Well-posedness of general 1D Initial Boundary Value Problems for scalar balance laws

Elena Rossi

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Abstract

We focus on the initial boundary value problem for a general scalar balance law in one space dimension. Under rather general assumptions on the flux and source functions, we prove the well-posedness of this problem and the stability of its solutions with respect to variations in the flux and in the source terms. For both results, the initial and boundary data are required to be bounded functions with bounded total variation. The existence of solutions is obtained from the convergence of a Lax–Friedrichs type algorithm with operator splitting. The stability result follows from an application of Kružkov's doubling of variables technique, together with a careful treatment of the boundary terms.

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1 Introduction

Consider the following general Initial-Boundary Value Problem (IBVP) for a one dimensional scalar balance law on the bounded interval $]a,b[\subset \mathbb{R}$

$$\begin{cases} \partial_t u + \mathrm{d}_x\, f(t,x,u) = g(t,x,u), & (t,x)\in I\times]a,b[,\\ u(0,x) = u_o(x), & x\in]a,b[,\\ u(t,a) = u_a(t), & t\in I,\\ u(t,b) = u_b(t), & t\in I, \end{cases} \tag{1.1}$$

where $I =]0,T[$ for a positive T and we introduce the notation

$$\mathrm{d}_x\, f\big(t,x,u(t,x)\big) = \partial_x f\big(t,x,u(t,x)\big) + \partial_u f\big(t,x,u(t,x)\big)\ \partial_x u(t,x). \tag{1.2}$$

Aim of the present work is to prove the well-posedness of (1.1) and the stability of its solutions with respect to variations in the flux and in the source functions.

IBVPs for balance laws in several space dimensions were originally studied by Bardos, le Roux and Nédélec [1]. However, the existence and uniqueness result proved in [1] is limited to rather smooth initial data, namely functions of class $\mathbf{C^2}$, while the boundary datum is assumed to be zero. An extension of this result to more general, although smooth, boundary data is carried out in [3]. Other contributions in the field are due to [12], see also [10,

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Chapter 2], and more recently [14] and [11]. In particular, in this latter article, the author proves the well-posedness of an IBVP for a multi-dimensional balance law with $\mathbf{L}^\infty$ data. However, a restrictive hypothesis on the flux and the source functions is needed, in order to get a maximum principle on the solution.

In all the above cited references, the vanishing viscosity technique is used to get the existence of solutions. Here, existence is obtained by proving the convergence of a Lax–Friedrichs type numerical scheme, together with operator splitting to account for the source term. The idea of the proof comes from [7, 14]. It is remarkable how the $\mathbf{L}^\infty$ and total variation estimates on the solution obtained in the present work (see Theorem 2.4) are more accurate with respect to those presented in [3], allowing moreover for less regular data.

As far as it concerns the uniqueness, the Lipschitz continuous dependence on initial and boundary data of solutions to general multiD IBVP, proved in [3, Theorem 4.3], applies to the present setting. Indeed, this result is valid in a generality wider than that assumed to prove existence of solutions in [3]: its proof follows directly form the definition of solution, which requires initial and boundary data of class $\mathbf{L}^\infty \cap \mathbf{BV}$.

The investigation of stability results for IBVPs for balance laws has begun only recently. At the present time, only partial results, namely considering particular classes of equations, are available. For instance, in the multi-dimensional case, [5] presents a stability estimate for a class of multiD linear conservation laws in a bounded domain, with homogeneous boundary conditions and initial data of class $\mathbf{L}^\infty \cap \mathbf{BV}$. In one space dimension, only conservation laws with a flux function not explicitly dependent on the space variable x have been considered, see [4, 6]. Our result is more general: a stability estimate for 1D IBVPs for balance laws with general flux and source functions. An adaptation of the doubling of variables technique by Kružkov, together with a careful treatment of the boundary terms, allows to obtain the desired result.

The paper is organised as follows. Section 2 presents the assumptions needed throughout the paper, the definitions of solution to problem (1.1) and the main analytic results, namely the well-posedness of the problem and the stability estimate. Section 3 is devoted to the introduction and analysis of the numerical scheme. The estimates necessary to prove its convergence, as well as the convergence result, constitute the contribution of this section. Finally, Section 4 contains the proofs of the main results.

2 Main Results

Throughout, we denote by I the time interval $]0, T[$, for a positive T, and we set

$$\Sigma = [0,T] \times]a,b[\times \mathbb{R}.$$

Following [11, 14], we set

$$\mathrm{sgn}^+(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0, \end{cases} \qquad \mathrm{sgn}^-(s) = \begin{cases} 0 & \text{if } s \geq 0, \\ -1 & \text{if } s < 0, \end{cases} \qquad \begin{array}{l} s^+ = \max\{s, 0\}, \\ s^- = \max\{-s, 0\}. \end{array}$$

In the rest of the paper, we will denote $\mathcal{I}(r,s) = [\min\{r,s\}, \max\{r,s\}]$, for any $r, s \in \mathbb{R}$.

We require the following assumptions

(f) $f \in \mathbf{C^2}(\Sigma; \mathbb{R})$; $\partial_u f$, $\partial^2_{xu} f \in \mathbf{L}^\infty(\Sigma; \mathbb{R})$.

(g) $g \in \mathbf{C^2}(\Sigma;\mathbb{R})$; $\partial_u g \in \mathbf{L}^\infty(\Sigma;\mathbb{R})$.

(D) $u_o \in (\mathbf{L}^\infty \cap \mathbf{BV})(]a,b[;\mathbb{R})$ and u_a, $u_b \in (\mathbf{L}^\infty \cap \mathbf{BV})(I;\mathbb{R})$.

We introduce the constants

$$L_f(t) = \|\partial_u f\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R};\mathbb{R})}, \qquad L_g(t) = \|\partial_u g\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R};\mathbb{R})}. \tag{2.1}$$

Concerning the definition of solution to problem (1.1), we refer below to the following extension of [14, Definition 1] presented in [11] for the multi dimensional case.

Definition 2.1. *An* MV–solution *to the IBVP* (1.1) *on the interval* $[0,T[$ *is a map* $u \in \mathbf{L}^\infty([0,T[\times]a,b[;\mathbb{R})$ *such that, for all* $\varphi \in \mathbf{C_c^1}(]-\infty,T[\times\mathbb{R};\mathbb{R}^+)$ *and* $k \in \mathbb{R}$,

$$\begin{aligned}
&\int_0^T \int_a^b \Big\{ \big(u(t,x)-k\big)^\pm \partial_t\varphi(t,x) + \operatorname{sgn}^\pm\big(u(t,x)-k\big) \Big[f\big(t,x,u(t,x)\big) - f\left(t,x,k\right)\Big] \partial_x\varphi(t,x) \\
&\qquad\qquad + \operatorname{sgn}^\pm\big(u(t,x)-k\big) \Big(g\big(t,x,u(t,x)\big) - \partial_x f\left(t,x,k\right)\Big)\varphi(t,x) \Big\} \mathrm{d}x\,\mathrm{d}t \\
&+ \int_a^b \big(u_o(x)-k\big)^\pm \varphi(0,x)\,\mathrm{d}x \\
&+ L_f(T) \left(\int_0^T \big(u_a(t)-k\big)^\pm \varphi(t,a)\,\mathrm{d}t + \int_0^T \big(u_b(t)-k\big)^\pm \varphi(t,b)\,\mathrm{d}t \right) \geq 0,
\end{aligned} \tag{2.2}$$

where $L_f(T)$ *is as in* (2.1).

We introduce a second definition of solution to problem (1.1). This is an adaptation of the definition of solution presented in [1] to the one dimensional case, where the domain is an open bounded interval.

Definition 2.2. *A BLN–solution to the IBVP* (1.1) *on the interval* $[0,T[$ *is a map* $u \in (\mathbf{L}^\infty \cap \mathbf{BV})([0,T[\times]a,b[;\mathbb{R})$ *such that, for all* $\varphi \in \mathbf{C_c^1}(]-\infty,T[\times\mathbb{R};\mathbb{R}^+)$ *and* $k \in \mathbb{R}$,

$$\begin{aligned}
&\int_0^T \int_a^b \Big\{ \big|u(t,x)-k\big| \partial_t\varphi(t,x) + \operatorname{sgn}\big(u(t,x)-k\big) \Big[f\big(t,x,u(t,x)\big) - f\left(t,x,k\right)\Big] \partial_x\varphi(t,x) \\
&\qquad\qquad + \operatorname{sgn}\big(u(t,x)-k\big) \Big(g\big(t,x,u(t,x)\big) - \partial_x f\left(t,x,k\right)\Big)\varphi(t,x) \Big\} \mathrm{d}x\,\mathrm{d}t \\
&+ \int_a^b \big|u_o(x)-k\big| \varphi(0,x)\,\mathrm{d}x \\
&+ \int_0^T \operatorname{sgn}\big(u_a(t)-k\big) \Big[f\Big(t,a,u(t,a^+)\Big) - f\left(t,a,k\right)\Big] \varphi(t,a)\,\mathrm{d}t \\
&- \int_0^T \operatorname{sgn}\big(u_b(t)-k\big) \Big[f\Big(t,b,u(t,b^-)\Big) - f\left(t,b,k\right)\Big] \varphi(t,b)\,\mathrm{d}t \geq 0.
\end{aligned} \tag{2.3}$$

We remark that, for functions in $(\mathbf{L}^\infty \cap \mathbf{BV})([0,T[\times]a,b[;\mathbb{R})$, Definition 2.1 and Definition 2.2 are equivalent, see [13] for further details.

Remark 2.3. In Definition 2.2, to ensure that the traces of u at the boundary $u(t,a^+)$ and $u(t,b^+)$ are well defined, we need the solution to be of bounded total variation. Moreover, we recall the well-known BLN–boundary conditions ([13, Lemma 5.6]), linking the boundary data to the traces of the solution at the boundary:

- at $x = a$: for all $k \in \mathbb{R}$, for a.e. $t \in]0, T[$

$$\Big(\text{sgn}(u(t, a^+) - k) - \text{sgn}(u_a(t) - k)\Big) \Big(f(t, a, u(t, a^+)) - f(t, a, k)\Big) \leq 0,$$

or, equivalently, for all $k \in \mathcal{I}(u(t, a^+), u_a(t))$ and a.e. $t \in]0, T[$

$$\text{sgn}(u(t, a^+) - u_a(t)) \Big(f(t, a, u(t, a^+)) - f(t, a, k)\Big) \leq 0;$$

- at $x = b$: for all $k \in \mathbb{R}$, for a.e. $t \in]0, T[$

$$\Big(\text{sgn}(u(t, b^-) - k) - \text{sgn}(u_b(t) - k)\Big) \Big(f(t, b, u(t, b^-)) - f(t, b, k)\Big) \geq 0,$$

or, equivalently, for all $k \in \mathcal{I}(u(t, b^-), u_b(t))$ and a.e. $t \in]0, T[$

$$\text{sgn}(u(t, b^-) - u_b(t)) \Big(f(t, b, u(t, b^-)) - f(t, b, k)\Big) \geq 0.$$

The following Theorem contains the existence and uniqueness result, as well as some *a priori* estimates on the solution to the IBVP (1.1).

Theorem 2.4. *Let* **(f)***,* **(g)** *and* **(D)** *hold. Then, for all $T > 0$, the IBVP* (1.1) *has a unique MV–solution $u \in (\mathbf{L}^\infty \cap \mathbf{BV})([0, T[\times]a, b[; \mathbb{R})$. Moreover, the following estimates hold: for any $t \in [0, T[$ and $\tau \in]0, t[$,*

$$\left\|u(t)\right\|_{\mathbf{L}^\infty(]a,b[)} \leq \Big(\max\Big\{\|u_o\|_{\mathbf{L}^\infty(]a,b[)}, \|u_a\|_{\mathbf{L}^\infty([0,t])}, \|u_b\|_{\mathbf{L}^\infty([0,t])}\Big\} + \mathcal{C}_1(t)\, t\Big)\, e^{\mathcal{C}_2(t)\, t}, \quad (2.4)$$

$$\text{TV}\,(u(t)) \leq e^{t\, \mathcal{C}_2(t)} \left(\text{TV}\,(u_o) + \text{TV}\,(u_a; [0, t]) + \text{TV}\,(u_b; [0, t]) + \mathcal{K}_2(t)\, t\right), \quad (2.5)$$

$$\left\|u(t) - u(t - \tau)\right\|_{\mathbf{L}^1(]a,b[)} \leq \tau \left(\mathcal{C}_t(t) + \|g\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{U}(t),\mathcal{U}(t)])}\right), \quad (2.6)$$

where $\mathcal{C}_1(t)$, $\mathcal{C}_2(t)$ are as in (3.8)*–*(3.9)*, $\mathcal{K}_2(t)$ is as in* (3.27) *and $\mathcal{C}_t(t)$ is as in* (3.33)*, with $\alpha = \|\partial_u f\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times[-\mathcal{U}(t),\mathcal{U}(t)])}$, $\mathcal{U}(t)$ being as in* (3.7)*.*

The proof of Theorem 2.4 is postponed to Section 4.

Remark 2.5. The $\mathbf{L}^\infty$ bound (2.4) provided in Theorem 2.4 is optimal.

- In the case $f = f(u)$ and $g = 0$, we get $\mathcal{C}_1(t) = 0$ and $\mathcal{C}_2(t) = 0$, so (2.4) reduces to the well-known maximum principle, compare with [10, Chapter 2, Remark 7.33].
- The functions $\mathcal{C}_1(t)$ and $\mathcal{C}_2(t)$ are clearly strictly related to the source term and to the space dependent flux. Consider, for instance, problem (1.1) with $u_o = 0$ and $u_b = 0$, $f(t, x, u) = -x$ and $g = 0$. The solution is $u(t, x) = t$, and from (3.8) we obtain $\mathcal{C}_1(t) = 1$, so that (2.4) now reads $\left\|u(t)\right\|_{\mathbf{L}^\infty(]a,b[)} \leq t$.
- Compare (2.4) with the $\mathbf{L}^\infty$ estimate on the solution presented in [3, Formula (2.5)]. At a glance, one could notice that the present estimate is more accurate. Moreover, here the boundary data should be of class $\mathbf{L}^\infty \cap \mathbf{BV}$, thus they are not required to be as regular as in [3, Theorem 2.7].

Consider, for instance, the following example:

$$f(u) = u(1-u), \qquad g(u) = 0, \qquad u_b(t) = 0, \qquad u_o(x) = 0,$$

and, as boundary datum in $x = a$, an increasing smooth function (at least of class $\mathbf{C^3}$, as required by [3, Theorem 2.7]) such that

$$u_a(0) = 0, \qquad u_a(t) = 0.4 \text{ for } t > 0.1.$$

Now compare the two $\mathbf{L}^\infty$ estimates at a time $t > 0.1$. As already observed in the first point of this remark, $\mathcal{C}_1(t) = \mathcal{C}_2(t) = 0$, and the estimate (2.4) reads

$$\left\|u(t)\right\|_{\mathbf{L}^\infty} \le \|u_a\|_{\mathbf{L}^\infty([0,t])} = 0.4.$$

On the other hand, with the notation of [3], $c_1 = 1$, $c_2 = 0$, $c_3 = \left\|\partial^2_{uu} f\right\|_{\mathbf{L}^\infty} = 2$, so that the estimate [3, Formula (2.5)] reads

$$\left\|u(t)\right\|_{\mathbf{L}^\infty} \le \left(t\,\|u_a\|_{\mathbf{L}^\infty} + \|\partial_t u_a\|_{\mathbf{L}^\infty} + 2\,\|u_a\|_{\mathbf{C^2}}\right) e^{t(1+2\|u_a\|_{\mathbf{C^2}})} + \|u_a\|_{\mathbf{L}^\infty},$$

where all the norms on the right hand side of the inequality are evaluated on $[0,t]$.

The following Theorem presents a stability estimate with respect to the flux and the source functions. A particular case of the IBVP (1.1) is considered, for instance, in [6], where a flux function independent of the space variable is taken into account. There, a stability estimate with respect to the flux function is provided.

