MULTIGRADED CAYLEY-CHOW FORMS

BRIAN OSSERMAN AND MATTHEW TRAGER

Abstract. We introduce a theory of multigraded Cayley-Chow forms associated to subvarieties of products of projective spaces. Two new phenomena arise: first, the construction turns out to require certain inequalities on the dimensions of projections; and second, in positive characteristic the multigraded Cayley-Chow forms can have higher multiplicities. The theory also provides a natural framework for understanding multifocal tensors in computer vision.

1. Introduction

Let $X \subseteq \mathbb{P}^n$ be a projective variety of dimension r and degree d. The set of all linear spaces of dimension $n-r-1$ meeting X is a hypersurface Z_X in the Grassmannian $\mathrm{Gr}(n-r-1,n)$. Any such hypersurface can be written as the zero set inside the Grassmannian of a polynomial H_X in the Plücker coordinates, which turns out to also be of degree d. The polynomial H_X is known as the "Chow form" or "Cayley form" of X, and we will refer to it as the **Cayley-Chow form**. From the Cayley-Chow form H_X we immediately recover the hypersurface Z_X, and one can then recover X as the set of points $P \in \mathbb{P}^n$ such that every $(n-r-1)$-dimensional linear space containing P corresponds to an element of Z_X. Thus, the Cayley-Chow form can be used to encode subvarieties of projective space, and this classical construction has played an important role in moduli space theory, especially in the guise of Chow varieties, but also for instance in Grothendieck's original construction of Quot and Hilbert schemes [Gro61]. See §2 of Chapter 3 of [GKZ94] for a presentation of this material.

The purpose of this paper is to generalize this classical theory to the case of subvarieties of products of projective spaces. We find that the generalization displays some interesting properties, particularly that certain dimensional inequalities have to be satisfied in order for it to work. We give necessary and sufficient conditions for the generalized theory to go through. Moreover, we explain that in cases where the necessary inequalities are not satisfied, the failure of the theory can in fact shed light on previously observed phenomena in computer vision. In addition, positive-characteristic phenomena arise in our more general setting, causing the Cayley-Chow form to sometimes have higher multiplicities.

Before we state our main results, we need to recall the notion of **multidegree** of a subvariety $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. If X has codimension c, we can represent its

The first author is partially supported by a grant from the Simons Foundation #279151. The second author was supported in part by the ERC advanced grant VideoWorld, the Institut Universitaire de France, the Inria-CMU associated team GAYA, and the ANR grant RECAP. This project arose out of discussions at the American Institute of Mathematics workshop on Algebraic Vision.

multidegree as a homogeneous polynomial

$$\sum_{\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_k)} a_{\boldsymbol{\gamma}} t_1^{\gamma_1}\cdots t_k^{\gamma_k}$$

of degree c, where $a_{\boldsymbol{\gamma}}$ is determined as the number of points of intersection (counting multiplicity) of X with $L_1\times\cdots\times L_k$, where each L_i is a general linear space of dimension γ_i. The multidegree is also equivalent to the Chow class of X, although we will not need this.

We can summarize our main results in the following theorem:

Theorem 1.1. *Let $X\subseteq \mathbb{P}^{n_1}\times\ldots\times\mathbb{P}^{n_k}$ be a projective variety of dimension r, and suppose that X is not of the form $X'\times\prod_{i\not\in I}\mathbb{P}^{n_i}$ for any $I\subsetneq\{1,\dots,k\}$ and $X'\subseteq\prod_{i\in I}\mathbb{P}^{n_i}$. Given also $\boldsymbol{\beta}=(\beta_1,\dots,\beta_k)$ with $0\le\beta_i\le n_i$ for $i=1,\dots,k$ and $\sum_{i=1}^k\beta_i=r+1$, write $\alpha_i=n_i-\beta_i$ for each i.*

Consider the closed subset

$$Z_{X,\boldsymbol{\beta}}=\{(L_1,\dots,L_k):X\cap(L_1\times\cdots\times L_k)\neq\emptyset\}\subseteq \mathrm{Gr}(\alpha_1,n_1)\times\cdots\times\mathrm{Gr}(\alpha_k,n_k).$$

Then $Z_{X,\boldsymbol{\beta}}$ is a hypersurface determining X if and only if for every nonempty $I\subsetneq\{1,\dots,k\}$ we have

$$\dim p_I(X)\ge\sum_{i\in I}\beta_i,$$

where $p_I(X)$ denotes the projection of X onto $\prod_{i\in I}\mathbb{P}^{n_i}$.

Assuming the above inequalities are satisfied, we have that $Z_{X,\boldsymbol{\beta}}$ is the zero set of a single multihomogeneous polynomial F_X under the product of the Plücker embeddings of the $\mathrm{Gr}(\alpha_i,n_i)$. There is a multiplicity $\epsilon_{X,\boldsymbol{\beta}}\ge 1$ such that if we write

$$H_{X,\boldsymbol{\beta}}:=F_X^{\epsilon_{X,\boldsymbol{\beta}}},$$

and X has multidegree

$$\sum_{\boldsymbol{\gamma}} a_{\boldsymbol{\gamma}} t_1^{\gamma_1}\cdots t_k^{\gamma_k},$$

then the multidegree of $H_{X,\boldsymbol{\beta}}$ (as a multihomogeneous polynomial) is given by

$$(a_{\alpha_1+1,\alpha_2,\dots,\alpha_k},\dots,a_{\alpha_1,\dots,\alpha_{k-1},\alpha_k+1}).$$

Finally, in characteristic 0, we always have $\epsilon_{X,\boldsymbol{\beta}}=1$.

In fact, we first show in Proposition 3.1 that $Z_{X,\boldsymbol{\beta}}$ is a hypersurface if and only if the slightly weaker inequalities (3.1) are satisfied, and then in Proposition 3.6 that when $Z_{X,\boldsymbol{\beta}}$ is a hypersurface, it determines X uniquely if and only if the above inequalities are satisfied. The condition that X not be a product with any of the projective spaces is just to simplify the statement; see Example 3.5 below. The multiplicity $\epsilon_{X,\boldsymbol{\beta}}$ is defined naturally in Definition 5.1 as the degree of the map from an incidence correspondence to $Z_{X,\boldsymbol{\beta}}$, and in positive characteristic, it may be strictly greater than 1; see Example 5.13 below. The remaining statements of the theorem are proved in Corollary 4.2, Proposition 5.3, and Corollary 5.10. An interesting aspect of Proposition 3.6 in comparison to the classical case is that if we define the set S_Z to consist of points $P\in\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ with the property that every $L_1\times\cdots\times L_k$ containing P must lie in $Z_{X,\boldsymbol{\beta}}$, then $X\subseteq S_Z$ always, but we will have in general that S_Z contains additional components.

In addition, while we have stated the fundamental inequalities in terms of the dimensions of projections of X, according to work of Castillo, Li and Zhang [CLZ] this may be reinterpreted with an equivalent formulation in terms of the multidegree of X. We carry this out in Corollary 5.11 below. Thus, while we do require non-trivial conditions on X in order for our theory to apply, when it applies it does so uniformly to all X of given multidegree.

It is natural to wonder how our construction compares to applying a Segre embedding together with the classical construction; we discuss this briefly in Remark 5.12 below.

We next make the following observation.

Remark 1.2. The inequalities given in Theorem 1.1 can only be satisfied if $k \leq r+1$, since otherwise the set of i such that $\beta_i \neq 0$ is necessarily proper in $\{1, \dots, k\}$, and will violate the necessary inequality.

We can now explain the relationship to computer vision, and specifically to the reconstruction of a configuration of unknown cameras. A basic model for a (positioned) camera is as a linear projection from the three-dimensional world to the two-dimensional film/sensor plane, which we consider as a $\mathbb{P}^3$ and a $\mathbb{P}^2$, respectively. Thus, a k-tuple of cameras corresponds to an k-tuple of linear projections, which together induce a rational map

$$\mathbb{P}^3 \dashrightarrow (\mathbb{P}^2)^k.$$

The closure of the image of this map is then a three-dimensional subvariety of $(\mathbb{P}^2)^k$, which is called the "multiview variety." This can be thought of as describing which k-tuples of points in $\mathbb{P}^2$ could come from a single point in $\mathbb{P}^3$. Knowing the multiview variety is equivalent to knowing the camera configuration, at least up to change of 'world coordinates' in $\mathbb{P}^3$. It is well known in computer vision that for $k = 2, 3, 4$, there exists a k-tensor which determines the camera configuration, called the "multifocal tensor." It is equally well known that this construction does not extend to $k > 4$.

The multifocal tensor is described in terms of incidences with k-tuples of linear spaces, and in fact this inspired our construction. On the other hand, we can reinterpret the theory of multifocal tensors in terms of Theorem 1.1 as follows. First, Aholt, Sturmfels and Thomas showed in Corollary 3.5 of [AST13] that all the coefficients of the multidegree of a multiview variety are equal to 1, so we see that when our Cayley-Chow form construction applies, the result is a multilinear polynomial in k variable sets, which is to say, a k-tensor. It is routine to check that for $k \leq 4$, our construction does apply for suitable choice of $\boldsymbol{\beta}$, and the Cayley-Chow form coming from the multivew variety is precisely the multifocal tensor. Conversely, if $k \geq 5$, then Remark 1.2 implies that no analogous construction exists for any choice of $\boldsymbol{\beta}$. See Examples 5.14 and 5.15 for further details. Beyond giving a new point of view on these known constructions, we also hope that Theorem 1.1 will provide new applications in computer vision, in the context of generalized cameras. Recent work of Ponce, Sturmfels and the second author [PST17], and of Escobar and Knutson [EK17] develops a theory of configurations of such cameras, including multidegree-type formulas, and we expect that Theorem 1.1 will provide a generalization of multifocal tensors to this setting, where the tensors will be replaced with higher-degree forms.

Remark 1.3. A different connection is to the notion of circuit polynomials in matroid theory, which we now describe. First observe that when all n_i are equal to 1, we must have $\beta_i = 1$ for some subset S of I of size $r+1$, and $\beta_i = 0$ for $i \notin S$. The inequalities of Theorem 1.1 are never satisfied except in the trivial case that $r = k-1$, and $\beta_i = 1$ for all i, so that $H_{X,\boldsymbol{\beta}}$ will simply recover the defining polynomial of X. However, the weaker inequalities (3.1) will be satisfied more generally: specifically, whenever we have $\dim p_I(X) = r$ for the I as above. Thus, in this case we can still define our multigraded Cayley-Chow form, although it will not suffice to recover X. This special case of our construction turns out to be connected to algebraic matroids.

