Optimality conditions for the minimal time problem for Complementarity Systems

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Abstract: In this paper, we tackle the minimal time problem for systems with complementarity constraints. A special focus is then made on LCS, and we investigate a bang-bang property. Finally, for sake of completeness, the results are completed by an HJB equation, giving necessary and sufficient conditions of optimality.

Keywords: Linear optimal control, Minimum-time control, Complementarity problems.

1. INTRODUCTION

This paper focuses on finding optimality conditions for the minimal time problem:

$$T^* = \min \ T(x,u) \tag{1}$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = \phi(x(t),u(t)), \\ g(x(t),u(t)) \leq 0, \\ h(x(t),u(t)) = 0, \quad \text{a.e. on } [0,T(x,u)] \\ 0 \leq G(x(t),u(t)) \perp H(x(t),u(t)) \geq 0, \\ u(t) \in \mathcal{U} \\ (x(0),x(T(x,u))) = (x_0,x_f), \end{cases} \tag{2}$$

with $\phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$, $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, $G, H : [t_0,t_1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^l$, $\mathcal{U} \subset \mathbb{R}^{m_u}$, $x_0, x_f \in \mathbb{R}^n$ given. We suppose that F and ϕ are $\mathcal{L} \times \mathcal{B}$-measurable, where $\mathcal{L} \times \mathcal{B}$ denotes the σ-algebra of subsets of appropriate spaces generated by product sets $M \times N$, where M is a Lebesgue ($\mathcal{L}$) measurable subset in $\mathbb{R}$, and N is a Borel ($\mathcal{B}$) measurable subset in $\mathbb{R}^n \times \mathbb{R}^m$.
We denote a solution of this problem by (T^*, x^*, u^*). If we don't bound u with the constraint $\mathcal{U}$, then the solution will most probably be $T^* = 0$ (i.e. an impulsive control, provided (2) is given a mathematical meaning). The condition

$$0 \leq G \perp H \geq 0 \tag{3}$$

means that G,$H \geq 0$ and $\langle G, H \rangle = 0$ for almost all $t \in [0,T]$. Systems like (2), despite their simple look, gives rise to several challenging questions, mainly because conditions (3) introduce non-differentiability at switching points and non-convexity of the set of constraints. It provides a modeling paradigm for many problems, as Nash equilibrium games, hybrid engineering systems (Brogliato (2003)), contact mechanics or electrical circuits (Acary et al. (2011)). Several problems have already been tackled, let us mention observer-based control (Çamlibel et al. (2006), Heemels et al. (2011)) and Zeno behavior (Çamlibel and Schumacher (2001), Pang and Shen (2007), Shen (2014)).

Another difficulty comes from the fact that the constraints involve both the control and the state. These *mixed constraints* make the analysis even more challenging. For instance, deriving a maximum principle with wide applicability involves the use of non-smooth analysis, even in the case of smooth and/or convex constraints (see e.g. Clarke and De Pinho (2010)). Some first order conditions were given in Guo and Ye (2016) for systems with complementarity constraints, but they do not tackle this problem, since therein, the final time T^* is fixed beforehand. However, slight changes in the proof made for (Vinter, 2010, Theorem 8.7.1) allow us to derive first order conditions for (1)(2). This paper is organized as follows: first, the necessary conditions for (1)(2) will be derived. Then, we will show how these results are adapted to the problem of minimal time control for Linear Complementarity Systems (LCS), and some cases where the hypothesis are met. The focus will be made on proving a bang-bang property of the minimal-time control, and a Hamilton-Jacobi-Bellman characterization of the minimal time function. The results will be illustrated for a certain class of one dimensional LCS, deriving the analytical solution.

2. NECESSARY CONDITIONS

Since we have to compare different trajectories that are defined on different time-intervals, it should be understood that for $T > T^*$, a function w defined on $[0,T^*]$ is extended to $[0,T]$ by assuming constant extension: $w(t) = w(T^*)$ for all $t \in [T^*,T]$.

Definition 1. We refer to any absolutely continuous function as an arc, and to any measurable function on $[0,T^*]$ as a control. An admissible pair for (1)(2) is a pair of functions (x,u) on $[t_0,t_1]$ for which u is a control and x is an arc, that satisfy all the constraints in (2). The complementarity cone is defined by $\mathcal{C}^l = \{(v,w) \in \mathbb{R}^m \mid 0 \leq v \perp w \geq 0\}$. We define the set constraint by:

$$S=\{(x,u)\in\mathbb{R}^n\times\mathcal{U}:\\ g(x,u)\le 0,\ h(x,u)=0,\ (G(x,u),H(x,u))\in\mathcal{C}^l\}.$$

We say that the local error bound condition holds (for the constrained system representing S) at $(\overline{x},\overline{u})\in S$ if there exist positive constants τ and δ such that for all $(x,u)\in\mathcal{B}_\delta(\overline{x},\overline{u})$

$$\mathrm{dist}_S(x,u)\le\tau(\|\max\{0,g(x,u)\}\|+\|h(x,u)\|+\\ \mathrm{dist}_{\mathcal{C}^l}(G(x,u),H(x,u))).$$

For every given $t\in[t_0,t_1]$ and a positive constants R and ε, we define a neighbourhood of the point $(x^*(t),u^*(t))$ as:

$$S_*^{\varepsilon,R}(t)=\{(x,u)\in S:\|x-x^*(t)\|\le\varepsilon,\ \|u-u^*(t)\|\le R\}. \quad (4)$$

(x^*,u^*) is a local minimizer of radius R if there exists ε such that for every pair (x,u) admissible for (1)(2) such that: $\|x^*-x\|_{W^{1,1}}=\|x^*(0)-x(0)\|+\int_0^{\min\{T,T^*\}}\|\dot{x}^*(t)-\dot{x}(t)\|dt\le\varepsilon$, $\|u(t)-u^*(t)\|\le R$ a.e. $[0,\min\{T,T^*\}]$, we have $T^*=T(x^*,u^*)\le T(x,u)=T$.

We will have to do the following assumptions on the problem:

Assumption 2. (1) There exist measurable functions k_x^ϕ, k_u^ϕ, such that for almost every $t\in[0,T^*]$ and for every (x^1,u^1), $(x^2,u^2)\in S_*^{\varepsilon,R}(t)$, we have:

$$\|\phi(t,x^1,u^1)-\phi(t,x^2,u^2)\|\\ \le k_x^\phi(t)\|x^1-x^2\|+k_u^\phi(t)\|u^1-u^2\|. \quad (5)$$

(2) There exists a positive measurable function k_S such that for almost every $t\in[0,T^*]$, the bounded slope condition holds:

$$(x,u)\in S_*^{\varepsilon,R}(t),(\alpha,\beta)\in\mathcal{N}^P_{S(t)}(x,u)\\ \implies \|\alpha\|\le k_S(t)\|\beta\|. \quad (6)$$

(3) The functions k_x^ϕ, and $k_S k_u^\phi$ are integrable, and there exists a positive number η such that $R\ge\eta k_S(t)$ a.e. $t\in[0,T^*]$.

(4) ϕ is $\mathcal{L}\times\mathcal{B}$-measurable, g, h, G and H are strictly differentiable in variable (x,u).