Theorem 2.6. *Let f_1, f_2 satisfy* **(f)***, g_1, g_2 satisfy* **(g)***, (u_o, u_a, u_b) satisfy* **(D)***. Call u_1 and u_2 the corresponding solutions to the IBVP* (1.1)*. Then, for all $t \in [0, T[$, the following estimate holds*

$$\begin{aligned}
&\left\|u_1(t) - u_2(t)\right\|_{\mathbf{L^1}(]a,b[)} \\
\le\ & \exp\left(t \min\left\{\|\partial_u g_1\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times U(t))}, \|\partial_u g_2\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times U(t))}\right\}\right) \\
& \times \left(\int_0^t \int_a^b \left\|\partial_x (f_2 - f_1)(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}x\,\mathrm{d}s\right. \\
& \quad + \int_0^t \int_a^b \left\|(g_1 - g_2)(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}x\,\mathrm{d}s \\
& \quad + \int_0^t \left\|\partial_u (f_2 - f_1)(s,\cdot,\cdot)\right\|_{\mathbf{L}^\infty(]a,b[\times U(s))} \min\left\{\mathrm{TV}\left(u_1(s)\right), \mathrm{TV}\left(u_2(s)\right)\right\} \mathrm{d}s \\
& \quad \left. +2 \int_0^t \left\|(f_2 - f_1)(s,a,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s + 2 \int_0^t \left\|(f_2 - f_1)(s,b,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s\right),
\end{aligned}$$

where $U(s)$ is as in (4.2)*.*

The proof is postponed to Section 4.

3 Existence of weak entropy solutions

Consider a space step Δx such that $b - a = N\Delta x$, $N \in \mathbb{N}$, and a time step Δt subject to a CFL condition which will be specified later. For $j = 1, \ldots, N$, introduce the following notation

$$y_k = (k - 1/2)\Delta x \qquad y_{k+1/2} = k\Delta x \qquad \text{for } k \in \mathbb{Z},$$

and let $x_{j+1/2} = a + j\Delta x = a + y_{j+1/2}$ be the cells interfaces, for $j = 0, \ldots, N$, and $x_j = a + (j - 1/2)\Delta x = a + y_j$ the cells centres, for $j = 1, \ldots, N$. Moreover, set $N_T = \lfloor T/\Delta t \rfloor$ and, for $n = 0, \ldots, N_T$, let $t^n = n\Delta t$ be the time mesh. Set $\lambda = \Delta t/\Delta x$ and let $\alpha \geq 1$ be the viscosity coefficient.

Approximate the initial datum u_o and the boundary data as follows:

$$u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_o(x)\,\mathrm{d}x\,, \qquad j = 1, \ldots, N,$$
$$u_a^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u_a(t)\,\mathrm{d}t\,, \quad u_b^n = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} u_b(t)\,\mathrm{d}t\,, \qquad n = 0, \ldots, N_T - 1.$$

Introduce moreover the notation $u_0^n = u_a^n$ and $u_{N+1}^n = u_b^n$.

We define a piecewise constant approximate solution u_Δ to (1.1) as

$$u_\Delta(t,x) = u_j^n \quad \text{for} \quad \begin{cases} t \in [t^n, t^{n+1}[\,, \\ x \in [x_{j-1/2}, x_{j+1/2}[\,, \end{cases} \quad \text{where} \quad \begin{array}{l} n = 0, \ldots, N_T - 1, \\ j = 1, \ldots, N, \end{array} \tag{3.1}$$

through a Lax–Friedrichs type scheme together with operator splitting, in order to treat the source term.

In particular, the algorithm is defined as follows:

Algorithm 3.1.

for $n = 0, \ldots N_T - 1$

for $j = 1, \ldots, N$

$$F_{j+1/2}^n(u_j^n, u_{j+1}^n) = \frac{1}{2}\left[f(t^n, x_{j+1/2}, u_j^n) + f(t^n, x_{j+1/2}, u_{j+1}^n)\right] - \frac{\alpha}{2}\left(u_{j+1}^n - u_j^n\right) \tag{3.2}$$
$$u_j^{n+1/2} = u_j^n - \lambda\left(F_{j+1/2}^n(u_j^n, u_{j+1}^n) - F_{j-1/2}^n(u_{j-1}^n, u_j^n)\right) \tag{3.3}$$
$$u_j^{n+1} = u_j^{n+1/2} + g\left(t^n, x_j, u_j^{n+1/2}\right)\Delta t \tag{3.4}$$

end

end

We require, moreover, the following CFL condition:

$$\alpha \geq L_f(T), \qquad \lambda \leq \frac{1}{3\,\alpha}. \tag{3.5}$$

The proof of the convergence of approximate solutions consists of several steps, whose aim is to show that the sequence verifies the hypotheses of Helly's compactness theorem.

3.1 $\mathbf{L}^\infty$ bound

Lemma 3.2. *Let* **(f)**, **(g)**, **(D)** *and* (3.5) *hold. Then, for all* $t \in]0,T[$, u_Δ *in* (3.1) *defined through Algorithm 3.1 satisfies*

$$\left\|u_\Delta(t,\cdot)\right\|_{\mathbf{L}^\infty(]a,b[)} \le \mathcal{U}(t) \tag{3.6}$$

where

$$\mathcal{U}(t) = \left(\max\left\{\|u_o\|_{\mathbf{L}^\infty(]a,b[)}, \|u_a\|_{\mathbf{L}^\infty([0,t])}, \|u_b\|_{\mathbf{L}^\infty([0,t])}\right\} + t\,\mathcal{C}_1(t)\right) e^{\mathcal{C}_2(t)\,t}, \tag{3.7}$$

with

$$\mathcal{C}_1(t) = \left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t]\times[a,b])} + \left\|g(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t]\times[a,b])}, \tag{3.8}$$

$$\mathcal{C}_2(t) = \left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R})} + \|\partial_u g\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times\mathbb{R})}. \tag{3.9}$$

Proof. Fix j between 1 and N, n between 0 and $N_T - 1$, and rewrite (3.3) as follows:

$$\begin{aligned}
&u_j^{n+1/2}\\
&= u_j^n - \lambda\left[F^n_{j+1/2}(u_j^n, u_{j+1}^n) \pm F^n_{j+1/2}(u_j^n, u_j^n) \pm F^n_{j-1/2}(u_j^n, u_j^n) - F^n_{j-1/2}(u_{j-1}^n, u_j^n)\right]\\
&= (1-\beta_j^n - \gamma_j^n)\,u_j^n + \beta_j^n\,u_{j-1}^n + \gamma_j^n\,u_{j+1}^n - \lambda\left(F^n_{j+1/2}(u_j^n,u_j^n) - F^n_{j-1/2}(u_j^n,u_j^n)\right),
\end{aligned} \tag{3.10}$$

with

$$\beta_j^n = \begin{cases} \lambda\dfrac{F^n_{j-1/2}(u_j^n,u_j^n) - F^n_{j-1/2}(u_{j-1}^n,u_j^n)}{u_j^n - u_{j-1}^n} & \text{if } u_j^n \neq u_{j-1}^n,\\ 0 & \text{if } u_j^n = u_{j-1}^n, \end{cases} \tag{3.11}$$

$$\gamma_j^n = \begin{cases} -\lambda\dfrac{F^n_{j+1/2}(u_j^n,u_{j+1}^n) - F^n_{j+1/2}(u_j^n,u_j^n)}{u_{j+1}^n - u_j^n} & \text{if } u_j^n \neq u_{j+1}^n,\\ 0 & \text{if } u_j^n = u_{j+1}^n. \end{cases} \tag{3.12}$$

Using the explicit expression of the numerical flux (3.2) and the hypotheses on f **(f)**, we observe that, whenever $u_j^n \neq u_{j-1}^n$,

$$\begin{aligned}
\beta_j^n &= \frac{\lambda}{2\,(u_j^n - u_{j-1}^n)}\left[f(t^n, x_{j-1/2}, u_j^n) - f(t^n, x_{j-1/2}, u_{j-1}^n) + \alpha\,(u_j^n - u_{j-1}^n)\right]\\
&= \frac{\lambda}{2}\left(\partial_u f(t^n, x_{j-1/2}, r^n_{j-1/2}) + \alpha\right),
\end{aligned}$$

with $r^n_{j-1/2} \in \mathcal{I}\left(u_{j-1}^n, u_j^n\right)$. Similarly, whenever $u_j^n \neq u_{j+1}^n$,

$$\begin{aligned}
\gamma_j^n &= -\frac{\lambda}{2\,(u_{j+1}^n - u_j^n)}\left[f(t^n, x_{j+1/2}, u_{j+1}^n) - f(t^n, x_{j+1/2}, u_j^n) - \alpha\,(u_{j+1}^n - u_j^n)\right]\\
&= \frac{\lambda}{2}\left(\alpha - \partial_u f(t^n, x_{j+1/2}, r^n_{j+1/2})\right),
\end{aligned}$$

with $r^n_{j+1/2} \in \mathcal{I}\left(u^n_j, u^n_{j+1}\right)$. By (3.5) we get

$$\beta^n_j,\ \gamma^n_j \in \left[0, \frac{1}{3}\right], \qquad\qquad (1-\beta^n_j-\gamma^n_j) \in \left[\frac{1}{3}, 1\right]. \tag{3.13}$$

Compute now

$$\begin{aligned}
&\left|F^n_{j+1/2}(u^n_j, u^n_j) - F^n_{j-1/2}(u^n_j, u^n_j)\right| \\
= &\left|f(t^n, x_{j+1/2}, u^n_j) - f(t^n, x_{j-1/2}, u^n_j)\right| \\
\le &\left|\partial_x f(t^n, \tilde{x}_j, u^n_j) \pm \partial_x f(t^n, \tilde{x}_j, 0)\right| |x_{j+1/2} - x_{j-1/2}| \\
\le &\ \Delta x \left|\partial_x f(t^n, \tilde{x}_j, 0)\right| + \Delta x \left|u^n_j\right| \left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(([0,t^n]\times[a,b]\times\mathbb{R};\mathbb{R})} \\
\le &\ \Delta x \left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} + \Delta x \left|u^n_j\right| \left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(([0,t^n]\times[a,b]\times\mathbb{R})},
\end{aligned}$$

where $\tilde{x}_j \in]x_{j-1/2}, x_{j+1/2}[$. Observe that, thanks to **(f)**, we have $\left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t]\times[a,b];\mathbb{R})} < +\infty$ for all $t \in I$. Inserting the above estimate into (3.10) and exploiting the bounds (3.13) on β^n_j and γ^n_j, we get

$$\begin{aligned}
u^{n+1/2}_j \le&\ (1-\beta^n_j-\gamma^n_j)\left|u^n_j\right| + \beta^n_j \left|u^n_{j-1}\right| + \gamma^n_j \left|u^n_{j+1}\right| + \lambda \left|F^n_{j+1/2}(u^n_j, u^n_j) - F^n_{j-1/2}(u^n_j, u^n_j)\right| \\
\le&\ (1-\beta^n_j-\gamma^n_j) \left\|u^n\right\|_{\mathbf{L}^\infty(]a,b[)} + \beta^n_j \max\left\{\left\|u^n\right\|_{\mathbf{L}^\infty(]a,b[)}, \left|u^n_a\right|\right\} \\
&+ \gamma^n_j \max\left\{\left\|u^n\right\|_{\mathbf{L}^\infty(]a,b[)}, \left|u^n_b\right|\right\} \\
&+ \lambda\,\Delta x \left(\left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} + \left|u^n_j\right| \left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times\mathbb{R})}\right) \\
\le&\ \max\left\{\left\|u^n\right\|_{\mathbf{L}^\infty(]a,b[)}, \left|u^n_a\right|, \left|u^n_b\right|\right\} \left(1 + \Delta t \left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times\mathbb{R})}\right) \\
&+ \Delta t \left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} \\
\le&\ \left(\max\left\{\left\|u^n\right\|_{\mathbf{L}^\infty(]a,b[)}, \left\|u_a\right\|_{\mathbf{L}^\infty([0,t^n])}, \left\|u_b\right\|_{\mathbf{L}^\infty([0,t^n])}\right\} + \Delta t \left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}\right) \\
&\times \exp\left(\Delta t \left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times\mathbb{R})}\right).
\end{aligned}$$

By (3.4), since, thanks to **(g)**, $\left\|g(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t]\times[a,b])} < +\infty$ for all $t \in I$, we have

$$\begin{aligned}
u^{n+1}_j =&\ u^{n+1/2}_j + \Delta t \left(g\left(t^n, x_j, u^{n+1/2}_j\right) \pm g\left(t^n, x_j, 0\right)\right) \\
\le&\ \left|u^{n+1/2}_j\right| + \Delta t \left|u^{n+1/2}_j\right| \left\|\partial_u g\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times\mathbb{R})} + \Delta t \left\|g(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} \\
\le&\ \left|u^{n+1/2}_j\right| \left(1 + \Delta t \left\|\partial_u g\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times\mathbb{R})}\right) + \Delta t \left\|g(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} \\
\le&\ \left(\left|u^{n+1/2}_j\right| + \Delta t \left\|g(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}\right) \exp\left(\Delta t \left\|\partial_u g\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times\mathbb{R})}\right) \\
\le&\ \left(\max\left\{\left\|u^n\right\|_{\mathbf{L}^\infty(]a,b[)}, \left\|u_a\right\|_{\mathbf{L}^\infty([0,t^n])}, \left\|u_b\right\|_{\mathbf{L}^\infty([0,t^n])}\right\}\right.
\end{aligned}$$

$$+ \Delta t \left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} + \Delta t \big\|g(\cdot,\cdot,0)\big\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}\Big)$$

$$\times \exp\left(\Delta t\left(\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times\mathbb{R})} + \left\|\partial_u g\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times\mathbb{R})}\right)\right).$$

An iterative argument yields the thesis. □

3.2 BV estimates

Proposition 3.3. (BV estimate in space) *Let* **(f)**, **(g)**, **(D)** *and* (3.5) *hold. Then, for n between* 1 *and N_T, the following estimate holds*

$$\sum_{j=0}^{N} \left|u^n_{j+1} - u^n_j\right| \le \mathcal{C}_x(t^n), \tag{3.14}$$

where

$$\mathcal{C}_x(t^n) = e^{\mathcal{C}_2(t^n)\,t^n}\left(\sum_{j=0}^{N}\left|u^0_{j+1} - u^0_j\right| + \sum_{m=1}^{n}\left|u^m_a - u^{m-1}_a\right| + \sum_{m=1}^{n}\left|u^m_b - u^{m-1}_b\right| + t^n\,\mathcal{K}_2(t^n)\right), \tag{3.15}$$

with $\mathcal{C}_2(t^n)$ defined as in (3.9) *and $\mathcal{K}_2(t^n)$ defined as in* (3.27).

