} \in [0, 1]$$

for all $0 \leq \alpha \leq \alpha_0$.

Remark 2.2 Let $j \in \{1, \dots, n\}$. Three situations:

1. if $\eta_j^* = 0$, then the admissible perturbations of η_j^* are all elements of $\mathbb{R}_+$;
2. if $\eta_j^* = 1$, then the admissible perturbations of η_j^* are all elements of $\mathbb{R}_-$;
3. if $\eta_j^* \in (0,1)$, then the admissible perturbations of η_j^* are all elements of $\mathbb{R}$.

Definition 2.2 Let $j \in \{2, \dots, n-1\}$. We say that $\tilde{t_j} \in \mathbb{R}$ is an admissible perturbation of t_j^* if there exists $\alpha_0 > 0$ such that

$$(t_j^* + \alpha \tilde{t_j}) - t_{j-1}^* \geq I_{\min} \quad \text{and} \quad t_{j+1}^* - (t_j^* + \alpha \tilde{t_j}) \geq I_{\min},$$

for all $0 \leq \alpha \leq \alpha_0$.

Remark 2.3 Let $j \in \{2, \dots, n-1\}$. Four situations:

1. if $t_j^* - t_{j-1}^* = I_{\min} = t_{j+1}^* - t_j^*$, then the admissible perturbations of t_j^* are all elements of $\{0\}$;
2. if $t_j^* - t_{j-1}^* > I_{\min} = t_{j+1}^* - t_j^*$, then the admissible perturbations of t_j^* are all elements of $\mathbb{R}_-$;
3. if $t_j^* - t_{j-1}^* = I_{\min} < t_{j+1}^* - t_j^*$, then the admissible perturbations of t_j^* are all elements of $\mathbb{R}_+$;
4. if $t_j^* - t_{j-1}^* > I_{\min} < t_{j+1}^* - t_j^*$, then the admissible perturbations of t_j^* are all elements of $\mathbb{R}$.

Definition 2.3 Let $j = n$. We say that $\tilde{t_n} \in \mathbb{R}$ is an admissible perturbation of t_n^* if there exists $\alpha_0 > 0$ such that

$$(t_n^* + \alpha \tilde{t_n}) - t_{n-1}^* \geq I_{\min},$$

for all $0 \leq \alpha \leq \alpha_0$.

Remark 2.4 Let $j = n$. Two situations:

1. if $t_n^* - t_{n-1}^* = I_{\min}$, then the admissible perturbations of t_j^* are all elements of $\mathbb{R}_+$;
2. if $t_n^* - t_{n-1}^* > I_{\min}$, then the admissible perturbations of t_j^* are all elements of $\mathbb{R}$.

Admissible perturbation of $\tilde{\eta}_j$. Let $j \in \{1, \dots, n\}$ and let $\tilde{\eta}_j \in \mathbb{R}$ be an admissible perturbation of η_j^*. From optimality of x^* and from the previous sensitivity analysis, it is easy to see that

$$\langle p_T, w(T) \rangle_{\mathbb{R}^q} \leq 0,$$

where the variation vector w is defined by (24). Since

$$w(T) = \int_{t_j^*}^T \Phi(T,s) B(s) G(t_{j-1}^*, t_j^*) \tilde{\eta}_j \, \mathrm{d}s,$$

we get that

$$\left(\int_{t_j^*}^T p_1(s) b(s) ds \right) \tilde{\eta}_j \leq 0.$$

Admissible perturbation of $\tilde{t_j}$. Let $j \in \{2, \dots, n\}$ and let $\tilde{t_j} \in \mathbb{R}$ be an admissible perturbation of t_j^*. From optimality of x^* and from the previous sensitivity analysis, it is easy to see that

$$\langle p_T, w(T) \rangle_{\mathbb{R}^q} \leq 0,$$

where the variation vector w is defined by (25). Since

$$\begin{aligned} w(T) = \Bigg(& B(-t_j^*)\eta_{j+1}^* (R_0 - 1) \int_{t_{j+1}^*}^T \Phi(T,s)B(s)\,\mathrm{d}s \\ & + B(-t_j^*)\eta_j^* \int_{t_j^*}^T \Phi(T,s)B(s)\,\mathrm{d}s \\ & - \Phi(T,t_j^*)B(t_j^*)G(t_{j-1}^*, t_j^*)\eta_j^* \Bigg)\tilde{t_j} \end{aligned}$$

we get that

$$\begin{aligned} & \Bigg(b(-t_j^*)\eta_{j+1}^*(R_0 - 1) \int_{t_{j+1}^*}^T p_1(s)b(s)\,\mathrm{d}s + b(-t_j^*)\eta_j^* \\ & \int_{t_j^*}^T p_1(s)b(s)\,\mathrm{d}s - p_1(t_j^*)b(t_j^*)G(t_{j-1}^*, t_j^*)\eta_j^* \Bigg)\tilde{t_j} \leq 0. \end{aligned}$$

2.4.3 First-order necessary optimality conditions

The main result is stated as follows.

