On the regulation of the state of a harvested fish population

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Abstract

The goal of this paper is to apply some tools of control theory to an age-structured fishery model in order to stabilize an exploited fish population around a nontrivial steady state.

1 Introduction

In this paper, we consider a population of exploited fish which is structured in n age classes ($n \geq 2$); under some assumptions on the population, we can represent the dynamics of the population by the following system of difference equations [6]:

$$\begin{cases} x_1(t+1) = f(\sum_{i=1}^{n} b_i x_i(t)), \\ x_2(t+1) = x_1(t)\exp(-M_1 - q_1 u(t)), \\ \quad \vdots \qquad\qquad \vdots \\ x_n(t+1) = x_{n-1}(t)\exp(-M_{n-1} - q_{n-1}u(t)). \end{cases} \tag{1}$$

Where: b_i is the number of individuals produced by individuals of the i^{th} age class, M_i is the natural mortality of the individuals of the i^{th} age class, q_i is the catchability of the individuals of the i^{th} age class, $u(t)$ is the fishing effort at time t and is regarded as an input, and $f : \mathbb{R}_+ \to \mathbb{R}_+$ is the stock-recruitment function. It is a continuous map such that $f(0) = 0$. We shall use in this paper the expression of f used in Beverton and Holt model [2] $f(x) = \frac{x}{1+\beta x}$, $\beta > 0$. The dynamic of the population can be be written as a discrete-time control system of the form

$$\begin{cases} x(t+1) = F(x(t), u(t)), \\ u(t) \in \mathbb{R}_+, \\ x(t) \in \Omega = \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > 0\}. \end{cases} \tag{2}$$

where $x(t)$ is the state variable at time $t = 0, 1, 2, \ldots$, $u(t)$ is the control (here it is the fishing effort). $F : \Omega \times \mathbb{R}_+ \to \Omega$ is a continuous function. We focus attention on the following problem: Is it possible to compute the fishing effort (as a feedback control) $u(x)$ in such a way, that for a given state $x^0 \neq 0$, one has :
(i) $F\big(x^0, u(x^0)\big) = x^0$ (x^0 is an equilibrium point).
(ii) x^0 is a globally asymptotically stable equilibrium point for the closed-loop system $x(t+1) = F(x(t), u\big(x(t)\big))$.
The considered system is nonlinear so the classical linear control techniques can not be used (unless to get local results but here we are interested in global stablization by means of a nonegative feedback control). Thus, we shall use a discrete version of Jurdjevic-Quinn theorem [3] that has been derived in ([1],[5]).

2 Main result

For a constant fishing effort u^0, system (1) has a non trivial equilibrium state x^0 whose coordinates are:

$$x_1^0 = \frac{K-1}{\beta K}, \text{ and for } j \geq 2,\ x_j^0 = \left(\prod_{i=1}^{j-1} a_i\right) x_1^0.$$

where $K = b_1 + a_1 b_2 + a_1 a_2 b_3 + \ldots + a_1 a_2 \ldots a_{n-1} b_{n-1}$ and $a_i = e^{-M_i - q_i u^0}$.
This steady state belongs to Ω provided that $K > 1$.

Proposition 2.1 *for any positive constant $\eta \leq u^0$, system (1) is globally asymptotically stabilizable by means of the continuous feedback law*

$$u(x) = u^0 + v(x) \tag{3}$$

which satisfies

$$\|v(x)\| \leq \eta, \quad \forall x \in \Omega.$$

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Remark 2.1 *The condition $\eta \leq u^0$ is needed only to ensure that $u(x) \geq 0$ for all $x \in \Omega$. The feedback $v(x)$ will be explicitly constructed in the proof.*

Proof. For the sake of clarity, we shall prove this result for $n = 3$ (the proof is the same for arbitrary n but the calculus are longer and more complicated).

Let V be the following candidate Lyapunov function $V(x) = (x_1 - x_1^0)^2 + \frac{K-b_1}{K}\left(\frac{x_2-x_2^0}{a_1}\right)^2 + \frac{a_1a_2b_3}{K}\left(\frac{x_3-x_3^0}{a_1a_2}\right)^2$. V satisfies: $V(x^0) = 0$ and $V(x) > 0$ for all $x \neq x^0$. Let $\tilde{F}(x) = F(x, u^0)$,

$$\begin{aligned} V(\tilde{F}(x)) = \; & (f(\sum b_i x_i) - x_1^0)^2 + \tfrac{K-b_1}{K}(x_1 - x_1^0)^2 \\ & + \tfrac{a_1a_2b_3}{K}\left(\tfrac{x_2-x_2^0}{a_1}\right)^2. \end{aligned}$$

We have

$$f(\sum b_i x_i) - x_1^0 = \frac{\sum b_i(x_i - x_i^0)}{(1+\beta\sum b_i x_i)(1+\beta\sum b_i x_i^0)}$$