Proof. For the sake of simplicity, introduce the space $\Sigma_n = [0,t^n]\times[a,b]\times[-\mathcal{U}(t^n),\mathcal{U}(t^n)]$, with the notation introduced in (3.7). By the definition of the scheme (3.4), observe that, for all $j = 1,\ldots,N-1$,

$$\begin{aligned}
\left|u^{n+1}_{j+1} - u^{n+1}_j\right| &= \left|u^{n+1/2}_{j+1} - u^{n+1/2}_j + \Delta t\left(g\left(t^n, x_{j+1}, u^{n+1/2}_{j+1}\right) - g\left(t^n, x_j, u^{n+1/2}_j\right)\right)\right| \\
&\le \left|u^{n+1/2}_{j+1} - u^{n+1/2}_j\right|\left(1 + \Delta t\left\|\partial_u g\right\|_{\mathbf{L}^\infty(\Sigma_n)}\right) + \Delta t\,\Delta x\left\|\partial_x g\right\|_{\mathbf{L}^\infty(\Sigma_n)},
\end{aligned}$$

where $\left\|\partial_x g\right\|_{\mathbf{L}^\infty(\Sigma_n)}$ is bounded, thanks to **(g)**. On the other hand, for $j=0$ and $j=N$ we have respectively

$$\begin{aligned}
\left|u^{n+1}_1 - u^{n+1}_a\right| &= \left|u^{n+1/2}_1 - u^{n+1}_a + \Delta t\, g\left(t^n, x_1, u^{n+1/2}_1\right) \pm \Delta t\, g\left(t^n, x_1, 0\right)\right| \\
&\le \left|u^{n+1/2}_1 - u^{n+1}_a\right| + \Delta t\left\|\partial_u g\right\|_{\mathbf{L}^\infty(\Sigma_n)}\left|u^{n+1/2}_1\right| + \Delta t\big\|g(\cdot,\cdot,0)\big\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}, \\
\left|u^{n+1}_b - u^{n+1}_N\right| &= \left|u^{n+1/2}_b - u^{n+1}_N - \Delta t\, g\left(t^n, x_N, u^{n+1/2}_N\right) \pm \Delta t\, g\left(t^n, x_N, 0\right)\right| \\
&\le \left|u^{n+1/2}_b - u^{n+1}_N\right| + \Delta t\left\|\partial_u g\right\|_{\mathbf{L}^\infty(\Sigma_n)}\left|u^{n+1/2}_N\right| + \Delta t\big\|g(\cdot,\cdot,0)\big\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}.
\end{aligned}$$

Therefore,

$$\sum_{j=0}^{N}\left|u^{n+1}_{j+1} - u^{n+1}_j\right|$$

$$
\begin{aligned}
&\leq e^{\Delta t\,\|\partial_u g\|_{\mathbf{L}^\infty(\Sigma_n)}} \sum_{j=1}^{N-1} \left|u_{j+1}^{n+1/2} - u_j^{n+1/2}\right| + \Delta t\,(b-a)\,\|\partial_x g\|_{\mathbf{L}^\infty(\Sigma_n)} \\
&\quad + \left|u_1^{n+1/2} - u_a^{n+1}\right| + \left|u_b^{n+1/2} - u_N^{n+1}\right| + 2\Delta t \left\|g(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} \\
&\quad + 2\,\Delta t\,\|\partial_u g\|_{\mathbf{L}^\infty(\Sigma_n)} \left\|u^{n+1/2}\right\|_{\mathbf{L}^\infty([a,b])} \\
&\leq e^{\Delta t\,\|\partial_u g\|_{\mathbf{L}^\infty(\Sigma_n)}} \left(\sum_{j=1}^{N-1} \left|u_{j+1}^{n+1/2} - u_j^{n+1/2}\right| + \left|u_1^{n+1/2} - u_a^{n+1}\right| + \left|u_b^{n+1/2} - u_N^{n+1}\right|\right) \\
&\quad + \Delta t\,(b-a)\,\|\partial_x g\|_{\mathbf{L}^\infty(\Sigma_n)} \qquad\qquad (3.16)\\
&\quad + 2\Delta t \left(\left\|g(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} + \|\partial_u g\|_{\mathbf{L}^\infty(\Sigma_n)} \left\|u^{n+1/2}\right\|_{\mathbf{L}^\infty([a,b])}\right).
\end{aligned}
$$

Focus first on $\sum_{j=1}^{N-1} \left|u_{j+1}^{n+1/2} - u_j^{n+1/2}\right|$: by (3.3) we obtain

$$
\begin{aligned}
& u_{j+1}^{n+1/2} - u_j^{n+1/2} \\
={}& u_{j+1}^n - u_j^n \\
& - \lambda \left[F_{j+3/2}^n(u_{j+1}^n, u_{j+2}^n) - F_{j+1/2}^n(u_j^n, u_{j+1}^n) - F_{j+1/2}^n(u_j^n, u_{j+1}^n) + F_{j-1/2}^n(u_{j-1}^n, u_j^n)\right] \\
& \pm \lambda\, F_{j+3/2}^n(u_j^n, u_{j+1}^n) \pm \lambda F_{j+1/2}^n(u_{j-1}^n, u_j^n) \\
={}& u_{j+1}^n - u_j^n \\
& - \lambda \left[F_{j+3/2}^n(u_{j+1}^n, u_{j+2}^n) - F_{j+1/2}^n(u_j^n, u_{j+1}^n) + F_{j+1/2}^n(u_{j-1}^n, u_j^n) - F_{j+3/2}^n(u_j^n, u_{j+1}^n)\right] \\
& - \lambda \left[F_{j+3/2}^n(u_j^n, u_{j+1}^n) - F_{j+1/2}^n(u_{j-1}^n, u_j^n) + F_{j-1/2}^n(u_{j-1}^n, u_j^n) - F_{j+1/2}^n(u_j^n, u_{j+1}^n)\right] \\
={}& \mathcal{A}_j^n - \lambda\,\mathcal{B}_j^n,
\end{aligned}
$$

where we set

$$
\begin{aligned}
\mathcal{A}_j^n &= u_{j+1}^n - u_j^n \\
&\qquad - \lambda \left[F_{j+3/2}^n(u_{j+1}^n, u_{j+2}^n) - F_{j+1/2}^n(u_j^n, u_{j+1}^n) + F_{j+1/2}^n(u_{j-1}^n, u_j^n) - F_{j+3/2}^n(u_j^n, u_{j+1}^n)\right], \\
\mathcal{B}_j^n &= F_{j+3/2}^n(u_j^n, u_{j+1}^n) - F_{j+1/2}^n(u_{j-1}^n, u_j^n) + F_{j-1/2}^n(u_{j-1}^n, u_j^n) - F_{j+1/2}^n(u_j^n, u_{j+1}^n).
\end{aligned}
$$

Rearrange $\mathcal{A}_j^n$ as follows:

$$
\begin{aligned}
\mathcal{A}_j^n = u_{j+1}^n - u_j^n &- \lambda\, \frac{F_{j+3/2}^n(u_{j+1}^n, u_{j+2}^n) - F_{j+3/2}^n(u_{j+1}^n, u_{j+1}^n)}{u_{j+2}^n - u_{j+1}^n} \left(u_{j+2}^n - u_{j+1}^n\right) \\
&- \lambda\, \frac{F_{j+3/2}^n(u_{j+1}^n, u_{j+1}^n) - F_{j+3/2}^n(u_j^n, u_{j+1}^n)}{u_{j+1}^n - u_j^n} \left(u_{j+1}^n - u_j^n\right) \\
&+ \lambda\, \frac{F_{j+1/2}^n(u_j^n, u_{j+1}^n) - F_{j+1/2}^n(u_j^n, u_j^n)}{u_{j+1}^n - u_j^n} \left(u_{j+1}^n - u_j^n\right) \\
&+ \lambda\, \frac{F_{j+1/2}^n(u_j^n, u_j^n) - F_{j+1/2}^n(u_{j-1}^n, u_j^n)}{u_j^n - u_{j-1}^n} \left(u_j^n - u_{j-1}^n\right)
\end{aligned}
$$

$$= \delta_j^n\,(u_j^n - u_{j-1}^n) + \gamma_{j+1}^n\,(u_{j+2}^n - u_{j+1}^n) + (1 - \gamma_j^n - \delta_{j+1}^n)(u_{j+1}^n - u_j^n),$$

where

$$\delta_j^n = \begin{cases} \lambda \dfrac{F_{j+1/2}^n(u_j^n, u_j^n) - F_{j+1/2}^n(u_{j-1}^n, u_j^n)}{u_j^n - u_{j-1}^n} & \text{if } u_j^n \neq u_{j-1}^n, \\ 0 & \text{if } u_j^n = u_{j-1}^n, \end{cases} \tag{3.17}$$

while γ_j^n is as in (3.12). It can be proven that $\delta_j^n \in \left[0, 1/3\right]$. Thus,

$$\begin{aligned} & \sum_{j=1}^{N-1} \left|\mathcal{A}_j^n\right| \\ \leq\ & \sum_{j=1}^{N-1} \left|u_{j+1}^n - u_j^n\right| + \sum_{j=0}^{N-2} \delta_{j+1}^n \left|u_{j+1}^n - u_j^n\right| - \sum_{j=1}^{N-1} \delta_{j+1}^n \left|u_{j+1}^n - u_j^n\right| \\ & + \sum_{j=2}^{N} \gamma_j^n \left|u_{j+1}^n - u_j^n\right| - \sum_{j=1}^{N-1} \gamma_j^n \left|u_{j+1}^n - u_j^n\right| \\ =\ & \sum_{j=1}^{N-1} \left|u_{j+1}^n - u_j^n\right| + \delta_1^n \left|u_1^n - u_a^n\right| - \delta_N^n \left|u_N^n - u_{N-1}^n\right| + \gamma_N^n \left|u_b^n - u_N^n\right| - \gamma_1^n \left|u_2^n - u_1^n\right|. \end{aligned} \tag{3.18}$$

Focus now on $\mathcal{B}_j^n$:

$$\begin{aligned} \mathcal{B}_j^n = & \frac{1}{2}\Big[f(t^n, x_{j+3/2}, u_j^n) + f(t^n, x_{j+3/2}, u_{j+1}^n) \\ & - f(t^n, x_{j+1/2}, u_j^n) - f(t^n, x_{j+1/2}, u_{j+1}^n) \\ & + f(t^n, x_{j-1/2}, u_{j-1}^n) + f(t^n, x_{j-1/2}, u_j^n) \\ & - f(t^n, x_{j+1/2}, u_{j-1}^n) - f(t^n, x_{j+1/2}, u_j^n)\Big] \\ = & \frac{1}{2}\Big(f(t^n, x_{j+3/2}, u_{j+1}^n) - f(t^n, x_{j+1/2}, u_{j+1}^n)\Big) \\ & + \frac{1}{2}\Big(f(t^n, x_{j-1/2}, u_{j-1}^n) - f(t^n, x_{j+1/2}, u_{j-1}^n)\Big) \\ & + \frac{1}{2}\Big(f(t^n, x_{j+3/2}, u_j^n) - 2\, f(t^n, x_{j+1/2}, u_j^n) + f(t^n, x_{j-1/2}, u_j^n)\Big) \\ = & \frac{\Delta x}{2}\, \partial_x f(t^n, \tilde{x}_{j+1}, u_{j+1}^n) - \frac{\Delta x}{2}\, \partial_x f(t^n, \tilde{x}_j, u_{j-1}^n) \\ & + \frac{\Delta x}{2}\Big(\partial_x f(t^n, \bar{x}_{j+1}, u_j^n) - \partial_x f(t^n, \bar{x}_j, u_j^n)\Big) \\ = & \frac{\Delta x}{2}\Bigg((\tilde{x}_{j+1} - \tilde{x}_j)\, \partial_{xx}^2 f(t^n, \hat{x}_{j+1/2}, u_{j+1}^n) + (u_{j+1}^n - u_{j-1}^n)\, \partial_{xu}^2 f(t^n, \tilde{x}_j, \tilde{u}_j^n) \\ & + (\bar{x}_{j+1} - \bar{x}_j)\, \partial_{xx}^2 f(t^n, \bar{x}_{j+1/2}, u_j^n)\Bigg), \end{aligned}$$

where

$$\tilde{x}_{j+1}, \bar{x}_{j+1} \in \left]x_{j+1/2}, x_{j+3/2}\right[, \qquad \tilde{x}_j, \bar{x}_j \in \left]x_{j-1/2}, x_{j+1/2}\right[,$$

$$\hat{x}_{j+1/2} \in \,]\tilde{x}_j, \tilde{x}_{j+1}[, \qquad \bar{x}_{j+1/2} \in \,]\bar{x}_j, \bar{x}_{j+1}[,$$
$$\tilde{u}_j^n \in \mathcal{I}\left(u_{j-1}^n, u_{j+1}^n\right).$$

Notice that

$$\left|\tilde{x}_{j+1} - \tilde{x}_j\right| \le 2\,\Delta x, \qquad \left|\bar{x}_{j+1} - \bar{x}_j\right| \le 2\,\Delta x.$$

Hence,

$$\left|\mathcal{B}_j^n\right| \le 2\,(\Delta x)^2 \left\|\partial_{xx}^2 f\right\|_{\mathbf{L}^\infty(\Sigma_n)} + \frac{\Delta x}{2} \left\|\partial_{xu}^2 f\right\|_{\mathbf{L}^\infty(\Sigma_n)} \left|u_{j+1}^n - u_{j-1}^n\right|,$$

so that

$$\begin{aligned}
\sum_{j=1}^{N-1} \lambda \left|\mathcal{B}_j^n\right| \le{}& \frac{\Delta t}{2} \left\|\partial_{xu}^2 f\right\|_{\mathbf{L}^\infty(\Sigma_n)} \sum_{j=1}^{N-1} \left|u_{j+1}^n - u_{j-1}^n\right| + 2\,\Delta t\,(b-a) \left\|\partial_{xx}^2 f\right\|_{\mathbf{L}^\infty(\Sigma_n)} \\
\le{}& \Delta t \left\|\partial_{xu}^2 f\right\|_{\mathbf{L}^\infty(\Sigma_n)} \sum_{j=0}^{N-1} \left|u_{j+1}^n - u_j^n\right| + 2\,\Delta t\,(b-a) \left\|\partial_{xx}^2 f\right\|_{\mathbf{L}^\infty(\Sigma_n)}.
\end{aligned} \tag{3.19}$$

Therefore, collecting together the estimates (3.18) and (3.19) we get:

$$\begin{aligned}
&\sum_{j=1}^{N-1} \left|u_{j+1}^{n+1/2} - u_j^{n+1/2}\right| \\
\le{}& \sum_{j=1}^{N-1} \left|u_{j+1}^n - u_j^n\right| + \delta_1^n \left|u_1^n - u_a^n\right| - \delta_N^n \left|u_N^n - u_{N-1}^n\right| + \gamma_N^n \left|u_b^n - u_N^n\right| - \gamma_1^n \left|u_2^n - u_1^n\right| \\
&+ \Delta t \left\|\partial_{xu}^2 f\right\|_{\mathbf{L}^\infty(\Sigma_n)} \sum_{j=0}^{N-1} \left|u_{j+1}^n - u_j^n\right| + 2\,\Delta t\,(b-a) \left\|\partial_{xx}^2 f\right\|_{\mathbf{L}^\infty(\Sigma_n)}.
\end{aligned} \tag{3.20}$$

Focus now on the terms involving the boundary data in (3.16). With the notation (3.11), (3.12) and (3.17), we observe that

$$\begin{aligned}
&\beta_1^n\,(u_1^n - u_a^n) + \lambda\left(F_{3/2}^n(u_1^n, u_1^n) - F_{1/2}^n(u_1^n, u_1^n)\right) \\
={}& \lambda\left[F_{1/2}^n(u_1^n, u_1^n) - F_{1/2}^n(u_a^n, u_1^n) + F_{3/2}^n(u_1^n, u_1^n) - F_{1/2}^n(u_1^n, u_1^n) \pm F_{3/2}^n(u_a^n, u_1^n)\right] \\
={}& \delta_1^n\,(u_1^n - u_a^n) + \lambda\left(F_{3/2}^n(u_a^n, u_1^n) - F_{1/2}^n(u_a^n, u_1^n)\right).
\end{aligned}$$

Hence, from the definition of the scheme (3.3),we have

$$\begin{aligned}
&u_1^{n+1/2} - u_a^{n+1} \\
={}& (1 - \beta_1^n - \gamma_1^n) u_1^n + \beta_1^n\, u_a^n + \gamma_1^n\, u_2^n - \lambda\left(F_{3/2}^n(u_1^n, u_1^n) - F_{1/2}^n(u_1^n, u_1^n)\right) - u_a^{n+1} \pm u_a^n \\
={}& \gamma_1^n (u_2^n - u_1^n) + (1 - \beta_1^n)(u_1^n - u_a^n) + (u_a^{n+1} - u_a^n) - \lambda\left(F_{3/2}^n(u_1^n, u_1^n) - F_{1/2}^n(u_1^n, u_1^n)\right) \\
={}& \gamma_1^n (u_2^n - u_1^n) + (1 - \delta_1^n)(u_1^n - u_a^n) + (u_a^{n+1} - u_a^n) - \lambda\left(F_{3/2}^n(u_a^n, u_1^n) - F_{1/2}^n(u_a^n, u_1^n)\right).
\end{aligned} \tag{3.21}$$

Since

$$\begin{aligned}
&\lambda \left(F^n_{3/2}(u^n_a, u^n_1) - F^n_{1/2}(u^n_a, u^n_1)\right) \\
&= \frac{\lambda}{2}\left[f(t^n, x_{3/2}, u^n_a) + f(t^n, x_{3/2}, u^n_1) - f(t^n, x_{1/2}, u^n_a) - f(t^n, x_{1/2}, u^n_1)\right] \\
&= \frac{\lambda}{2}\left[\Delta x\, \partial_x f(t^n, \tilde{x}_1, u^n_a) + \Delta x\, \partial_x f(t^n, \overline{x}_1, u^n_1) \pm \Delta x\, \partial_x f(t^n, \tilde{x}_1, 0) \pm \Delta x\, \partial_x f(t^n, \overline{x}_1, 0)\right] \\
&= \frac{\Delta t}{2}\left(\partial^2_{xu} f(t^n, \tilde{x}_1, \tilde{u}^n_a)\, u^n_a + \partial^2_{xu} f(t^n, \overline{x}_1, \overline{u}^n_1)\, u^n_1 + \partial_x f(t^n, \tilde{x}_1, 0) + \partial_x f(t^n, \overline{x}_1, 0)\right),
\end{aligned}$$

where $\tilde{x}_1, \overline{x}_1 \in]x_{1/2}, x_{3/2}[$, $\tilde{u}^n_a \in \mathcal{I}(0, u^n_a)$ and $\overline{u}^n_1 \in \mathcal{I}(0, u^n_1)$, we conclude

$$\begin{aligned}
&\lambda \left|F^n_{3/2}(u^n_a, u^n_1) - F^n_{1/2}(u^n_a, u^n_1)\right| \\
&\le \Delta t \left(\frac{1}{2}\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(\Sigma_n)}(|u^n_a| + |u^n_1|) + \left\|\partial_x f(\cdot, \cdot, 0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}\right).
\end{aligned}$$