Specifically, one approach to algebraic matroids is as follows: in order to construct a matroid on $\{1,\dots,k\}$ of rank r, choose a variety $X \subseteq \mathbb{A}^k = (\mathbb{A}^1)^k$ of dimension r, with prime ideal $\mathfrak{p} \subseteq K[x_1,\dots,x_k]$. Define independent sets by algebraic independence of the images of the x_i modulo $\mathfrak{p}$. In this context, a circuit $C \subseteq \{1,\dots,k\}$ will have the property that the closure of the projection of X to $\prod_{i\in C}\mathbb{A}^1$ has codimension 1, and hence is cut out by a single polynomial H in $r+1$ variables. This polynomial is called the "circuit polynomial," and precisely cuts out the closure of the locus of $(P_i)_{i\in C} \in (\mathbb{A}^1)^{r+1}$ such that there exists $(P_i)_{i\notin C}$ with $(P_1,\dots,P_k) \in X$. See §5 of Király-Rosen-Theren [KRT13]. Replacing X by its closure in $(\mathbb{P}^1)^k$ and H by its multihomogenization, we recover the Cayley-Chow form construction with $\beta_i = 1$ for $i \in C$ and $\beta_i = 0$ otherwise (at least, up to omission of $\epsilon_{X,\boldsymbol{\beta}}$).

Finally, we mention that the inequalities arising both in Theorem 1.1 and in the condition for $Z_{X,\boldsymbol{\beta}}$ to be a hypersurface (see Proposition 3.1 below) are closely related to concepts arising in polymatroid theory; see Remark 2.4 below.

Acknowledgements. We would like to thank Bernd Sturmfels for bringing to our attention various connections to the literature, particularly the notion of matroid circuit polynomials.

Conventions. We work throughout over an algebraically closed field K. A variety is always assumed irreduible.

Given $\boldsymbol{\beta} = (\beta_1,\dots,\beta_k) \in \mathbb{Z}^k$, we will write $|\boldsymbol{\beta}| := \sum_{i=1}^k \beta_i$, and for $I \subseteq \{1,\dots,k\}$, we will write $\boldsymbol{\beta}_I := (\beta_i)_{i\in I}$, and $|\boldsymbol{\beta}_I| := \sum_{i\in I}\beta_i$. We also write $I^c := \{1,\dots,k\} \smallsetminus I$.

2. Multidegrees and dimensions of projections

We begin by collecting some background results on the relationship between multidegree and dimensions of projections. If we have a subvariety X of $\prod_{i=1}^k \mathbb{P}^{n_i}$ with multidegree

$$\sum_{\boldsymbol{\gamma}} a_{\boldsymbol{\gamma}} t_1^{\gamma_1}\cdots t_k^{\gamma_k},$$

we say the **support** of the multidegree is the set of $\boldsymbol{\gamma}$ for which $a_{\boldsymbol{\gamma}} \neq 0$. Note that by definition, this is contained in the subset of $\boldsymbol{\gamma}$ with $\gamma_i \geq 0$ for all i, and $\sum_i \gamma_i = \operatorname{codim} X$.

The main theorem of [CLZ] asserts:

Theorem 2.1 (Castillo-Li-Zhang)**.** *If*

$$X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$$

is irreducible, the support of its multidegree is

$$\left\{\boldsymbol{\gamma} : \sum_i (n_i - \gamma_i) = \dim X, \text{ and } \sum_{i\in I}(n_i - \gamma_i) \leq \dim p_I(X) \quad \forall I \subsetneq \{1,\dots,k\}\right\},$$

where p_I denotes projection onto the product of the subset of the $\mathbb{P}^n$s indexed by I.

Moreover, the function $\delta(I) = \dim p_I X$ satisfies the following conditions:

- $\delta(\emptyset) = 0$*;*
- *for $I \subseteq J$ we have $\delta(I) \leq \delta(J)$;*
- *and for any I, J we have $\delta(I \cap J) + \delta(I \cup J) \leq \delta(I) + \delta(J)$.*

In fact, they treat the case that all n_i are equal, but one reduces immediately to this case by linearly embedding each $\mathbb{P}^{n_i}$ into a larger projective space of fixed dimension. Note that the first part of the theorem is equivalent to saying that a general choice of L_i of dimension γ_i will yield $X \cap (L_1 \times \cdots \times L_k) \neq \emptyset$ if and only if $\sum_{i\in I}(n_i - \gamma_i) \leq \dim p_I(X)$ for all $I \subsetneq \{1,\dots,k\}$. The second part of the theorem connects multidegrees to polymatroid theory, and says in particular that the function δ is "submodular."

The first part of the theorem implies that the dimensions of the projections determine the support of the multidegree. We see using some standard facts in polymatroid theory that the converse also holds.

Corollary 2.2. *Given*

$$X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$$

irreducible, the data of the support of the multidegree of X is equivalent to the data of the dimensions of $p_I(X)$ for all $I \subseteq \{1,\dots,k\}$.

Proof. Let $\mathcal{P}$ be the polytope cut out by the inequalities of Theorem 2.1, and $\bar{\mathcal{P}}$ the face of $\mathcal{P}$ cut out by the hyperplane $\sum_i(n_i - \gamma_i) = \dim X$. Then $\mathcal{P}$ is known as a "polymatroid", and $\bar{\mathcal{P}}$ is the set of "bases;" see §1 of [HH02]. Moreover, from Proposition 1.3 of [HH02] we see that the vertices of $\bar{\mathcal{P}}$ are integral, and for every I there is some vertex lying in the corresponding bounding hyperplane (in their notation, we take any π such that $I = \{i_1,\dots,i_{|I|}\}$). It follows that we can recover the $\dim p_I(X)$ from the integral points of $\bar{\mathcal{P}}$. Since Theorem 2.1 says that the support of the multidegree is equal to the set of lattice points in $\bar{\mathcal{P}}$, we conclude that it determines the $\dim p_I(X)$, as desired. □

The following standard fact from polymatroid theory will also be helpful. Since the proof is quite short, we include it.

Proposition 2.3. *Suppose we have a function δ from subsets of $\{1,\dots,k\}$ to $\mathbb{Z}_{\geq 0}$ satisfying the conditions in Theorem 2.1, and write $r = \delta(\{1,\dots,k\})$. Suppose also that we are given $\boldsymbol{\beta} = (\beta_1,\dots,\beta_k) \in (\mathbb{Z}_{\geq 0})^k$ with $|\boldsymbol{\beta}| = r+1$, and satisfying that for all $I \subseteq \{1,\dots,k\}$, we have*

$$|\boldsymbol{\beta}_I| \leq \delta(I) + 1. \tag{2.1}$$

Then there exists a nonempty $J \subseteq \{1,\dots,k\}$ such that for all $I \subseteq \{1,\dots,k\}$, we have

$$|\boldsymbol{\beta}_I| = \delta(I) + 1 \tag{2.2}$$

if and only if $I \supseteq J$.

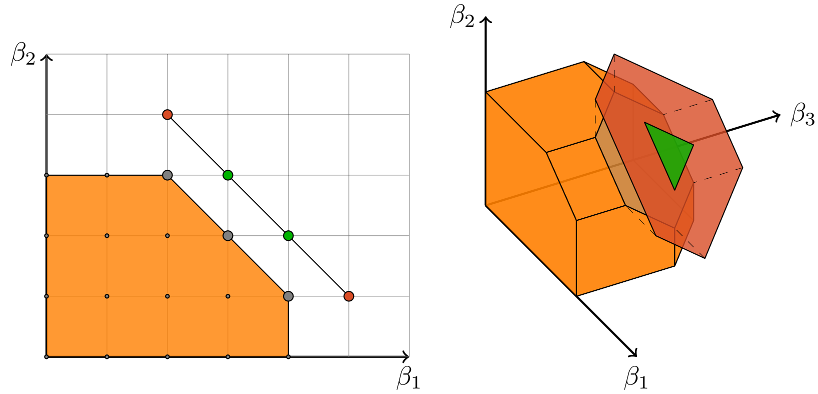

FIGURE 1. Two polymatroids. The sets of bases (corresponding to our multidegree supports) are in gray; while the sets of circuits and of non-circuit 1-deficient vectors (both satisfying $|\boldsymbol{\beta}| = r+1$) are in green and red, respectively.

Proof. First note that if such a J exists, it is necessarily nonempty, since we have assumed $\delta(\emptyset) = 0$. Now, if we have I_1 and I_2 satisfying (2.2), then we see that

$$\begin{aligned}\delta(I_1 \cap I_2) &\leq \delta(I_1) + \delta(I_2) - \delta(I_1 \cup I_2) \\ &\leq |\beta_{I_1}| - 1 + |\boldsymbol{\beta}_{I_2}| - 1 - |\boldsymbol{\beta}_{I_1 \cup I_2}| - 1 \\ &= |\boldsymbol{\beta}_{I_1 \cap I_2}| - 1,\end{aligned}$$

so $I_1 \cap I_2$ also satisfies (2.2). The result follows. □

Remark 2.4. In polymatroid theory, there is a notion of "1-deficient" vectors, and among vectors $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)$ satisfying the conditions that $|\boldsymbol{\beta}| = r + 1$, those satisfying (2.1) are precisely the 1-deficient vectors. This is exactly the condition that arises for us in order for $Z_{X,\boldsymbol{\beta}}$ to be a hypersurface – see Proposition 3.1 below. Now, given such a $\boldsymbol{\beta}$, note that the J of Proposition 2.3 is equal to all of $\{1, \ldots, k\}$ if and only if $\boldsymbol{\beta}$ satisfies the stronger inequalities of Theorem 1.1. This requires that all β_i are strictly positive, and if we restrict our attention to $\boldsymbol{\beta}$ with all $\beta_i > 0$, then the condition that $J = \{1, \ldots, k\}$ is equivalent in the polymatroid language to saying that $\boldsymbol{\beta}$ is a "circuit." Thus, among the vectors $\boldsymbol{\beta}$ with $|\boldsymbol{\beta}| = r + 1$ and all β_i strictly positive, the set of vectors satisfying the inequalities of Theorem 1.1 is exactly the set of circuits of the polymatroid determined by the multidegree of X. See Figure 1 for examples, and §1.2 of [MPS07] for more on the polymatroid terminology.

