FEEDBACK STABILIZATION OF HOMOGENEOUS POLYNOMIAL SYSTEMS

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Abstract. The propose of this paper is to drive a simple necessary and sufficient stabilizability condition for a class of multi-inputs polnomial systems. This result represents a generalization of the main theorems of (Iggidr and Vivalda, 1992). The stabilizing feedbacks are explicitly given.

Key Words. Stabilization, feedback, dilations, homogeneous systems.

1. INTRODUCTION

The stabilization problem of nonlinear control systems by means of smooth feedback is one of the most important problem in control theory. It has attracted the interest of an increasing number of authors in the last decay. The local stabilizability of nonlinear systems has been extensively studied and a necessary and sufficient stabilizability condition for two-dimensional systems has been given in (Dayawansa *et al.*, 1990). However, the global problem, even for low dimensional systems, is still an open problem. Many interesting results have been published (Andreini *et al.*, 1988; Byrnes and Isidori, 1989; Jurdjevic and Quinn, 1978; Kokotovic and Sussmann, 1989; Saberi *et al.*, 1990; Sontag, 1989; Sontag, 1990; Seibert and Suarez, 1991) but they are, generally, difficult to apply for a given system because they are based on the construction of auxiliary functions such Lyapunov functions or Lyapunov control function. In this paper we solve, completely, the stabilization problem for the class of systems considered.

More precisely we consider a nonlinear control system

$$\begin{cases} \dot{x} = X(x) + Bu \\ x \in \mathbb{R}^n, \ u \in \mathbb{R}^{n-1} \end{cases} \qquad (1)$$

where X is a polynomial vector field, which is homogeneous of degree $k \in \mathbb{N}$ with respect to a family of dilations (see definition below) and B is an $n \times (n-1)$ matrix of rank $n-1$. We shall say that (1) is globally smoothly asymptotically stabilizable (G.S.A.S) if there exists a smooth (C^∞) feedback such that $u(0) = 0$ and the origin is a globally asymptotically stable equilibrium point for the (closed-loop) system

$$\dot{x} = X(x) + Bu(x) \qquad (2)$$

In this communication, we give a necessary and sufficient condition for (1) to be G.S.A.S and we construct explicitly the stabilizing feedback which turns to be also polynomial. This result is a generalization of (Iggidr and Vivalda, 1992) that concerns the case where X is homogeneous with respect to the standard dilation. We can remark that for this class of nonlinear control systems, the stabilizability by feedback is equivalent to the null controllability. Homogeneous systems are also considered in (Hermes, 1990; Hermes, 1991; Kawski, 1990; Kawski, 1989).

To illustrate the methodology developed in this article, we consider the following examples :

Example 1

$$\begin{cases} \dot{x} = x^8 + x^6 y - x^2 y^3 - y^4 \\ \dot{y} = u \\ (x, y) \in \mathbb{R}^2 \ , \ u \in \mathbb{R} \end{cases} \qquad (3)$$

is G.S.A.S thanks to the feedback law

$$\begin{aligned} u(x,y) &= x^2 + y + \left(x^4 + x^2\,y + 2\,y^2\right) \times \\ &\quad \left(x^2 - y\right)\,\left(x + 2\,x^3 + 2\,x\,y\right) \end{aligned}$$

One can remark that for this system the condition of (Boothby and Marino, 1989; Boothby, 1990) is not met so the result of Boothby-Marino (Boothby and Marino, 1989; Boothby, 1990) can not be applied even to achieve the local stabilization.

Example 2

$$\begin{cases} \dot{x} = x^6 + x^2y^2 + y^3 \\ \dot{y} = u \\ (x,y) \in \mathbb{R}^2 \ , \ u \in \mathbb{R} \end{cases} \tag{4}$$

This system satisfies the condition of theorem 1 so it can be stabilized by the feedback law given in proposition 1.

Example 3

$$\begin{cases} \dot{x} = x^6 + x^3y + y^2 \\ \dot{y} = u \\ (x,y) \in \mathbb{R}^2 \ , \ u \in \mathbb{R} \end{cases} \tag{5}$$

is not stabilizable.

2. NOTATIONS AND PRELIMINARIES

In this paper, $< x, y >$ is the usual scalar product and for $n-1$ vectors $x_1, \dots, x_{n-1}$ of $\mathbb{R}^n$, we will denote by $x_1 \wedge \dots \wedge x_{n-1}$ their vectorial product i.e. the unique vector satisfying $\forall x \in \mathbb{R}^n$

$$det(x_1, \dots, x_{n-1}, x) =< x_1 \wedge \dots \wedge x_{n-1}, x >$$

Let $b_2, \dots, b_n$ be the column vectors of B and define $b_1 = b_2 \wedge \dots \wedge b_n$. Since B is of rank $n-1$, $(b_1, b_2, \dots, b_n)$ is a basis of $\mathbb{R}^n$.