By the positivity of the coefficients involved in (3.21), we obtain

$$\begin{aligned}
\left|u^{n+1/2}_1 - u^{n+1}_a\right| \le{}& \gamma^n_1\, |u^n_2 - u^n_1| + (1 - \delta^n_1)|u^n_1 - u^n_a| + |u^{n+1}_a - u^n_a| \\
&+ \Delta t \left(\frac{1}{2}\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(\Sigma_n)}(|u^n_a| + |u^n_1|) + \left\|\partial_x f(\cdot, \cdot, 0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}\right).
\end{aligned} \tag{3.22}$$

Similarly as before, compute

$$\begin{aligned}
&\beta^n_N(u^n_N - u^n_{N-1}) + \lambda\left(F^n_{N+1/2}(u^n_N, u^n_N) - F^n_{N-1/2}(u^n_N, u^n_N)\right) \\
&= \lambda\Big[F^n_{N-1/2}(u^n_N, u^n_N) - F^n_{N-1/2}(u^n_{N-1}, u^n_N) + F^n_{N+1/2}(u^n_N, u^n_N) \\
&\qquad - F^n_{N-1/2}(u^n_N, u^n_N) \pm F^n_{N+1/2}(u^n_{N-1}, u^n_N)\Big] \\
&= \delta^n_N\,(u^n_N - u^n_{N-1}) + \lambda\left(F^n_{N+1/2}(u^n_{N-1}, u^n_N) - F^n_{N-1/2}(u^n_{N-1}, u^n_N)\right).
\end{aligned}$$

Therefore, concerning the other boundary term, we have

$$\begin{aligned}
u^{n+1}_b - u^{n+1/2}_N ={}& u^{n+1}_b \pm u^n_b - (1 - \beta^n_N - \gamma^n_N)\, u^n_N + \beta^n_N\, u^n_{N-1} + \gamma^n_N\, u^n_b \\
&+ \lambda\left(F^n_{N+1/2}(u^n_N, u^n_N) - F^n_{N-1/2}(u^n_N, u^n_N)\right) \\
={}& (u^{n+1}_b - u^n_b) + (1 - \gamma^n_N)(u^n_b - u^n_N) + \beta^n_N\,(u^n_N - u^n_{N-1}) \\
&+ \lambda\left(F^n_{N+1/2}(u^n_N, u^n_N) - F^n_{N-1/2}(u^n_N, u^n_N)\right) \\
={}& (u^{n+1}_b - u^n_b) + (1 - \gamma^n_N)(u^n_b - u^n_N) + \delta^n_N\,(u^n_N - u^n_{N-1}) \qquad (3.23) \\
&+ \lambda\left(F^n_{N+1/2}(u^n_{N-1}, u^n_N) - F^n_{N-1/2}(u^n_{N-1}, u^n_N)\right), \qquad (3.24)
\end{aligned}$$

Observing that

$$\begin{aligned}
&\lambda\left(F^n_{N+1/2}(u^n_{N-1}, u^n_N) - F^n_{N-1/2}(u^n_{N-1}, u^n_N)\right) \\
&= \frac{\lambda}{2}\left[f(t^n, x_{N+1/2}, u^n_{N-1}) + f(t^n, x_{N+1/2}, u^n_N) - f(t^n, x_{N-1/2}, u^n_{N-1}) - f(t^n, x_{N-1/2}, u^n_N)\right]
\end{aligned}$$

$$= \frac{\lambda}{2}\left[\Delta x\,\partial_x f(t^n,\tilde{x}_N,u^n_{N-1}) + \Delta x\,\partial_x f(t^n,\overline{x}_N,u^n_N) \pm \Delta x\,\partial_x f(t^n,\tilde{x}_N,0) \pm \Delta x\,\partial_x f(t^n,\overline{x}_N,0)\right]$$
$$= \frac{\Delta t}{2}\left(\partial^2_{xu} f(t^n,\tilde{x}_N,\tilde{u}^n_{N-1})\,u^n_{N-1} + \partial^2_{xu} f(t^n,\overline{x}_N,\overline{u}^n_N)\,u^n_N + \partial_x f(t^n,\tilde{x}_N,0) + \partial_x f(t^n,\overline{x}_N,0)\right),$$

where $\tilde{x}_N,\overline{x}_N \in]x_{N-1/2},x_{N+1/2}[$, $\tilde{u}^n_{N-1} \in \mathcal{I}(0,u^n_{N-1})$ and $\overline{u}^n_N \in \mathcal{I}(0,u^n_N)$, we conclude

$$\begin{aligned}
&\lambda\left|F^n_{N+1/2}(u^n_{N-1},u^n_N) - F^n_{N-1/2}(u^n_{N-1},u^n_N)\right| \\
&\le \Delta t\left(\frac{1}{2}\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(\Sigma_n)}(|u^n_{N-1}| + |u^n_N|) + \left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}\right).
\end{aligned}$$

By the positivity of the coefficients involved in (3.23)–(3.24), we obtain

$$\begin{aligned}
\left|u^{n+1}_b - u^{n+1/2}_N\right| \le{}& |u^{n+1}_b - u^n_b| + (1-\gamma^n_N)|u^n_b - u^n_N| + \delta^n_N|u^n_N - u^n_{N-1}| \\
&+ \Delta t\left(\frac{1}{2}\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(\Sigma_n)}(|u^n_{N-1}| + |u^n_N|) + \left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}\right).
\end{aligned} \tag{3.25}$$

Insert (3.20), (3.22) and (3.25) into (3.16):

$$\begin{aligned}
&\sum_{j=0}^{N}\left|u^{n+1}_{j+1} - u^{n+1}_j\right| \\
&\le e^{\Delta t\,\|\partial_u g\|_{\mathbf{L}^\infty(\Sigma_n)}}\Bigg[\gamma^n_1|u^n_2 - u^n_1| + (1-\delta^n_1)|u^n_1 - u^n_a| + |u^{n+1}_a - u^n_a| \\
&\quad + \Delta t\left(\frac{1}{2}\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(\Sigma_n)}(|u^n_a| + |u^n_1|) + \left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}\right) \\
&\quad + \sum_{j=1}^{N-1}\left|u^n_{j+1} - u^n_j\right| + \delta^n_1\left|u^n_1 - u^n_a\right| - \delta^n_N\left|u^n_N - u^n_{N-1}\right| + \gamma^n_N\left|u^n_b - u^n_N\right| - \gamma^n_1\left|u^n_2 - u^n_1\right| \\
&\quad + \Delta t\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(\Sigma_n)}\sum_{j=0}^{N-1}\left|u^n_{j+1} - u^n_j\right| + 2\,\Delta t\,(b-a)\left\|\partial^2_{xx} f\right\|_{\mathbf{L}^\infty(\Sigma_n)} \\
&\quad + |u^{n+1}_b - u^n_b| + (1-\gamma^n_N)|u^n_b - u^n_N| + \delta^n_N|u^n_N - u^n_{N-1}| \\
&\quad + \Delta t\left(\frac{1}{2}\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(\Sigma_n)}(|u^n_{N-1}| + |u^n_N|) + \left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])}\right)\Bigg] \\
&\quad + \Delta t\,(b-a)\|\partial_x g\|_{\mathbf{L}^\infty(\Sigma_n)} + 2\Delta t\left\|g(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} + 2\Delta t\,\|\partial_u g\|_{\mathbf{L}^\infty(\Sigma_n)}\left\|u^{n+1/2}\right\|_{\mathbf{L}^\infty([a,b])} \\
&\le e^{\Delta t\,\|\partial_u g\|_{\mathbf{L}^\infty(\Sigma_n)}}\Bigg[|u^{n+1}_a - u^n_a| + |u^{n+1}_b - u^n_b| + \left(1 + \Delta t\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(\Sigma_n)}\right)\sum_{j=0}^{N}\left|u^n_{j+1} - u^n_j\right| \\
&\quad + \mathcal{K}_1(t^n)\,\Delta t + \frac{1}{2}\left(3\,\mathcal{U}(t^n) + \|u_a\|_{\mathbf{L}^\infty([0,t^n])}\right)\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(\Sigma_n)}\Delta t\Bigg] \\
&\quad + \Delta t\,(b-a)\|\partial_x g\|_{\mathbf{L}^\infty(\Sigma_n)} + 2\Delta t\left\|g(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} + 2\,\Delta t\,\|\partial_u g\|_{\mathbf{L}^\infty(\Sigma_n)}\,\mathcal{U}(t^n),
\end{aligned}$$

where

$$\mathcal{K}_1(t) = 2\left(\left\|\partial_x f(\cdot,\cdot,0)\right\|_{\mathbf{L}^\infty([0,t^n]\times[a,b])} + (b-a)\left\|\partial^2_{xx} f\right\|_{\mathbf{L}^\infty(\Sigma_n)}\right). \tag{3.26}$$

Setting

$$
\begin{aligned}
\mathcal{K}_2(t) = {}& 2\,\mathcal{C}_1(t) + (b-a)\left(2\left\|\partial^2_{xx} f\right\|_{\mathbf{L}^\infty(\Sigma_t)} + \|\partial_x g\|_{\mathbf{L}^\infty(\Sigma_t)}\right)\\
&+ \frac{1}{2}\left(3\,\mathcal{U}(t) + \|u_a\|_{\mathbf{L}^\infty([0,t])}\right)\left\|\partial^2_{xu} f\right\|_{\mathbf{L}^\infty(\Sigma_t)} \Delta t + 2\,\|\partial_u g\|_{\mathbf{L}^\infty(\Sigma_t)}\,\mathcal{U}(t),
\end{aligned}
\tag{3.27}
$$

with $\mathcal{U}(t)$ as in (3.7), $\Sigma_t = [0,t]\times[a,b]\times[-\mathcal{U}(t),\mathcal{U}(t)]$ and $\mathcal{C}_1(t)$ as in (3.8), we deduce by a standard iterative procedure the following estimate

$$
\sum_{j=0}^{N}\left|u^n_{j+1}-u^n_j\right| \le e^{\mathcal{C}_2(t^n)\,t^n}\left(\sum_{j=0}^{N}\left|u^0_{j+1}-u^0_j\right| + \sum_{m=1}^{n}\left|u^m_a-u^{m-1}_a\right| + \sum_{m=1}^{n}\left|u^m_b-u^{m-1}_b\right| + t^n\,\mathcal{K}_2(t^n)\right),
$$

where actually the norms appearing in $\mathcal{C}_2(t)$ in (3.9) can now be computed on Σ_t instead of on $[0,t]\times[a,b]\times\mathbb{R}$. □

Corollary 3.4. (BV estimate in space and time) *Let* **(f)**, **(g)**, **(D)** *and* (3.5) *hold. Then, for n between* 1 *and N_T, the following estimate holds*

$$
\sum_{m=0}^{n-1}\sum_{j=0}^{N}\Delta t\left|u^m_{j+1}-u^m_j\right| + \sum_{m=0}^{n-1}\sum_{j=0}^{N+1}\Delta x\left|u^{m+1}_j-u^m_j\right| \le \mathcal{C}_{xt}(t^n),
\tag{3.28}
$$

where $\mathcal{C}_{xt}(t^n)$ is given by (3.35).

Proof. By Proposition 3.3, we have

$$
\sum_{m=0}^{n-1}\sum_{j=0}^{N}\Delta t\left|u^m_{j+1}-u^m_j\right| \le n\,\Delta t\,\mathcal{C}_x(t^n).
\tag{3.29}
$$

By the definition of the scheme (3.3)–(3.4), for $m\in\{0,\ldots,n-1\}$ and $j\in\{1,\ldots,N\}$, we have

$$
\left|u^{m+1}_j-u^m_j\right| \le \left|u^{m+1}_j-u^{m+1/2}_j\right| + \left|u^{m+1/2}_j-u^m_j\right|.
\tag{3.30}
$$

In particular, the first term on the r.h.s. of (3.30) can be estimated as follows

$$
\left|u^{m+1}_j-u^{m+1/2}_j\right| \le \Delta t\,\|g\|_{\mathbf{L}^\infty([0,t^m]\times[a,b]\times[-\mathcal{U}(t^m),\mathcal{U}(t^m)])},
\tag{3.31}
$$

and the norm of g appearing above is bounded thanks to **(g)**. Concerning the second term on the r.h.s. of (3.30), by (3.2) and (3.3), we obtain

$$
\begin{aligned}
&\left|u^{m+1/2}_j-u^m_j\right|\\
\le{}& \frac{\lambda\,\alpha}{2}\left(\left|u^m_{j+1}-u^m_j\right| + \left|u^m_j-u^m_{j-1}\right|\right)\\
&+\frac{\lambda}{2}\left|f(t^m,x_{j+1/2},u^m_j)+f(t^m,x_{j+1/2},u^m_{j+1})-f(t^m,x_{j-1/2},u^m_{j-1})-f(t^m,x_{j-1/2},u^m_j)\right|\\
\le{}& \frac{\lambda\,\alpha}{2}\left(\left|u^m_{j+1}-u^m_j\right| + \left|u^m_j-u^m_{j-1}\right|\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda}{2}\left[\left|\partial_x f(t^m,\tilde{x}_j,u_j^m)\right|\Delta x+\left|\partial_u f(t^m,x_{j-1/2},\tilde{u}_{j-1/2}^m)\right|\left|u_j^m-u_{j-1}^m\right|\right.\\
&\qquad\left.+\left|\partial_x f(t^m,\overline{x}_j,u_{j+1}^m)\right|\Delta x+\left|\partial_u f(t^m,x_{j-1/2},\tilde{u}_{j+1/2}^m)\right|\left|u_{j+1}^m-u_j^m\right|\right]\\
\leq{}&\frac{\lambda}{2}\left(\alpha+\|\partial_u f\|_{\mathbf{L}^\infty([0,t^m]\times[a,b]\times[-\mathcal{U}(t^m),\mathcal{U}(t^m)])}\right)\left(\left|u_{j+1}^m-u_j^m\right|+\left|u_j^m-u_{j-1}^m\right|\right)\\
&+\Delta t\,\|\partial_x f\|_{\mathbf{L}^\infty([0,t^m]\times[a,b]\times[-\mathcal{U}(t^m),\mathcal{U}(t^m)])},
\end{aligned}
$$

where $\tilde{x}_j,\overline{x}_j\in\,]x_{j-1/2},x_{j+1/2}[$, $\tilde{u}_{j-1/2}^m\in\mathcal{I}(u_{j-1}^m,u_j^m)$ and $\tilde{u}_{j+1/2}^m\in\mathcal{I}(u_j^m,u_{j+1}^m)$. Remark that, by **(f)**, the norm of $\partial_x f$ appearing above is bounded. Therefore, by Proposition 3.3,

$$
\begin{aligned}
\sum_{j=1}^{N}\Delta x\left|u_j^{m+1/2}-u_j^m\right|\leq{}&\Delta t\left(\alpha+\|\partial_u f\|_{\mathbf{L}^\infty([0,t^m]\times[a,b]\times[-\mathcal{U}(t^m),\mathcal{U}(t^m)])}\right)\sum_{j=0}^{N}\left|u_{j+1}^m-u_j^m\right|\\
&+\Delta t\,(b-a)\,\|\partial_x f\|_{\mathbf{L}^\infty([0,t^m]\times[a,b]\times[-\mathcal{U}(t^m),\mathcal{U}(t^m)])}\\
={}&\Delta t\,\mathcal{C}_t(t^m),
\end{aligned}
\tag{3.32}
$$

where we set

$$
\mathcal{C}_t(\tau)=\left(\alpha+\|\partial_u f\|_{\mathbf{L}^\infty([0,\tau]\times[a,b]\times[-\mathcal{U}(\tau),\mathcal{U}(\tau)])}\right)\mathcal{C}_x(\tau)+(b-a)\,\|\partial_x f\|_{\mathbf{L}^\infty([0,\tau]\times[a,b]\times[-\mathcal{U}(\tau),\mathcal{U}(\tau)])}.
\tag{3.33}
$$

