Stabilization of periodic discrete-time nonlinear systems

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Abstract

This paper provides sufficient conditions for stabilization of periodic discrete systems. These conditions are derived from a discrete version of a Theorem due to Krasovski for continuous-time systems. This tool allows to give a systematic design of time-varying stabilizing control for autonomous discrete systems that can not be stabilized by time-invariant feedback.

Keywords: Lyapunov function, stability, stabilization, time-varying feedback, periodic discrete systems.

1 Introduction

The aim of this article is to design stabilizer feedback for a class of periodic discrete-time nonlinear control systems. More precisely, we consider systems that can be written $x_{k+1} = F(x_k, u_k, k)$, F being time-periodic and the unforced system being Lyapunov stable. It is also assumed that there exists a positive definite periodic function $V(x, k)$ in such a way $V(F(x, 0, k), k + 1) - V(x, k) \leq 0$. The paper is organized as follows. In Section 2, we give a discrete version of Krasovski theorem [6] and a sufficient condition under which the attractivity of the equilibrium point is guaranteed. In Section 3, this will be used to derive a stabilization result of periodic systems. As an application, we derive a sufficient condition for time-varying feedback stabilization for a class of autonomous systems that can not be stabilized by time-invariant feedback [1, 2, 3, 5]. Due to the lack of space, the proofs are not reproduced here, but can be found in [4].

2 Stability of periodic systems

Let us consider a discrete nonlinear system

$$x_{k+1} = f(x_k, k),\ f(0, k) = 0,\ x(k) \in \mathbb{R}^n,\ k \in \mathbb{N} \quad (1)$$

We assume that f is continuous and periodic with respect to k, i.e. there exists $T \in \mathbb{N}$ such that $f(x, k + T) = f(x, k)$, $\forall (x, k) \in \mathbb{R}^n \times \mathbb{N}$. We shall denote by $x(k, x_{k_0}, k_0)$ the solution of (1) with initial state x_{k_0} at initial time k_0. Let $a : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous function. We shall say that a belongs to $\mathcal{K}$ if it is strictly increasing, vanishes at zero and $\lim_{r\to+\infty} a(r) = +\infty$.

Theorem 1 *Suppose there exists a continuous function $V : \mathbb{R}^n \times \mathbb{N} \to \mathbb{R}$, periodic in k with period $T \geq 2$ such that for some $a \in \mathcal{K}$, $\forall x \in \mathbb{R}^n$, $\forall k \in \mathbb{N}$:*

(1) $V(x, k) \geq a(\|x\|)$, $V(0, k) = 0$.

(2) $\Delta V(x, k) = V(f(x, k), k + 1) - V(x, k) \leq 0$.

(3) *The set $\{(x, k) \in \mathbb{R}^n \times \mathbb{N} : \Delta V(x, k) = 0\}$ contains no complete solution of (1) except the trivial one.*

Then the origin is globally uniformly asymptotically stable.

For the sequel, we define for all $0 \leq k \leq T$ a function $\hat{f}_k$ in the following way : ${\hat{f}_k}^{\,k}(x) = x$ and by induction ${\hat{f}_k}^{\,p+1}(x) = f({\hat{f}_k}^{\,p}(x), p)$ for all $p \geq k$. Actually ${\hat{f}_k}^{\,p}(x)$ is nothing but the value at time p of the solution of (1) with initial state x at initial time k. With this notation we can prove the following result which gives a sufficient condition to get the condition **(3)** of the above theorem.

Proposition 1 *Suppose* **(1)** *and* **(2)** *hold. If the set $\{x \in \mathbb{R}^n : V({\hat{f}_k}^{\,p+1}(x), p+1) - V({\hat{f}_k}^{\,p}(x), p) = 0, p \geq k\}$ is reduced to the origin for all $k = 0, \ldots, T$, then assumption* **(3)** *of Theorem (1) is satisfied and so the origin is globally uniformly asymptotically stable.*