Families of dilations.

For positive integers $r_1 \le r_2 \le \dots \le r_n$ and for a given coordinates $(x_1, \dots, x_n)$ on $\mathbb{R}^n$, a one-parameter family of dilations parametrized by $\lambda \in \mathbb{R}$ is

$$\begin{array}{rcl} \Delta_\lambda : \mathbb{R}^n & \to & \mathbb{R}^n \\ (x_1, \dots, x_n) & \mapsto & (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n) \end{array}$$

A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is Δ_λ-homogeneous of order k if $\psi \, o \, \Delta_\lambda = \lambda^k \psi$.

Throught this paper, we shall say that a vector field

$$X(x) = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$$

is Δ_λ-homogeneous of order k, written $X \in \mathcal{H}_k$, if all its compements X_i are homogeneous functions of degree k with respect to the same dilation.

In this paper, we consider only dilations for which $r_1 = 1$. A complete study for arbitrary integers $r_1, \dots, r_n$ as well as the application to the local stabilization of analytic systems will be published in a forthcoming paper.

3. STABILIZATION OF ODD HOMOGENEOUS VECTOR FIELDS

Theorem 1 The control system (1), where X is an odd homogeneous vector field, is G.S.A.S if and only if, the following assumption holds :

(a) there exists $\alpha = (\alpha_2, \dots, \alpha_n)^T \in \mathbb{R}^{n-1}$ such that $< X(b_1 + B\alpha), b_1 > < 0$

*Proof:*Let $(x_1, \dots, x_n)$ be the coordinates of x in the basis $(b_1, b_2, \dots, b_n)$. In this basis, the system (1) is given by :

$$\begin{cases} \dot{x}_1 =< X(x) \, , \, b_1 >= P_1(x) \\ \dot{x}_2 = P_2(x) + u_2 \\ \vdots \\ \dot{x}_n = P_n(x) + u_n \end{cases} \tag{6}$$

where P_i are homogeneous polynomial functions of degree k.

The condition (a) is sufficient, to prove this we estabishe the following result

Proposition 1 If (a) holds then the system (6) is G.S.A.S and the stabilizing feedback is given by :

$$\begin{aligned} u_i(x) &= \alpha_i x_1^{r_i - 1} P_1(x) - x_1 g_i(x) \\ & \quad -(x_i - \alpha_i x_1^{r_i}) - P_i(x) \end{aligned} \tag{7}$$

for $i \in \{2, \dots, n\}$, where:

$$\begin{aligned} g_i(x) = \int_0^1 \frac{\partial P_1}{\partial x_i} (x_1, t x_2 + (1-t)\alpha_2 x_1^{r_2}, \dots \\ \dots, t x_n + (1-t)\alpha_n x_1^{r_n}) dt \end{aligned} \tag{8}$$

*Proof:*Let

$$\begin{aligned} \varphi(t) &= P_1\Big(x_1, t x_2 + (1-t)\alpha_2 x_1^{r_2}, \dots \\ & \qquad \dots, t x_n + (1-t)\alpha_n x_1^{r_n}\Big) \\ &= P_1\Big(x_1, t(x_2 - \alpha_2 x_1^{r_2}) + \alpha_2 x_1^{r_2}, \dots \\ & \qquad \dots, t(x_n - \alpha_n x_1^{r_n}) + \alpha_n x_1^{r_n}\Big) \end{aligned}$$

we have:

$$\begin{aligned} &\varphi(1) = P_1(x_1, x_2, \dots, x_n) = P_1(x) \\ &\varphi(0) = P_1(x_1, \alpha_2 x_1^{r_2}, \dots, \alpha_n x_1^{r_n}) \\ &\varphi(1) = \varphi(0) + \textstyle\int_0^1 \varphi'(t) dt \end{aligned}$$

So, we can write :

$$\begin{aligned} P_1(x_1, x_2, \dots, x_n) = P_1(x_1, \alpha_2 x_1^{r_2}, \dots, \alpha_n x_1^{r_n}) \\ + \sum_{i=2}^n (x_i - \alpha_i x_1^{r_i}) g_i(x) \end{aligned}$$