Exploiting (3.30), (3.31) and (3.32), we obtain

$$
\begin{aligned}
\sum_{j=0}^{N+1}\Delta x\left|u_j^{m+1}-u_j^m\right|={}&=\Delta x\left|u_a^{m+1}-u_a^m\right|+\Delta x\left|u_b^{m+1}-u_b^m\right|+\sum_{j=1}^{N}\Delta x\left|u_j^{m+1}-u_j^m\right|\\
\leq{}&\Delta x\left|u_a^{m+1}-u_a^m\right|+\Delta x\left|u_b^{m+1}-u_b^m\right|\\
&+\Delta t\left(\mathcal{C}_t(t^m)+\|g\|_{\mathbf{L}^\infty([0,t^m]\times[a,b]\times[-\mathcal{U}(t^m),\mathcal{U}(t^m)])}\right),
\end{aligned}
$$

which, summed over $m=0,\ldots,n-1$, yields

$$
\begin{aligned}
\sum_{m=0}^{n-1}\sum_{j=0}^{N+1}\Delta x\left|u_j^{m+1}-u_j^m\right|\leq{}&\Delta x\sum_{m=0}^{n-1}\left(\left|u_a^{m+1}-u_a^m\right|+\left|u_b^{m+1}-u_b^m\right|\right)\\
&+n\,\Delta t\left(\mathcal{C}_t(t^n)+\|g\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times[-\mathcal{U}(t^n),\mathcal{U}(t^n)])}\right).
\end{aligned}
\tag{3.34}
$$

Summing (3.29) and (3.34) we obtain the desired estimate (3.28), with

$$
\begin{aligned}
\mathcal{C}_{xt}(t^n)={}&n\,\Delta t\,(1+\alpha+\|\partial_u f\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times[-\mathcal{U}(t^n),\mathcal{U}(t^n)])})\,\mathcal{C}_x(t^n)\\
&+n\,\Delta t\left((b-a)\,\|\partial_x f\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times[-\mathcal{U}(t^n),\mathcal{U}(t^n)])}+\|g\|_{\mathbf{L}^\infty([0,t^n]\times[a,b]\times[-\mathcal{U}(t^n),\mathcal{U}(t^n)])}\right)\\
&+\Delta x\sum_{m=0}^{n-1}\left(\left|u_a^{m+1}-u_a^m\right|+\left|u_b^{m+1}-u_b^m\right|\right),
\end{aligned}
\tag{3.35}
$$

concluding the proof. □

3.3 Discrete entropy inequality

We introduce the following notation: for $j = 1, \ldots, N$, $n = 0, \ldots, N_T - 1$, $k \in \mathbb{R}$,

$$
\begin{aligned}
F^n_{j+1/2}(u, v) &= \frac{1}{2}\left[f(t^n, x_{j+1/2}, u) + f(t^n, x_{j+1/2}, v)\right] - \frac{\alpha}{2}\,(v - u)\,,\\
H^n_j(u, v, z) &= v - \lambda\,\left(F^n_{j+1/2}(v, z) - F^n_{j-1/2}(u, v)\right),\\
G^{n,k}_{j+1/2}(u, v) &= F^n_{j+1/2}(u \wedge k, v \wedge k) - F^n_{j+1/2}(k, k),\\
L^{n,k}_{j+1/2}(u, v) &= F^n_{j+1/2}(k, k) - F^n_{j+1/2}(u \vee k, v \vee k).
\end{aligned}
$$

Observe that, due to the definition of the scheme, $u_j^{n+1/2} = H^n_j(u^n_{j-1}, u^n_j, u^n_{j+1})$. Notice moreover that the following equivalences hold true: $(s-k)^+ = s \wedge k - k$ and $(s-k)^- = k - s \vee k$.

Lemma 3.5. *Let* **(f)**, **(g)**, **(D)** *and* (3.5) *hold. Then the approximate solution* u_Δ *in* (3.1)*, defined through Algorithm 3.1, satisfies the following discrete entropy inequalities: for* $j = 1, \ldots, N$, $n = 0, \ldots, N_T - 1$, $k \in \mathbb{R}$,

$$
\begin{aligned}
(u^{n+1}_j - k)^+ - (u^n_j - k)^+ &+ \lambda\,\left(G^{n,k}_{j+1/2}(u^n_j, u^n_{j+1}) - G^{n,k}_{j-1/2}(u^n_{j-1}, u^n_j)\right)\\
&+ \lambda\ \mathrm{sgn}^+(u^{n+1}_j - k)\,\left(f(t^n, x_{j+1/2}, k) - f(t^n, x_{j-1/2}, k)\right)\\
&- \Delta t\ \mathrm{sgn}^+(u^{n+1}_j - k)\, g\!\left(t^n, x_j, u_j^{n+1/2}\right) \le 0,
\end{aligned}
\tag{3.36}
$$

and

$$
\begin{aligned}
(u^{n+1}_j - k)^- - (u^n_j - k)^- &+ \lambda\,\left(L^{n,k}_{j+1/2}(u^n_j, u^n_{j+1}) - L^{n,k}_{j-1/2}(u^n_{j-1}, u^n_j)\right)\\
&+ \lambda\ \mathrm{sgn}^-(u^{n+1}_j - k)\,\left(f(t^n, x_{j+1/2}, k) - f(t^n, x_{j-1/2}, k)\right)\\
&- \Delta t\ \mathrm{sgn}^-(u^{n+1}_j - k)\, g\!\left(t^n, x_j, u_j^{n+1/2}\right) \le 0.
\end{aligned}
\tag{3.37}
$$

Proof. Consider the map $(u, v, z) \mapsto H^n_j(u, v, z)$. By the CFL condition (3.5), it holds

$$
\begin{aligned}
\frac{\partial H^n_j}{\partial u}(u, v, z) &= \frac{\lambda}{2}\left(\partial_u f(t^n, x_{j-1/2}, u) + \alpha\right) \ge 0,\\
\frac{\partial H^n_j}{\partial v}(u, v, z) &= 1 - \lambda\,\alpha - \frac{\lambda}{2}\left(\partial_u f(t^n, x_{j+1/2}, v) - \partial_u f(t^n, x_{j-1/2}, v)\right) \ge 0,\\
\frac{\partial H^n_j}{\partial z}(u, v, z) &= \frac{\lambda}{2}\left(\alpha - \partial_u f(t^n, x_{j+1/2}, z)\right) \ge 0.
\end{aligned}
$$

Notice that

$$
H^n_j(k, k, k) = k - \lambda\left(f(t^n, x_{j+1/2}, k) - f(t^n, x_{j-1/2}, k)\right).
$$

By the monotonicity properties obtained above, we have

$$
\begin{aligned}
&H^n_j(u^n_{j-1} \wedge k, u^n_j \wedge k, u^n_{j+1} \wedge k) - H^n_j(k, k, k)\\
\ge\ &H^n_j(u^n_{j-1}, u^n_j, u^n_{j+1}) \wedge H^n_j(k, k, k) - H^n_j(k, k, k)\\
=\ &\left(H^n_j(u^n_{j-1}, u^n_j, u^n_{j+1}) - H^n_j(k, k, k)\right)^+
\end{aligned}
$$

$$
\begin{aligned}
&= \left(u_j^{n+1/2} - k + \lambda \left(f(t^n, x_{j+1/2}, k) - f(t^n, x_{j-1/2}, k)\right)\right)^+ \\
&= \left(u_j^{n+1} - k - \Delta t\, g\Big(t^n, x_j, u_j^{n+1/2}\Big) + \lambda \left(f(t^n, x_{j+1/2}, k) - f(t^n, x_{j-1/2}, k)\right)\right)^+ . \qquad (3.38)
\end{aligned}
$$

Moreover, we also have

$$
\begin{aligned}
&H_j^n(u_{j-1}^n \wedge k, u_j^n \wedge k, u_{j+1}^n \wedge k) - H_j^n(k,k,k) \\
&= (u_j^n \wedge k) - k \\
&\quad - \lambda \Big[F_{j+1/2}^n(u_j^n \wedge k, u_{j+1}^n \wedge k) - F_{j-1/2}^n(u_{j-1}^n \wedge k, u_j^n \wedge k) - F_{j+1/2}^n(k,k) + F_{j-1/2}^n(k,k)\Big] \\
&= (u_j^n - k)^+ - \lambda \left(G_{j+1/2}^{n,k}(u_j^n, u_{j+1}^n) - G_{j-1/2}^{n,k}(u_{j-1}^n, u_j^n)\right) . \qquad (3.39)
\end{aligned}
$$

Hence, by (3.38) and (3.39),

$$
\begin{aligned}
&(u_j^n - k)^+ - \lambda \left(G_{j+1/2}^{n,k}(u_j^n, u_{j+1}^n) - G_{j-1/2}^{n,k}(u_{j-1}^n, u_j^n)\right) \\
&\geq \left(u_j^{n+1} - k - \Delta t\, g\Big(t^n, x_j, u_j^{n+1/2}\Big) + \lambda \left(f(t^n, x_{j+1/2}, k) - f(t^n, x_{j-1/2}, k)\right)\right)^+ \\
&= \operatorname{sgn}^+ \left(u_j^{n+1} - k - \Delta t\, g\Big(t^n, x_j, u_j^{n+1/2}\Big) + \lambda \left(f(t^n, x_{j+1/2}, k) - f(t^n, x_{j-1/2}, k)\right)\right) \\
&\quad \times \left(u_j^{n+1} - k - \Delta t\, g\Big(t^n, x_j, u_j^{n+1/2}\Big) + \lambda \left(f(t^n, x_{j+1/2}, k) - f(t^n, x_{j-1/2}, k)\right)\right) \\
&\geq \left(u_j^{n+1} - k\right)^+ - \Delta t \operatorname{sgn}^+ \left(u_j^{n+1} - k\right) g\Big(t^n, x_j, u_j^{n+1/2}\Big) \\
&\quad + \lambda \operatorname{sgn}^+ \left(u_j^{n+1} - k\right) \left(f(t^n, x_{j+1/2}, k) - f(t^n, x_{j-1/2}, k)\right),
\end{aligned}
$$

proving (3.36), while (3.37) is proven in an entirely similar way. □

3.4 Convergence towards an entropy weak solution

The uniform $\mathbf{L}^\infty$-bound provided by Lemma 3.2 and the total variation estimate in Corollary 3.4 allow to apply Helly's compactness theorem, ensuring the existence of a subsequence of u_Δ, still denoted by u_Δ, converging in $\mathbf{L^1}$ to a function $u \in \mathbf{L}^\infty([0,T[\times]a,b[)$, for all $T > 0$. We need to prove that this limit function is indeed an MV–solution to the IBVP (1.1), in the sense of Definition 2.1.

Lemma 3.6. *Let* **(f)**, **(g)**, **(D)** *and* (3.5) *hold. Then the piecewise constant approximate solutions u_Δ resulting from Algorithm 3.1 converge, as $\Delta x \to 0$, towards an MV–solution of the IBVP* (1.1).

Proof. We consider the discrete entropy inequality (3.36), for the positive semi-entropy, and we follow [7], see also [14]. The proof for the negative semi-entropy is done analogously. Add and subtract $G_{j+1/2}^{n,k}(u_j^n, u_j^n)$ in (3.36) and rearrange it as follows

$$
0 \geq (u_j^{n+1} - k)^+ - (u_j^n - k)^+ + \lambda \left(G_{j+1/2}^{n,k}(u_j^n, u_{j+1}^n) - G_{j+1/2}^{n,k}(u_j^n, u_j^n)\right)
$$

$$+\lambda\left(G^{n,k}_{j+1/2}(u^n_j,u^n_j)-G^{n,k}_{j-1/2}(u^n_{j-1},u^n_j)\right)-\Delta t\ \mathrm{sgn}^+(u^{n+1}_j-k)\,g\Big(t^n,x_j,u^{n+1/2}_j\Big)$$
$$+\lambda\ \mathrm{sgn}^+(u^{n+1}_j-k)\ \left(f(t^n,x_{j+1/2},k,u^n_{j+1/2})-f(t^n,x_{j-1/2},k,u^n_{j-1/2})\right).$$

Let $\varphi\in\mathbf{C^1_c}([0,T[\times[a,b];\mathbb{R}^+)$ for some $T>0$, multiply the inequality above by $\Delta x\,\varphi(t^n,x_j)$ and sum over $j=1,\ldots,N$ and $n=0,\ldots,N_T-1$, so to get

$$0\geq\Delta x\sum_{n=0}^{N_T-1}\sum_{j=1}^{N}\Big[(u^{n+1}_j-k)^+-(u^n_j-k)^+\Big]\ \varphi(t^n,x_j) \tag{3.40}$$
$$+\Delta t\sum_{n=0}^{N_T-1}\sum_{j=1}^{N}\Bigg[\Big(G^{n,k}_{j+1/2}(u^n_j,u^n_{j+1})-G^{n,k}_{j+1/2}(u^n_j,u^n_j)\Big) \tag{3.41}$$
$$-\Big(G^{n,k}_{j-1/2}(u^n_{j-1},u^n_j)-G^{n,k}_{j+1/2}(u^n_j,u^n_j)\Big)\Bigg]\ \varphi(t^n,x_j) \tag{3.42}$$
$$-\Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_{j=1}^{N}\mathrm{sgn}^+(u^{n+1}_j-k)\,g\Big(t^n,x_j,u^{n+1/2}_j\Big)\ \varphi(t^n,x_j) \tag{3.43}$$
$$+\Delta t\sum_{n=0}^{N_T-1}\sum_{j=1}^{N}\mathrm{sgn}^+(u^{n+1}_j-k)\Big(f(t^n,x_{j+1/2},k,u^n_{j+1/2})-f(t^n,x_{j-1/2},k,u^n_{j-1/2})\Big)\,\varphi(t^n,x_j). \tag{3.44}$$

Consider each term separately. Summing by parts, we obtain

$$\begin{aligned}
[(3.40)]=&\ -\Delta x\sum_{j=1}^{N}(u^0_j-k)^+\,\varphi(0,x_j)-\Delta x\,\Delta t\sum_{n=1}^{N_T-1}\sum_{j=1}^{N}(u^n_j-k)^+\frac{\varphi(t^n,x_j)-\varphi(t^{n-1},x_j)}{\Delta t}\\
\underset{\Delta x\to0^+}{\longrightarrow}&\ -\int_a^b(u_o(x)-k)^+\,\varphi(0,x)\,\mathrm{d}x-\int_0^T\int_a^b(u(t,x)-k)^+\,\partial_t\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,,\\
[(3.43)]=&\ -\Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_{j=1}^{N}\mathrm{sgn}^+(u^{n+1}_j-k)\,g\Big(t^n,x_j,u^{n+1/2}_j\Big)\ \varphi(t^n,x_j)\\
\underset{\Delta x\to0^+}{\longrightarrow}&\ -\int_0^T\int_a^b\mathrm{sgn}^+(u(t,x)-k)\,g\,\big(t,x,u(t,x)\big)\ \varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,,\\
[(3.44)]=&\ \Delta x\,\Delta t\sum_{n=0}^{N_T-1}\sum_{j=1}^{N}\mathrm{sgn}^+(u^{n+1}_j-k)\ \frac{f(t^n,x_{j+1/2},k)-f(t^n,x_{j-1/2},k)}{\Delta x}\ \varphi(t^n,x_j)\\
\underset{\Delta x\to0^+}{\longrightarrow}&\ \int_0^T\int_a^b\mathrm{sgn}^+(u(t,x)-k)\,\partial_xf(t,x,k)\,\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}$$

by the Dominated Convergence Theorem. Concerning (3.41)–(3.42), we get

$$\begin{aligned}
[(3.41)-(3.42)]=\Delta t&\sum_{n=0}^{N_T-1}\sum_{j=1}^{N}\Big(G^{n,k}_{j+1/2}(u^n_j,u^n_{j+1})-G^{n,k}_{j+1/2}(u^n_j,u^n_j)\Big)\ \varphi(t^n,x_j)\\
&-\Delta t\sum_{n=0}^{N_T-1}\sum_{j=0}^{N-1}\Big(G^{n,k}_{j+1/2}(u^n_j,u^n_{j+1})-G^{n,k}_{j+3/2}(u^n_{j+1},u^n_{j+1})\Big)\ \varphi(t^n,x_{j+1})
\end{aligned}$$

$$
\begin{aligned}
&= \Delta t \sum_{n=0}^{N_T-1} \sum_{j=1}^{N-1} \Big[\Big(G^{n,k}_{j+1/2}(u^n_j, u^n_{j+1}) - G^{n,k}_{j+1/2}(u^n_j, u^n_j) \Big)\ \varphi(t^n, x_j) \\
&\qquad - \Big(G^{n,k}_{j+1/2}(u^n_j, u^n_{j+1}) - G^{n,k}_{j+3/2}(u^n_{j+1}, u^n_{j+1}) \Big)\ \varphi(t^n, x_{j+1}) \Big] \\
&\quad + \Delta t \sum_{n=0}^{N_T-1} \Big[\Big(G^{n,k}_{N+1/2}(u^n_N, u^n_b) - G^{n,k}_{N+1/2}(u^n_N, u^n_N) \Big)\ \varphi(t^n, x_N) \\
&\qquad - \Big(G^{n,k}_{1/2}(u^n_a, u^n_1) - G^{n,k}_{3/2}(u^n_1, u^n_1) \Big)\ \varphi(t^n, x_1) \Big] \\
&= T^{int} + T^b = T,
\end{aligned}
$$

where we set

$$
\begin{aligned}
T^{int} &= \Delta t \sum_{n=0}^{N_T-1} \sum_{j=1}^{N-1} \Big[\Big(G^{n,k}_{j+1/2}(u^n_j, u^n_{j+1}) - G^{n,k}_{j+1/2}(u^n_j, u^n_j) \Big)\ \varphi(t^n, x_j) \\
&\qquad - \Big(G^{n,k}_{j+1/2}(u^n_j, u^n_{j+1}) - G^{n,k}_{j+3/2}(u^n_{j+1}, u^n_{j+1}) \Big)\ \varphi(t^n, x_{j+1}) \Big], \\
T^b &= \Delta t \sum_{n=0}^{N_T-1} \Big[\Big(G^{n,k}_{N+1/2}(u^n_N, u^n_b) - G^{n,k}_{N+1/2}(u^n_N, u^n_N) \Big)\ \varphi(t^n, x_N) \\
&\qquad - \Big(G^{n,k}_{1/2}(u^n_a, u^n_1) - G^{n,k}_{3/2}(u^n_1, u^n_1) \Big)\ \varphi(t^n, x_1) \Big].
\end{aligned}
$$