Let (x^*,u^*) be a local minimizer of (1)(2). In order to compute the first order condition of this problem, one introduces a new state variable (as inspired by Vinter (2010)), absolutely continuous, which will represent time. For any $T>0$, denote this variable $\tau:[0,T]\to[0,T^*]$, and let us introduce $\tilde{x}=x\circ\tau$, $\tilde{u}=u\circ\tau$. Then, for any $t\in[0,T]$:

$$\dot{\tilde{x}}(t)=\dot{\tau}(t)\dot{x}(\tau(t))=\dot{\tau}(t)\phi(\tilde{x}(t),\tilde{u}(t)),\\ 0\le G(\tilde{x}(t),\tilde{u}(t))\perp H(\tilde{x}(t),\tilde{u}(t))\ge 0.$$

This method is at the core of the proof for the following Theorem. Define the sets $I_t^-(x,u)=\{i\in\overline{q}:g_i(x(t),u(t))<0\}$, $I_t^{+0}(x,u)=\{i:G_i(x(t),u(t))>0=H_i(x(t),u(t))\}$, $I_t^{0+}(x,u)=\{i:G_i(x(t),u(t))=0<H_i(x(t),u(t))\}$, $I_t^{00}(x,u)=\{i:G_i(x(t),u(t))=0=H_i(x(t),u(t))\}$, and for any $(\lambda^g,\lambda^h,\lambda^G,\lambda^H)\in\mathbb{R}^{p+q+2m}$, denote:

$$\Psi(x,u;\lambda^g,\lambda^h,\lambda^G,\lambda^H)=g(x,u)^\intercal\lambda^g+h(x,u)^\intercal\lambda^h\\ -G(x,u)^\intercal\lambda^G-H(x,u)^\intercal\lambda^H. \quad (7)$$

Theorem 3. Suppose Assumption 2 holds. Let (x^*,u^*) be a local minimizer for (1)(2). If for almost every $t\in[0,T^*]$ the local error bound condition for the system representing S holds at $(x^*(t),u^*(t))$ (see Definition 1), then (x^*,u^*) is W-stationary; i.e. there exist an arc $p:[0,T^*]\to\mathbb{R}^n$, a scalar $\lambda_0\in\{0,1\}$ and multipliers $\lambda^g:[0,T^*]\to\mathbb{R}^q$, $\lambda^h:[0,T^*]\to\mathbb{R}^p$, $\lambda^G,\lambda^H:[0,T^*]\to\mathbb{R}^m$ such that:

$$(\lambda_0,p(t))\ne 0\ \forall t\in[0,T^*], \quad (8a)$$

$$(\dot{p}(t),0)\in\partial^C\{\langle -p(t),\phi(\cdot,\cdot)\rangle\}(x^*(t),u^*(t))\\ +\nabla_{x,u}\Psi(x^*(t),u^*(t);\lambda^g(t),\lambda^h(t),\lambda^G(t),\lambda^H(t))\\ +\{0\}\times\mathcal{N}^C_{\mathcal{U}}(u^*(t)), \quad (8b)$$

$$\lambda^g(t)\ge 0,\ \lambda_i^g(t)=0,\ \forall i\in I_t^-(x^*,u^*), \quad (8c)$$

$$\lambda_i^G(t)=0,\ \forall i\in I_t^{+0}(x^*,u^*), \quad (8d)$$

$$\lambda_i^H(t)=0,\ \forall i\in I_t^{0+}(x^*,u^*), \quad (8e)$$

$$\lambda_0=\langle p(t),\phi(x^*(t),u^*(t))\rangle. \quad (8f)$$

Moreover, the Weierstrass condition of radius R holds: for almost every $t\in[0,T^*]$:

$$(x^*(t),u)\in S,\ \|u-u^*(t)\|<R\implies\\ \langle p(t),\phi(x^*(t),u)\rangle\le\langle p(t),\phi(x^*(t),u^*(t))\rangle. \quad (8g)$$

Proof. Let $a\in\mathcal{C}^2([0,T^*],\mathbb{R}^n)$ be such that $a(0)=x^*(0)$ and $\|a-x^*\|_{W^{1,1}}=\int_0^{T^*}\|\dot{a}(t)-\dot{x}^*(t)\|dt<\frac{\varepsilon}{2}$. Let $b:[0,T^*]\to\mathbb{R}^m$ be a function such that $\|u^*(t)-b(t)\|<\frac{R}{2}$. Let us introduce the following fixed-end time optimal control problem:

$$\min\ \tau(T^*) \quad (9)$$

$$\text{s.t.}\begin{cases}\dot{\tilde{x}}(t)=\alpha(t)\phi(\tilde{x}(t),\tilde{u}(t)),\\ \dot{\tau}(t)=\alpha(t),\\ \dot{z}(t)=\alpha(t)\|\phi(\tilde{x}(t),\tilde{u}(t))-\dot{a}(\tau(t))\|\\ g(\tilde{x}(t),\tilde{u}(t))\le 0,\\ h(\tilde{x}(t),\tilde{u}(t))=0, & \text{a.e. on } [0,T^*]\\ 0\le G(\tilde{x}(t),\tilde{u}(t))\perp H(\tilde{x}(t),\tilde{u}(t))\ge 0,\\ \alpha(t)\in\left[\frac{1}{2},\frac{3}{2}\right],\\ \tilde{u}(t)\in\mathcal{U},\\ \|\tilde{u}(t)-b(\tau(t))\|\le\frac{R}{2},\\ (\tilde{x}(0),\tilde{x}(T^*))=(x_0,x_f),\\ \tau(0)=0,\ |z(0)-z(T^*)|\le\frac{\varepsilon}{2}.\end{cases} \quad (10)$$

Denote by $(\tilde{x},\tau,z,\tilde{u},\tilde{v},\alpha)$ an admissible trajectory for (9)(10), where $(\tilde{x},\tau,z)$ are state variables and $(\tilde{u},\tilde{v},\alpha)$ are controls. We claim that a minimizer for this problem is

$$\left(x^*,\ \tau^*:t\mapsto t,\ z:t\mapsto\int_0^t\|\dot{x}^*(s)-\dot{a}(s)\|ds,\ u^*,\ v^*,\ \alpha^*\equiv 1\right) \quad (11)$$

with minimal cost $\tau^*(T^*)=T^*$. To prove this, let us assume that another admissible trajectory $(\tilde{x},\tau,z,\tilde{u},\tilde{v},\alpha)$ has a lower cost $T=\tau(T^*)<T^*$. Therefore, $\tau(0)=0$, $\tau(t)=\int_0^t\alpha(s)ds$, and since $\alpha>0$, τ is a continuous strictly increasing function from $[0,T^*]$ to $[0,T]$. Hence it admits an inverse τ^{-1}. Define on $[0,T]$:

$$x=\tilde{x}\circ\tau^{-1},u=\tilde{u}\circ\tau^{-1}$$

and extend these functions to $[0,T^*]$ by assuming that $x(t)=x(T)$ for all $t\in[T,T^*]$ (the same goes for u). Obviously, (x,u) is an admissible trajectory for (1)(2), with minimal time T. Also, it is in the neighborhood of (x^*,u^*), since for almost all $t\in[0,T]$,

$$\begin{aligned}\|u^*(t)-u(t)\| &\le \|u^*(t)-b(t)\| + \|u(t)-b(t)\| \\ &\le \frac{R}{2} + \|\tilde{u}(\sigma) - b(\tau(\sigma))\| \text{ where } t=\tau(\sigma) \\ &\le R\end{aligned}$$

and:

$$\begin{aligned}\|x-x^*\|_{W^{1,1}} &= \int_0^T \|\dot{x}(t)-\dot{x}^*(t)\|dt \\ &\le \int_0^T \|\dot{x}(t)-\dot{a}(t)\|dt + \int_0^T \|\dot{x}^*(t)-\dot{a}(t)\|dt \\ &\le \int_0^T \|\phi(x(t),u(t))-\dot{a}(t)\|dt + \frac{\varepsilon}{2} \\ &\le \int_0^{T^*} \|\phi(\tilde{x}(\sigma),\tilde{u}(\sigma))-\dot{a}(\tau(\sigma))\|\dot{\tau}(\sigma)d\sigma + \frac{\varepsilon}{2} \\ &\le \int_0^{T^*} \|\phi(\tilde{x}(\sigma),\tilde{u}(\sigma))-\dot{a}(\tau(\sigma))\|\alpha(\sigma)d\sigma + \frac{\varepsilon}{2} \\ &\le z(T^*)-z(0)+\frac{\varepsilon}{2} \\ &\le \varepsilon.\end{aligned}$$

Therefore, since they are in the same neighbourhood, the two trajectories can be compared. Since (x^*, u^*) is supposed to be a local minimizer for (1)(2), we should have $T^* \le T$. This is a contradiction, so the claim that (11) is the minimizer was right.

