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ON THE CONTROL OF AN EXPLOITED POPULATION OF FISH *

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Abstract

The aim of this paper is to show how some tools of control theory can be helpfull to regulate the exploitation of a population of fish.

1 Introduction

In this paper, we are interested in the stabilization of an exploited population of fish arround a non trivial steady state.

The dynamic of the population is supposed to be described by a discrete-time system of the form

$$x(t+1) = F(x(t), u(t)), \tag{1}$$

where $x(t)$ is the state variable at time $k = 0, 1, 2, \dots$ and $u(t)$ is the control (here it is the fishing effort).

The problem addressed here is how to compute the fishing effort (as a feedback control) $u(x)$ in such a way, that for a given state $x^0 \neq 0$, one has

(i) $F(x^0, u(x^0)) = x^0$ (x^0 is an equilibrium point).

(ii) x^0 is a globally asymptotically stable equilibrium point for the closed-loop system

$$x(t+1) = F\Big(x(t), u\big(x(t)\big)\Big).$$

More precisely, we consider a density-dependent model of a population of exploited fish which is structured in n age classes [liu, mag, ric] :

$$\begin{cases} x_1(t+1) = f(\sum_{i=1}^{n} b_i x_i(t)) \\ x_2(t+1) = x_1(t) exp(-M_1 - q_1 u(t)) \\ x_3(t+1) = x_2(t) exp(-M_2 - q_2 u(t)) \\ \vdots \\ x_n(t+1) = x_{n-1}(t) exp(-M_{n-1} - q_{n-1} u(t)) \end{cases} \tag{2}$$

Where:

- $b_i \geq 0$ is the number of individuals produced by individuals of the i^{th} age class.
- $M_i \geq 0$ is the natural mortality of individuals of age i.
- $q_i \geq 0$ is the catchability of individuals of age i.
- $u(t)$ is the fishing effort at time t and is regarded as an input.
- f is the stock-recruitment function. It is a continuous function satisfying $f(0) = 0$.

Several authors have proposed different kind of functions f (see [bev, mag, ric]. We shall use in this paper the expression of f used in Beverton and Holt model [bev]

$$f(x) = \frac{x}{1 + \beta x}, \ \beta > 0.$$

To construct the stabilizing feedback law, we shall use and adapt a machinery developped in [ben, igg].

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2 Main result

For the sake of simplicity we shall give the result for $n = 3$ (the calculus are exactly the same for an arbitrary n but the expression of the feedback is longer). We also suppose that only individuals of age n and over are reproductive ($b_1 = b_2 = 0$). So that, we consider the following system

$$\begin{cases} x_1(t+1) = f(b_3 x_3(t)) \\ x_2(t+1) = x_1(t) exp(-M_1 - q_1 u(t)) \\ x_3(t+1) = x_2(t) exp(-M_2 - q_2 u(t)) \end{cases} \quad (3)$$

For a constant fishing effort u^0, system (3) has a non trivial equilibrium state

$$x_1^0 = \frac{b_3 a_1 a_2 - 1}{\beta b_3 a_1 a_2} = \frac{b_3 x_3^0}{1 + \beta b_3 x_3^0}, \; x_2^0 = a_1 x_1^0, \; x_3^0 = a_1 a_2 x_1^0$$

Where $a_i = exp(-M_i - q_i u^0)$.

This steady state belongs to $\Omega = \mathbb{R}_+^3$ provided that

$$b_3 > \frac{1}{a_1 a_2}.$$

Theorem 2.1 *for any positive constant $\eta \leq u^0$, system (3) is globally asymptotically stabilizable by means of the continuous feedback law*

$$u(x) = u^0 + v(x) \quad (4)$$

which satisfies

$$||v(x)|| \leq \eta, \quad \forall x \in \Omega.$$

Proof. Let V be the following candidate Lyapunov function

$$V(x) = (x_1 - x_1^0)^2 + \frac{(x_2 - x_2^0)^2}{a_1^2} + \frac{(x_3 - x_3^0)^2}{(a_1 a_2)^2}.$$

and define

$$\tilde{F}(x) = F(x, u^0). \quad (5)$$

$\tilde{V} : \Omega \times \mathbb{R} \to \mathbb{R}$ and

$$\tilde{V}(x, u) = V(F(x, u)), \quad (6)$$

We evaluate the variation of V along the closed-loop system (3-4):

$$\begin{aligned} \Delta V(x) &= V(F(x, u(x))) - V(x) \\ &= \tilde{V}(x, u(x)) - V(x) \\ &= \tilde{V}(x, u^0 + v(x)) - V(x) \\ &= \tilde{V}(x, u^0) - V(x) + \frac{\partial \tilde{V}}{\partial u}(x, u^0)\, v(x) \\ &\quad + \int_0^1 (1-t) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, u^0 + t\, v(x))\, v^2(x)\, dt. \end{aligned}$$

Notice that

$$\tilde{V}(x, u^0) = V(F(x, u^0)), \quad (7)$$

and

$$\frac{\partial \tilde{V}}{\partial u}(x, u^0) = \frac{\partial V}{\partial x}(F(x, u^0)) \frac{\partial F}{\partial u}(x, u^0).$$

So,

$$\begin{aligned} \Delta V(x) = \; & V(F(x, u^0)) - V(x) \\ & + \tfrac{\partial V}{\partial x}(F(x, u^0)) \tfrac{\partial F}{\partial u}(x, u^0) v(x) \\ & + \textstyle\int_0^1 (1-t) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, u^0 + t\, v(x))\, v^2(x)\, dt. \end{aligned} \quad (8)$$