Define

$$
\begin{aligned}
S = &\ -\Delta x\, \Delta t \sum_{n=0}^{N_T-1} \sum_{j=1}^{N} G^{n,k}_{j+1/2}(u^n_j, u^n_j)\, \frac{\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)}{\Delta x} \\
&\ - \alpha\, \Delta t \sum_{n=0}^{N_T-1} \Big((u^n_a - k)^+ \varphi(t^n, a) + (u^n_b - k)^+ \varphi(t^n, b) \Big).
\end{aligned}
\tag{3.45}
$$

Observe that

$$
\begin{aligned}
G^{n,k}_{j+1/2}(u^n_j, u^n_j) &= F^n_{j+1/2}(u^n_j \wedge k, u^n_j \wedge k) - F^n_{j+1/2}(k,k) \\
&= f(t^n, x_{j+1/2}, u^n_j \wedge k) - f(t^n, x_{j+1/2}, k) \\
&= \operatorname{sgn}^+(u^n_j - k) \Big(f(t^n, x_{j+1/2}, u^n_j) - f(t^n, x_{j+1/2}, k) \Big).
\end{aligned}
$$

It follows then easily that

$$
\begin{aligned}
S \underset{\Delta x \to 0^+}{\longrightarrow} &\ -\int_0^T \int_a^b \operatorname{sgn}^+(u(t,x) - k)\ \big(f(t,x,u - f(t,x,k) \big)\ \partial_x \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t \\
&\ - \alpha \left(\int_0^T (u_a(t) - k)^+ \varphi(t,a)\, \mathrm{d}t + \int_0^T (u_b(t) - k)^+ \varphi(t,b)\, \mathrm{d}t \right).
\end{aligned}
$$

Let us rewrite S in (3.45) as follows:

$$
S = -\Delta t \sum_{n=0}^{N_T-1} \sum_{j=1}^{N} G^{n,k}_{j+1/2}(u^n_j, u^n_j) \left(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j) \right)
$$

$$
\begin{aligned}
&\quad - \alpha\, \Delta t \sum_{n=0}^{N_T-1} \Big((u_a^n - k)^+ \, \varphi(t^n, a) + (u_b^n - k)^+ \, \varphi(t^n, b) \Big) \\
&= \Delta t \sum_{n=0}^{N_T-1} \left(\sum_{j=1}^{N} G^{n,k}_{j+1/2}(u_j^n, u_j^n)\, \varphi(t^n, x_{j+1}) - \sum_{j=0}^{N-1} G^{n,k}_{j+3/2}(u_{j+1}^n, u_{j+1}^n)\, \varphi(t^n, x_{j+1}) \right) \\
&\quad - \alpha\, \Delta t \sum_{n=0}^{N_T-1} \Big((u_a^n - k)^+ \, \varphi(t^n, a) + (u_b^n - k)^+ \, \varphi(t^n, b) \Big) \\
&= \, - \Delta t \sum_{n=0}^{N_T-1} \sum_{j=1}^{N-1} \Big(G^{n,k}_{j+1/2}(u_j^n, u_j^n) - G^{n,k}_{j+3/2}(u_{j+1}^n, u_{j+1}^n) \Big)\ \varphi(t^n, x_{j+1}) \\
&\quad - \Delta t \sum_{n=0}^{N_T-1} \Big(G^{n,k}_{N+1/2}(u_N^n, u_N^n)\, \varphi(t^n, x_{N+1}) - G^{n,k}_{3/2}(u_1^n, u_1^n))\, \varphi(t^n, x_1) \Big) \\
&\quad - \alpha\, \Delta t \sum_{n=0}^{N_T-1} \Big((u_a^n - k)^+ \, \varphi(t^n, a) + (u_b^n - k)^+ \, \varphi(t^n, b) \Big) \\
&= S^{int} + S^b,
\end{aligned}
$$

where we set

$$
\begin{aligned}
S^{int} &= \, - \Delta t \sum_{n=0}^{N_T-1} \sum_{j=1}^{N-1} \Big(G^{n,k}_{j+1/2}(u_j^n, u_j^n) - G^{n,k}_{j+3/2}(u_{j+1}^n, u_{j+1}^n) \Big)\ \varphi(t^n, x_{j+1}), \\
S^b &= \, - \Delta t \sum_{n=0}^{N_T-1} \Big(G^{n,k}_{N+1/2}(u_N^n, u_N^n)\, \varphi(t^n, x_{N+1}) - G^{n,k}_{3/2}(u_1^n, u_1^n))\, \varphi(t^n, x_1) \Big) \\
&\quad - \alpha\, \Delta t \sum_{n=0}^{N_T-1} \Big((u_a^n - k)^+ \, \varphi(t^n, a) + (u_b^n - k)^+ \, \varphi(t^n, b) \Big) .
\end{aligned}
$$

Focus on S^{int}: by adding and subtracting $G^{n,k}_{j+1/2}(u_j^n, u_j^n)$ in the first brackets, we can rewrite this term as

$$
\begin{aligned}
S^{int} &= \, - \Delta t \sum_{n=0}^{N_T-1} \sum_{j=1}^{N-1} \Big(G^{n,k}_{j+1/2}(u_j^n, u_{j+1}^n) - G^{n,k}_{j+1/2}(u_j^n, u_j^n) \Big)\ \varphi(t^n, x_{j+1}) \\
&\quad - \Delta t \sum_{n=0}^{N_T-1} \sum_{j=1}^{N-1} \Big(G^{n,k}_{j+1/2}(u_j^n, u_{j+1}^n) - G^{n,k}_{j+3/2}(u_{j+1}^n, u_{j+1}^n) \Big)\ \varphi(t^n, x_{j+1}).
\end{aligned}
$$

We evaluate now the distance between T^{int} and S^{int}:

$$
\Big| T^{int} - S^{int} \Big| \le \Delta t \sum_{n=0}^{N_T-1} \sum_{j=1}^{N-1} \Big| G^{n,k}_{j+1/2}(u_j^n, u_{j+1}^n) - G^{n,k}_{j+1/2}(u_j^n, u_j^n) \Big|\, \Big| \varphi(t^n, x_{j+1}) - \varphi(t^n, x_j) \Big|.
$$

Observe that

$$
\Big| G^{n,k}_{j+1/2}(u_j^n, u_{j+1}^n) - G^{n,k}_{j+1/2}(u_j^n, u_j^n) \Big|
$$

$$
\begin{aligned}
&= \left| F^n_{j+1/2}(u^n_j \wedge k, u^n_{j+1} \wedge k) - F^n_{j+1/2}(u^n_j \wedge k, u^n_j) \wedge k \right| \\
&= \frac{1}{2} \Big| f(t^n, x_{j+1/2}, u^n_j \wedge k) + f(t^n, x_{j+1/2}, u^n_{j+1} \wedge k) \\
&\qquad\quad -2\, f(t^n, x_{j+1/2}, u^n_j \wedge k) - \alpha\, (u^n_{j+1} \wedge k - u^n_j \wedge k) \Big| \\
&= \frac{1}{2} \Big| f(t^n, x_{j+1/2}, u^n_{j+1} \wedge k) - f(t^n, x_{j+1/2}, u^n_j \wedge k) \\
&\qquad\qquad -\alpha\, (u^n_{j+1} \wedge k - u^n_j \wedge k) \Big| \\
&\le \frac{1}{2} \left(L_f(t^n) + \alpha \right) \left| u^n_{j+1} \wedge k - u^n_j \wedge k \right| \\
&\le \alpha \left| u^n_{j+1} - u^n_j \right|.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left| T^{int} - S^{int} \right| &\le \alpha\, \Delta x\, \Delta t\, \| \partial_x \varphi \|_{\mathbf{L}^\infty} \sum_{n=0}^{N_T-1} \sum_{j=1}^{N-1} \left| u^n_{j+1} - u^n_j \right| \\
&\le \alpha\, \Delta x\, T\, \| \partial_x \varphi \|_{\mathbf{L}^\infty} \max_{0 \le n \le N_T-1} \mathrm{TV}\left(u_\Delta(t^n) \right) = \mathcal{O}(\Delta x),
\end{aligned} \tag{3.46}
$$

thanks to the uniform BV estimate (3.14). Pass now to the terms T^b and S^b:

$$
\begin{aligned}
S^b - T^b = &- \Delta t \sum_{n=0}^{N_T-1} \Big(G^{n,k}_{N+1/2}(u^n_N, u^n_N)\, \varphi(t^n, x_{N+1}) - G^{n,k}_{3/2}(u^n_1, u^n_1))\, \varphi(t^n, x_1) \Big) \\
&- \alpha\, \Delta t \sum_{n=0}^{N_T-1} \Big((u^n_a - k)^+\, \varphi(t^n, a) + (u^n_b - k)^+\, \varphi(t^n, b) \Big) \\
&- \Delta t \sum_{n=0}^{N_T-1} \Big(G^{n,k}_{N+1/2}(u^n_N, u^n_b) - G^{n,k}_{N+1/2}(u^n_N, u^n_N) \Big)\, \varphi(t^n, x_N) \\
&+ \Delta t \sum_{n=0}^{N_T-1} \Big(G^{n,k}_{1/2}(u^n_a, u^n_1) - G^{n,k}_{3/2}(u^n_1, u^n_1) \Big)\, \varphi(t^n, x_1) \\
= &\,\Delta t \sum_{n=0}^{N_T-1} \Big(G^{n,k}_{1/2}(u^n_a, u^n_1)\, \varphi(t^n, x_1) - \alpha\, (u^n_a - k)^+\, \varphi(t^n, a) \Big) && (3.47) \\
&- \Delta t \sum_{n=0}^{N_T-1} \Big(\alpha\, (u^n_b - k)^+\, \varphi(t^n, b) + G^{n,k}_{N+1/2}(u^n_N, u^n_b)\, \varphi(t^n, x_N) \Big) && (3.48) \\
&- \Delta t \sum_{n=0}^{N_T-1} G^{n,k}_{N+1/2}(u^n_N, u^n_N) \left(\varphi(t^n, x_{N+1}) - \varphi(t^n, x_N) \right). && (3.49)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\frac{\partial F^n_{j+1/2}}{\partial u}(u, v) &= \frac{1}{2} \Big(\partial_u f(t^n, x_{j+1/2}, u) + \alpha \Big) \ge \frac{1}{2} \left(-L_f(t^n) + \alpha \right) \ge 0, \\
\frac{\partial F^n_{j+1/2}}{\partial v}(u, v) &= \frac{1}{2} \Big(\partial_u f(t^n, x_{j+1/2}, v) - \alpha \Big) \le \frac{1}{2} \left(L_f(t^n) - \alpha \right) \le 0,
\end{aligned}
$$

meaning that the numerical flux is increasing with respect to the first variable and decreasing with respect to the second one. Thus,

$$\begin{aligned}
G^{n,k}_{j+1/2}(u,v) &= F^n_{j+1/2}(u \wedge k, v \wedge k) - F^n_{j+1/2}(k,k) \\
&\geq F^n_{j+1/2}(k, v \wedge k) - F^n_{j+1/2}(k,k) \\
&= \frac{1}{2}\Big(f(t^n, x_{j+1/2}, v \wedge k) - f(t^n, x_{j+1/2}, k) - \alpha(v \wedge k - k)\Big) \\
&\geq -\frac{L_f(t^n)+\alpha}{2}\,|v \wedge k - k| \\
&\geq -\alpha\,(v-k)^+
\end{aligned}$$

and

$$\begin{aligned}
G^{n,k}_{j+1/2}(u,v) &= F^n_{j+1/2}(u \wedge k, v \wedge k) - F^n_{j+1/2}(k,k) \\
&\leq F^n_{j+1/2}(u \wedge k, k) - F^n_{j+1/2}(k,k) \\
&= \frac{1}{2}\Big(f(t^n, x_{j+1/2}, u \wedge k) - f(t^n, x_{j+1/2}, k) - \alpha(k - u \wedge k)\Big) \\
&\leq \frac{L_f(t^n)+\alpha}{2}\,|u \wedge k - k| \\
&\leq \alpha\,(u-k)^+.
\end{aligned}$$

Hence,

$$\begin{aligned}
[(3.47)] &= \Delta t \sum_{n=0}^{N_T-1} G^{n,k}_{1/2}(u^n_a, u^n_1)\,\big(\varphi(t^n, x_1) - \varphi(t^n, a)\big) \\
&\quad + \Delta t \sum_{n=0}^{N_T-1} \Big(G^{n,k}_{1/2}(u^n_a, u^n_1) - \alpha\,(u^n_a - k)^+\Big)\,\varphi(t^n, a) \\
&\leq \alpha\, T\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sup_{n \in [0, N_T-1]} (u^n_a - k)^+ + \alpha\, \Delta t \sum_{n=0}^{N_T-1} \Big((u^n_a - k)^+ - (u^n_a - k)^+\Big)\,\varphi(t^n, a) \\
&\leq \alpha\, T\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \|u_a\|_{\mathbf{L}^\infty([0,T])} = \mathcal{O}(\Delta x), \\
[(3.48)] &= -\Delta t \sum_{n=0}^{N_T-1} \Big(\alpha\,(u^n_b - k)^+ + G^{n,k}_{N+1/2}(u^n_N, u^n_b)\Big)\,\varphi(t^n, b) \\
&\quad - \Delta t \sum_{n=0}^{N_T-1} G^{n,k}_{N+1/2}(u^n_N, u^n_b)\,\big(\varphi(t^n, x_N) - \varphi(t^n, b)\big) \\
&\leq -\alpha\, \Delta t \sum_{n=0}^{N_T-1} \Big((u^n_b - k)^+ - (u^n_b - k)^+\Big)\,\varphi(t^n, b) + \alpha\, T\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sup_{n \in [0, N_T-1]} (u^n_b - k)^+ \\
&\leq \alpha\, T\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \|u_b\|_{\mathbf{L}^\infty([0,T])} = \mathcal{O}(\Delta x), \\
[(3.49)] &= \Delta t \left| \sum_{n=0}^{N_T-1} G^{n,k}_{N+1/2}(u^n_N, u^n_N)\,\big(\varphi(t^n, x_{N+1}) - \varphi(t^n, x_N)\big) \right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \Delta t\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sum_{n=0}^{N_T-1} \left| G^{n,k}_{N+1/2}(u^n_N, u^n_N) \right| \\
&= \Delta t\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sum_{n=0}^{N_T-1} \left| F^n_{N+1/2}(u^n_N \wedge k, u^n_N \wedge k) - F^N_{N+1/2}(k,k) \right| \\
&= \Delta t\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sum_{n=0}^{N_T-1} \left| f(t^n, x_{N+1/2}, u^n_N \wedge k) - f(t^n, x_{N+1/2}, k) \right| \\
&\leq L_f(T)\, \Delta t\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sum_{n=0}^{N_T-1} \left| u^n_N \wedge k - k \right| \\
&= L_f(T)\, \Delta t\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sum_{n=0}^{N_T-1} (u^n_N - k)^+ \\
&\leq L_f(T)\, T\, \Delta x\, \|\partial_x \varphi\|_{\mathbf{L}^\infty} \sup_{0 \leq n \leq N_T-1} \|u^n\|_{\mathbf{L}^\infty} = \mathcal{O}(\Delta x),
\end{aligned}
$$

thanks to the uniform $\mathbf{L}^\infty$ estimate (3.6). Hence, $S^b - T^b \leq \mathcal{O}(\Delta x)$, so that we finally get

$$
\begin{aligned}
0 &\geq [(3.40) \dots (3.44)] \\
&= [(3.40)] + [(3.44)] + T \pm S \\
&\geq [(3.40)] + [(3.44)] + S - \mathcal{O}(\Delta x),
\end{aligned}
$$

concluding the proof. □

3.5 Uniqueness

The uniqueness of the solution to the IBVP (1.1) follows from the Lipschitz continuous dependence of the solution on initial and boundary data, proved for the multidimensional case in [3, Theorem 4.3].