Remark that since we supposed that Assumption 2 is verified, the same assumptions adapted for problem (9)(10) are also valid. Therefore, the results of (Guo and Ye, 2016, Theorem 3.2) for (9)(10) state that there exists an arc $p : [0,T^*] \to \mathbb{R}^n$, a scalar $\lambda_0 \in \{0,1\}$ and multipliers $\lambda^g : [0,T^*] \to \mathbb{R}^q$, $\lambda^h : [0,T^*] \to \mathbb{R}^p$, $\lambda^G, \lambda^H : [0,T^*] \to \mathbb{R}^m$ such that (8a)-(8e) hold, along with the Weierstrass condition (8g).

Notice that since $z^*(T^*) - z^*(0) < \frac{\varepsilon}{2}$ and $\|u^*(t) - b(t)\| < \frac{R}{2}$ (the constraints are inactive), these inequalities do not appear in the first order conditions (the normal cone associated with these constraints reduces to $\{0\}$). Also, one could argue that there should be an adjoint state associated with z, but simple calculations show that it is identically 0.

Moreover, we should have another arc p_τ associated with τ, but it must comply with $\dot{p_\tau} \equiv 0$, $p_\tau(T^*) = -\lambda_0$, such that $p_\tau \equiv -\lambda_0$. Also, the stationary inclusion associated with α leads to $0 \in -\langle p, \phi(x^*(t), u^*(t))\rangle - p_\tau(t) + \mathcal{N}^C_{[\frac{1}{2},\frac{3}{2}]}(\alpha(t))$. But since $\alpha \equiv 1$ and $1 \in \left]\frac{1}{2},\frac{3}{2}\right[$, $\mathcal{N}^C_{[\frac{1}{2},\frac{3}{2}]}(\alpha(t)) = \{0\}$ for almost every $t \in [0,T^*]$, and so, it yields (8f).

3. APPLICATION TO LCS

3.1 Sufficient condition for the bounded slope condition

These results still rely on assumptions, among which the bounded slope condition is a stringent, non-intuitive, and hard to verify condition. A sufficient condition for the bounded slope condition to hold is given by (Guo and Ye, 2016, Proposition 3.7). We give some cases for which this condition holds when the underlying system is an LCS:

$$T^* = \min\ T(x,u,v) \tag{12}$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = Ax(t) + Bv(t) + Fu(t), \\ 0 \le v(t) \perp Cx(t) + Dv(t) + Eu(t) \ge 0, \\ u(t) \in \mathcal{U}, \quad \text{a.e. on } [0,T^*] \\ (x(0), x(T^*)) = (x_0, x_f), \end{cases} \tag{13}$$

where $A \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{m\times m}$, $E \in \mathbb{R}^{m\times m_u}$, $B \in \mathbb{R}^{n\times m}$, $F \in \mathbb{R}^{n\times m_u}$, $C \in \mathbb{R}^{m\times n}$, $\mathcal{U} \subseteq \mathbb{R}^{m_u}$. However, a direct application of (Guo and Ye, 2016, Theorem 3.5(b)) proves the following proposition.

Proposition 4. Suppose Assumption 2 for the problem (12)(13) holds. Suppose also that $\mathcal{U}$ is a union of finitely many polyhedral sets. Let (x^*, u^*, v^*) be a local minimizer for (12)(13). Then, (x^*, u^*, v^*) is M-stationary, meaning it is W-stationary with arc p, and moreover, there exist measurable functions $\eta^G, \eta^H : [0,T^*] \to \mathbb{R}^m$ such that:

$$\begin{gathered} 0 = B^\intercal p + D^\intercal \eta^H + \eta^G, \\ 0 \in -F^\intercal p - E^\intercal \eta^H + \mathcal{N}^C_{\mathcal{U}}(u^*(t)), \\ \eta^G_i(t) = 0,\ \forall i \in I^{+0}_t(x^*,u^*,v^*), \\ \eta^H_i(t) = 0,\ \forall i \in I^{0+}_t(x^*,u^*,v^*), \\ \eta^G_i \eta^H_i = 0 \text{ or } \eta^G_i > 0,\ \eta^H_i > 0,\ \forall i \in I^{00}_t(x^*,u^*,v^*). \end{gathered}$$

Since the system is linear, (Guo and Ye, 2016, Proposition 2.3) asserts that the local error bound condition holds at every admissible point. There is one case for which one can check that the bounded slope condition hold: when $\mathcal{U} = \mathbb{R}^{m_u}$, as proved in Vieira et al. (2018). However, this case of unbounded $\mathcal{U}$ is rather unrealistic, since it could lead to $T^* = 0$ (in the sense that the target x_f can be reached from x_0 given any positive time $T^* > 0$; see for instance Lohéac et al. (2018) for an example). When one attempts to add a constraint $\mathcal{U}$ to the previous proof, (Guo and Ye, 2016, Proposition 3.7) adds a normal cone that prevents checking the inequality, unless one supposes that the optimal trajectory is inside an R-neighbourhood which lies in the interior of $\mathcal{U}$. Nonetheless, there are two cases when (13) verifies the bounded slope condition, even with constraints on u.

Proposition 5. Suppose either $C = 0$, or D is a diagonal matrix with positive entries. Then the bounded slope condition for (13) holds.

Proof. The case when $C = 0$ is obvious, when one applies directly Proposition (Guo and Ye, 2016, Proposition 3.7). Assume $D = \text{diag}(d_1, ..., d_m)$, where the $d_i > 0$, $i \in \overline{m}$, are the diagonal entries of D, and diag means that D is a diagonal matrix built with these entries. First of all, remark that:

$$\forall \lambda^H \in \mathbb{R}^m,\ \|C^\intercal \lambda^H\| \le \|C^\intercal D^{-1}\| \|D\lambda^H\|.$$

Now, for $t \in [0,T^*]$ and $(x,u) \in S^{\varepsilon,R}_*(t)$, take λ^G and λ^H in $\mathbb{R}^m$ such that:

$$\begin{gathered} \lambda^G_i = 0,\ \forall i \in I^{+0}_t(x,u),\ \lambda^H_i = 0,\ \forall i \in I^{0+}_t(x,u), \\ \lambda^G_i > 0,\ \lambda^H_i > 0, \text{ or } \lambda^G_i\lambda^H_i = 0,\ \forall i \in I^{00}_t(x,u). \end{gathered}$$

It yields:

$$\|D\lambda^H\|^2 = \sum_{i\in I^{+0}_t(x,u)} (D_{ii}\lambda^H_i)^2 + \sum_{i\in I^{0+}_t(x,u)} (D_{ii}\lambda^H_i)^2 + \sum_{i\in I^{00}_t(x,u)} (D_{ii}\lambda^H_i)^2 \tag{14}$$

One can easily see that:

$$\sum_{i\in I_t^{+0}(x,u)} (D_{ii}\lambda_i^H)^2 = \sum_{i\in I_t^{+0}(x,u)} (D_{ii}\lambda_i^H+\lambda_i^G)^2,$$

$$0 = \sum_{i\in I_t^{0+}(x,u)} (D_{ii}\lambda_i^H)^2 \leq \sum_{i\in I_t^{0+}(x,u)} (D_{ii}\lambda_i^H+\lambda_i^G)^2,$$

and, $\forall i \in I_t^{00}(x,u)$,

$(D_{ii}\lambda_i^H)^2 \leq (D_{ii}\lambda_i^H)^2+(\lambda_i^G)^2+2D_{ii}\lambda_i^H\lambda_i^G = (D_{ii}\lambda_i^H+\lambda_i^G)^2$.

