Time-varying stabilizing feedback for a class of nonlinear discrete-time control systems

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Abstract

The goal of this paper is to introduce and to use time-varying control for the stabilization of nonlinear discrete-time control systems which can not be stabilized by state feedback.

Keywords: stabilization, nonautonomous systems, Lyapunov functions.

1. Introduction

The stabilization problem of nonlinear continuous systems has attracted the interest of an increasing number of authors in the last decay (see [8] and references therein). For discrete-time systems there is a few results concerning the stabilization [1, 2, 3, 9]. Recently in [1] we have given the following necessary condition for the existence of a continuous stabilizing state feedback :

Theorem 1. *Given a discrete-time nonlinear system*

$$x(k+1) = f(x(k), u(k)) \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, *and* f *is a continuous function in a neighborhood* $A \times U$ *of* $(0,0)$ *satisfying* $f(0,0) = 0$, *a necessary condition for the existence of a continuous feedback* $u(x)$, *with* $u(0) = 0$, *stabilizing the system (1) is that the image of the map* $\gamma : A \times U \to \mathbb{R}^n$ *defined by* $\gamma(x,u) = f(x,u) - x$ *contains some neighborhood of the origin.*

In this paper we are interested in the stabilization of nonlinear discrete-time systems of the form

$$x(k+1) = x(k) + \sum_{i=1}^{m} u_i(k) g_i(x(k)) \tag{2}$$

where $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n$, $u = (u_1, \ldots, u_m)^{\mathrm{T}} \in \mathbb{R}^m$, and $g_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, m$ $(m \geq 2)$, are continuous functions on $\mathbb{R}^n$. Those systems are not generally stabilizable by continuous state feedback:

Proposition. *If* $m < n$ *then system (2) can not be stabilized by means of continuous state feedback.*

Proof: Since $\operatorname{rank}\{g_1(0), \ldots, g_m(0)\} \leq m < n$, the map $\gamma : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$\gamma(x,u) = f(x,u) - x = \sum_{i=1}^{m} u_i(k) g_i(x(k))$$

is not locally onto. So the necessary condition given by theorem 1 is violated. ■

As for the continuous systems [4, 6, 7] this impossibilty to stabilize by state feedback motivates the introduction of time-varying control.

2. Main result

In the sequel, we limit ourself to the class of non linear systems of the form (2) with

$$g_1 = \frac{\partial}{\partial x_1} \tag{3}$$

For all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$, set $\tilde{x} = (x_2, \ldots, x_n)^{\mathrm{T}}$ and let

$$V(x,t) = \frac{1}{2}\left[(x_1 + h(\tilde{x},t))^2 + \sum_{i=2}^{n} x_i^2\right] \tag{4}$$

where h is any $\mathcal{C}^2$ function satisfying

$$h(\tilde{x}, t+2\pi) = h(\tilde{x}, t), \quad \forall (\tilde{x}, t) \tag{5}$$

$$h(0,t) = 0, \quad \forall t \tag{6}$$

A possible choice for h satisfying (5-6) is

$$h(\tilde{x}, t) = \psi(\tilde{x}) \cos t \tag{7}$$

where ψ is linear. So V is 2π-periodic with respect to time t, vanishes if and only if $x = 0$ and satisfy

$$\{(x,t) \in \mathbb{R}^{n+1} \,|\, V(x,t) \leq K \text{ and } |t| \leq K\}$$

is bounded $\forall\, K > 0$. Besides, define $v : \mathbb{R}^{n+1} \to \mathbb{R}$ and $f, \tilde{g}_i : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ $(1 \leq i \leq m)$ by

$$v(x,t) = -\frac{1}{2}\big(x_1 - h(\tilde{x},t)\big) - h(\tilde{x}, t+1) \tag{8}$$

$$f(x,t) = \begin{pmatrix} x + v(x,t)g_1(x) \\ t+1 \end{pmatrix} = \begin{pmatrix} x_1 + v(x,t) \\ \tilde{x} \\ t+1 \end{pmatrix} \quad (9)$$

$$\tilde{g}_i(x,t) = \begin{pmatrix} g_i(x) \\ 0 \end{pmatrix}$$

Now we can state and prove the main theorem of this paper.