3. SLICING BY PRODUCTS OF LINEAR SPACES

In this section, we carry out our fundamental analysis of the behavior of slicing with products of general linear spaces. We begin with the following.

Proposition 3.1. *Let $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ be a projective variety of dimension r, and suppose we have $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k)$ with $0 \leq \beta_i \leq n_i$ for $i = 1, \ldots, k$ and $|\boldsymbol{\beta}| = r + 1$. Write $\alpha_i = n_i - \beta_i$ for each i.*

Consider the closed subset

$$Z_{X,\boldsymbol{\beta}} = \{(L_1, \ldots, L_k) : X \cap (L_1 \times \cdots \times L_k) \neq \emptyset\} \subseteq \mathrm{Gr}(\alpha_1, n_1) \times \cdots \times \mathrm{Gr}(\alpha_k, n_k).$$

Then $Z_{X,\boldsymbol{\beta}}$ is a hypersurface if and only if for every nonempty $I \subseteq \{1, \ldots, k\}$ we have

$$\dim p_I(X) \geq |\boldsymbol{\beta}_I| - 1, \tag{3.1}$$

where $p_I(X)$ denotes the projection of X onto $\prod_{i\in I} \mathbb{P}^{n_i}$.

Remark 3.2. Since $|\boldsymbol{\beta}| = r + 1$, we have that (3.1) is equivalent to having $r - \dim p_I(X) \leq |\boldsymbol{\beta}_{I^c}|$, where $I^c = \{1, \ldots, k\} \setminus I$. Hence, when the conditions from the previous Proposition are satisfied, we have that $r - \dim p_{I^c}(X) \leq |\boldsymbol{\beta}_I| \leq \dim p_I(X) + 1$. The former inequality has the geometric interpretation that the generic fiber of X under p_{I^c} has dimension at most $|\boldsymbol{\beta}_I|$.

Proof. Define the incidence correspondence $V_X \subseteq X \times \mathrm{Gr}(\alpha_1, n_1) \times \cdots \times \mathrm{Gr}(\alpha_k, n_k)$ given by

$$V_X = \{(P, L_1, \ldots, L_k) : P \in L_1 \times \cdots \times L_k\} \tag{3.2}$$

Thus, $Z_{X,\boldsymbol{\beta}}$ is the image of V_X under projection to the product of Grassmannians. Considering the projection of V_X to X, we see that V_X is irreducible of dimension $d_1 = r + \sum_{i=1}^{k} \alpha_i(n_i - \alpha_i)$. In particular, $Z_{X,\boldsymbol{\beta}}$ is automatically irreducible. The dimension of the product of Grassmannians is $d_2 = \sum_{i=1}^{k}(\alpha_1 + 1)(n_1 - \alpha_1)$. Because $d_2 - d_1 = 1$, we will have that $Z_{X,\boldsymbol{\beta}}$ is a hypersurface if and only if V_X has generic fiber dimension 0 under the projection to the product of Grassmannians, or equivalently, if there exist $L_1, \ldots, L_k$ such that $X \cap (L_1 \times \cdots \times L_k)$ is finite and nonempty.

First suppose that we have $\dim p_I(X) < |\boldsymbol{\beta}_I| - 1$ for some $I \subseteq \{1, \ldots, k\}$. Let $L_1, \ldots, L_k$ be such that $X \cap (L_1 \times \cdots \times L_k)$ is not empty, and write $L_I = \prod_{i\in I} L_i$, and similarly for L_{I^c}, so that $X \cap (L_1 \times \cdots \times L_k) = (X \cap p_I^{-1} L_I) \cap p_{I^c}^{-1} L_{I^c}$. Since $X \cap p_I^{-1} L_I$ is not empty, its dimension is at least the dimension of the generic fiber of X under p_I, that is $r - \dim p_I(X)$. Hence, $\dim(X \cap p_I^{-1} L_I) \geq r - \dim p_I(X) > r + 1 - |\boldsymbol{\beta}_I| = |\boldsymbol{\beta}_{I^c}|$, while $\mathrm{codim}(p_{I^c}^{-1} L_{I^c}) = |\boldsymbol{\beta}_{I^c}|$. We conclude that $X \cap (L_1 \times \cdots \times L_k)$ has positive dimension, and thus that all the inequalities are necessary in order for $Z_{X,\boldsymbol{\beta}}$ to be a hypersurface.

Conversely, suppose that the stated inequalities are satisfied, and fix a point $P = (P_1, \ldots, P_k) \in X$ such that for all $I \subsetneq \{1, \ldots, k\}$, the fiber $p_I^{-1}(p_I(P)) \cap X$ has dimension less than or equal to $\sum_{i\notin I}(n_i - \alpha_i)$. Note that this is always possible since our inequalities are equivalent to assuming that the minimal fiber dimension $r - \dim p_I(X)$ is less than or equal to the desired value. Then we claim that a general choice of $L_1, \ldots, L_k$ with $P_i \in L_i$ for each i will have $X \cap (L_1 \times \cdots \times L_k)$ nonempty and finite. We prove this by considering the incidence correspondence $Y \subseteq X \times \mathrm{Gr}(\alpha_1, n_1) \times \cdots \times \mathrm{Gr}(\alpha_k, n_k)$ consisting of $(Q, L_1, \ldots, L_k)$ with $L_i \ni P_i$ for all i, and $Q \in L_1 \times \cdots \times L_k$. Considering the case $Q = P$, we see that the image of Y under projection to $\mathrm{Gr}(\alpha_1, n_1) \times \cdots \times \mathrm{Gr}(\alpha_k, n_k)$ is exactly the tuples of L_i containing P_i, which is itself a product of Grassmannians, having dimension $\sum_i \alpha_i(n_i - \alpha_i)$. The finiteness statement we want amounts to showing that Y has generically finite fibers under this projection, or equivalently, that the dimension of Y is no larger than the dimension of its image. We do this by decomposing Y into locally closed subsets Y_I, defined to be the subset of Y on which $Q_i = P_i$ precisely when $i \in I$ (here we allow $I = \emptyset$). We then consider the projection of Y_I onto X. First, $Y_I \neq \emptyset$ if and only if $\alpha_i \geq 1$ for all $i \notin I$. In this case, Y_I maps into $p_I^{-1}(p_I(P)) \cap X \subseteq X$, and every fiber will have dimension equal to

$$\sum_{i \in I} \alpha_i(n_i - \alpha_i) + \sum_{i \notin I} (\alpha_i - 1)(n_i - \alpha_i).$$

We conclude that Y_I has dimension less than or equal to

$$\sum_{i \notin I}(n_i - \alpha_i) + \sum_{i \in I} \alpha_i(n_i - \alpha_i) + \sum_{i \notin I}(\alpha_i - 1)(n_i - \alpha_i) = \sum_i \alpha_i(n_i - \alpha_i),$$

as desired. □

Example 3.3. Consider a product of varieties $X = X_1 \times \cdots \times X_k$, so that each $X_i \subseteq \mathbb{P}^{n_i}$ has dimension r_i, and $\sum_i r_i = r$. If $\boldsymbol{\beta}$ is such that $|\boldsymbol{\beta}| = r + 1$, and $Z_{X,\boldsymbol{\beta}} \subseteq \mathrm{Gr}(\alpha_1, n_1) \times \cdots \times \mathrm{Gr}(\alpha_k, n_k)$ is as defined above, then Proposition 3.1 states that $Z_{X,\boldsymbol{\beta}}$ has codimension 1 if and only if $\sum_{i \in I} r_i \leq |\boldsymbol{\beta}_I| \leq \sum_{i \in I} r_i + 1$ for all I. This can occur if and only if $\beta_j = r_j + 1$ for one index and $\beta_i = r_i$ otherwise (a condition that can also be deduced directly from $Z_{X,\boldsymbol{\beta}}$). In this case, $Z_{X,\boldsymbol{\beta}} = \mathrm{Gr}(\alpha_1, n_1) \times \cdots \times Z_{X_j} \times \cdots \times \mathrm{Gr}(\alpha_k, n_k)$, where Z_{X_j} is the hypersurface arising in the classical Cayley-Chow construction.

Example 3.4. Consider a variety $X_0 \subseteq \mathbb{P}^{n_0}$, of dimension r_0, and let X be the image of X_0 in the diagonal embedding $\mathbb{P}^{n_0} \to (\mathbb{P}^{n_0})^k$. In this case, the conditions in Proposition 3.1 require that $0 \leq |\boldsymbol{\beta}_I| \leq r_0 + 1$. Hence, $Z_{X,\boldsymbol{\beta}}$ is a hypersurface for any choice of $(\beta_1, \ldots, \beta_k)$ summing to $r_0 + 1$. More specifically, we have a surjective rational map

$$\mathrm{Gr}(\alpha_1, n_1) \times \cdots \times \mathrm{Gr}(\alpha_k, n_k) \dashrightarrow \mathrm{Gr}(n_0 - r_0 - 1, n_0)$$

given by intersection of linear spaces, and it is clear that on the open subset where this map is defined, we have that $Z_{X,\boldsymbol{\beta}}$ is the preimage of the hypersurface Z_{X_0} arising in the classical Cayley-Chow construction. By irreducibility of $Z_{X,\boldsymbol{\beta}}$, we see that it must simply be the closure of the preimage of Z_{X_0}.

Example 3.5. Suppose that for some $I \subsetneq \{1, \ldots, k\}$, we have $X' \subseteq \prod_{i \in I} \mathbb{P}^{n_i}$ such that $X = X' \times \prod_{i \notin I} \mathbb{P}^{n_i}$ (equivalently, $X = p_I^{-1}(p_I(X))$). Write r' for the dimension of X', and suppose we have $\boldsymbol{\beta}$ satisfying (3.1). Then we have $|\boldsymbol{\beta}_I| \leq \dim p_I(X) + 1 = r' + 1$, so we must have $|\beta_{I^c}| \geq \sum_{i \in I^c} n_i$, and then it follows that $\beta_i = n_i$ for all $i \in I^c$, and also that $|\beta_I| = r' + 1$. In particular, the inequalities of Theorem 1.1 are violated. However, one easily verifies that

$$Z_{X,\boldsymbol{\beta}} = Z_{X',\boldsymbol{\beta}_I} \times \prod_{i \notin I} \mathrm{Gr}(\alpha_i, n_i),$$

so the study of $Z_{X,\boldsymbol{\beta}}$ in this case reduces to the study of $Z_{X',\boldsymbol{\beta}_I}$, and in particular X can be recovered from $Z_{X,\boldsymbol{\beta}}$ if and only if X' can be recovered from $Z_{X',\boldsymbol{\beta}_I}$.