Therefore, it yields:

$$\|D\lambda^H\|^2 \leq \sum_{i=1}^{m}(D_{ii}\lambda_i^H+\lambda_i^G)^2 = \|D^\intercal\lambda^H+\lambda^G\|^2.$$

One finally proves: $\forall\zeta\in\mathcal{N}_\mathcal{U}(u)$,

$$\begin{aligned}\|C^\intercal\lambda^H\| &\leq \|C^\intercal D^{-1}\|\|D^\intercal\lambda^H+\lambda^G\| \\ &\leq \|C^\intercal D^{-1}\| \left\| \begin{matrix} D^\intercal\lambda^H+\lambda^G \\ E^\intercal\lambda^H+\zeta \end{matrix} \right\|.\end{aligned}$$

Using (Guo and Ye, 2016, Proposition 3.7), we see that (13) complies with the bounded slope condition.

3.2 A bang-bang property

Reachable set for linear systems We turn ourselves to the reachability set of linear systems in order to state a result that will be useful in order to prove a bang-bang property for LCS. Consider the following system:

$$\begin{cases}\dot{x}(t) = Mx(t)+Nu(t), \\ u(t)\in\mathcal{V},\end{cases} \tag{15}$$

for some matrices $M\in\mathbb{R}^{n\times n}$ and $N\in\mathbb{R}^{n\times m}$. We define the reachable (or accessible) set from $x_0\in\mathbb{R}^n$ at time $t\geq 0$, with controls taking values in $\mathcal{V}$, denoted by $\mathrm{Acc}_\mathcal{V}(x_0,t)$, the set of points $x(t)$, where $x:[0,t]\to\mathbb{R}^n$ is a solution of (15), with $u(s)\in\mathcal{V}$ for almost all $s\in[0,t]$ and $x(0)=x_0$. As stated in (Trélat, 2005, Corollary 2.1.2), which is proved using Aumann's theorem (see for instance Clarke (1981)), the following Proposition shows that the set of constraints $\mathcal{V}$ can be embedded in its convex hull:

Proposition 6. (Trélat, 2005, Corollary 2.1.2) Suppose that $\mathcal{V}$ is compact. Then:

$$\mathrm{Acc}_\mathcal{V}(x_0,t) = \mathrm{Acc}_{\mathrm{conv}(\mathcal{V})}(x_0,t)$$

where $\mathrm{conv}(\mathcal{V})$ denotes the convex hull of $\mathcal{V}$.

Thanks to Krein-Milman's Theorem (see Appendix A), this justifies that minimal-time optimal controls can be searched as bang-bang controls (meaning, u only takes values that are extremal points of $\mathcal{V}$ if one supposes in addition that $\mathcal{V}$ is convex).

Extremal points for LCS For this section, let us state the following Assumption:

Assumption 7. In (13), $C=0$, D is a **P**-matrix, and $\mathcal{U}$ is a finite union of polyhedral compact convex sets.

As it can be expected for a minimal time problem with linear dynamics, a bang-bang property can be proved, where the *bang-bang controls* have to be properly defined. Let us define first some notions. Denote by Ω the constraints on the controls (u,v) in (13), meaning:

$$\Omega = \{(u,v)\in\mathcal{U}\times\mathbb{R}^m | 0\leq v\perp Dv+Eu\geq 0\}. \tag{16}$$

The set $\mathrm{Acc}_\Omega(x_0,t)$ denotes the reachable set from $x_0\in\mathbb{R}^n$ at time $t\geq 0$ with controls with values in Ω. For a convex set $\mathfrak{C}$, a point $c\in\mathfrak{C}$ is called an extreme point if $\mathfrak{C}\backslash\{c\}$ is still convex. The set of extreme points of $\mathfrak{C}$ will be denoted $\mathrm{Ext}(\mathfrak{C})$. Suppose Ω is compact (which is not necessarily the case: take for instance $D=0$ with $0\in\mathcal{U}$). Applying Proposition 6, one proves that: $\mathrm{Acc}_\Omega(x_0,t) = \mathrm{Acc}_{\mathrm{conv}(\Omega)}(x_0,t)$. The set Ω is not convex and has empty interior; finding its boundary or extreme points is not possible in this case. However, Krein-Milman's Theorem (see Appendix A) proves that $\mathrm{conv}(\Omega)$ can be generated by its extreme points. In what follows, we will prove that the extreme points of $\mathrm{conv}(\Omega)$ are actually points of Ω that can be easily identified from the set $\mathcal{U}$. For an index set $\alpha\subseteq\overline{m}$, denote by $\mathbb{R}^m_\alpha$ the set of points q in $\mathbb{R}^m$ such that $q_\alpha\geq 0$, $q_{\overline{m}\backslash\alpha}\leq 0$, and define $E^{-1}\mathbb{R}^m_\alpha = \{\tilde{u}\in\mathbb{R}^{m_u} | E\tilde{u}\in\mathbb{R}^m_\alpha\}$ (E is not necessarily invertible).

Lemma 8. Suppose Assumption 7 holds true. For a certain $\alpha\subseteq\overline{m}$, denote by $\mathcal{P}_\alpha$ the set:

$$\begin{aligned}\mathcal{P}_\alpha = \{&(u,v)\in\left(\mathcal{U}\cap E^{-1}\mathbb{R}^m_\alpha\right)\times\mathbb{R}^m | \\ &v_\alpha=0,\ D_{\overline{\alpha}\bullet}v+E_{\overline{\alpha}\bullet}u=0,\ v\geq 0, Dv+Eu\geq 0\},\end{aligned}$$

and by $\mathcal{E}_\alpha$ the set:

$$\begin{aligned}\mathcal{E}_\alpha = \{&(u,v)\in\mathrm{Ext}\left(\mathcal{U}\cap E^{-1}\mathbb{R}^m_\alpha\right)\times\mathbb{R}^m | \\ &v_\alpha=0,\ D_{\overline{\alpha}\bullet}v+E_{\overline{\alpha}\bullet}u=0,\ v\geq 0, Dv+Eu\geq 0\}.\end{aligned}$$

Then $\mathrm{Ext}(\mathcal{P}_\alpha)=\mathcal{E}_\alpha$.

Proof. If $\mathcal{U}\cap E^{-1}\mathbb{R}^m_\alpha$ is empty, then the equality is obvious. Choose α such that $\mathcal{U}\cap E^{-1}\mathbb{R}^m_\alpha$ is not empty.