Theorem 2. *Assume that the functions $\tilde{g}_1, \ldots, \tilde{g}_m$ and f satisfy for any $(x,t) \in \mathbb{R}^n \times S^1$*

$$\operatorname{rank}(\{\tilde{g}_i(f^k(x,t)),\ k \in \mathbb{N},\ 2 \le i \le m\}) = n-1 \quad (10)$$

Then the origin is a globally uniformly asymptotically stable equilibrium point of the time-varying periodic closed-loop system (2) with

$$u(x,t) = \begin{pmatrix} v(x,t) \\ 0 \end{pmatrix} - M^{-1}(x,t)\big(\nabla V(f(x,t))\tilde{g}(x,t)\big)^{\mathrm{T}} \quad (11)$$

where $M(x,t) = I + \frac{1}{2} g^{\mathrm{T}}(x) \frac{\partial^2 V}{\partial x^2}(f(x,t))\, g(x)$.

Proof: The functions V and v from $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}$ are both 2π-periodic with respect to time t. So they induce naturally some functions from $\mathbb{R}^n \times S^1$ to $\mathbb{R}$. Hence, by taking the change of control

$$\begin{aligned} u_1 &= v(x,t) + \tilde{u}_1 \\ u_i &= \tilde{u}_i, \quad 2 \le i \le m \end{aligned}$$

the time-varying closed-loop system can be considered as an autonomous control system on $\mathbb{R}^n \times S^1$ defined by

$$\begin{cases} x(k+1) = x(k) + v(x(k),\theta(k))g_1(x(k)) \\ \qquad\qquad + \sum_{i=1}^{m} \tilde{u}_i(k) g_i(x(k)) \\ \theta(k+1) = \theta(k) + 1 \end{cases} \quad (12)$$

where $\theta \in S^1$. The difference of the Lyapunov function V along the trajectories of system (12) is given by

$$\Delta V(x,\theta) = V(f(x,\theta) + \tilde{g}(x,\theta)\tilde{u}) - V(x,\theta)$$

where $\tilde{g}(x,\theta) = \big(\tilde{g}_1(x,\theta), \ldots, \tilde{g}_m(x,\theta)$. If one chooses h such that $V(f(x,\theta) + \tilde{g}(x,\theta)\tilde{u})$ is quadratic on $\tilde{u}$ (of the form (7) for instance), one get

$$\begin{aligned} \Delta V(x,\theta) &= V(f(x,\theta)) - V(x,\theta) \\ &\quad + \tilde{u}^{\mathrm{T}}\big(\nabla V(f(x,\theta))\tilde{g}(x,\theta)\big)^{\mathrm{T}} \\ &\quad + \frac{1}{2}\tilde{u}^{\mathrm{T}} g^{\mathrm{T}}(x) \frac{\partial^2 V}{\partial x^2}(f(x,\theta))\, g(x)\tilde{u} \end{aligned}$$

Furthermore, since V is positive definite on $\mathbb{R}^n \times S^1$ then, for any $(x,\theta) \in \mathbb{R}^n \times S^1$,

$$N(x,\theta) = g^{\mathrm{T}}(x) \frac{\partial^2 V}{\partial x^2}(f(x,\theta))\, g(x) \ge 0 \quad (13)$$