We now analyze when X can be recovered from $Z_{X,\boldsymbol{\beta}}$.

Proposition 3.6. *In the situation of Proposition 3.1 (and in particular assuming (3.1)), we have that X is uniquely determined by $Z_{X,\boldsymbol{\beta}}$ if*

$$\dim p_I(X) \geq |\boldsymbol{\beta}_I| \tag{3.3}$$

for all $I \subsetneq \{1, \ldots, k\}$.

Conversely, if X is not of the form of Example 3.5, and X is uniquely determined by $Z_{X,\boldsymbol{\beta}}$, then (3.3) *is satisfied.*

Proof. Let S_Z be the set of points $(P_1,\dots,P_k)$ with the property that every $L_1\times\cdots\times L_k$ containing $(P_1,\dots,P_k)$ has $X\cap(L_1\times\cdots\times L_k)\neq\emptyset$. Then obviously $X\subseteq S_Z$. We claim that if (3.3) is satisfied, then

$$S_Z\subseteq X\cup\bigcup_{I\subsetneq\{1,\dots,k\}}p_I^{-1}(X_I) \tag{3.4}$$

where $X_I\subseteq p_I(X)$ is the closed subset over which the fibers of X under p_I have dimension greater than or equal to $\sum_{i\notin I}(n_i-\alpha_i)$. Note that (3.3) implies that $X_I\neq p_I(X)$. From the claim we see that – given the multidegree of X – we can recover X from S_Z: indeed, by Corollary 2.2 the multidegree of X determines $\dim p_I(X)$ for all I, and we see that every other potential component of S_Z has dimension strictly smaller than X under at least one projection.

To prove the claim, fix $P=(P_1,\dots,P_k)\notin X$ and with $p_I(P)\notin X_I$ for all $I\subsetneq\{1,\dots,k\}$. We wish to show that $P\notin S_Z$, or equivalently, that there exist $L_1,\dots,L_k$ with $P_i\in L_i$ for all i, and with $X\cap(L_1\times\cdots\times L_k)=\emptyset$. The proof is similar to the proof of Proposition 3.1: consider the incidence correspondence $Y\subseteq X\times\mathrm{Gr}(\alpha_1,n_1)\times\cdots\times\mathrm{Gr}(\alpha_k,n_k)$ consisting of $(Q,L_1,\dots,L_k)$ with $L_i\ni P_i$ for all i, and $Q\in L_1\times\cdots\times L_k$. In this case, we wish to show that the image of Y under projection to $\mathrm{Gr}(\alpha_1,n_1)\times\cdots\times\mathrm{Gr}(\alpha_k,n_k)$ does not contain all tuples of L_i containing P_i, so it will suffice to show that Y has dimension strictly smaller than $\sum_i\alpha_i(n_i-\alpha_i)$. We decompose Y into the subsets Y_I as before. For $I\subsetneq\{1,\dots,k\}$, just as before Y_I has image contained in $p_I^{-1}(p_I(P))\cap X\subseteq X$, with every fiber having dimension equal to

$$\sum_{i\in I}\alpha_i(n_i-\alpha_i)+\sum_{i\notin I}(\alpha_i-1)(n_i-\alpha_i).$$

We conclude that Y_I has dimension less than or equal to

$$\sum_{i\notin I}(n_i-\alpha_i)-1+\sum_{i\in I}\alpha_i(n_i-\alpha_i)+\sum_{i\notin I}(\alpha_i-1)(n_i-\alpha_i)=\sum_i\alpha_i(n_i-\alpha_i)-1,$$

as desired.

Conversely, suppose that we have some $I\subsetneq\{1,\dots,k\}$ such that $|\boldsymbol{\beta}_I|>\dim p_I(X)$. Then we have that for a general choice of L_i for $i\in I$, the intersection $X\cap\bigcap_{i\in I}p_i^{-1}L_i$ is empty. Thus, $p_I(Z_{X,\boldsymbol{\beta}})$ is a proper subset of $\prod_{i\in I}\mathrm{Gr}(\alpha_i,n_i)$. Since $Z_{X,\boldsymbol{\beta}}$ is a hypersurface, it must be equal to $p_I(Z_{X,\boldsymbol{\beta}})\times\prod_{i\notin I}\mathrm{Gr}(\alpha_i,n_i)$. We conclude that $Z_{X,\boldsymbol{\beta}}$ is invariant under applying automorphisms of $\mathbb{P}^{n_i}$ for $i\notin I$, and since we have assumed that X is not of the form of Example 3.5, it follows that X cannot be recovered from $Z_{X,\boldsymbol{\beta}}$. □

Note that (unlike in the classical case), it may really be the case that the S_Z in the above proof contains components other than X; see Example 5.15 below.

Remark 3.7. In the case that the inequalities (3.1) are satisfied, we can understand the hypersurface $Z_{X,\boldsymbol{\beta}}$ as follows: according to Proposition 2.3, there a nonempty $J\subseteq\{1,\dots,k\}$ which is minimal – in the strong sense – satisfying

$$|\boldsymbol{\beta}_J|=\dim p_J(X)+1.$$

In this case, $X' = p_J(X)$ will satisfy our setup with the stronger inequalities (3.3), so X' can be recovered from $Z_{X',\boldsymbol{\beta}_J}$. Furthermore, we will have $Z_{X,\boldsymbol{\beta}}$ equal to the product of $Z_{X',\boldsymbol{\beta}_J}$ with all the $\mathrm{Gr}(\alpha_i, n_i)$ for $i \notin J$, so the information in $Z_{X,\boldsymbol{\beta}}$ is exactly equal to the data of $p_J(X)$.

4. Tensor products of unique factorization domains

In the classical setting, the fact that the hypersurface Z_X is the zero set of a single polynomial F_X in Plücker coordinates is a consequence of the fact that the homogeneous coordinate ring of a Grassmannian in Plücker coordinates is a unique factorization domain (UFD). We will make use of this to conclude the same statement in our case, but this requires a certain amount of care, as the condition of being a UFD is not very stable (for instance, there are examples where R is a UFD, but the power series right $R[[t]]$ is not). We address this with the following proposition, which states that under relatively mild additional hypotheses, Gauss' argument for unique factorization in a polynomial ring over a UFD extends to more general tensor products.

For the following proposition, we temporarily drop the hypothesis that we are working over a field K.

Proposition 4.1. *Let A be a ring, B and C algebras over A, and suppose that B is a Noetherian UFD, C is flat and finitely generated over A, and for every field K over A, we have that $K \otimes_A C$ is a UFD, with unit group equal to $K^{\times}$. Then $B \otimes_A C$ is a UFD.*

Proof. First observe that under our hypotheses, if B' is an A-algebra with fraction field K', then we have injections $K' \to K' \otimes_A C$ and $B' \to B' \otimes_A C \to K' \otimes_A C$. Indeed, injectivity of the last map follows from flatness of C, while injectivity of the first is a consequence of the implicit hypothesis that $K' \otimes_A C$, being a domain, is not the zero ring. Injectivity of the map $B' \to B' \otimes_A C$ follows from the injectivity of the first map. Note also that $B \otimes_A C$ is finitely generated over a Noetherian ring, hence Noetherian, so it suffices to show that every irreducible element is prime.

Our first claim is that if $x \in B$ is a prime element, then $x \otimes 1$ is prime in $B \otimes_A C$, and given $y \in B$ we have that y is a multiple of x if and only if $y \otimes 1$ is a multiple of $x \otimes 1$ in $B \otimes_A C$. Indeed, if we let $B' = B/(x)$ and apply the above, the second statement follows immediately from the injectivity of $B/(x) \to B/(x) \otimes_A C = (B \otimes_A C)/(x \otimes 1)$, while the injection $B/(x) \otimes_A C \to K' \otimes_A C$ together with the hypothesis that $K' \otimes_A C$ is an integral domain implies that $x \otimes 1$ is prime. From the second part of the claim, we can conclude that if K is the fraction field of B, then the intersection of K with $B \otimes_A C$ inside of $K \otimes_A C$ is equal to B. We also see that conversely if $x \otimes 1$ is irreducible in $B \otimes_A C$ then x must be irreducible in B. In this case, x is prime in B, and hence $x \otimes 1$ is prime in $B \otimes_A C$.

Now, suppose x is irreducible in $B \otimes_A C$, and consider the image of x in $K \otimes_A C$. If x becomes a unit in $K \otimes_A C$, then by hypothesis it is of the form $y \otimes 1$ for some $y \in K^{\times}$, so we have from the above that $y \in B$, so $x = y \otimes 1$ is prime in $B \otimes_A C$.

Next, we see that any nonzero element of $K \otimes_A C$ can be written uniquely up to $B^{\times}$ in the form αf, for $\alpha \in K^*$, and $f \in B \otimes_A C$ with the property that f is not a multiple of any non-unit in B. Indeed, every element of $K \otimes_A C$ can be multiplied by an element of B to clear denominators, so is of the form αf where $f \in B \otimes_A C$. Obviously, if f is a multiple of a non-unit in B, we can absorb it into α, so we can write the element in the desired form. Uniqueness follows by taking two such representations, clearing denominators, and using that primes in B remain prime in $B \otimes_A C$.

It then follows that if we have an irreducible element of $B \otimes_A C$ which does not become a unit in $K \otimes_A C$, then it must remain irreducible in $K \otimes_A C$. Indeed, a nontrivial factorization in $K \otimes_A C$ can be represented as $(\alpha_1 f_1)(\alpha_2 f_2)$ as above, and then we see that since f_1 and f_2 are not multiples of any non-units of B, the same is true of $f_1 f_2$. Thus, the hypothesis that $\alpha_1\alpha_2 f_1 f_2 \in B \otimes_A C$ implies that $\alpha_1\alpha_2$ can't have any denominators, and then $(\alpha_1\alpha_2 f_1) f_2$ gives a nontrivial factorization in $B \otimes_A C$. Finally, we conclude that any such irreducible element must be prime: it is prime in $K \otimes_A C$ by hypothesis, so if it divides a product in $B \otimes_A C$, it divides one of the factors in $K \otimes_A C$. But again using the above representation, we see that it must also divide the same factor in $B \otimes_A C$. We thus conclude that every irreducible element of $B \otimes_A C$ is prime, and hence that $B \otimes_A C$ is a UFD. □

Returning to varieties over K, we then conclude the desired statement on hypersurfaces in products of Grassmannians.