Theorem 2.1 *Let $\delta^* = (\eta_1^*, \dots, \eta_n^*, t_2^*, \dots, t_n^*) \in \mathbb{R}^{2n-1}$ be an optimal solution and let x^* stand for the corresponding optimal state. Then, the adjoint vector p defined as the unique solution to the backward linear Cauchy problem given by*

$$\begin{cases} \dot{p}(t) = -\nabla f(x^*(t))^\top \times p(t), & a.e.\ t \in [0,T], \\ p(T) = p_T, \end{cases}$$

is such that:

1. *the inequality*

$$\left(\int_{t_j^*}^T p_1(s)b(s)ds \right) \tilde{\eta}_j \leq 0,$$

   holds true for all $j = 1, \dots, n$ and for all admissible perturbation $\tilde{\eta}_j$ of η_j^*;

2. *the inequality*

$$\begin{aligned} \Bigg(& (R_0 - 1) \int_{t_{j+1}^*}^T p_1(s)b(s)\ ds\ b(-t_j^*)\eta_{j+1}^* \\ & + \int_{t_j^*}^T p_1(s)b(s)\ ds\ b(-t_j^*)\eta_j^* \\ & - p_1(t_j^*)b(t_j^*)G(t_{j-1}^*, t_j^*)\eta_j^* \Bigg)\tilde{t_j} \leq 0, \end{aligned} \tag{26}$$

holds true for all $j = 2, \ldots, n$ *and for all admissible perturbation* $\tilde{t_j}$ *of* t_j^**.*

3 Numerical results

The aim of this section is to implement the necessary optimality conditions to derive the optimal pulses using the 3-dimensional reduced force-fatigue model. The Mayer cost is taken as $\varphi(x(T)) = (F(T) - F_{\text{ref}})^2$. We fix the two remaining parameters to $K_{m,\text{rest}}$ and $\tau_{1,\text{rest}}$ respectively and the values of other constants are specified in Table 1. We fix $\eta_i = 1$, $i = 1, \ldots, 4$ (maximal amplitudes). Hence only the interpulses have to be calculated using (26). We relax the constraint $t_n^* - t_{n-1}^* > I_{\min}$ so that $\tilde{t_j}$ can be taken positive or negative and relation (26) becomes

$$\begin{aligned} \Theta_j :=& (R_0 - 1) \int_{t_{j+1}^*}^{T} p_1(s) b(s)\, ds\, b(-t_j^*) \eta_{j+1}^* \\ &+ \int_{t_j^*}^{T} p_1(s) b(s)\, ds\, b(-t_j^*) \eta_j^* - p_1(t_j^*) b(t_j^*) G(t_{j-1}^*, t_j^*) \eta_j^* = 0. \end{aligned} \tag{27}$$

This leads to a standard shooting algorithm where $p(0) = p_0$ has to be determined using the transversality condition: $p_T = -\nabla\varphi(x(T, x_0, p_0))$. The shooting equation is solved using a Newton-like algorithm where $x(T, x_0, p_0)$, $p(T, x_0, p_0)$ are computed using integrator. Numerical results are presented in Fig. 1 and Table 2 taking $F_{\text{ref}} = 250$ N and $T = 200$ ms and we assume the sampling times to be only t_2, t_2, t_4; hence taking $t_1 = 0 < t_2 < t_3 < t_4 < t_5 = T$ in our relation (27).

The result is neat, proving the ability of our method to compute the optimal pulse sequences to be compared to suboptimal policies using MPC methods used in previous calculations.

Θ_1 = 9e-24	Θ_2 = -2e-24	Θ_3 = 1e-25
$p_1(T)$ = 4.9e-28	$p_2(T)$ = 9.7e-24	$p_3(T)$ = 4.5e-26

Table 2 Final numerical values of the constraints Θ_j and the adjoint vector $p(T)$.

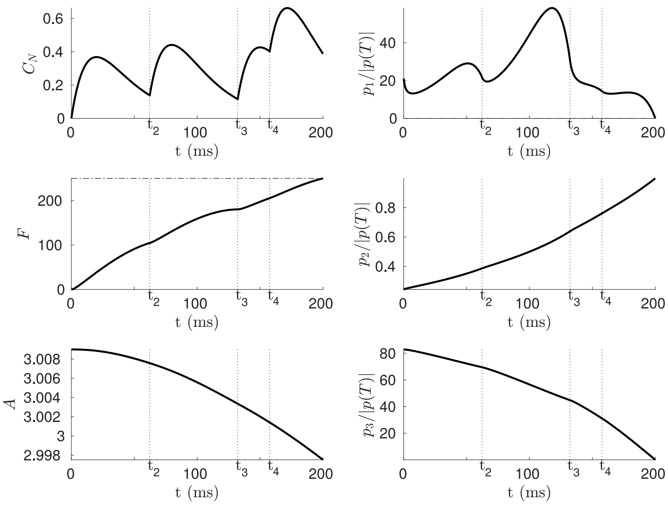

Fig. 1 Time evolution of the state and adjoint vectors for the problem of minimization of the cost $(F(T)-F_{\text{ref}})^2$. Optimal sampling times $t_2 < t_3 < t_4$ are computed using conditions (27) and the amplitudes η_i, $i=1,\dots 4$ of impulsions are fixed to 1.

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