3 Stabilization

Consider a discrete-time nonlinear control system

$$x_{k+1} = F(x_k, u_k, k),\ F(0, 0, k) = 0,\ k \in \mathbb{N} \quad (2)$$

where $x_k \in \mathbb{R}^n$ is the state at time k and $u \in \mathbb{R}^m$ is the control. $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}$ is assumed to be continuous in (x, u) and periodic with respect to time k with period $T \geq 2$. The problem addressed here is the following: does a periodic feedback $u(x, k)$, $u(0, k) = 0$, exists such that the origin is a globally asymptotically stable equilibrium point for the closed-loop system

$$x_{k+1} = F(x_k, u(x_k, k), k)$$

We will give a sufficient condition under which such a feedback exists. To do this we need the following notations. For a Lyapunov function $V(x, k)$ which is C^2 in x and T-periodic in k let $\tilde{V} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{N} \to \mathbb{R}$ be defined by

$$\tilde{V}(x, u, k) = V(F(x, u, k), k + 1) \quad (3)$$

$$\varphi(x, u, v, k) = \int_0^1 (1 - t) v^T \frac{\partial^2 \tilde{V}}{\partial u^2}(x, tu, k)\, v\, dt \quad (4)$$

Set $F_0(x, k) = F(x, 0, k)$. For a fixed $\eta > 0$, let $K_1(x, k)$ and $K_2(x, k)$ be any T-periodic and continuous real valued functions satisfying for all $(x, k) \in \mathbb{R}^n \times \mathbb{N}$, $K_1(x, k) + K_2(x, k) \neq 0$ and

$$K_1(x,k) \geq \sup_{\|u\|\leq\eta, \|v\|=1} |\varphi(x,u,v,k)| \quad (5)$$

$$K_2(x,k) \geq \left\| \frac{\partial V}{\partial x}\big(F_0(x,k), k+1\big)\frac{\partial F}{\partial u}(x,0,k) \right\| \quad (6)$$

and set
$$K(x,k) = \frac{\eta}{\eta K_1(x) + K_2(x)} \quad (7)$$

One can notice that K is T-periodic with respect to k. Now, we can state the main result of this section :

Theorem 2 *Assume that $x_{k+1} = F_0(x_k,k)$ is stable and there exists a C^2 Lyapunov function $V(x,k)$ T-periodic in k such that V and F_0 satisfy the assumptions* **(1)** *and* **(2)** *of Theorem (1). If for all $k \in \{0,\dots,T\}$ the set*

$$\{x \in \mathbb{R}^n : V(\hat{F_0}_k^{p+1}(x), p+1) = V(\hat{F_0}_k^{p}(x), p),$$
$$\tfrac{\partial V}{\partial x}\big(\hat{F_0}_k^{p+1}(x), p+1\big)\tfrac{\partial F}{\partial u}(\hat{F_0}_k^{p}(x), 0, p) = 0, p \geq k\}$$

is reduced to the origin then, for any positive constant η, system (2) is globally asymptotically stabilizable by means of the continuous feedback law

$$u = -K(x,k)\left(\frac{\partial V}{\partial x}\big(F_0(x,k), k+1\big)\frac{\partial F}{\partial u}(x,0,k)\right)^T$$

which satisfies

$$\begin{cases} \|u(x,k)\| \leq \eta \quad \forall\,(x,k) \in \mathbb{R}^n \times \mathbb{N}, \\ u(0,k) = 0 \quad \forall\, k \in \mathbb{N}, \\ u(x,k+T) = u(x,k) \quad \forall\,(x,k) \in \mathbb{R}^n \times \mathbb{N}. \end{cases}$$

Now, we apply the above theorem to the following nonlinear control system

$$x_{k+1} = x_k + \Phi(x_k, u_k) \quad (8)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $\Phi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^2$ function satisfying $\Phi(x,0) = 0$, $\forall x \in \mathbb{R}^n$. It is known that if the map Φ fails to be locally onto then (8) cannot be stabilized by means of continuous state feedback $u_k = u(x_k)$. In what follows we give a sufficient condition for system (8) to be stabilizable by a time-varying feedback $u_k = u(x_k,k)$.