Corollary 4.2. *Let $G := G_1 \times \cdots \times G_k \subseteq \mathbb{P}^{N_1} \times \cdots \times \mathbb{P}^{N_k}$ be a product of Plücker embeddings of Grassmannians, and let $Z \subseteq G$ be a hypersurface. Then $Z = Z(F)$ for some multihomogeneous form F.*

Proof. First, we have that Z corresponds to a multihomogeneous prime ideal of height 1 in $S(G)$, the multihomogeneous coordinate ring of G. Now, we claim that $S(G)$ is a UFD. Since $S(G) = S(G_1) \otimes_K \cdots \otimes_K S(G_k)$, we will prove this by induction on k, using Proposition 4.1. The base case is exactly the classical case; see Proposition 2.1 of Chapter 3 of [GKZ94]. Thus, we need only observe that $S(G_k)$ over K satisfies the hypotheses of the proposition, most of which are immediate: flatness is automatic over K, the hypothesis on the units comes from the fact that $S(G_k)$ is the homogeneous coordinate ring of a projective variety, and the hypothesis that for any field extension K' over K we have that $K' \otimes_K S(G_k)$ is a UFD also follows from the classical case, since $K' \otimes_K S(G_k)$ is simply the homogeneous coordinate ring of the Plücker embedding over K'. We thus conclude that $S(G)$ is a UFD, and therefore that $Z = Z(F)$ for some $F \in S(G)$. Finally, F must be multihomogeneous, or it could not generate a multihomogeneous prime ideal. □

5. (Multi)degrees and Cayley-Chow forms

We are now ready to define multigraded Cayley-Chow forms. In order to obtain good behavior of multidegrees, there is one additional twist to consider. Namely, unlike in the classical case, it is possible that the Cayley-Chow form naturally has a multiplicity greater than 1.

Definition 5.1. In the situation of Proposition 3.1, suppose also that (3.1) is satisfied. Let $\epsilon_{X,\boldsymbol{\beta}}$ be the degree of the map $V_X \to Z_{X,\boldsymbol{\beta}}$, where V_X is as in (3.2).

Then let F_X be a multihomogeneous polynomial in multi-Plücker coordinates with $Z(F_X) = Z_{X,\boldsymbol{\beta}}$ (Corollary 4.2), and define the **multigraded Cayley-Chow form** of X to be

$$H_{X,\boldsymbol{\beta}} := F_X^{\epsilon_{X,\boldsymbol{\beta}}}.$$

Recall that V_X is the incidence correspondence consisting of a point of X together with a tuple of linear spaces containing the coordinates of the point. Then the map to $Z_{X,\boldsymbol{\beta}}$ is simply the map forgetting the point of X.

Without further hypotheses, it may certainly be the case that $\epsilon_{X,\boldsymbol{\beta}} > 1$.

Example 5.2. Let $X = C_1 \times C_2 \subseteq \mathbb{P}^2 \times \mathbb{P}^2$, where each C_i is the curve defined by a homogeneous form F_i of degree d_i. In order for (3.1) to be satisfied, we need to have either $\boldsymbol{\beta} = (1,2)$ or $\boldsymbol{\beta} = (2,1)$. In the first case, we see that if we have fixed L, P with $X \cap (L \times P) \neq \emptyset$, then in fact $X \cap (L \times P)$ contains d_1 points, so $\epsilon_{X,\boldsymbol{\beta}} = d_1$. Meanwhile $Z_{X,\boldsymbol{\beta}}$ depends only on P, and the F_X of Definition 5.1 is simply F_2. Thus, $H_{X,\boldsymbol{\beta}} = F_2^{d_1}$. Similarly, if $\boldsymbol{\beta} = (2,1)$, we find $H_{X,\boldsymbol{\beta}} = F_1^{d_2}$.

For a more interesting example in positive characteristic, see Example 5.13 below. However, we have the following.

Proposition 5.3. *In the situation of Definition 5.1, suppose further that the inequalities of* (3.3) *are satisfied. Then the map $V_X \to Z_{X,\boldsymbol{\beta}}$ is generically injective, and if K has characteristic* 0*, we have $\epsilon_{X,\boldsymbol{\beta}} = 1$.*

Proof. The generic injectivity amounts to saying that if $P = (P_1, \dots, P_k) \in X$ is general, then general choices of L_i containing P_i will have $X \cap (L_1 \times \cdots \times L_k) = \{P\}$. We recall that (3.3) implies that for all $I \subsetneq \{1, \dots, k\}$, the generic fiber dimension of X under p_I is strictly less than $|\boldsymbol{\beta}_{I^c}|$. We prove by induction on k the following slightly more general statement: suppose that $Y \subseteq \prod_i \mathbb{P}^{n_i}$ is a pure-dimensional algebraic set such that for every $I \subsetneq \{1, \dots, k\}$, every component of Y has generic fiber dimension under p_I strictly less than $|\boldsymbol{\beta}_{I^c}|$. Then there exists a dense open subset U of Y such that for every $P = (P_1, \dots, P_k) \in U$, a general choice of L_i containing P_i will have $Y \cap (L_1 \times \cdots \times L_k) = \{P\}$. Note that we allow $I = \emptyset$ in our hypotheses, which says simply that Y has dimension strictly smaller than $|\boldsymbol{\beta}|$.

We first observe that the desired statement reduces to the case that Y is irreducible. Indeed, if $Y_1, \dots, Y_n$ are the components of Y, and if we construct $S_{Z,i}$ from each Y_i as in the proof of Proposition 3.6 (with Y_i in place of X), then (3.4) together with our hypotheses on the Y_i implies that we cannot have $Y_i \subseteq S_{Z,i}$ for any distinct i, j. To see this, if we write $Y_{I,i}$ in place of X_I, and write $r = \dim Y = \dim Y_i$, we see that each $Y_{I,i}$ must have dimension strictly less than $r - |\boldsymbol{\beta}_{I^c}|$, while our hypotheses imply that $\dim p_I(Y_j) > r - |\boldsymbol{\beta}_{I^c}|$, so we cannot have $p_I(Y_j) \subseteq Y_{I,i}$. Thus, if we suppose that U_i satisfies the desired conditions for each Y_i separately, we then see that

$$\bigcup_i \left(U_i \smallsetminus \bigcup_{j \neq i} S_{Z,j} \right)$$

satisfies the desired condition for all of Y. Consequently, for simplicity we will henceforth assume Y is irreducible.

Then the base case our of induction is $k = 1$, which is exactly the classical situation, and is proved by considering projection from any point in Y. For induction, we claim that the general fiber of Y under p_k satisfies our hypotheses for $k-1$. First, since Y is irreducible the general fiber will be pure-dimensional. Next, for any fixed $P_k \in p_k(Y)$, and $I' \subsetneq \{1, \dots, k-1\}$, if Y_{P_k} denotes the fiber of Y over P_k, and if we set $I = I' \cup \{k\}$, we see that for any $(P_i)_{i \in I'} \in p_{I'}(Y_{P_k})$, the fiber of Y_{P_k} over $(P_i)_{i \in I'}$ is equal to the fiber of Y over $(P_i)_{i \in I}$. Note that $|\boldsymbol{\beta}_{I^c}|$ is the desired dimension bound also for $k-1$, since now we take the complement of I'

in $\{1,\ldots,k-1\}$. By hypothesis, there is an open subset $U_I \subseteq p_I(Y)$ such that the fiber dimension of Y over any point of U_I is strictly less than $|\boldsymbol{\beta}_{I^c}|$. Then we observe that there is an open subset V_I of $p_k(Y)$ such that $p_I^{-1}(U_I)$ is dense in every fiber $Y \cap p_k^{-1}(Q)$ for $Q \in V_I$: indeed, this follows from constructibility of images, together with semicontinuity of fiber dimension, since the only way that $p_I^{-1}(U_I)$ can fail to be dense in the fiber over some $Q \in p_k(Y)$ is if the dimension of $Y \smallsetminus p_I^{-1}(U_I)$ over Q is strictly larger than the generic fiber dimension of $Y \smallsetminus p_I^{-1}(U_I)$ over $p_k(Y)$. Then a general P_k will not only yield Y_{P_k} pure-dimensional, but will also lie in every V_I, so we see that every component of Y_{P_k} necessarily satisfies the desired generic fiber dimension bound.

Now, we note that the set of $(P, L_1, \ldots, L_k) \in Y \times \mathrm{Gr}(\alpha_1, n_1) \times \cdots \times \mathrm{Gr}(\alpha_k, n_k)$ such that $Y \cap (L_1 \times \cdots \times L_k) = \{P\}$ is constructible: indeed, this follows from semicontinuity of fiber dimension, properness, and constructibility of connected fibers (see Theorem 9.7.7 of [GD66]).[1] Thus, to prove the desired statement, it suffices to show that this set is Zariski dense inside the locus of points satisfying $P \in L_1 \times \cdots \times L_k$. If this were not the case, it would be contained in a proper Zariski closed subset Z. We would then have that a dense open subset of Y has the property that Z does not fully contain any of the fibers over that subset: this is, there would be a dense open subset of points $P \in Y$ such that the choices of L_i containing P_i and having $Y \cap (L_1 \times \cdots \times L_k) = \{P\}$ are contained in a proper closed subset.

On the other hand, by the above claim and the induction hypothesis, if we fix a general $P_k \in p_k(Y)$, and let Y_{P_k} be the corresponding fiber, then we know that for $(P_1, \ldots, P_{k-1})$ general in Y_{P_k}, and L_i general containing P_i for $i = 1, \ldots, k-1$, we have $Y_{P_k} \cap (L_1 \times \cdots \times L_{k-1}) = \{(P_1, \ldots, P_{k-1})\}$. We also know that

$$\dim p_k(Y \cap (L_1 \times \cdots \times L_{k-1} \times \mathbb{P}^{n_k})) < \beta_k,$$

so $Y \cap (L_1 \times \cdots \times L_{k-1} \times \mathbb{P}^{n_k})$ will meet a general L_k containing P_k only in P_k. Thus, we will have $Y \cap (L_1 \times \cdots \times L_k) = \{P\}$. Given the generality of P_i and L_i, this proves the desired statement.