Now, introduce the positive definite function :

$$V(x_1, x_2, \ldots, x_n) = \frac{1}{2}\left(x_1^2 + \sum_{i=2}^{n}(x_i - \alpha_i x_1^{r_i})^2\right)$$

the derivative of V along trajectories of (6-7) is

$$\begin{aligned}\dot{V} &= x_1 P_1(x) + \sum_{i=2}^{n}(x_i - \alpha_i x_1^{r_i})\Big(P_i(x) + \\ &\qquad + u_i(x) - \alpha_i x_1^{r_i-1} P_1(x)\Big) \\ &= x_1 P_1(x) - x_1 \sum_{i=2}^{n}(x_i - \alpha_i x_1^{r_i}) g_i(x) \\ &\qquad - \sum_{i=2}^{n}(x_i - \alpha_i x_1^{r_i})^2 \\ &= x_1 P_1(x_1, \alpha_2 x_1^{r_2}, \ldots, \alpha_n x_1^{r_n}) \\ &\qquad - \sum_{i=2}^{n}(x_i - \alpha_i x_1^{r_i})^2 \\ &= x_1^{k+1} P_1(1, \alpha_2, \ldots, \alpha_n) \\ &\qquad - \sum_{i=2}^{n}(x_i - \alpha_i x_1^{r_i})^2\end{aligned}$$

(where k is the degree of P_1, k is odd). So, $\dot{V}(x) < 0$ for all nonzero x and global stability is assured and this completes the proof of the proposition.

Let us prove that (a) is a necessary condition for the existence of a stabilizing feedback.

Suppose that $P_1(1, x_2, \ldots, x_n) \geq 0$ for any $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ then for any x_1 positive and any $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$:

$$P_1(x_1, x_2, \ldots, x_n) = x_1^k P_1\left(1, \frac{x_2}{x_1^{r_2}}, \ldots, \frac{x_n}{x_1^{r_n}}\right) \geq 0$$

This shows the origin cannot be reachable from any point of the open half space $\{x_1 > \alpha > 0\}$. The proof of theorem 1 is now completed. ■

4. STABILIZATION OF EVEN HOMOGENEOUS VECTOR FIELDS

Theorem 2 The control system (1) is G.S.A.S if and only if the even homogeneous polynomial $< X(x), b_1 >$ takes both positive and negative values.

Proof: X is a homogeneous polynomial vector field of degree $k = 2p \in \mathbb{N}^*$

Let $(x_1, \ldots, x_n)$ be the coordinates of x in the basis $(b_1, b_2, \ldots, b_n)$. In this basis, the system (1) is given by :

$$\begin{cases} \dot{x}_1 = < X(x), b_1 > = P_1(x) \\ \dot{x}_2 = P_2(x) + u_2 \\ \vdots \\ \dot{x}_n = P_n(x) + u_n \end{cases} \qquad (9)$$

where P_i are homogeneous polynomial functions of degree 2p.

If $P_1(x)$ does not change its sign, then the map $f : \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ defined by :

$$\begin{aligned} f(x, u) &= \Big(< X(x), b_1 >, P_2(x) + u_2, \ldots \\ &\qquad \ldots, P_n(x) + u_n\Big)^T \end{aligned}$$

cannot be onto an open neighborhood of the origin in $\mathbb{R}^n$, so (9) cannot be stabilizable by virtue of BROCKETT'S theorem (Brockett, 1983), moreover the origin cannot be reachable from any point of the half-space $x_1 > \alpha > 0$ (or $x_1 < \beta < 0$).

Now suppose that $P_1(x)$ changes its sign, without loss of generality we can assume that P_1 takes both positive and negative values in the plane (x_1, x_2) (if not one has to make first a linear change of coordinates, see (Iggidr and Vivalda, 1992)).

we can write

$$P_1(x) = Q(x_1, x_2) + R(x_1, \ldots, x_n)$$

where

$$R(x) = \sum_{i=3}^{n} x_i R_i(x_1, \ldots, x_n)$$

and

$$Q(x_1, x_2) = P_1(x_1, x_2, , 0, \ldots, 0)$$

are homogeneous of degree $2p$.

$Q(x_1, x_2)$ changes its sign, so it can be written :

$$Q(x_1, x_2) = L_1^{s_1} L_2^{s_2} \cdots L_q^{s_q} \Phi(x_1, x_2)$$

where the polynomial functions L_i are of the form $L_i = a_i x_1^{r_2} + b_i x_2$ and are linearly independant and $\Phi(x_1, x_2)$ is a definite polynomial function.