- $\mathcal{E}_\alpha\subseteq\mathrm{Ext}(\mathcal{P}_\alpha)$: Let $(u,v)\in\mathcal{E}_\alpha$. Suppose that $(u,v)\notin\mathrm{Ext}(\mathcal{P}_\alpha)$. Thus, there exist (u^1,v^1) and (u^2,v^2) in $\mathcal{P}_\alpha$, both different than (u,v), such that $(u,v) = \frac{1}{2}[(u^1,v^1)+(u^2,v^2)]$. But this implies that $u=\frac{1}{2}(u^1+u^2)$, and since $u\in\mathrm{Ext}(\mathcal{U}\cap E^{-1}\mathbb{R}^m_\alpha)$, $u=u_1=u_2$. Therefore, since D is a **P**-matrix, $v=\mathrm{SOL}(D,Eu) = \mathrm{SOL}(D,Eu^i)=v^i$ for $i\in\{1,2\}$. Therefore, $(u,v) = (u^1,v^1)=(u^2,v^2)$, and (u,v) is an extremal point of $\mathcal{P}_\alpha$. This is a contradiction.
- $\mathrm{Ext}(\mathcal{P}_\alpha)\subseteq\mathcal{E}_\alpha$: Let $(u,v)\in\mathrm{Ext}(\mathcal{P}_\alpha)$. Suppose that $u\notin\mathrm{Ext}(\mathcal{U}\cap E^{-1}\mathbb{R}^m_\alpha)$. Therefore, there exists u^1 and u^2 in $\mathcal{U}\cap E^{-1}\mathbb{R}^m_\alpha$, different than u, such that $u=\frac{1}{2}(u^1+u^2)$. Define for $i\in\{1,2\}$ v^i =SOL(D,Eu^i). Since Eu^1 and Eu^2 are membe of $\mathbb{R}^m_\alpha$, for $i\in\{1,2\}$:
$$v^i_{\overline{\alpha}} = -(D_{\overline{\alpha\alpha}})^{-1}(Eu^i)_{\overline{\alpha}},\ v^i_\alpha=0.$$
So:
$$v_{\overline{\alpha}} = -(D_{\overline{\alpha\alpha}})^{-1}(Eu)_{\overline{\alpha}} = \frac{1}{2}\left(v^1_{\overline{\alpha}}+v^2_{\overline{\alpha}}\right),$$
$$v_\alpha = 0 = \frac{1}{2}\left(v^1_\alpha+v^2_\alpha\right).$$
So $(u,v)=\frac{1}{2}\left[(u^1,v^1)+(u^2,v^2)\right]$ with $(u^i,v^i)\in\mathcal{P}_\alpha$, $i\in\{1,2\}$. But since $(u,v)\in\mathrm{Ext}(\mathcal{P}_\alpha)$, $u=u^1=u^2$. This is a contradiction.

Remark 9. If $\ker(E)=\{0\}$ (and in particular, if E is invertible), it may be easier to search for extreme points of the set $E\mathcal{U}\cap\mathbb{R}^m_\alpha$, as one can prove easily that: $\mathrm{Ext}(\mathcal{U}\cap E^{-1}\mathbb{R}^m_\alpha) = E^{-1}\mathrm{Ext}(E\mathcal{U}\cap\mathbb{R}^m_\alpha)$.

Proposition 10. Suppose Assumption 7 holds true. Denote by $\mathcal{E}$ the set $\mathcal{E}=\bigcup_{\alpha\subseteq\overline{m}}\mathcal{E}_\alpha$, where $\mathcal{E}_\alpha$ is defined in Lemma 8. Then, for all $t>0$ and all $x_0\in\mathbb{R}^n$, $\mathrm{Acc}_\Omega(x_0,t) = \mathrm{Acc}_\mathcal{E}(x_0,t)$, where Ω is defined in (16).

Proof. The function SOL$(D,\cdot)$: $q \mapsto v$ =SOL(D,q) is piecewise linear and continuous, as stated in Cottle et al. (2009). The pieces of SOL$(D,\cdot)$ are the sets $\mathbb{R}^m_\alpha$, for α ranging over the subsets of $\overline{m}$. Therefore, for each $\alpha \subseteq \overline{m}$, the set $\mathcal{U} \cap E^{-1}\mathbb{R}^m_\alpha$ is the union of compact convex polyhedra (possibly empty), and therefore it admits a finite number of extreme points. Thus, each $\mathcal{P}_\alpha = \{(u, \mathrm{SOL}(D, Eu)) \mid u \in \mathcal{U} \cap E^{-1}\mathbb{R}^m_\alpha\}$ in Lemma 8 is the union of compact convex polyhedra.
In order to simplify the proof, suppose that $\mathcal{U}$ (and therefore, each non empty $\mathcal{P}_\alpha$) is a single compact convex polyhedron (and not a union of several; the proof would still be the same by reasoning on each of them). Therefore, by Lemma 8 and Krein-Milman's theorem (see Appendix A), $\mathcal{P}_\alpha = \mathrm{conv}(\mathcal{E}_\alpha)$ for each subset α of $\overline{m}$. Since it can be shown that $\Omega = \bigcup_{\alpha\subseteq\overline{m}} \mathcal{P}_\alpha$, it proves:

$$\mathrm{conv}(\Omega) = \mathrm{conv}\left(\bigcup_{\alpha\subseteq\overline{m}} \mathcal{P}_\alpha\right).$$

Let us prove that conv(Ω) =conv$(\mathcal{E})$:

- conv(Ω) $\subseteq$conv$\left(\bigcup_{\alpha\subseteq\overline{m}} \mathcal{E}_\alpha\right)$: Remark that for all $\beta \subseteq \overline{m}$, $\mathcal{E}_\beta \subseteq \bigcup_{\alpha\subseteq\overline{m}} \mathcal{E}_\alpha$. Therefore, $\mathcal{P}_\beta$ =conv$(\mathcal{E}_\beta) \subseteq$ conv$\left(\bigcup_{\alpha\subseteq\overline{m}} \mathcal{E}_\alpha\right)$, and thus, since β was arbitrary, $\Omega = \bigcup_{\alpha\subseteq\overline{m}} \mathcal{P}_\alpha \subseteq$ conv$\left(\bigcup_{\alpha\subseteq\overline{m}} \mathcal{E}_\alpha\right)$. It then leads to: conv$(\Omega)$ $\subseteq$conv$\left(\bigcup_{\alpha\subseteq\overline{m}} \mathcal{E}_\alpha\right)$.
- conv$\left(\bigcup_{\alpha\subseteq\overline{m}} \mathcal{E}_\alpha\right)$ $\subseteq$conv(Ω):

$$\begin{aligned}\forall \beta \subseteq \overline{m},\ \mathcal{E}_\beta \subseteq \mathrm{conv}(\mathcal{E}_\beta) = \mathcal{P}_\beta \\ \implies \bigcup_{\alpha\subseteq\overline{m}} \mathcal{E}_\alpha \subseteq \bigcup_{\alpha\subseteq\overline{m}} \mathcal{P}_\alpha = \Omega \\ \implies \mathrm{conv}\left(\bigcup_{\alpha\subseteq\overline{m}} \mathcal{E}_\alpha\right) \subseteq \mathrm{conv}(\Omega).\end{aligned}$$

Applying now Proposition 6, it proves the following equalities:

$$\begin{aligned}\mathrm{Acc}_{\mathcal{E}}(x_0,t) &= \mathrm{Acc}_{\mathrm{conv}(\mathcal{E})}(x_0,t) \\ &= \mathrm{Acc}_{\mathrm{conv}(\Omega)}(x_0,t) \\ &= \mathrm{Acc}_{\Omega}(x_0,t).\end{aligned}$$

The interest of Proposition 10 is twofold: first, the complementarity constraints does not affect the bang-bang property that is shared with linear system (it is preserved even for this kind of piecewise linear system); secondly, it is actually sufficient to search for the extreme points of $\mathcal{U} \cap E^{-1}\mathbb{R}^m_\alpha$, as proved in Lemma 8 with the sets $\mathcal{E}_\alpha$. This result is illustrated in the next examples.

Example 11.

$$T^* = \min\ T(x,u,v) \tag{17}$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = ax(t) + bv(t) + fu(t), \\ 0 \le v(t) \perp dv(t) + eu(t) \ge 0, \quad \text{a.e. on } [0,T^*] \\ u(t) \in \mathcal{U} = [-1,1] \\ (x(0), x(T^*)) = (x_0, x_f), \end{cases} \tag{18}$$

where a, b, d, f, e are scalars, and we suppose $d > 0$ and $e \neq 0$. We suppose also that there exist at least one trajectory stirring x_0 to x_f.