We have thus proved the generic injectivity statement in general. To prove that $\epsilon_{X,\boldsymbol{\beta}} = 1$ in characteristic 0, we use the Bertini theorem given as Corollary 5 of [Kle74]: a general divisor in a basepoint-free linear system is smooth at all smooth points of the ambient scheme. In particular, intersecting a generically reduced scheme with the preimage of a general hyperplane in any of the $\mathbb{P}^{n_i}$ will yield another generically reduced scheme. Applying this inductively, if we fix general $L_1, \ldots, L_{k-1}$, and general L'_k of dimension $\alpha_k + 1$, then

$$X \cap (L_1 \times \cdots \times L_{k-1} \times L'_k)$$

will consist of a finite number of reduced points, and the same will still be true if we further intersect with any L_k of codimension 1 in L'_k. Since we have already shown that such an intersection consists of a single point, we conclude that it is in fact a single reduced point, and $\epsilon_{X,\boldsymbol{\beta}} = 1$. □

Example 5.4. The following example demonstrates the delicacy of the inductive statement proved in Proposition 5.3: given $d_1, d_2 > 1$, let S_1, S_2 be surfaces of

[1] Here we are considering only classical points; the correct statement for schemes involves geometrically connected fibers, but since we work over an algebraically closed field and consider only classical points, this amounts to the same thing.

degrees d_1, d_2 respectively in $\mathbb{P}^3$, and C_1, C_2 curves in S_1, S_2. Set

$$Y = (C_1 \times S_2) \cup (S_1 \times C_2),$$

and $\beta_1 = \beta_2 = 2$. Then $p_1(Y)$ and $p_2(Y)$ are both irreducible, and the "generic fibers" of both p_i have dimension 1 (which is strictly smaller than either β_i), in the sense that there are open dense subsets of each $p_i(Y)$ on which the fiber dimension is 1. Thus, Y satisfies a slight variant of the induction statement, but we see that any $L_1 \times L_2$ meeting Y has to contain at least $\min(d_1, d_2)$ points.

In order to complete the proofs of the basic properties of our generalized Cayley-Chow construction, it will be helpful to extend it to cycles. The main reason for this is that even if X is irreducible, its intersections with general products of linear spaces are not always irreducible (Bertini theorems imply that they usually are, but they will not be if for instance some projection has 1-dimensional image). Some preliminary notation is the following.

Notation 5.5. If $X \subseteq Y$ is a pure-dimensional closed subscheme of a smooth variety, denote by $[X]$ the associated cycle. If Ξ, Ξ' are cycles on Y which meet in the expected codimension, write $\Xi \cdot \Xi'$ for the induced intersection cycle (see for instance Serre's definition of intersection multiplicity on p. 427 of [Har77]).

Note that we do not work up to rational (or other) equivalence; the point of introducing the notation is that it is not always true that $[X] \cdot [X'] = [X \cap X']$, even when X and X' intersect in the expected dimension. In our situations, we will have $[X] \cdot [X'] = [X \cap X']$ due to generality hypotheses, but we will have to justify this point.

Definition 5.6. Given $n_1, \dots, n_k$, r and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)$ with $|\boldsymbol{\beta}| = r + 1$ and $0 \leq \beta_i \leq n_i$ for all i, linearly extend the construction $X \mapsto \epsilon_{X,\boldsymbol{\beta}}[Z_{X,\boldsymbol{\beta}}]$ to effective r-cycles on $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ by using our previous construction for those components satisfying (3.1), and extending by zero for any additional components. Extend the resulting Cayley-Chow form construction multiplicatively. For an effective r-cycle Ξ, denote the resulting constructions by $Z_{\Xi,\boldsymbol{\beta}}$ and $H_{\Xi,\boldsymbol{\beta}}$ respectively.

Note that we are incorporating the multiplicities into our new notation, so that if $\Xi = [X]$, we have $Z_{\Xi,\boldsymbol{\beta}} = \epsilon_{X,\boldsymbol{\beta}}[Z_{X,\boldsymbol{\beta}}]$.

Remark 5.7. Observe that we can rephrase Definition 5.1 as saying that we are taking the form cutting out $\epsilon_{X,\boldsymbol{\beta}}[Z_{X,\boldsymbol{\beta}}]$, and the latter is $\pi_*[V_X]$, where π is the projection to the product of Grassmannians. Definition 5.6 allows us to extend this as follows: if Ξ is an effective r-cycle, we can construct an incidence correspondence cycle V_Ξ on $\left(\prod_{i \in I^c} \mathbb{P}^{n_i}\right) \times \left(\prod_{i \in I^c} \mathrm{Gr}(\alpha_i, n_i)\right)$ by linearly extending our previous construction, and our extension by zero in Definition 5.6 means that we will still have $Z_{\Xi,\boldsymbol{\beta}} = \pi_* V_\Xi$. Indeed, according to the proof of Proposition 3.1, a component of Ξ fails to satisfy (3.1) precisely when the corresponding component of V_Ξ drops dimension under π.

We then have the following description of the behavior of multigraded Cayley-Chow forms under partial evaluation.

Proposition 5.8. *Let Ξ be an effective r-cycle as in Definition 5.6, and $H_{\Xi,\boldsymbol{\beta}}$ its associated multigraded Cayley-Chow form. For any $I \subsetneq \{1, \dots, k\}$, given general $(L_i)_{i \in I}$ of codimensions β_i, set $L_I := \prod_{i \in I} L_i$. Then the partial evaluation of*

$H_{\Xi,\boldsymbol{\beta}}$ at the L_i for $i \in I$ yields the multigraded Cayley-Chow form associated to $p_{I^c*}([p_I^{-1}(L_I)] \cdot \Xi)$ and $\boldsymbol{\beta}_{I^c}$.

In the above, if $p_I^{-1}(L_I)$ does not meet the support of Ξ, we should interpret the associated multigraded Cayley-Chow form to be constant.

Remark 5.9. Note that if X is a subvariety, the generality of L_I implies that $p_I^{-1}(L_I) \cap X$ is necessarily pure-dimensional, of codimension $|\boldsymbol{\beta}_I|$ in X, so it is reasonable to pass to the associated cycle, and apply pushforward of cycles. If X is not Cohen-Macaulay, then even when $p_I^{-1}(L_I)$ meets X in the expected codimension, we could *a priori* have that $[p_I^{-1}(L_I) \cap X] \neq [p_I^{-1}(L_I)] \cdot [X]$, so we have to be slightly careful with our arguments. However, we see that with L_I general, this will not be the case: the non-smooth locus of X is strictly smaller-dimensional, so again using generality of L_I, we see that every component of $p_I^{-1}(L_I) \cap X$ must have a dense open subset inside the smooth locus of X. But $p_I^{-1}(L_I)$ is also smooth, so we conclude that in this case, the intersection multiplicities of every component of $p_I^{-1}(L_I) \cap X$ are simply determined by the lengths of the intersected scheme, which is to say that $[p_I^{-1}(L_I) \cap X] = [p_I^{-1}(L_I)] \cdot [X]$.

We also note that the proof of Proposition 3.1 shows that if X satisfies (3.1), and if $p_I^{-1}(L_I) \cap X \neq \emptyset$ (still assuming L_I general), then we will have that $p_I^{-1}(L_I) \cap X$ has generically finite fibers under p_{I^c}, so that $\dim p_{I^c}(p_I^{-1}(L_I) \cap X) = \dim p_I^{-1}(L_I) \cap X$. Indeed, if $p_I^{-1}(L_I)$ is nonempty for a general L_I, this means that V_X maps dominantly to $\prod_{i \in I} \mathrm{Gr}(\alpha_i, n_i)$ under $p_I \circ \pi$, where π is projection to the product of Grassmannians. On the other hand, the proof of Proposition 3.1 implies that there is a dense open subset $U \subseteq V_X$ on which projection to $\prod_i \mathrm{Gr}(\alpha_i, n_i)$ is finite. Thus, a general L_I is in the image of U, meaning that there exist L_i for $i \in I^c$ such that $X \cap (L_1 \times \cdots \times L_k)$ is (nonempty and) finite. In particular, we must have that $p_I^{-1}(L_I) \cap X$ has finite fiber over $(L_i)_{i \in I^c}$, as claimed.

Proof of Proposition 5.8. Both sides being multiplicative, the desired identity reduces immediately to the case that $\Xi = [X]$, with X a subvariety. Let $\widetilde{Y} = p_I^{-1}(L_I) \cap X$, let $\Psi = p_{I^c*}[\widetilde{Y}]$, and let $\bar{L}_I \in \prod_{i \in I} \mathrm{Gr}(\alpha_i, n_i)$ be the point determined by the L_i. Then on the level of underlying sets, we see that $Z_{\Psi,\boldsymbol{\beta}_{I^c}}$ is given by the fiber of $Z_{X,\boldsymbol{\beta}}$ over $\bar{L}_I$, which in turn is the vanishing cycle of partial evaluation of $H_{X,\beta}$ at $\bar{L}_I$. Thus, we need to see that the associated multiplicities behave as expected. Let V_X be the incidence correspondence in $(\prod_i \mathbb{P}^{n_i}) \times (\prod_i \mathrm{Gr}(\alpha_i, n_i))$, and V_Ψ be the incidence correspondence cycle as in Remark 5.7, so that we have $Z_{[X],\boldsymbol{\beta}} = \pi_* V_X$ and $Z_{\Psi,\boldsymbol{\beta}_{I^c}} = \pi_* V_\Psi$ (although note that the two maps π are onto different products of Grassmannians).

Write $\widetilde{L}_I \subseteq \prod_i \mathrm{Gr}(\alpha_i, n_i)$ for the fiber $p_I^{-1}(\bar{L}_I)$, and let $V_{\widetilde{Y}} \subseteq (\prod_i \mathbb{P}^{n_i}) \times \left(\prod_{i \in I^c} \mathrm{Gr}(\alpha_i, n_i)\right)$ be the scheme-theoretic incidence correspondence, which we will consider as lying in $(\prod_i \mathbb{P}^{n_i}) \times (\prod_i \mathrm{Gr}(\alpha_i, n_i))$ using the point $\bar{L}_I$. One then checks easily that $V_{\widetilde{Y}} = V_X \cap \pi^{-1}(\widetilde{L}_I)$, for instance by comparing the functors of points. Next, we note that because of the generality of L_I, we have $[V_X \cap \pi^{-1}(\widetilde{L}_I)] = [V_X] \cdot [\pi^{-1}(\widetilde{L}_I)]$. Indeed, $V_X \cap \pi^{-1} \widetilde{L}_I$ is simply the fiber of V_X over a general point of $\prod_{i \in I} \mathrm{Gr}(\alpha_i, n_i)$, and the non-Cohen-Macaulay locus of V_X is a proper algebraic subset, hence of strictly smaller dimension. Semicontinuity of fiber dimension then implies that no component of a general fiber is entirely contained in the non-Cohen-Macaulay locus of V_X, and we obtain the desired identity as in Remark 5.9. The same argument shows that $[Z_{X,\boldsymbol{\beta}} \cap \widetilde{L}_I] = [Z_{X,\boldsymbol{\beta}}] \cdot [\widetilde{L}_I]$.