As a matter of fact, if there exists $(x_0,\theta_0) \in \mathbb{R}^n \times S^1$ such that the matrix $N(x_0,\theta_0)$ has a negative eigenvalue $\lambda(x_0,\theta_0)<0$ associted with an eigenvector $v(x_0,\theta_0) \in \mathbb{R}^m$, then, for any $\alpha \in \mathbb{R}$, one has from the Taylor expansion formula

$$\begin{aligned} &V(f(x_0,\theta_0) + \tilde{g}(x_0,\theta_0)\alpha v(x_0,\theta_0)) = \\ &\quad V(f(x_0,\theta_0)) + \nabla V(f(x_0,\theta_0))\tilde{g}(x_0,\theta_0)v(x_0,\theta_0)\alpha \\ &\quad + \frac{1}{2}\lambda(x_0,\theta_0)\left\|v(x_0,\theta_0)\right\|^2 \alpha^2 \end{aligned}$$

It turns out that

$$\lim_{\alpha\to\infty} V(f(x_0,\theta_0) + \tilde{g}(x_0,\theta_0)\alpha v(x_0,\theta_0)) = -\infty$$

that provides a contradiction with the fact that V is positive definite on $\mathbb{R}^n \times S^1$, and so (13) is established. Hence, the matrix

$$M(x,\theta) = I + \frac{1}{2} g^{\mathrm{T}}(x) \frac{\partial^2 V}{\partial x^2}(f(x,\theta))\, g(x)$$

is invertible. It follows from (4), (8) and (9) that with the feedback law defined on $\mathbb{R}^n \times S^1$ by

$$\tilde{u}(x,\theta) = -M^{-1}(x,\theta)\big(\nabla V(f(x,\theta))\tilde{g}(x,\theta)\big)^{\mathrm{T}} \quad (14)$$

the difference of the Lyapunov function V along the trajectories of the closed-loop system (12-14) becomes

$$\begin{aligned} \Delta V(x,\theta) &= V(f(x,\theta)) - V(x,\theta) - \tilde{u}^{\mathrm{T}}(x,\theta)\tilde{u}(x,\theta) \\ &= -\frac{3}{8}(x_1 + h(\tilde{x},\theta))^2 - \tilde{u}^{\mathrm{T}}(x,\theta)\tilde{u}(x,\theta) \le 0 \end{aligned}$$

which implies that V decreases along the solutions of the closed-loop system (12-14) and then the origin is Lyapunov stable. To show that zero is unifomly globally asymptotically stable equilibrium point of the closed-loop system (2-11) or that the invariant set $\{0\} \times S^1$ is globally attractive, set

$$\begin{aligned} E &= \{(x,\theta) \in \mathbb{R}^n \times S^1 \mid \Delta V(x,\theta) = 0\} \\ &= \{(x,\theta) \mid V(f(x,\theta)) - V(x,\theta) = 0,\ \tilde{u}(x,\theta) = 0\} \quad (15) \end{aligned}$$

According to LaSalle's invariance principle all the solutions of the closed-loop system (12-14) tend to Ω the largest invariant set contained in E. In order to prove the global attractivity let us show that $\Omega = \{0\} \times S^1$. Using (14) one has

$$\tilde{u}(x,\theta) = 0 \Rightarrow \nabla V(f(x,\theta))\tilde{g}(x,\theta) = 0 \quad (16)$$

Let $(x(k),\theta(k))$ be a solution of the closed-loop system with $(x(0),\theta(0)) = (x,\theta) \in \Omega$. Since Ω is invariant for the closed-loop system we have $(x(k),\theta(k)) \in \Omega$ for all $k \ge 0$. But, $\tilde{u}(x,\theta)$ vanishing on Ω, one has $(x(k),\theta(k)) = f^k(x,\theta)$ and so, by (15) and (16)

$$V(f^{k+1}(x,\theta)) - V(f^k(x,\theta)) = 0, \quad \forall k \in \mathbb{N} \quad (17)$$

and

$$\nabla V(f^{k+1}(x,\theta))\tilde{g}(f^k(x,\theta)) = 0, \quad \forall\, k \in \mathbb{N} \qquad (18)$$