In this case, there are two index sets α as described in Lemma 8: $\emptyset$ or $\{1\}$. Therefore, we should have a look at the extreme points of $\mathcal{U} \cap \mathbb{R}^1_\emptyset = \mathcal{U} \cap \mathbb{R}_- = [-1,0]$ and of $\mathcal{U} \cap \mathbb{R}^1_{\{1\}} = \mathcal{U} \cap \mathbb{R}_+ = [0,1]$. Thus, it is sufficient to look at input functions u with values in $\{-1,0,1\}$. Suppose that the constants in (18) (with $u(t)$ supposed unconstrained for the moment) is completely controllable, which means:

If $e > 0$ **:** if $f < 0$, then $b \ge 0$ or $[b < 0$ and $b - \frac{fd}{e} > 0]$.
if $f > 0$, then $b \le 0$ or $[b > 0$ and $b - \frac{fd}{e} < 0]$.
If $e < 0$ **:** the same cases as with $e > 0$ hold by inverting the sign of f.

All other cases (like $f = 0$ or $e = 0$) are discarded. Let us now deduce from Theorem 3 the only stationary solution. First of all, the equation (8b) tells us that the adjoint state complies with the ODE $\dot{p} = -ap$. Therefore there exists p_0 such that, for all $t \in [0,T^*]$, $p(t) = p_0 e^{-at}$. Could we have $p_0 = 0$? It would imply that $p \equiv 0$ and then, $\lambda_0 = \langle p(t), ax(t) + bv(t) + fu(t)\rangle = 0$, so $(p(t), \lambda_0) = 0$ for almost all t in $[0,T^*]$. This is not allowed, so $p_0 \neq 0$. Moreover, there exist multipliers λ^G and λ^H such that, for almost all t in $[0,T^*]$:

$$\lambda^G(t) = -bp(t) - d\lambda^H(t), \tag{19}$$

$$\lambda^G(t) = 0 \text{ if } v(t) > 0 = dv(t) + eu(t), \tag{20}$$

$$\lambda^H(t) = 0 \text{ if } v(t) = 0 < dv(t) + eu(t), \tag{21}$$

$$fp(t) + e\lambda^H(t) \in \mathcal{N}_{[-1,1]}(u(t)) = \begin{cases} \{0\} & \text{if } |u(t)| \neq 1, \\ \mathbb{R}^+ & \text{if } u(t) = 1, \\ -\mathbb{R}^+ & \text{if } u(t) = -1, \end{cases} \tag{22}$$

$$\begin{aligned}\lambda_0 = \quad &\max\ \langle p(t), ax(t) + b\tilde{v} + f\tilde{u}\rangle, \\ &\text{s.t. } 0 \le \tilde{v} \perp d\tilde{v} + e\tilde{u} \ge 0\end{aligned} \tag{23}$$

(all these equations are derived from (8) with $g \equiv h \equiv 0$, $G \equiv v$, $H \equiv Cx + Dv + Eu$, $\phi \equiv Ax + Bv + Fu$). Since $d > 0$, one can easily prove that: $v = \frac{1}{d}\max(0, -eu)$. Let us suppose for now that $e > 0$ (all subsequent work is easily adapted for $e < 0$ by replacing u by $-u$). In this case:

$$v = \begin{cases} 0 & \text{if } u \in [0,1], \\ -\dfrac{eu}{d} & \text{if } u \in [-1,0]. \end{cases} \tag{24}$$

Let us now discuss all possible cases for u. If $u(t) = 0$, then $v(t) = 0 = dv(t) + eu(t)$. We use stationarity conditions for (23): since the MPEC Linear Condition holds, there exists multipliers η^H and η^G such that $\eta^H = -\frac{f}{e}p(t)$, $\eta^G = \left(\frac{df}{e} - b\right)p(t)$, and $\eta^G\eta^H = 0$ or $\eta^H > 0, \eta^G > 0$ (one has M-stationarity, see Proposition 4). However, $\eta^H \neq 0$ and $\eta^G \neq 0$. Furthermore, η^H has the same sign as $-fp_0$ and η^G has the same sign as fp_0. Therefore, the two have opposite signs, and $u(t) = 0$ can not be an M-stationary solution. Therefore, it proves that necessarily, the optimal control u^* complies with $|u^*(t)| = 1$ for almost all t on $[0,T^*]$. If $u(t) = 1$, then $v(t) = 0 < dv(t) + eu(t)$, and by (21), $\lambda^H(t) = 0$. Then by (22), $fp_0 \ge 0$. If $u(t) = -1$, then $v(t) > 0 = dv(t) + eu(t)$, and by (19)(20), $\lambda^H(t) = -\frac{b}{d}p(t)$. Then by (22), $\left(f - \frac{eb}{d}\right)p_0 \le 0$. It is impossible to have

$fp_0 \geq 0$ and $\left(f - \frac{eb}{d}\right) p_0 \leq 0$ at the same time, since f and $f - \frac{eb}{d}$ have the same sign by the complete controllability conditions. Therefore, u^* take only one value along $[0, T^*]$: 1 or -1. Then we have two possible optimal state x^* starting from x_0:

if $a \neq 0$**:**

$$x^*(t) = \begin{cases} \left(x_0 + \dfrac{f}{a}\right) \exp(at) - \dfrac{f}{a} \\ \qquad\qquad \text{if } u^*(t) = 1, \\ \left(x_0 + \dfrac{be - fd}{ad}\right) \exp(at) - \dfrac{be - fd}{ad} \\ \qquad\qquad \text{if } u^*(t) = -1. \end{cases}$$

One must then find the solution that complies with $x(T^*) = x_f$. One can isolate the optimal time T^*:

$$T^* = \begin{cases} \dfrac{1}{a} \ln\left(\dfrac{ax_f + f}{ax_0 + f}\right) \\ \qquad \text{if } ax_0 + f \neq 0 \text{ and } \dfrac{ax_f + f}{ax_0 + f} > 0, \\ \dfrac{1}{a} \ln\left(\dfrac{adx_f + be - fd}{adx_0 + be - fd}\right) \\ \qquad \text{if } adx_0 + be - fd \neq 0 \\ \qquad \text{and } \dfrac{adx_f + be - fd}{adx_0 + be - fd} > 0. \end{cases} \tag{25}$$

Since we supposed that there exists at least one trajectory stirring x_0 to x_f, one of these two expressions of T^* must be positive. Therefore, one can infer that:

$$u^* \equiv \begin{cases} 1 \text{ if } ax_0 + f \neq 0 \text{ and } \dfrac{ax_f + f}{ax_0 + f} > 0, \\ -1 \text{ if } adx_0 + be - fd \neq 0 \\ \qquad\qquad \text{and } \dfrac{adx_f + be - fd}{adx_0 + be - fd} > 0. \end{cases}$$

if $a = 0$**:**

$$x^*(t) = \begin{cases} ft & \text{if } u^*(t) = 1, \\ \dfrac{be - fd}{d} t & \text{if } u^*(t) = -1. \end{cases}$$

With the same calculations made in the case $a \neq 0$, one proves that:

$$u^* \equiv \begin{cases} 1 \text{ if } f \neq 0 \text{ and } \dfrac{x_f}{f} > 0, \\ -1 \text{ if } be - fd \neq 0 \text{ and } \dfrac{dx_f}{be - fd} > 0. \end{cases}$$

The proof of Proposition 10 relies on the fact that when D is a **P**-matrix, Ω is the union of compact convex polyhedra. However, some examples show that even when D is not a **P**-matrix, then this property may hold.