We next claim that $V_\Psi = (p_{I^c} \times p_{I^c})_*[V_{\widetilde{Y}}]$. This is clear again on the level of underlying sets, so we just need to verify that the multiplicities agree. By construction, $V_{\widetilde{Y}}$ is smooth over $\widetilde{Y}$, so inherits the same multiplicities, and the two pushforwards under p_{I^c} visibly have the same fibers, so the claim follows.

We then have

$$\begin{aligned} Z_{\Psi,\boldsymbol{\beta}_{I^c}} &= \pi_* V_\Psi = \pi_*(p_{I^c} \times p_{I^c})_*[V_{\widetilde{Y}}] = p_{I^c*}\pi_*[V_X \cap \pi^{-1}(\widetilde{L}_I)] \\ &= p_{I^c*}\pi_*([V_X] \cdot [\pi^{-1}(\widetilde{L}_I)]) = p_{I^c*}((\pi_*[V_X]) \cdot [\widetilde{L}_I]) \\ &= p_{I^c*}([Z_{[X],\boldsymbol{\beta}}] \cdot [\widetilde{L}_I]) = p_{I^c*}[Z_{[X],\boldsymbol{\beta}} \cap \widetilde{L}_I], \end{aligned}$$

where the fifth equality is the projection formula on the level of cycles; see Proposition 8.1.1(c) of [Ful98]. Note that in the final expression, we are applying p_{I^c*} to a cycle already supported in a fiber of p_I, so this is just a formality, and we obtain the desired expression. □

We now conclude the desired assertion on multidegrees of Cayley-Chow forms. We can extend multidegree linearly to cycles, so we state the result in that context.

Corollary 5.10. *Under the hypotheses of Proposition 5.8, suppose that* Ξ *has multidegree* $\sum_{\boldsymbol{\gamma}} a_{\boldsymbol{\gamma}} t_1^{\gamma_1} \cdots t_k^{\gamma_k}$. *Then given* $\boldsymbol{\beta}$, *the Cayley-Chow form* $H_{\Xi,\boldsymbol{\beta}}$ *has multidegree*

$$(a_{\alpha_1+1,\alpha_2,\dots,\alpha_k}, \dots, a_{\alpha_1,\dots,\alpha_{k-1},\alpha_k+1}).$$

Proof. By linearity, it suffices to treat the case that $\Xi = [X]$ for a subvariety X. For each $j \in \{1, \dots, k\}$, we wish to show that the degree of $H_{X,\boldsymbol{\beta}}$ in the jth set of variables is equal to

$$a_{\alpha_1,\dots,\alpha_{j-1},\alpha_j+1,\alpha_{j+1},\dots,\alpha_k}. \tag{5.1}$$

Setting $I = \{1, \dots, k\} \smallsetminus \{j\}$ and applying Proposition 5.8, it suffices to show that the classical Cayley-Chow form of $p_{j*}(p_I^{-1}(L_I) \cap X))$ has degree given by (5.1) (note that it necessarily has the expected dimension; see Remark 5.9). By the classical theory (Proposition 2.1 and 2.2 of [GKZ94]), we thus want to show that $p_{j*}(p_I^{-1}(L_I) \cap X)$ has degree in $\mathbb{P}^{n_j}$ given by (5.1). But this follows from the definitions and the projection formula in intersection theory (see for instance p. 426 of [Har77]). □

Finally, using the work of Castillo, Li and Zhang [CLZ], we can also translate the inequalities (3.1) and (3.3) into multidegree-based criteria as follows.

Corollary 5.11. *If* X *has multidegree* $\sum a_{\boldsymbol{\gamma}} t_1^{\gamma_1} \cdots t_k^{\gamma_k}$, *then the* $Z_{X,\boldsymbol{\beta}}$ *associated to* $(\beta_1, \dots, \beta_k) = (n_1 - \alpha_1, \dots, n_k - \alpha_k)$ *is a hypersurface if and only if*

$$a_{\alpha_1+1,\dots,\alpha_k} t_1 + \cdots + a_{\alpha_1,\dots,\alpha_k+1} t_k \tag{5.2}$$

is not identically zero. Moreover, $Z_{X,\boldsymbol{\beta}}$ *determines* X *if and only if every term of* (5.2) *is nonzero.*

Proof. If some n-tuple $(\alpha_1, \dots, \alpha_j + 1, \dots, \alpha_k)$ is in the support of the multi-degree, then Theorem 2.1 implies that $|\boldsymbol{\beta}_I| = \sum_{i \in I}(n_i - \alpha_i) \leq \dim p_I(X) + 1$ for all I, so according to Proposition 3.1 we have that $Z_{X,\boldsymbol{\beta}}$ is a hypersurface. Conversely, if

$|\boldsymbol{\beta}_I| \leq \dim p_I(X)+1$ for all I, then we claim that there exists j such that for any I with

$$|\boldsymbol{\beta}_I| = \dim p_I(X)+1,$$

we necessarily have $j \in I$. Indeed, this follows immediately from Proposition 2.3, by choosing any $j \in J$. We then have that $(\alpha_1,\dots,\alpha_j+1,\dots,\alpha_k)$ is in the support of the multidegree of X.

Next, if $|\boldsymbol{\beta}_I| \leq \dim p_I(X)$ for all $I \subsetneq \{1,\dots,k\}$, it is clear that for all j and I, we will have $\sum_{i\in I}(n_i-\gamma_i) \leq \dim p_I(X)$, where as before $(\gamma_1,\dots,\gamma_k) = (\alpha_1,\dots,\alpha_j+1,\dots,\alpha_k)$. Thus, for each j we have $\boldsymbol{\gamma}$ in the support of the multidegree of X. Conversely, if for some $I \subseteq \{1,\dots,k\}$ we have $|\boldsymbol{\beta}_I| > \dim p_I(X)$, then for any $j \notin I$, with $\boldsymbol{\gamma}$ as above we will have $\sum_{i\in I}(n_i-\gamma_i) = |\beta_I| > \dim p_I(X)$, so $\boldsymbol{\gamma}$ is not in the support of the multidegree of X. □

Remark 5.12. Given our basic setup, it is of course possible to re-embed X into a high-dimensional projective space via the Segre embedding, and then apply the classical Cayley-Chow construction. This works canonically and unconditionally to characterize X, but it doesn't reflect the geometry of the embedding of X into the original product of projective spaces, and it will typically require a great deal more data. For instance, if a 3-fold is embedded in $\mathbb{P}^2 \times \mathbb{P}^2$, then our construction will give a bihomogeneous form in two sets of three variables. On the other hand, the Segre embedding gives a 3-fold in $\mathbb{P}^8$, so the relevent Grassmannian will be $\mathrm{Gr}(4,8)$, whose Plücker embedding lands in $\mathbb{P}^{125}$. Thus, the classical Cayley-Chow form is in 126 variables in this case!

We conclude with further examples. The first shows that in positive characteristic, our multigraded Cayley-Chow form may indeed come with multiplicity strictly greater than 1.

Example 5.13. Let K have characteristic p, and let $X \subseteq \mathbb{P}^2 \times \mathbb{P}^2$ be the graph of the Frobenius morphism φ. Then X has multidegree $p^2t_1^2+pt_1t_2+t_2^2$. If $\boldsymbol{\beta}=(2,1)$, then

$$Z_{X,\boldsymbol{\beta}} = \{(P,L) : \varphi(P) \in L\} \subseteq \mathbb{P}^2 \times (\mathbb{P}^2)^*.$$

If $L = Z(G)$ for a linear form G and $P=(u_0,u_1,u_2)$, then $\varphi(P) \in L$ if and only if $G(u_0^p,u_1^p,u_2^p)=0$, so we see that $Z_{X,\boldsymbol{\beta}}$ is cut out by a $(p,1)$-form, as it should be.

On the other hand, if $\boldsymbol{\beta}=(1,2)$, then

$$Z_{X,\boldsymbol{\beta}} = \{(L,P) : P \in \varphi(L)\} \subseteq (\mathbb{P}^2)^* \times \mathbb{P}^2.$$

If $L=Z(G)$ and $P=(u_0,u_1,u_2)$ as above, then we observe that $\varphi(L)$ is cut out by $\widehat{G}$, the linear form obtained from G by raising the coefficients to the pth power. Then $P \in \varphi(L)$ if and only if $\widehat{G}(u_0,u_1,u_2)=0$, so in this case $Z_{X,\boldsymbol{\beta}}$ is still cut out by a $(p,1)$-form. Thus, in order to get the right degree, we have to take the cycle $pZ_{X,\boldsymbol{\beta}}$ in place of $Z_{X,\boldsymbol{\beta}}$. We see this geometrically by observing that if we intersect X with $p_2^{-1}(Y)$ for any point Y, we get a length-p^2 subscheme which can be identified under the first projection with the fiber of φ over Y. Intersecting with a line in $\mathbb{P}^2$ will then reduce the length to p, but cannot reduce it to 1. Thus, the forgetful map from the incidence correspondence has degree p in this case.

The following examples come from computer vision.

Example 5.14. [Multifocal tensors] We expand on the discussion of multiview varieties from the introduction. Given $k \geq 2$ linear projections $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ (viewed as positioned pinhole cameras, with the centers of projection being camera centers), we let $X \subseteq (\mathbb{P}^2)^k$ be the closure of the image of the induced rational map. This is called the "multiview variety" associated to the camera configuration, and determines the configuration (up to linear change of coordinates on $\mathbb{P}^3$). If we assume that the camera centers are all distinct, then $\dim p_I(X) = 3$ whenever $|I| \geq 2$, so Proposition 3.1 guarantees that $Z_{X,\boldsymbol{\beta}}$ is a hypersurface for all $(\beta_1, \ldots, \beta_k)$ such that $|\boldsymbol{\beta}| = 4$.