Notice that, by (4),

$$\nabla V(x,\theta) = \begin{pmatrix} x_1 + h(\tilde{x},\theta) \\ \tilde{x} + (x_1 + h(\tilde{x},\theta))\dfrac{\partial h}{\partial \tilde{x}}(\tilde{x},\theta) \\ (x_1 + h(\tilde{x},\theta))\dfrac{\partial h}{\partial \theta}(\tilde{x},\theta) \end{pmatrix}$$

and, by (8) and (9), for any $k \in \mathbb{N}$

$$f^k(x,\theta) = \begin{pmatrix} \dfrac{1}{2^k}(x_1 + h(\tilde{x},\theta)) - h(\tilde{x},\theta+k) \\ \tilde{x} \\ \theta + k \end{pmatrix}$$

so that

$$\nabla V(f^k(x,\theta)) = \begin{pmatrix} \dfrac{1}{2^k}(x_1 + h(\tilde{x},\theta)) \\ \tilde{x} + \dfrac{1}{2^k}(x_1 + h(\tilde{x},\theta))\dfrac{\partial h}{\partial \tilde{x}}(\tilde{x},\theta+k) \\ \dfrac{1}{2^k}(x_1 + h(\tilde{x},\theta))\dfrac{\partial h}{\partial \theta}(\tilde{x},\theta+k) \end{pmatrix}$$

Now, a simple computation shows that (17) implies

$$x_1 + h(\tilde{x},\theta) = 0 \qquad (19)$$

and so

$$\nabla V(f^{k+1}(x,\theta))\tilde{g}(f^k(x,\theta)) =$$

$$\left(\left\langle \begin{pmatrix} 0 \\ \tilde{x} \\ 0 \end{pmatrix}, \tilde{g_1}(f^k(x,\theta)) \right\rangle, \ldots, \left\langle \begin{pmatrix} 0 \\ \tilde{x} \\ 0 \end{pmatrix}, \tilde{g_m}(f^k(x,\theta)) \right\rangle\right)$$

It turns out from (3) and (18) that $(0, \tilde{x}^{\mathrm{T}}, 0)^{\mathrm{T}}$ belongs to

$$\big(\mathrm{Span}(\{\tilde{g}_2(f^k(x,\theta)), \ldots, \tilde{g}_m(f^k(x,\theta)),\ k \in \mathbb{N}\})\big)^{\perp}$$

Hence, it follows from (10) that $\tilde{x} = 0$, and so, by (19), $x_1 = -h(0,\theta) = 0$ which finally implies $x = 0$. ■

Example. Consider the system evolving in $\mathbb{R}^3$ defined by

$$\begin{cases} x(k+1) = x(k) + u_1(k) \\ y(k+1) = y(k) + x(k)u_2(k) \\ z(k+1) = z(k) + u_2(k) \end{cases}$$

This system doesn't satisfy the necessary stabilizability condition of theorem 1. So the best we can do is to design a time-varying feedback stabilizer. The asumptions of theorem 2 are satisfied with the Lyapunov function

$$V(x,y,z,t) = (x + y\,\cos(t))^2 + y^2 + z^2$$

So, one may easily check that this system can be stabilized thanks to the following periodic time-varying feedback:

$$u_1\big(x(k),y(k),z(k),k\big) = -\frac{1}{2}\big[x(k) - y(k)\cos(k) + 2y(k)\cos(k+1)\big] + \frac{\varphi_1\big(x(k),y(k),z(k),k\big)}{4 + 2x^2(k) + x^2(k)\cos^2(k+1)}$$

$$u_2\big(x(k),y(k),z(k),k\big) = \frac{\varphi_2\big(x(k),y(k),z(k),k\big)}{4 + 2\,x^2(k) + x^2(k)\cos^2(k+1)}$$

with

$$\varphi_1(x,y,z,t) = -2x - x^3 - 6y\cos(t+1) - x^2y\cos(t+1) + 2xz\cos(t+1) + 4y\cos(t+2) + 2x^2y\cos(t+2)$$

$$\varphi_2(x,y,z,t) = -4xy - 4z - x^2\cos(t+1) - 3xy\cos^2(t+1) + 2xy\cos(t+1)\cos(t+2)$$

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