Example 12.

$$T^* = \min \; T(x, u, v) \tag{26}$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = ax(t) + bv(t) + fu(t), \\ 0 \leq v(t) \perp -v(t) + u(t) \geq 0, \\ u(t) \in \mathcal{U} = [-1, 1], \text{ a.e. on } [0, T^*], \\ (x(0), x(T^*)) = (x_0, x_f). \end{cases} \tag{27}$$

It is clear that there exists no solution to the LCP$(-1, u)$ appearing in (27) for $u \in [-1, 0)$, so $\mathcal{U}$ can actually be restricted to $[0, 1]$. A graphic showing the shape of Ω and its convex hull is shown in Figure 1.

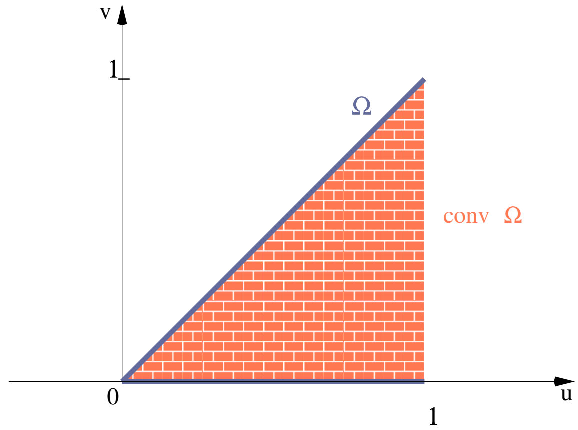

Fig. 1. Ω and its convex hull for (27)

It clearly appears that conv(Ω) is generated by three extreme points: $(u, v) \in \mathcal{E} = \{(0,0), (1,0), (1,1)\}$. Exactly the same way as in the proof of Proposition of 10, one can simply show that for all $t \geq 0$ and for all $x_0 \in \mathbb{R}$, $\text{Acc}_\Omega(x_0, t) = \text{Acc}_\mathcal{E}(x_0, t)$. Therefore, the optimal trajectory can be searched with controls (u, v) with values in $\mathcal{E}$. It is also interesting to note that this bang-bang property can be guessed from the condition of maximisation of the Hamiltonian in (8g) and from Figure 1. Indeed, (8g) state that at almost all time t, the linear function $\Lambda : (u, v) \mapsto \langle p(t), Bv + Fu \rangle$ must be maximized with variables (u, v) in Ω. When one tries to maximize Λ over conv(Ω) it becomes a Linear Program (LP) over a simplex. It is well known that linear functions reach their optimum over simplexes at extreme points; in this case, the extreme points are the points of $\mathcal{E}$.

3.3 Characterisation through HJB equation

An other way to solve the minimal time optimal control problem is through the Dynamic Programming Principle and the Hamilton-Jacobi-Bellman (HJB) equation. The theory needs pure control constraints, but not convexity of the set of constraints. Therefore, the Assumption that $C = 0$ in (13) still holds. However, one doesn't need D to be a **P**-matrix anymore. The only necessary Assumption needed is an assumption of compactness.

Assumption 13. In (13), $C = 0$, and the set Ω defined in (16) is a compact subset of $\mathbb{R}^{m_u} \times \mathbb{R}^m$.

The HJB equation is a non-linear PDE that the objective cost must comply with. In this framework, the minimal time T^* is seen as a function of the target x_f. However, the equation will not be directly met by $T^*(x_f)$, but by a discounted version of it, called the Kružkov transform, and defined by:

$$z^*(x_f) = \begin{cases} 1 - e^{-T^*(x_f)} & \text{if } T^*(x_f) < +\infty \\ 1 & \text{if } T^*(x_f) = +\infty \end{cases} \tag{28}$$

This transformation comes immediately when one tries to solve this optimal control problem with the running cost $C(t(x_f)) = \int_0^{t(x_f)} e^{-t} dt = 1 - e^{-t(x_f)}$ where $t(x_f)$ is a free variable. Minimizing $C(t(x_f))$ amounts to minimizing T^*.

Once one finds the optimal solution $z(x_f)$, it is easy to recover T^*, since $T^*(x_f) = -\ln(1 - z(x_f))$.

The concept of solution for the HJB equation needs the concept of viscosity solution. A reminder of the definitions of sub- and supersolutions appears in Appendix B. But the most useful definitions are recalled here. First of all, one needs the notion of lower semicontinuous envelope.

Definition 14. Denote $z : X \to [-\infty, +\infty]$, $X \subseteq \mathbb{R}^n$. We call lower semicontinuous envelope of z the function $\underset{\sim}{z}$ defined pointwise by:

$$\underset{\sim}{z}(x) = \liminf_{y \to x} z(y) = \lim_{r \to 0^+} \inf\{z(y) : y \in X,\ |y - x| \leq r\}$$

One can see easily that $\underset{\sim}{z} = z$ at every point where z is (lower semi-)continuous. Secondly, one needs the definition of an envelope solution.

Definition 15. Consider the Dirichlet problem

$$\begin{cases} F(x, z(x), \nabla z(x)) = 0 & x \in \kappa \\ z(x) = g(x) & x \in \partial\kappa \end{cases} \tag{29}$$

with $\kappa \subseteq \mathbb{R}^n$ open, $F : \kappa \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ continuous, and $g : \partial\kappa \to \mathbb{R}$. Denote $\underline{\mathcal{S}} = \{\text{subsolutions of (29)}\}$ and $\overline{\mathcal{S}} = \{\text{supersolutions of (29)}\}$. Let $z : \overline{\kappa} \to \mathbb{R}$ be locally bounded.

(1) z is an envelope viscosity subsolution of (29) if there exists $\underline{\mathcal{S}}(z) \subseteq \underline{\mathcal{S}}$, $\underline{\mathcal{S}}(z) \neq \emptyset$, such that:
$$z(x) = \sup_{w \in \underline{\mathcal{S}}(z)} w(x),\ x \in \overline{\kappa}$$
(2) z is an envelope viscosity supersolution of (29) if there exists $\overline{\mathcal{S}}(z) \subseteq \overline{\mathcal{S}}$, $\overline{\mathcal{S}}(z) \neq \emptyset$, such that:
$$z(x) = \inf_{w \in \overline{\mathcal{S}}(z)} w(x),\ x \in \overline{\kappa}$$
(3) z is an envelope viscosity solution of (29) if it is an envelope viscosity sub- and supersolution.

With these definitions, one can formulate the next Theorem, stating the HJB equation for z^*:

Theorem 16. z^* is the envelope viscosity solution o the Dirichlet problem:

$$\begin{cases} z + H(x, \nabla z) = 1 & \text{in } \mathbb{R}^n \backslash \{x_f\}, \\ z = 0 & \text{on } \{x_f\}, \end{cases} \tag{30}$$

where $H(x,p) = \sup_{(u,v)\in\Omega} \langle -p, Ax + Bv + Fu \rangle$. In case Assumption 7 is met, then H can be defined as:

$$H(x,p) = \sup_{(u,v)\in\widetilde{\Omega}} \langle -p, Ax + Bv + Fu \rangle, \tag{31}$$

where $\widetilde{\Omega} = \{(u,v) \in \Omega | u \in \mathcal{E}\}$, and $\mathcal{E}$ has been defined in Proposition 10.

Proof. By (Bardi and Capuzzo-Dolcetta, 2008, Chapter V.3.2, Theorem 3.7), the lower semicontinuous envelope of z^*, $\underset{\sim}{z}^*$, is the envelope viscosity solution of the Dirichlet problem (30). Thanks to Proposition 6, one can analyse the problem equivalently on Ω or on convΩ. Reasoning on convΩ rather than on Ω, one can prove using (Wolenski and Zhuang, 1998, Proposition 2.6) that $T^*(\cdot)$ is a lower semicontinuous function; therefore, so is z^*. It proves that $z^* = \underset{\sim}{z}^*$ and therefore, z^* is the envelope viscosity solution of (30).
Finally, Proposition 10 justifies the expression of H in (31).