Since $1+\beta\sum b_i x_i \geq 1$ and $1+\beta\sum b_i x_i^0 = K$, we get

$$|f(\sum b_i x_i) - x_1^0| \leq \frac{|\sum b_i(x_i - x_i^0)|}{K}$$

Hence,

$$\begin{aligned} V(\tilde{F}(x)) \leq & \left(\frac{\sum b_i(x_i - x_i^0)}{K}\right)^2 \\ & + \frac{K-b_1}{K}(x_1 - x_1^0)^2 + \frac{a_1a_2b_3}{K}\left(\frac{x_2 - x_2^0}{a_1}\right)^2 \\ \leq & \left(\frac{b_1}{K}(x_1 - x_1^0) + \frac{a_1b_2}{K}\frac{x_2 - x_2^0}{a_1} + \frac{a_1a_2b_3}{K}\frac{x_3 - x_3^0}{a_1a_2}\right)^2 \\ & + \frac{K-b_1}{K}(x_1 - x_1^0)^2 + \frac{a_1a_2b_3}{K}\left(\frac{x_2 - x_2^0}{a_1}\right)^2. \end{aligned}$$

Using the convexity of the map $x \longmapsto x^2$ and the fact that $K = b_1 + a_1b_2 + a_1a_2b_3$, we get $V(\tilde{F}(x)) \leq V(x)$. Hence, the equilibruim x^0 is Lyapunov stable for the system $x(t+1) = F\Big(x(t), u^0\Big) = \tilde{F}(x(t))$. Now, using ([4], Th. 4.1), we shall compute $v(x)$. We define $\tilde{V}(x,u) = V(F(x,u))$. Using the Taylor expansion formula, we evalute the variation of V along the solutions of the closed-loop system $x(t+1) = F\Big(x(t), u^0 + v(x)\Big)$ in the following way:

$$\begin{aligned} \Delta V(x) = & V\big(F(x, u^0 + v(x))\big) - V(x) \\ = & V\big(\tilde{F}(x)\big) - V(x) + v(x)\tfrac{\partial V}{\partial x}\big(\tilde{F}(x)\big)\tfrac{\partial F}{\partial u}(x, u^0) \\ & + v^2(x)\underbrace{\int_0^1 (1-t)\frac{\partial^2 \tilde{V}}{\partial u^2}(x, u^0 + tv(x))dt}_{\varphi(x,v)}. \end{aligned}$$

For a fixed number η satisfying $0 < \eta \leq u^0$, let $\lambda_1(x)$ and $\lambda_2(x)$ be any nonnegative continuous real valued functions satisfying $\lambda_1(x) + \lambda_2(x) \neq 0, \quad \forall x \in \Omega$ and

$$\lambda_1(x) \geq \sup_{|v|\leq\eta} |\varphi(x,v)|, \lambda_2(x) \geq \left|\frac{\partial V}{\partial x}\big(F(x,u^0)\big)\frac{\partial F}{\partial u}(x,u^0)\right|$$

and set $\lambda(x) = \frac{\eta}{\eta\lambda_1(x) + \lambda_2(x)}$. We construct the feedback control according to the following formula :

$$v(x) = -\lambda(x)\left(\frac{\partial V}{\partial x}\big(F(x,u^0)\big)\frac{\partial F}{\partial u}(x,u^0)\right) \quad (4)$$

which satisfies

$$|v(x)| \leq \eta, \quad \forall x \in \Omega. \quad (5)$$

With this feedback and the expression of φ, the variation of V along the solutions of the closed-loop system can be written :

$$\Delta V = V\big(\tilde{F}(x)\big) - V(x) - v^2(x)\left(\frac{1}{\lambda(x)} - \varphi\big(x, v(x)\big)\right). \quad (6)$$

Taking into account the definition of λ, λ_1, and λ_2 we have

$$\frac{1}{\lambda(x)} - \varphi\big(x, v(x)\big) \geq \frac{\lambda_2(x)}{\eta} \geq 0.$$

Hence,

$$\Delta V(x) = V\big(F(x, u^0 + v(x))\big) - V(x) \leq 0 \quad \forall x \in \Omega.$$

The closed-loop system is then Lyapunov stable and since V is proper, all the solutions are bounded. It remains to show the attractivity of the equilibrium x^0. Let L be the largest invariant set contained in $\{x \in \Omega \,|\, \Delta V(x) = V\big(F(x, u(x))\big) - V(x) = 0\}$. By Lasalle Invariance Principle, all the solutions tend to L. Here, we have

$$\Delta V(x) = 0 \;\Rightarrow\; v(x) = 0 \Rightarrow x_1 = x_1^0 \text{ and } x_2 = x_2^0.$$

It is easy to see that $L = \{x^0\}$ which ends the proof.

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