Feedback stabilization of discrete-time nonlinear systems via the Control Lyapunov functions

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Abstract

In this paper we state sufficient conditions for the existence of feedback laws which render the equilibrium solution of a class of discrete-time nonlinear systems globally asymptotically stable.

Keywords: Control Lyapunov Functions, Nonlinear discrete-time systems, Feedback stabilization.

1 Introduction:

Feedback stabilization of various classes of discrete-time nonlinear systems have been studied in the past few years; see ([1, 2, 4, 5]) and references therein.

We consider the single-input discrete-time nonlinear systems of the form

$$x_{k+1} = f(x_k) + u_k g(x_k), \quad x_k \in \mathbb{R}^n, \quad u \in \mathbb{R} \tag{1}$$

where f and g are smooth on $\mathbb{R}^n$ and $f(0) = 0$.

We say that (1) is globally asymptotically stable at the origin, if there exists a map $x \longmapsto K(x)$ such that the resulting system

$$x_{k+1} = f(x_k) + K(x_k) g(x_k) \tag{2}$$

is globally asymptotically stable at $0 \in \mathbb{R}^n$.

The main object of this paper is to provide explicit feedback $K : \mathbb{R}^n \longrightarrow \mathbb{R}$, smooth on $\mathbb{R}^n \setminus \{0\}$ and $K(0) = 0$ such that $u_k = K(x_k)$ globally asymptotically stabilizes the origin of the system (1), under the assumption that a 'quadratic Control Lyapunov Function' is known. This problem has been addressed for stochastic systems in [3].

This result represents a continuation of a line of work started in [7], in which, under the same condition, the author has studied the local stabilization for the multi-inputs discrete-time systems affine in the control, and the practical stabilization for the single-input systems.

2 Stabilization and clf's

In this section, we will state and prove an analogous of Sontag's result [6] for discete-time systems.

Definition 1 *A smooth, proper and definite positive function V mapping $\mathbb{R}^n$ into $\mathbb{R}$ is said to be a control Lyapunov function (henceforth just 'clf') for the discrete systems (1) if and only if,*

$$\inf_{u_k \in \mathbb{R}} \Big(\Delta V(x_k, u_k) = V(x_{k+1}) - V(x_k) \Big) < 0, \ \ \forall x \in \mathbb{R}^n \setminus \{0\}$$

In the following, we will assume that V is a quadratic clf for system (1) (*i.e.* there exists a positive definite matrix P such that $V(x) = x^T P x = \|x\|_p^2 = \langle x, x \rangle_p$

Remark: In [1], the authors have studied the same problem and have given, under the same assumptions, the following feedback

$$u(x) = \begin{cases} 0 & \text{for } \ x \in \Omega_0 = \{\in \mathbb{R}^n \ : \ g(x) = 0\} \\ -\frac{\langle f(x), g(x) \rangle_p}{\|g(x)\|_p^2} & \text{for } x \in \Omega_1 = \{x \in \mathbb{R}^n \ : \ g(x) \neq 0\} \end{cases}$$

but it is easy to show that $\lim_{x \in \Omega_1 \to x_0} u(x) = \infty$ for all $x_0 \in \partial\Omega_0 \setminus \{0\}$ such that $f(x_0) \neq 0$ and so $u(x)$ is not only discontinuous but is also unbounded in a neighbourhood of such a point.

Here, we will construct a stabilizing feedback which is smooth on $\mathbb{R}^n \setminus \{0\}$.

Theorem 1 *If there is a quadratic 'clf' for the discrete-time system (1), then there is a stabilizing feedback $K : \mathbb{R}^n \longrightarrow \mathbb{R}$, smooth on $\mathbb{R}^n \setminus \{0\}$ and $K(0) = 0$.*

Proof: For every $(x, u) \in \mathbb{R}^n \times \mathbb{R}$, we let

$$h_x(u) = u^2 \|g(x)\|_p^2 + 2u \langle f(x), g(x) \rangle_p + \|f(x)\|_p^2 - \|x\|_p^2$$

and we introduce the sets $\Omega_0 = \{\, x \in \mathbb{R}^n \setminus \{0\} \,|\, g(x) = 0 \,\}$ and $\Omega_1 = \{\, x \in \mathbb{R}^n \setminus \{0\} \,|\, g(x) \neq 0 \,\}$.