Remark 17. The target $\{x_f\}$ could be changed to any closed nonempty set $\mathcal{T}$ with compact boundary.

Example 18. Example 11 revisited.
Let us check that the Kružkov transform of T^* found in (25) complies with (30). The verification will be carried in the case when $\frac{ax_f+f}{ax_0+f} > 0$, the other cases being treated with the same calculations. In this case, the Kružkov transform of T^* defined in (28) amounts to: $z^*(x_f) = 1 - \left(\frac{ax+f}{ax_0+f}\right)^{-\frac{1}{a}}$. Therefore, one must check that

$$\begin{aligned} 1 - z^*(x_f) &= -\left(\frac{ax_f + f}{ax_0 + f}\right)^{-\frac{1}{a}} \\ &= \sup_{(u,v)\in\widetilde{\Omega}} \left\{(ax_f + bv + fu)\frac{dz^*}{dx}(x_f)\right\} \end{aligned} \tag{32}$$

where Ω is defined as $\widetilde{\Omega} = \{(u,v) \in \{-1,0,1\} \times \mathbb{R} \mid 0 \leq v \perp dv + eu \geq 0\}$.

As it has been shown in Example 11, the sup in (32) is attained at $u = 1$, $v = 0$. Therefore:

$$\begin{aligned} &\sup_{(u,v)\in\Omega} \left\{(ax_f + bv + fu)\frac{dz^*}{dx}(x_f)\right\} \\ &= (ax_f + f)\left(-\frac{1}{ax_0 + f}\left(\frac{ax_f + f}{ax_0 + f}\right)^{-\frac{1}{a}-1}\right) \\ &= -\left(\frac{ax_f + f}{ax_0 + f}\right)^{-\frac{1}{a}} = 1 - z^* \end{aligned}$$

Therefore, using the same definition of H made in (31), it is proven that z^* complies with the equation $z^* + H\left(x_f, \frac{dz^*}{dx}\right) = 1$, which is the HJB Equation (30).

4. CONCLUSION

The necessary conditions for optimality exposed in Guo and Ye (2016) were extended to the case of minimal time problem. These results were precised for LCS, and some special properties that the optimum possesses in the case of LCS, were also shown. As future work, one could extend the class of LCS complying for the Bounded Slope Condition, and also prove the bang-bang property for a broader class of LCS, as Example 12 suggests.

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REFERENCES

Acary, V., Bonnefon, O., and Brogliato, B. (2011). *Nonsmooth Modeling and Simulation for Switched Circuits*, volume 69 of *Lecture Notes in Electrical Engineering*. Springer Science & Business Media.

Bardi, M. and Capuzzo-Dolcetta, I. (2008). *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Springer Science & Business Media.

Brogliato, B. (2003). Some perspectives on the analysis and control of complementarity systems. *IEEE Transactions on Automatic Control*, 48(6), 918–935.

Çamlibel, M. and Schumacher, J. (2001). On the Zeno behavior of linear complementarity systems. *Proceedings of the 40th IEEE Conference on Decision and Control*, 1, 346–351.

Çamlibel, M.K., Pang, J.S., and Shen, J. (2006). Conewise linear systems: non-Zenoness and observability. *SIAM Journal on Control and Optimization*, 45(5), 1769–1800.

Clarke, F. and De Pinho, M. (2010). Optimal control problems with mixed constraints. *SIAM Journal on Control and Optimization*, 48(7), 4500–4524.

Clarke, F.H. (1981). A variational proof of Aumann's theorem. *Applied Mathematics and Optimization*, 7(1), 373–378.

Cottle, R., Pang, J.S., and Stone, R. (2009). *The Linear Complementarity Problem*. SIAM.

Guo, L. and Ye, J.J. (2016). Necessary optimality conditions for optimal control problems with equilibrium constraints. *SIAM Journal on Control and Optimization*, 54(5), 2710–2733.

Heemels, W., Çamlibel, M., Schumacher, J., and Brogliato, B. (2011). Observer-based control of linear complementarity systems. *International Journal of Robust and Nonlinear Control*, 21(10), 1193–1218.

Lohéac, J., Trélat, E., and Zuazua, E. (2018). Minimal controllability time for finite-dimensional control systems under state constraints. Working paper or preprint.

Pang, J.S. and Shen, J. (2007). Strongly regular differential variational systems. *IEEE Transactions on Automatic Control*, 52(2), 242–255.

Shen, J. (2014). Robust non-zenoness of piecewise affine systems with applications to linear complementarity systems. *SIAM Journal on Optimization*, 24(4), 2023–2056.

Trélat, E. (2005). *Contrôle optimal : Théorie & Applications*. Vuibert. URL `https://hal.archives-ouvertes.fr/hal-00086625`. ISBN 2 7117 7175 X.

Vieira, A., Brogliato, B., and Prieur, C. (2018). Quadratic Optimal Control of Linear Complementarity Systems: First order necessary conditions and numerical analysis. URL `https://hal.inria.fr/hal-01690400`. Https://hal.inria.fr/hal-01690400.

Vinter, R. (2010). *Optimal Control*. Springer Science & Business Media.

Wolenski, P. and Zhuang, Y. (1998). Proximal analysis and the minimal time function. *SIAM Journal on Control and Optimization*, 36(3), 1048–1072.

Appendix A. KREIN-MILMAN THEOREM

Since the Krein-Milman Theorem is used in this paper, it is worth recalling its statement. Let us start with a definition.

Definition 19. Let C be a convex compact subset of a Hausdorff locally convex set. Let $c \in C$. The point c is called an extremal point of C if $C\backslash\{c\}$ is still convex. Equivalently, c is an extreme point of C if the following implication holds: $c_1, c_2 \in C,\ c = \frac{1}{2}(c_1 + c_2) \implies c = c_1 = c_2$ The set of extreme points of C is denoted by $\mathrm{Ext}(C)$.

Theorem 20. (Krein-Milman). Let C be a convex compact subset of a Hausdorff locally convex set. Then

$$C = \text{cl conv } (\mathrm{Ext}(C))$$

Appendix B. VISCOSITY SOLUTIONS

In order to understand some results concerning the HJB equation, one needs to know some definitions related to the concept of viscosity solutions. The definitions given here, extracted from Bardi and Capuzzo-Dolcetta (2008), are only the ones useful for this manuscript. In particular, the definitions given here are the ones useful to handle the concept of discontinuous viscosity solutions. The interested reader can find broader results in Bardi and Capuzzo-Dolcetta (2008) and the references therein. Let us first define the notion of subsolution and supersolution of a first order equation

$$F(x, u, \nabla u) = 0 \text{ in } \Omega, \tag{B.1}$$

with $\Omega \subseteq \mathbb{R}^n$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ continuous. For this, let us fix some notations: for $E \subseteq \mathbb{R}^n$, denote $\mathrm{USC}(E) = \{u : E \to \mathbb{R} \text{ upper semicontinuous}\}$, $\mathrm{LSC}(E) = \{u : E \to \mathbb{R} \text{ lower semicontinuous}\}$.

Definition 21. A function $u \in \mathrm{USC}(\Omega)$ (resp. $\mathrm{LSC}(\Omega)$) is a viscosity subsolution (resp. supersolution) of (B.1) if, for any $\phi \in \mathscr{C}^1(\Omega)$ and $x \in \Omega$ such that $u - \phi$ has a local maximum (resp. minimum) at x,

$$F(x, u(x), \nabla\phi(x)) \leq 0 \text{ (resp. } \geq 0).$$