Theorem 3 *Assume that for all $1 \leq i,j \leq m$*

$$\frac{\partial^2 \Phi_i}{\partial u_1 \partial u_j}(x,u) = 0,\ \forall\,(x,u) \in \mathbb{R}^n \times \mathbb{R}^m \quad (9)$$

and that there exist a C^2 Lyapunov function $V(x,k)$ and a continuous real valued function $\alpha(x,k)$ which are T-peridic with respect to time k such that V and the function F_0 defined by $F_0(x,k) = x + \alpha(x,k)g_1(x)$ satify the conditions of theorem 2. Then, for any positive constant η, system (8) is globally asymptotically stabilizable by means of the continuous feedback

$$u(x,k) = \nu(x,k) + \tilde{u}(x,k) \quad (10)$$
$$\nu(x,k) = [\,\alpha(x,k), 0, \dots, 0\,]^T$$
$$\tilde{u}(x,k) = -K(x,k)\left[\frac{\partial V}{\partial x}\big(F_0(x,k), k+1\big)\frac{\partial \Phi}{\partial u}\,(x, \nu(x,k))\right]^T$$

where $K(x,k)$ is got from (3)–(7) with

$$\tilde{V}(x,u,k) = V\left(x + \Phi\left(x, \nu(x,k) + u\right), k+1\right)$$

Since the conditions and the design of the control laws in theorem 3 make use of the functions V and α, it is natural to look for particular systems of the form (8) for which V and α can be explicited. Setting

$$g(x) = \big(g_1(x), \dots, g_m(x)\big) = \frac{\partial \Phi}{\partial u}(x,0)$$

it turns out, as in continuous-time (see [7]), that if

$$g_1 = \frac{\partial \Phi}{\partial u_1}(x,0) = \frac{\partial}{\partial x_1} \quad (11)$$

it is actually possible to give an explicit design of V and α, and by the way of the control laws.

Hereafter, for all $x \in \mathbb{R}^n$, set $\bar{x} = (x_2, \dots, x_n)^T$ and for a fixed integer $T \geq 2$ let

$$V(x,k) = \frac{1}{2}\left([x_1 + h(\bar{x},k)]^2 + \|\bar{x}\|^2\right)$$
$$\alpha(x,k) = -\frac{1}{2}[x_1 - h(\bar{x},k)] - h(\bar{x},k+1)$$

where h is a $\mathcal{C}^2$ function satisfying $h(\bar{x},k+T) = h(\bar{x},k)$, $\forall(\bar{x},k) \in \mathbb{R}^{n-1} \times \mathbb{N}$ and $h(0,k) = 0$, $\forall k \in \mathbb{N}$. A possible choice for h is $h(\bar{x},k) = \psi(\bar{x})H(k)$ where ψ is a definite function with respect to $\bar{x}$, and $H(k)$ is a time periodic function with period T. So V and α are T-periodic with respect to time k, and V vanishes if and only if $x = 0$. With these notations we have:

Proposition 2 *Assume that (9) and (11) hold. If there exists a function h as specified above such that for all $\bar{x} \in \mathbb{R}^{n-1}$,*

$$\operatorname{rank}\{\tilde{g}_i\left(h(\bar{x},k), \bar{x}\right),\ 2 \leq i \leq m,\ 0 \leq k \leq T-1\} = n-1$$

where $\tilde{g}_i(x) = (0, g_i^2(x), \dots, g_i^n(x))$, then, for any positive constant η, system (8) is globally asymptotically stabilizable by means of the feedback (10).

References

[1] R.W. Brockett. Asymptotic stability and feedback stabilization, in *Differential Geometric Control Theory*, Birkhauser, Boston, 1983, p. 181-191.

[2] M. Bensoubaya, A. Ferfera and A. Iggidr. Stabilization of discrete-time nonlinear systems. *Comtes rendus de l'Academie des Sciences, Paris,* Tome 321, Série I, No. 3, Août 1995, p. 371-374.

[3] M. Bensoubaya, A. Ferfera and A. Iggidr. Time-varying stabilizing feedback for a class of nonlinear discrete-time control systems. *34th CDC* (1995), p. 512-514.

[4] M. Bensoubaya, A. Ferfera and A. Iggidr. Stabilization of periodic discrete-time nonlinear systems. INRIA report, 1997.

[5] W. Lin and C.I. Byrnes Design of discrete-time nonlinear control systems via smooth feedback. *IEEE Trans. Autom. Control*, 39, No 11, November 1994, p. 2340-2346.

[6] N. N. Krasovski. Problems of the theory of stability of motion. *Stanford Univ. Press*, Stanford, California 1963 ; translation of the Russian edition, Moskow, 1959.

[7] J.B. Pomet. Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift *Systems & Control Letters*, (18), (1992), p. 147-158.