An easy computation shows that, for a given command law u, the difference $\Delta V = V(x_{k+1}) - V(x_k)$ is equal to $h_{x_k}(u_k)$. If $x \in \Omega_0$, this difference does not depend on u ($\Delta V = \|f(x_k)\|_p^2 - \|x_k\|_p^2$ in this case) and is therefore negative since V is a clf for system (1).

If $x \in \Omega_1$, for the same reason, the exists $u \in \mathbb{R}$ such that $h_x(u) < 0$, this proves that $h_x(u)$ regarded as a polynomial in u admits two separate roots $\lambda_1(x)$ and $\lambda_2(x)$. In this case our task is to find a function $K : x \longmapsto K(x)$ such that $\lambda_1(x) < K(x) < \lambda_2(x)$ for all $x \in \Omega_1$.

First we take two smooth functions φ and ψ defined by:

$$\varphi(t) = \begin{cases} 0 & \text{if } t \leq 2, \\ 1 & \text{if } t \geq 3. \end{cases} \qquad \psi(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t \geq 1. \end{cases}$$

and satisfying $0 \leq \varphi(t),\ \psi(t) \leq 1$ for all $t \in \mathbb{R}$.

Next, for $x \in \Omega_1$, we introduce the following quantities,

$$\begin{aligned} A(x) &= (\lambda_1(x)+1)\psi(\lambda_1(x)+1) \\ &\quad + (\lambda_2(x)-1)(1-\psi(\lambda_2(x)-1) \\ B(x) &= \varphi(\lambda_2(x)-\lambda_1(x)) \\ C(x) &= \frac{\lambda_1(x)+\lambda_2(x)}{2}. \end{aligned}$$

Finally, we claim that the following mapping K defined by:

$$K(x) = \begin{cases} 0 & \text{if } x \in \Omega_0 \\ A(x)B(x) + (1-B(x))C(x) & \text{if } x \in \Omega_1 \end{cases}$$

is smooth on $\mathbb{R}^n \setminus \{0\}$ and globally asymptotically stabilizes system (1).

We first show that $K(x) \in]\lambda_1(x), \lambda_2(x)[$ if $x \in \Omega_1$:

- if $\lambda_2(x) - \lambda_1(x) \leq 2$, then $B(x) = 0$ and so $K(x) = C(x) \in]\lambda_1(x), \lambda_2(x)[$;
- if $\lambda_2(x) - \lambda_1(x) > 2$, by considering all the possible placements of λ_1 and λ_2 regarding 0 and 1, we can see that $A(x) \in]\lambda_1(x), \lambda_2(x)[$ which in turn implies that $K(x) \in]\lambda_1(x), \lambda_2(x)[$.

In order to show that K is smooth on $\mathbb{R}^n \setminus \{0\}$, we remark first that this is obviously the case in the interior of Ω_0 and in Ω_1; we will next prove that if $x_0 \in \partial\Omega_0$, there exists a neighborhood U of x_0 on which K vanishes.

Let $\varepsilon_0 = -(\|f(x_0)\|_p^2 - \|x_0\|_p^2)$, V is a clf implies $\varepsilon_0 > 0$. Let $\varepsilon > 0$, since $g(x_0) = 0$ ($x_0 \in \Omega_0$), by continuity there exists a neighborhood U of x_0 such that $\|f(x)\|_p^2 - \|x\|_p^2 < -\varepsilon_0/2$, $\|g(x)\|_p^2 < \varepsilon$ and $2|\langle f(x), g(x)\rangle_p| < \varepsilon$. From these inequalities, we deduce that

$$h_x(u) \leq \varepsilon u^2 + \varepsilon \mid u \mid -\varepsilon_0/2, \quad \forall (x,u) \in U \times \mathbb{R}. \qquad (3)$$

Now, in order to have $K(x) = 0$ for all $x \in U \cap \Omega_1$, it is sufficient that $A(x) = 0$ and $B(x) = 1$ for all $x \in U \cap \Omega_1$. To this end it is sufficient to have $\lambda_1(x) \leq -1$ and $\lambda_2(x) \geq 4$ which is equivalent to $h_x(-1) \leq 0$ and $h_x(4) \leq 0$; these last inequalities are satisfied in U if ε is chosen such that $\varepsilon \leq \varepsilon_0/40$ and so the theorem is proven.

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