Now we shall construct a feedback control $v(x)$ in order to get $\Delta V(x) \leq 0$ for all $x \in \Omega$. To this end, we introduce some notations. Let $\varphi : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\varphi(x, v, w) = \int_0^1 (1-t) v^T(x) \frac{\partial^2 \tilde{V}}{\partial u^2}(x, u^0 + t\, v(x))\, w^2(x)\, dt. \quad (9)$$

For a fixed number η satisfying $0 < \eta < u^0$, let $K_1(x)$ and $K_2(x)$ be any nonnegative continuous real valued functions satisfying $K_1(x) + K_2(x) \neq 0, \quad \forall x \in \Omega$ and

$$K_1(x) \geq \sup_{|v| \leq \eta, |w|=1} |\varphi(x, u, w)|, \quad \forall x \in \Omega. \quad (10)$$

$$K_2(x) \geq \left| \frac{\partial V}{\partial x}(F(x, u^0)) \frac{\partial f}{\partial u}(x, u^0) \right|, \quad \forall x \in \Omega. \quad (11)$$

and set

$$K(x) = \frac{\eta}{\eta K_1(x) + K_2(x)} > 0 \; \forall x \in \Omega. \quad (12)$$

We construct the feedback control according to the following formula :

$$v(x) = -K(x) \left(\frac{\partial V}{\partial x}(F(x, u^0)) \frac{\partial F}{\partial u}(x, u^0) \right) \quad (13)$$

which satisfies

$$|v(x)| \leq \eta, \quad \forall x \in \Omega. \quad (14)$$

Tacking into account (8-13-9), the variation of V along the solutions of the closed-loop system can be written :

$$\begin{aligned} \Delta V(x) = \; & V(F(x, u^0)) - V(x) \\ & - \tfrac{1}{K(x)} v^2(x) + \varphi(x, v(x), v(x)). \end{aligned} \quad (15)$$

On the one hand, we have

$$V(F(x, u^0)) - V(x) = (f(b_3 x_3) - x_1^0)^2 - \frac{(x_3 - x_3^0)^2}{(a_1 a_2)^2}$$

$$= \left(\frac{b_3 x_3}{1 + \beta b_3 x_3} - x_1^0 \right)^2 - \left(\frac{x_3 - x_3^0}{a_1 a_2} \right)^2$$

$$= \left(\frac{b_3 x_3}{1 + \beta b_3 x_3} - \frac{b_3 x_3^0}{1 + \beta b_3 x_3^0} \right)^2 - \left(\frac{x_3 - x_3^0}{a_1 a_2} \right)^2$$

Hence

$$V(F(x,u^0)) - V(x) =$$

$$\left(\frac{b_3(x_3-x_3^0)}{(1+\beta b_3x_3)(1+\beta b_3x_3^0)}\right)^2 - \left(\frac{x_3-x_3^0}{a_1a_2}\right)^2$$

$$= \frac{(b_3a_1a_2)^2-(1+\beta b_3x_3)^2(1+\beta b_3x_3^0)^2}{(a_1a_2)^2(1+\beta b_3x_3)^2(1+\beta b_3x_3^0)^2}(x_3-x_3^0)^2$$

We have $1+\beta b_3x_3^0 = 1+\beta b_3\frac{b_3a_1a_2-1}{\beta b_3a_1a_2}a_1a_2 = b_3a_1a_2$. This yields,

$$V(F(x,u^0)) - V(x) = \frac{1-(1+\beta b_3x_3)^2}{(1+\beta b_3x_3)^2(1+\beta b_3x_3^0)^2}b_3^2(x_3-x_3^0)^2 \leq 0 \tag{16}$$

On the other hand, $\varphi(x,v,w)$ being homogeneous of degree 2 with respect to w, we have for all $x\in\Omega$ such that $v(x)\neq 0$:

$$\varphi(x,v(x),v(x)) = v^2(x)\varphi(x,v(x),\frac{v(x)}{|v(x)|})$$

From this and (15-16), we get

$$\Delta V(x) = V(F(x,u^0)) - V(x) \leq 0 \ \text{ if } v(x)=0.$$

And for $v(x)\neq 0$,

$$\begin{aligned}\Delta V(x) = \ & V(F(x,u^0)) - V(x)\\ & -v^2(x)\Big(\tfrac{1}{K(x)} - \varphi(x,v(x),\tfrac{v(x)}{|v(x)|})\Big).\end{aligned}$$

Thanks to the construction of $v(x)$ and $K(x)$, we have $\frac{1}{K(x)} > \varphi(x,v(x),\frac{v(x)}{|v(x)|})$. This allows to conclude that

$$\Delta V(x)\leq 0 \ \ \forall x\in\Omega.$$

The closed-loop system is then Lyapunov stable. On the other hand,

$$\Delta V(x) = 0 \ \Leftrightarrow \ x_3 = x_3^0 \text{ and } v(x)=0.$$

It is easy to show that the largest invariant set contained in

$$\{x\in\Omega \mid \Delta V(x) = V(F(x,u(x))) - V(x) = 0\}$$

is reduced to $\{x^0\}$ so, by Lasalle Invariance Principle [las], the equilibrium x^0 is a globally asymptotically stable equilibrium point for the closed-loop system.

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