On the other hand, $Z_{X,\boldsymbol{\beta}}$ determines X if and only if $|\boldsymbol{\beta}_I| \leq 3$ for all $I \subsetneq \{1, \ldots, k\}$, that is, if and only if $\beta_i \neq 0$ for all i. This clearly requires $k \leq 4$, and we see that if $k = 2$ the vector $\boldsymbol{\beta}$ must be $(2, 2)$; if $k = 3$ then it is a permutation of $(2, 1, 1)$; if $k = 4$ then it is $(1, 1, 1, 1)$. Moreover, the multidegree of X is computed in Corollary 3.5 of Aholt-Sturmfels-Thomas [AST13] to be

$$t_1^2 \cdots t_k^2 \left(\sum_{1 \leq i_1 < i_2 < i_3 \leq k} \frac{1}{t_{i_1} t_{i_2} t_{i_3}} + \sum_{1 \leq i_1, i_2 \leq k} \frac{1}{t_{i_1}^2 t_{i_2}} \right),$$

so according to Corollary 5.10, we find that in the allowed cases, $H_{X,\boldsymbol{\beta}}$ is multilinear, so it is associated with a tensor.

We thus recover the multifocal tensor construction when $k \leq 4$; these are known as the "fundamental matrix," the "trifocal tensor," and the "quadrifocal tensor," respectively. On the other hand, for $k \geq 5$, we see that the constructed form never suffices to recover X.

Example 5.15. Consider the $k = 3$ (i.e., trifocal) case of the previous example, and let P_i be the centers of projection. It is helpful to observe that we can think of X as consisting of triples (ℓ_1, ℓ_2, ℓ_3) where each ℓ_i is a line through P_i in $\mathbb{P}^3$, and $\ell_1 \cap \ell_2 \cap \ell_3 \neq \emptyset$ (see Proposition 2.1 of [Li]). To avoid having to discuss too many cases, we will assume that the P_i are not collinear (and in particular are distinct). As discussed in the previous example, we will have to have up to permutation that $\boldsymbol{\beta} = (2, 1, 1)$, so that in $\mathbb{P}^2$, we will have L_1 a point, and L_2 and L_3 lines. In the ambient $\mathbb{P}^3$, they will correspond to lines and planes containing P_i, respectively. We will analyze the set S_Z from the proof of Proposition 3.6. In fact, the second author, Hebert and Ponce give a description of S_Z in Proposition 9 of [THP15], observing that it does indeed contain extra components beyond X itself.

To describe S_Z, suppose we have fixed (ℓ_1, ℓ_2, ℓ_3), not necessarily in X, so that we want to know under what conditions every L_1, L_2, L_3 containing ℓ_1, ℓ_2, ℓ_3 must meet X, or equivalently, under what conditions every L_2, L_3 contain some choice of ℓ_2', ℓ_3' such that $\ell_1 \cap \ell_2' \cap \ell_3' \neq \emptyset$ (note that $L_1 = \ell_1$ necessarily). Obviously, this is the case if $\ell_1 \cap \ell_2 \cap \ell_3 \neq \emptyset$ already, so that $(\ell_1, \ell_2, \ell_3) \in X$. However, there are two other cases in which this occurs: if $\ell_1 = \ell_2 = \overline{P_1 P_2}$, or $\ell_1 = \ell_3 = \overline{P_1 P_3}$. Indeed, in the former case, we have that any plane L_3 must meet the line $\ell_1 = \ell_2$, yielding a choice of ℓ_3', and similarly for the latter case. On the other hand, one can check directly that in any other situation, we can always find L_2, L_3 such that no ℓ_2', ℓ_3' will have nonempty simultaneous intersection with ℓ_1. Indeed, we will be always be able to choose L_2 and L_3 so that $L_1 \cap L_2$ and $L_1 \cap L_3$ are distinct points, and $L_1 \cap L_2 \cap L_3$ is therefore empty. We thus find that S_Z consists of X together with two additional 2-dimensional components.

To compare this to the containment in (3.4), we describe the geometry of the projections of X; by symmetry, it suffices to look at p_1 and $p_{\{1,2\}}$. We have that p_1 is surjective, and if we fix ℓ_1, the fiber of X over ℓ_1 consists of pairs of lines ℓ_2, ℓ_3 which intersect ℓ_1 in a common point. If $\ell_1 \neq \overline{P_1P_2}, \overline{P_1P_3}$, then each of ℓ_2 and ℓ_3 can meet ℓ_1 in only a single point, so the choice of ℓ_2 is determined by the choice of a point on ℓ_1, and then ℓ_3 is determined as well. Thus, on this set the fibers are 1-dimensional. However, if $\ell_1 = \overline{P_1P_2}$ (so that $P_3 \notin \ell_1$), then every choice of ℓ_2 meets ℓ_1, and as long as $\ell_2 \neq \ell_1$, then $\ell_1 \cap \ell_2 = \{P_2\}$, and $\ell_3 = \overline{P_2P_3}$ is uniquely determined. On the other hand, we could also have $\ell_2 = \ell_1$, in which case we have a 1-dimensional set of choices of ℓ_3. Thus, in this case the fiber has two components, one of dimension 2, and one of dimension 1. The same holds if $\ell_1 = \overline{P_1P_3}$. We conclude that the general fiber is 1-dimensional, but there are two fibers which are (non-purely) 2-dimensional, corresponding to $\overline{P_1P_2}$ and $\overline{P_1P_3}$, respectively. The analogous description holds for p_2 and p_3.

Next, the image of $p_{\{1,2\}}$ is precisely the set of pairs (ℓ_1, ℓ_2) which have nonempty intersection, which forms a 3-dimensional set. Provided $\ell_1 \neq \ell_2$, we will have that $\ell_1 \cap \ell_2$ is a single point. If this point is not P_3, then ℓ_3 is uniquely determined, and thus $p_{\{1,2\}}$ is injective over such pairs. If $\ell_1 = \ell_2 = \overline{P_1P_2}$, so that P_3 is not on ℓ_1 or ℓ_2, then ℓ_3 is determined by a choice of point of ℓ_1, and we have a 1-dimensional fiber. Finally, if $\ell_1 \neq \ell_2$ but both go through P_3 (so that they are necessarily $\overline{P_1P_3}$ and $\overline{P_2P_3}$ respectively), then any choice of ℓ_3 is valid, and we obtain a 2-dimensional fiber. To summarize, the general fiber is 0-dimensional, but the fiber corresponding to $(\overline{P_1P_2}, \overline{P_1P_2})$ is 1-dimensional, and the fiber corresponding to $(\overline{P_1P_3}, \overline{P_2P_3})$ is 2-dimensional. The analogous description holds for $p_{\{2,3\}}$ and $p_{\{1,3\}}$.

We compare this to the proof of Proposition 3.6 as follows: if $I = \{i\}$, the set X_I is where the fibers of p_i have dimension at least 2 if $i = 1$, and at least 3 if $i = 2, 3$. The latter two cases give the empty set, but the former consists of the two points where ℓ_1 is either $\overline{P_1P_2}$ or $\overline{P_1P_3}$. Note that the two extra components of S_Z we have identified are contained in $p_1^{-1}(X_{\{1\}})$, but strictly. On the other hand, if $I = \{i, j\}$, the set X_I is where the fibers of $p_{\{i,j\}}$ have dimension at least 2 if $I = \{2, 3\}$, and at least 1 otherwise. In the first case, we get that X_I is the single point where $\ell_2 = \overline{P_1P_2}$ and $\ell_3 = \overline{P_1P_3}$. In this case, we already have that $p_{\{2,3\}}^{-1}(\ell_2, \ell_3)$ is contained in X, so we do not get any new component of S_Z. For $I = \{1, 2\}$, we have the same behavior over $(\overline{P_1P_3}, \overline{P_2P_3})$, but X_I also includes the point $(\overline{P_1P_2}, \overline{P_1P_2})$, and one additional component of S_Z is equal to the fiber of $p_{\{1,2\}}$ over this point. Considering $I = \{1, 3\}$ gives the other additional component of S_Z. Thus, in this case we have strict containment in (3.4), although we obtain equality if we restrict I to two-element sets.

References

[AST13] Chris Aholt, Bernd Sturmfels, and Rekha Thomas, *A Hilbert scheme in computer vision*, Canadian Journal of Mathematics **65** (2013), 961–988.

[CLZ] Federico Castillo, Binglin Li, and Naizhen Zhang, *Representable Chow classes of a product of projective spaces*, preprint.

[EK17] Laura Escobar and Allen Knutson, *The multidegree of the multi-image variety*, preprint, 2017.

[Ful98] William Fulton, *Intersection theory*, second ed., Springer-Verlag, 1998.

[GD66] Alexander Grothendieck and Jean Dieudonné, *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, troisième partie*, Publications mathématiques de l'I.H.É.S., vol. 28, Institut des Hautes Études Scientifiques, 1966.

[GKZ94] Israel Gelfand, Mikhail Kapranov, and Andrei Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Mathematics: Theory and Applications, Birkäuser, 1994.

[Gro61] Alexandre Grothendieck, *Techniques de construction et theoremes d'existence en geometrie algebrique, IV: Les schemas de Hilbert*, Seminaire Bourbaki (1961), no. 221.

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, 1977.

[HH02] Jürgen Herzog and Takayuki Hibi, *Discrete polymatroids*, Journal of Algebraic Combinatorics **16** (2002), no. 3, 239–268.

[Kle74] Steven L. Kleiman, *The transversality of a general translate*, Compositio Mathematica **28** (1974), 287–297.

[KRT13] Franz Király, Zvi Rosen, and Louis Theran, *Algebraic matroids with graph symmetry*, preprint, 2013.

[Li] Binglin Li, *Images of rational maps of projective spaces*, International Mathematics Research Notices, to appear.

[MPS07] Márton Makai, Gyula Pap, and Jácint Szabó, *Matching problems in polymatroids without double circuits*, Integer Programming and Combinatorial Optimization (Matteo Fischetti and David Williamson, eds.), Lecture Notes in Computer Science, vol. 4513, Springer-Verlag, 2007, pp. 167–181.

[PST17] Jean Ponce, Bernd Sturmfels, and Matthew Trager, *Congruences and concurrent lines in multi-view geometry*, Advances in Applied Mathematics **88** (2017), 62–91.

[THP15] Matthew Trager, Martial Hebert, and Jean Ponce, *The joint image handbook*, International Conference on Computer Vision, 2015.