Proposition 3.7. (Lipschitz continuous dependence on initial and boundary data) *Let* **(f)** *and* **(g)** *hold. Let* $(u_o,\, u_a,\, u_b)$ *and* $(v_o,\, v_a,\, v_b)$ *satisfy* **(D)**. *Call* u *and* v *the corresponding solutions to the IBVP* (1.1). *Then, for all* $t > 0$, *the following estimate holds*

$$
\left\| u(t) - v(t) \right\|_{\mathbf{L}^1(]a,b[)} \leq e^{L_g(t)\, t} \left[\|u_o - v_o\|_{\mathbf{L}^1(]a,b[)} + L_f(t) \Big(\|u_a - v_a\|_{\mathbf{L}^1([0,t])} + \|u_b - v_b\|_{\mathbf{L}^1([0,t])} \Big) \right],
$$

where $L_f(t)$ *and* $L_g(t)$ *are defined in* (2.1).

4 Proofs of Theorem 2.4 and Theorem 2.6

Proof of Theorem 2.4. The existence of a unique solution to the IBVP (1.1) is ensured by the results presented in Section 3, see in particular § 3.4 and § 3.5.

The estimates on the solution to the IBVP (1.1) are obtained by passing to the limit in the corresponding discrete estimates, namely (3.6) for the $\mathbf{L}^\infty$–bound, (3.14) for the bound on the total variation, and (3.30)–(3.32) for the Lipschitz continuity in time. □

Proof of Theorem 2.6. By Lemma 3.2 we have, for all $t > 0$

$$\left\|u_1(t)\right\|_{\mathbf{L}^\infty(]a,b[)} \leq \mathcal{U}_1(t), \qquad \left\|u_2(t)\right\|_{\mathbf{L}^\infty(]a,b[)} \leq \mathcal{U}_2(t), \tag{4.1}$$

the definition of $\mathcal{U}_i(t)$, $i = 1, 2$, following from (3.7). Introduce, moreover, the following notation: for all $t > 0$

$$U_i(t) = [-\mathcal{U}_i(t), \mathcal{U}_i(t)], \quad i = 1, 2 \qquad U(t) = U_1(t) \cup U_2(t). \tag{4.2}$$

We apply the *doubling of variables method* introduced by Kružkov [9], following also [3, proof of Theorem 4.3]. Let $\varphi \in \mathbf{C}^1_\mathbf{c}(]0, T[\times\mathbb{R}; \mathbb{R}^+)$ be a test function as in Definition 2.2 with

$$\varphi(t, x) = 0 \text{ for all } t \in]0, T[\text{ and } x \in [a, a + h_*] \cup [b - h_*, b] \tag{4.3}$$

for a positive h_*. Clearly, $\varphi(0, x) = 0$ for all $x \in \mathbb{R}$.

Let $Y \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}; \mathbb{R}^+)$ be such that

$$Y(-z) = Y(z), \qquad Y(z) = 0 \text{ for } |z| \geq 1, \qquad \int_\mathbb{R} Y(z)\,\mathrm{d}z = 1,$$

and define, for $h \in]0, h_*[$, $Y_h(z) = \frac{1}{h} Y\left(\frac{z}{h}\right)$. Clearly, $Y_h \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}; \mathbb{R}^+)$, $Y_h(-z) = Y_h(z)$, $Y_h(z) = 0$ for $|z| \geq h$ and $\int_\mathbb{R} Y_h(z)\,\mathrm{d}z = 1$. Moreover, $Y_h \to \delta_0$ as $h \to 0$, δ_0 being the Dirac delta in 0. Define

$$\psi_h(t, x, s, y) = \varphi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) Y_h(t-s)\, Y_h(x-y).$$

For the sake of simplicity, introduce the space $\Pi_T =]0, T[\times]a, b[$. Since $u_1 = u_1(t, x)$ and $u_2 = u_2(s, y)$ are solutions to the IBVPs (1.1), we derive the following entropy inequalities

$$\begin{aligned}
&\iiiint\limits_{\Pi_T\times\Pi_T} \Big\{ \left|u_1(t,x) - u_2(s,y)\right| \partial_t \psi_h(t,x,s,y) \\
&\qquad + \operatorname{sgn}\left(u_1(t,x) - u_2(s,y)\right) \Big[f_1\left(t,x,u_1(t,x)\right) - f_1\left(t,x,u_2(s,y)\right)\Big]\, \partial_x \psi_h(t,x,s,y) \\
&\qquad + \operatorname{sgn}\left(u_1(t,x) - u_2(s,y)\right) \Big(g_1\left(t,x,u_1(t,x)\right) - \partial_x f_1\left(t,x,u_2(s,y)\right)\Big)\, \psi_h(t,x,s,y) \Big\} \\
&\qquad\qquad \times \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\
&+ \iint\limits_{\Pi_T} \int_a^b \left|u_o(x) - u_2(s,y)\right| \psi_h(0,x,s,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\iiiint\limits_{\Pi_T\times\Pi_T} \Big\{ \left|u_2(s,y) - u_1(t,x)\right| \partial_s \psi_h(t,x,s,y) \\
&\qquad + \operatorname{sgn}\left(u_2(s,y) - u_1(t,x)\right) \Big[f_2\left(s,y,u_2(s,y)\right) - f_2\left(s,y,u_1(t,x)\right)\Big]\, \partial_y \psi_h(t,x,s,y) \\
&\qquad + \operatorname{sgn}\left(u_2(s,y) - u_1(t,x)\right) \Big(g_2\left(s,y,u_2(s,y)\right) - \partial_y f_2\left(s,y,u_1(t,x)\right)\Big)\, \psi_h(t,x,s,y) \Big\} \\
&\qquad\qquad \times \mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

$$+\iiint_{\Pi_T}\int_a^b \left|u_o(y)-u_1(t,x)\right|\psi_h(t,x,0,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Now combine the two inequalities above and rearrange the terms therein: setting $\psi_h = \psi_h(t,x,s,y)$, we obtain

$$\begin{aligned}
&\iiiint_{\Pi_T\times\Pi_T} \Big\{ \left|u_1(t,x)-u_2(s,y)\right| (\partial_t\psi_h+\partial_s\psi_h)\\
&\qquad + \operatorname{sgn}\big(u_1(t,x)-u_2(s,y)\big)\Big[f_1\big(t,x,u_1(t,x)\big)-f_1\big(s,y,u_2(s,y)\big)\Big]\ (\partial_x\psi_h+\partial_y\psi_h)\\
&\qquad + \operatorname{sgn}\big(u_1(t,x)-u_2(s,y)\big)\\
&\qquad \times\Bigg[\Big(f_1\big(s,y,u_2(s,y)\big)-f_1\big(t,x,u_2(s,y)\big)\Big)\ \partial_x\psi_h-\partial_x f_1\big(t,x,u_2(s,y)\big)\,\psi_h\\
&\qquad\quad +\Big(f_2\big(s,y,u_1(t,x)\big)-f_2\big(t,x,u_1(t,x)\big)\Big)\ \partial_y\psi_h+\partial_y f_2\big(s,y,u_1(t,x)\big)\,\psi_h\Bigg]\\
&\qquad + \operatorname{sgn}\big(u_1(t,x)-u_2(s,y)\big)\Big(f_1\big(s,y,u_2(s,y)\big)-f_1\big(t,x,u_1(t,x)\big)\\
&\qquad\quad +f_2\big(t,x,u_1(t,x)\big)-f_2\big(s,y,u_2(s,y)\big)\Big)\ \partial_y\psi_h\\
&\qquad + \operatorname{sgn}\big(u_1(t,x)-u_2(s,y)\big)\Big(g_1\big(t,x,u_1(t,x)\big)-g_1(s,y,u_2(s,y))\Big)\ \psi_h\\
&\qquad + \operatorname{sgn}\big(u_1(t,x)-u_2(s,y)\big)\Big(g_1(s,y,u_2(s,y))-g_2\big(s,y,u_2(s,y)\big)\Big)\ \psi_h\Big\}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
&+\iiint_{\Pi_T}\int_a^b \left|u_o(x)-u_2(s,y)\right|\psi_h(0,x,s,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&+\iiint_{\Pi_T}\int_a^b \left|u_o(y)-u_1(t,x)\right|\psi_h(t,x,0,y)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\\
&=\iiiint_{\Pi_T\times\Pi_T}\sum_{j=1}^{6} I_j\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s+\iiint_{\Pi_T}\int_a^b J_1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s+\iiint_{\Pi_T}\int_a^b J_2\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\geq 0,
\end{aligned}\tag{4.4}$$

where

$$I_1 = \left|u_1(t,x)-u_2(s,y)\right| (\partial_t\psi_h+\partial_s\psi_h)\,, \tag{4.5}$$

$$I_2 = \operatorname{sgn}\big(u_1(t,x)-u_2(s,y)\big)\Big[f_1\big(t,x,u_1(t,x)\big)-f_1\big(s,y,u_2(s,y)\big)\Big]\ (\partial_x\psi_h+\partial_y\psi_h)\,, \tag{4.6}$$

$$\begin{aligned}
I_3 = {}& \operatorname{sgn}\big(u_1(t,x)-u_2(s,y)\big)\\
&\times\Bigg[\Big(f_1\big(s,y,u_2(s,y)\big)-f_1\big(t,x,u_2(s,y)\big)\Big)\ \partial_x\psi_h-\partial_x f_1\big(t,x,u_2(s,y)\big)\,\psi_h\\
&\quad+\Big(f_2\big(s,y,u_1(t,x)\big)-f_2\big(t,x,u_1(t,x)\big)\Big)\ \partial_y\psi_h+\partial_y f_2\big(s,y,u_1(t,x)\big)\,\psi_h\Bigg],
\end{aligned}\tag{4.7}$$

$$\begin{aligned}
I_4 = {}& \operatorname{sgn}\big(u_1(t,x)-u_2(s,y)\big)\Big(f_1\big(s,y,u_2(s,y)\big)-f_1\big(t,x,u_1(t,x)\big)\\
&+f_2\big(t,x,u_1(t,x)\big)-f_2\big(s,y,u_2(s,y)\big)\Big)\ \partial_y\psi_h,
\end{aligned}\tag{4.8}$$

$$I_5 = \operatorname{sgn}\left(u_1(t,x) - u_2(s,y)\right)\left(g_1\left(t,x,u_1(t,x)\right) - g_1(s,y,u_2(s,y))\right)\,\psi_h, \tag{4.9}$$

$$I_6 = \operatorname{sgn}\left(u_1(t,x) - u_2(s,y)\right)\left(g_1(s,y,u_2(s,y)) - g_2\left(s,y,u_2(s,y)\right)\right)\,\psi_h, \tag{4.10}$$

$$J_1 = \left|u_o(x) - u_2(s,y)\right|\psi_h(0,x,s,y), \tag{4.11}$$

$$J_2 = \left|u_o(y) - u_1(t,x)\right|\psi_h(t,x,0,y). \tag{4.12}$$

Now we let h go to 0 in (4.4). The integrals of J_1 (4.11) and J_2 (4.12), are treated exactly as in [3, proof of Theorem 4.3]:

$$\lim_{h\to 0}\left(\iint\limits_{\Pi_T}\int_a^b J_1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s + \iint\limits_{\Pi_T}\int_a^b J_2\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t\right) = 0. \tag{4.13}$$

The integral of $I_1 + I_2 + I_5$, see (4.5), (4.6) and (4.9), is treated exactly as in [9, Theorem 1], leading to

$$\begin{aligned}
&\lim_{h\to 0}\iiiint\limits_{\Pi_T\times\Pi_T}\{I_1 + I_2 + I_5\}\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
= &\iint\limits_{\Pi_T}\Big\{\left|u_1(t,x) - u_2(t,x)\right|\partial_t\varphi(t,x)\\
&+\operatorname{sgn}\left(u_1(t,x) - u_2(t,x)\right)\left(f_1\left(t,x,u_1(t,x)\right) - f_1\left(t,x,u_2(t,x)\right)\right)\partial_x\varphi(t,x)\\
&+\operatorname{sgn}\left(u_1(t,x) - u_2(t,x)\right)\left(g_1\left(t,x,u_1(t,x)\right) - g_1(t,x,u_2(t,x))\right)\varphi(t,x)\Big\}\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned} \tag{4.14}$$

The integral of I_3 (4.7) can be as well treated as in [9, Theorem 1], bearing in mind that now we have two different fluxes, namely f_1 and f_2. Nevertheless, this does not constitute an issue in the proof. Thus we have

$$\lim_{h\to 0}\iiiint\limits_{\Pi_T\times\Pi_T} I_3\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s = 0. \tag{4.15}$$

Consider now the integral of I_4 (4.8):

$$\begin{aligned}
&\iiiint\limits_{\Pi_T\times\Pi_T} I_4\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
= &\iiiint\limits_{\Pi_T\times\Pi_T}\operatorname{sgn}\left(u_1(t,x) - u_2(s,y)\right)\\
&\qquad\times\Big[(f_1 - f_2)\left(s,y,u_2(s,y)\right) - (f_1 - f_2)\left(t,x,u_1(t,x)\right)\Big]\partial_y\psi_h(t,x,s,y)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
= &-\iiiint\limits_{\Pi_T\times\Pi_T}\mathrm{d}_y\Big\{\operatorname{sgn}\left(u_1(t,x) - u_2(s,y)\right)\\
&\qquad\times\Big[(f_1 - f_2)\left(s,y,u_2(s,y)\right) - (f_1 - f_2)\left(t,x,u_1(t,x)\right)\Big]\Big\}\,\psi_h(t,x,s,y)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\,.
\end{aligned} \tag{4.16}$$

Recall that for a Lipschitz function h and an $\mathbf{L}^\infty \cap \mathbf{BV}$ function z it holds $\mathrm{d}_y\, h(y, z(y)) = \partial_y h(y, z(y)) + \partial_z h(y, z(y))\, z'(y)$, where we apply the chain rule. There, $z'(y)$ is a finite measure, see [2, Lemma A2.1] and also [8, Lemma 4.1]. Thus, the integral in (4.16) splits in two parts. Let us analyse them in detail. The first part involves the partial derivative with respect to y:

$$\begin{aligned}
&\lim_{h\to 0} \iiiint\limits_{\Pi_T\times\Pi_T} \operatorname{sgn}\big(u_1(t,x) - u_2(s,y)\big)\, \partial_y (f_2 - f_1)\big(s, y, u_2(s,y)\big)\ \psi_h(t,x,s,y)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\
\le\ &\lim_{h\to 0} \iiiint\limits_{\Pi_T\times\Pi_T} \left|\partial_y (f_2 - f_1)\big(s, y, u_2(s,y)\big)\right| \psi_h(t,x,s,y)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\
=\ &\iint\limits_{\Pi_T} \left|\partial_y (f_2 - f_1)\big(s, y, u_2(s,y)\big)\right| \varphi(s,y)\,\mathrm{d}y\,\mathrm{d}s\,.
\end{aligned}\tag{4.17}$$