At last, two exponents s_i are odd (because $Q(x_1, x_2)$ takes positive and negative values); we can assume that $i = 1$ and $j = 2$. Since L_1 and L_2 are linearly independant, we can suppose that $L_1 = a_1 x_1^{r_2} + b_1 x_2$ with $b_1 \neq 0$

Let us consider the following feedback :

$$\begin{cases} u_2(x) = -P_2(x) - \dfrac{1}{b_1}\Big(a_1 r_2 x_1^{r_2-1} P_1(x) \\ \qquad +x_1 L_2^{s_2} L_3^{s_3} \cdots L_q^{s_q} \Phi(x_1,x_2) + L_1^{s_1}\Big) \\ u_3(x) = -x_3 - P_3(x) - x_1 R_3(x) \\ \vdots \\ u_n(x) = -x_n - P_n(x) - x_1 R_n(x) \end{cases} \tag{10}$$

and the function

$$V = \frac{1}{2}x_1^2 + \frac{1}{s_1+1}L_1^{s_1+1} + \frac{1}{2}x_3^2 + \cdots + \frac{1}{2}x_n^2$$

V is smooth (polynomial), positive definite and proper.

Along the trajectories of the closed-loop system, one has

$$\dot{V} = \frac{dV(x(t))}{dt} = < \nabla V(x) \, , \, X(x) >$$

where $X(x)$ is the closed-loop system with the feedback above(10)

$$\nabla V(x) = \begin{pmatrix} x_1 + r_2 a_1 x_1^{r_2-1} L_1^{s_1} \\ b_1 L_1^{s_1} \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{aligned} \dot{V} &= (x_1 + r_2 a_1 x_1^{r_2-1} L_1^{s_1}) P_1(x) \\ &\quad + b_1 L_1^{s_1}(P_2(x) + u_2(x)) \\ &\quad + \sum_{i=3}^{n} x_i (P_i(x) + u_i(x)) \\ &= (x_1 + a_1 r_2 x_1^{r_2-1} L_1^{s_1}) P_1(x) \\ &\quad - L_1^{s_1}\Big(a_1 r_2 x_1^{r_2-1} P_1(x) \\ &\quad + x_1 L_2^{s_2} L_3^{s_3} \cdots L_q^{s_q} \, \Phi(x_1,x_2) + L_1^{s_1}\Big) \\ &\quad - \sum_{i=3}^{n} x_i^2 - x_1 \sum_{i=3}^{n} x_i R_i \\ &= x_1 P_1(x) \\ &\quad - L_1^{s_1}\Big(x_1 L_2^{s_2} L_3^{s_3} \cdots L_q^{s_q} \, \Phi(x_1,x_2)\Big) \\ &\quad - L_1^{2s_1} - \sum_{i=3}^{n} x_i^2 - x_1 \sum_{i=3}^{n} x_i R_i \\ &= x_1 P_1(x) - x_1 Q(x) - x_1 R(x) \\ &\quad - L_1^{2s_1} - \sum_{i=3}^{n} x_i^2 \end{aligned}$$

So

$$\dot{V} = -L_1^{2s_1} - \sum_{i=3}^{n} x_i^2 \leq 0 \tag{11}$$

If we let $E = \{x \in \mathbb{R}^n \;/\; \dot{V}(x) = 0\}$ then by LASALLE'S invariance theorem ((Lasalle and Lefschetz, 1961), pp.66) all integral curves of the closed-loop system tend to the largest subset Ω of E wich is invariant by the vector field corresponding to the closed-loop system, the proof will be finished if we show that $\Omega = \{0\}$.

We have

$$\begin{aligned} E \;=\; \{x \in \mathbb{R}^n \mid a_1 x_1^{r_2} + b_1 x_2 = 0 \, , \\ x_3 = 0 \, , \, \ldots, x_n = 0\} \end{aligned}$$

is a curve in $\mathbb{R}^n$. For $x \in E$, the closed-loop system is given by :

$$\begin{cases} \dot{x}_1 &= 0 \\ \dot{x}_2 &= -\dfrac{1}{b_1} x_1 L_2^{s_2} L_3^{s_3} \cdots L_q^{s_q} \Phi(x_1,x_2) \\ \dot{x}_3 &= -x_1 R_3(x_1,x_2,0,\ldots,0) \\ \vdots & \\ \dot{x}_n &= -x_1 R_n(x_1,x_2,0,\ldots,0) \end{cases} \tag{12}$$

and it is easy to show that $\Omega = \{0\}$. Thus, the theorem is established.

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