Concerning the second part, the function

$$H(t,x,s,y,u_1,u_2) = \operatorname{sgn}(u_1 - u_2)\ \big[(f_2 - f_1)\,(s,y,u_2) - (f_2 - f_1)\,(t,x,u_1)\big]$$

is clearly Lipschitz with respect to u_2, with Lipschitz constant $\left\|\partial_u (f_2 - f_1)(s,y,\cdot)\right\|_{\mathbf{L}^\infty(U_2(s))}$, with the notation introduced in (4.2). We can thus apply [2, Lemma A2.1], see also [8, Lemma 4.1], to get

$$\left|\partial_{u_2} H\,\big(t,x,s,y,u_1(t,x),u_2(s,y)\big)\ \partial_y u_2(s,y)\right| \le \left\|\partial_u (f_2 - f_1)(s,y,\cdot)\right\|_{\mathbf{L}^\infty(U_2(s))} \left|\partial_y u_2(s,y)\right|.$$

Therefore we obtain

$$\begin{aligned}
&\lim_{h\to 0} \iiiint\limits_{\Pi_T\times\Pi_T} \left|\partial_{u_2} H\,\big(t,x,s,y,u_1(t,x),u_2(s,y)\big)\ \partial_y u_2(s,y)\right| \psi_h(t,x,s,y)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\
\le\ &\iint\limits_{\Pi_T} \left\|\partial_u (f_2 - f_1)(t,x)\right\|_{\mathbf{L}^\infty(U_2(t))} \left|\partial_x u_2(t,x)\right| \varphi(t,x)\,\mathrm{d}t\,\mathrm{d}x\,.
\end{aligned}\tag{4.18}$$

As far as the integral of I_6 (4.10) is concerned, we obtain

$$\iiiint\limits_{\Pi_T\times\Pi_T} I_6\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \le \iint\limits_{\Pi_T} \left|(g_1 - g_2)\big(t,x,u_2(t,x)\big)\right| \varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,. \tag{4.19}$$

Therefore, in the limit $h \to 0$, collecting together (4.13), (4.14), (4.15), (4.17), (4.18) and (4.19), we obtain that (4.4) becomes

$$\begin{aligned}
\iint\limits_{\Pi_T} \Big\{ &\left|u_1(t,x) - u_2(t,x)\right| \partial_t \varphi(t,x) \\
&+ \operatorname{sgn}\big(u_1(t,x) - u_2(t,x)\big) \Big(f_1\,\big(t,x,u_1(t,x)\big) - f_1\,\big(t,x,u_2(t,x)\big)\Big)\, \partial_x \varphi(t,x) \\
&+ \operatorname{sgn}\big(u_1(t,x) - u_2(t,x)\big) \Big(g_1\,\big(t,x,u_1(t,x)\big) - g_1(t,x,u_2(t,x))\Big)\, \varphi(t,x) \\
&+ \left|\partial_x (f_2 - f_1)\big(t,x,u_2(t,x)\big)\right| \varphi(t,x) \\
&+ \left\|\partial_u (f_2 - f_1)(t,x)\right\|_{\mathbf{L}^\infty(U_2(t))} \left|\partial_x u_2(t,x)\right| \varphi(t,x)
\end{aligned}\tag{4.20}$$

$$+\left|(g_1 - g_2)\big(t, x, u_2(t,x)\big)\right| \varphi(t,x) \Big\} \,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Following the proof of [3, Theorem 4.3], introduce a function $\Phi_\varepsilon \in \mathbf{C}^{\mathbf{2}}(\mathbb{R};[0,1])$ such that $\Phi_\varepsilon(a) = \Phi_\varepsilon(b) = 1$, $\Phi_\varepsilon(x) = 0$ for $x \in [a+\varepsilon, b-\varepsilon]$ and $\left\|\Phi'_\varepsilon\right\|_{\mathbf{L}^\infty} \leq 1/\varepsilon$. Let $\Psi \in \mathbf{C}^{\mathbf{2}}_{\mathbf{c}}(]0,T[;\mathbb{R}^+)$, with $\Psi(0) = 0$. For any $\varepsilon > 0$ sufficiently small, the function

$$\varphi_\varepsilon(t,x) = \Psi(t)\ \big(1 - \Phi_\varepsilon(x)\big)$$

satisfies (4.3), being thus an admissible test function. Use it in (4.20) and pass to the limit as ε goes to 0: we get

$$\begin{aligned}
\iint_{\Pi_T} & \Big\{ \left|u_1(t,x) - u_2(t,x)\right| \Psi'(t) \\
& + \mathrm{sgn}\,\big(u_1(t,x) - u_2(t,x)\big) \Big(g_1\,\big(t,x,u_1(t,x)\big) - g_1(t,x,u_2(t,x))\Big)\,\Psi(t) \\
& + \left|\partial_x(f_2 - f_1)\,\big(t,x,u_2(t,x)\big)\right| \Psi(t) \qquad\qquad (4.21)\\
& + \left\|\partial_u(f_2 - f_1)(t,x)\right\|_{\mathbf{L}^\infty(U_2(t))} \left|\partial_x u_2(t,x)\right| \Psi(t) + \left|(g_1 - g_2)\big(t,x,u_2(t,x)\big)\right| \Psi(t) \Big\} \,\mathrm{d}x\,\mathrm{d}t \\
& + \int_0^T \mathrm{sgn}\,\Big(u_1(t,a^+) - u_2(t,a^+)\Big)\Big(f_1\,\Big(t,a,u_1(t,a^+)\Big) - f_1\,\Big(t,a,u_2(t,a^+)\Big)\Big)\,\Psi(t)\,\mathrm{d}t \\
& - \int_0^T \mathrm{sgn}\,\Big(u_1(t,b^-) - u_2(t,b^-)\Big)\Big(f_1\,\Big(t,b,u_1(t,b^-)\Big) - f_1\,\Big(t,b,u_2(t,b^-)\Big)\Big)\,\Psi(t)\,\mathrm{d}t \geq 0,
\end{aligned}$$

where we used [3, Lemma A.6 and Lemma A.4], recalling that the exterior normal in a has negative sign.

Introduce τ, t such that $0 < \tau < t < T$. Define, for $\ell > 0$, the map

$$\Psi_\ell(s) = \alpha_\ell(s - \tau - \ell) - \alpha_h(s - t - \ell), \text{ with } \alpha_\ell(\xi) = \int_{-\infty}^{z} Y_\ell(\zeta)\,\mathrm{d}\zeta\,.$$

This function is clearly in $\mathbf{C}^{\mathbf{2}}_{\mathbf{c}}(]0,T[;\mathbb{R}^+)$ and such that $\Psi_\ell(0) = 0$. Moreover, $\Psi_\ell \to \chi_{[\tau,t]}$ and $\Psi'_\ell \to \delta_\tau - \delta_t$ as ℓ tends to 0. Substitute Ψ_ℓ in (4.21) and pass to the limit $\ell \to 0$:

$$\begin{aligned}
0 \leq & \int_a^b \left|u_1(\tau,x) - u_2(\tau,x)\right| \mathrm{d}x - \int_a^b \left|u_1(t,x) - u_2(t,x)\right| \mathrm{d}x & (4.22)\\
& + \int_\tau^t \int_a^b \left\|\partial_u g_1(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \left|u_1(s,x) - u_2(s,x)\right| \mathrm{d}x\,\mathrm{d}s & (4.23)\\
& + \int_\tau^t \int_a^b \left|\partial_x(f_2 - f_1)\big(s,x,u_2(s,x)\big)\right| \mathrm{d}x\,\mathrm{d}s & (4.24)\\
& + \int_\tau^t \left\|\partial_u(f_2 - f_1)(s,\cdot,\cdot)\right\|_{\mathbf{L}^\infty(]a,b[\times U_2(s))} \mathrm{TV}\,\big(u_2(s)\big)\,\mathrm{d}s & (4.25)\\
& + \int_\tau^t \int_a^b \left|(g_1 - g_2)\big(s,x,u_2(s,x)\big)\right| \mathrm{d}x\,\mathrm{d}s & (4.26)\\
& + \int_\tau^t \mathrm{sgn}\,\Big(u_1(s,a^+) - u_2(s,a^+)\Big)\Big(f_1\,\Big(s,a,u_1(s,a^+)\Big) - f_1\,\Big(s,a,u_2(s,a^+)\Big)\Big)\,\mathrm{d}s & (4.27)
\end{aligned}$$

$$- \int_\tau^t \operatorname{sgn}\Big(u_1(s,b^-) - u_2(s,b^-)\Big) \Big(f_1\Big(s,b,u_1(s,b^-)\Big) - f_1\Big(s,b,u_2(s,b^-)\Big)\Big)\, \mathrm{d}s\,. \tag{4.28}$$

Now we aim to find an estimate for (4.27) and (4.28). Focus in particular on (4.28), the procedure being analogous for (4.27). The only contribution from this term come from the negative part of its argument, that is

$$B = \left[\operatorname{sgn}\Big(u_1(s,b^-) - u_2(s,b^-)\Big) \Big(f_1\Big(s,b,u_1(s,b^-)\Big) - f_1\Big(s,b,u_2(s,b^-)\Big)\Big)\right]^-. \tag{4.29}$$

Fix $s \in [\tau, t]$. To ease readability, in the following we will denote $u_1 = u_1(s,b^-)$, $u_2 = u_2(s,b^-)$, $u_b = u_b(s)$ and $f_i(z) = f_i(s,b,z)$, $i = 1,2$. We have the following

$$B = \begin{cases} \big(f_1(u_1) - f_1(u_2)\big)^- & \text{if } u_1 > u_2,\\ \big(f_1(u_1) - f_1(u_2)\big)^+ & \text{if } u_1 < u_2. \end{cases}$$

Before analysing each case, recall the BLN–boundary conditions (see Remark 2.3):

$$\forall k \in \mathcal{I}(u_b, u_1): \qquad \operatorname{sgn}(u_1 - u_b)\big(f_1(u_1) - f_1(k)\big) \geq 0, \tag{4.30}$$

$$\forall k \in \mathcal{I}(u_b, u_2): \qquad \operatorname{sgn}(u_2 - u_b)\big(f_2(u_2) - f_2(k)\big) \geq 0. \tag{4.31}$$

Case $\boldsymbol{u_1 > u_2}$. In this case we have

$$\begin{aligned} B = {} & \big(f_1(u_1) - f_1(u_2)\big)^- = \big(f_1(u_1) - f_2(u_2) + f_2(u_2) - f_1(u_2)\big)^- \\ \leq {} & \big(f_2(u_2) - f_1(u_1)\big)^+ + \|f_2 - f_1\|_{\mathbf{L}^\infty(U_2(s))}. \end{aligned}$$

We have three sub-cases:

1) $\boldsymbol{u_b < u_2 < u_1}$. The BLN–condition (4.30) now reads $f_1(u_1) \geq f_1(k)$ for all $k \in [u_b, u_1]$. The choice $k = u_2$ is admissible: $f_1(u_1) \geq f_1(u_2)$. Thus

$$f_2(u_2) - f_1(u_1) \leq f_2(u_2) - f_1(u_2) \leq \|f_2 - f_1\|_{\mathbf{L}^\infty(U_2(s))}.$$

2) $\boldsymbol{u_2 < u_b < u_1}$. The BLN–conditions (4.30)–(4.31) now reads

$$\begin{aligned} \forall k \in [u_b, u_1]: &\qquad f_1(u_1) \geq f_1(k),\\ \forall k \in [u_2, u_b]: &\qquad f_2(u_2) \leq f_2(k). \end{aligned}$$

In both cases, the choice $k = u_b$ is admissible, yielding $f_1(u_1) \geq f_1(u_b)$ and $f_2(u_2) \leq f_2(u_b)$. Thus

$$f_2(u_2) - f_1(u_1) \leq f_2(u_b) - f_1(u_b).$$

3) $\boldsymbol{u_2 < u_1 < u_b}$. The BLN–condition (4.31) now reads $f_2(u_2) \leq f_2(k)$ for all $k \in [u_2, u_b]$. The choice $k = u_1$ is admissible: $f_2(u_2) \leq f_2(u_1)$. Thus

$$f_2(u_2) - f_1(u_1) \leq f_2(u_1) - f_1(u_1) \leq \|f_2 - f_1\|_{\mathbf{L}^\infty(U_1(s))}.$$

Case $u_1 < u_2$. The proof is done in a similar way.

We can thus conclude that, for $s \in [\tau, t]$, $B \leq 2 \left\|(f_2 - f_1)(s, b, \cdot)\right\|_{\mathbf{L}^\infty(U(s))}$, where $U(s)$ is as in (4.2), so that, going back to (4.28),

$$-\int_\tau^t B \, \mathrm{d}s \leq \int_\tau^t 2 \left\|(f_2 - f_1)(s, b, \cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s \, .$$

Similarly, we obtain

$$[(4.27)] \leq \int_\tau^t 2 \left\|(f_2 - f_1)(s, a, \cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s \, .$$

Therefore, we get the following estimate for (4.22)–(4.28):

$$\begin{aligned}
&\int_a^b \left|u_1(t,x) - u_2(t,x)\right| \mathrm{d}x \\
\leq\ & \int_a^b \left|u_1(\tau,x) - u_2(\tau,x)\right| \mathrm{d}x + \int_\tau^t \int_a^b \left\|\partial_u g_1(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \left|u_1(s,x) - u_2(s,x)\right| \mathrm{d}x \, \mathrm{d}s \\
&+ \int_\tau^t \int_a^b \left\|\partial_x (f_2 - f_1)(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U_2(s))} \mathrm{d}x \, \mathrm{d}s + \int_\tau^t \int_a^b \left\|(g_1 - g_2)(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U_2(s))} \mathrm{d}x \, \mathrm{d}s \\
&+ \int_\tau^t \left\|\partial_u (f_2 - f_1)(s,\cdot,\cdot)\right\|_{\mathbf{L}^\infty(]a,b[\times U_2(s))} \mathrm{TV}\left(u_2(s)\right) \mathrm{d}s \\
&+ 2\int_\tau^t \left\|(f_2 - f_1)(s,a,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s + 2\int_\tau^t \left\|(f_2 - f_1)(s,b,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s \, .
\end{aligned}$$

Let now τ go to 0, recalling that the initial datum is the same, and apply Gronwall lemma:

$$\begin{aligned}
&\int_a^b \left|u_1(t,x) - u_2(t,x)\right| \mathrm{d}x \\
\leq\ & e^{t \left\|\partial_u g_1\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times U(t))}} \left(\int_0^t \int_a^b \left\|\partial_x (f_2 - f_1)(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U_2(s))} \mathrm{d}x \, \mathrm{d}s \right. \\
&+ \int_0^t \int_a^b \left\|(g_1 - g_2)(s,x,\cdot)\right\|_{\mathbf{L}^\infty(U_2(s))} \mathrm{d}x \, \mathrm{d}s \\
&+ \int_0^t \left\|\partial_u (f_2 - f_1)(s,\cdot,\cdot)\right\|_{\mathbf{L}^\infty(]a,b[\times U_2(s))} \mathrm{TV}\left(u_2(s)\right) \mathrm{d}s \\
&\left. +2\int_0^t \left\|(f_2 - f_1)(s,a,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s + 2\int_0^t \left\|(f_2 - f_1)(s,b,\cdot)\right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s \right) .
\end{aligned}$$

Exchanging the role u_1 and u_2, and thus that of f_1, g_1 and f_2, g_2, we get a symmetric estimate. Therefore, recalling the definition of $U(t)$ (4.2), the final estimate reads

$$\begin{aligned}
&\int_a^b \left|u_1(t,x) - u_2(t,x)\right| \mathrm{d}x \\
\leq\ & \exp\left(t \, \min\left\{ \left\|\partial_u g_1\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times U(t))}, \left\|\partial_u g_2\right\|_{\mathbf{L}^\infty([0,t]\times[a,b]\times U(t))} \right\} \right)
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\int_0^t \int_a^b \left\| \partial_x (f_2 - f_1)(s,x,\cdot) \right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}x \, \mathrm{d}s \right. \\
&\quad + \int_0^t \int_a^b \left\| (g_1 - g_2)(s,x,\cdot) \right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}x \, \mathrm{d}s \\
&\quad + \int_0^t \left\| \partial_u (f_2 - f_1)(s,\cdot,\cdot) \right\|_{\mathbf{L}^\infty(]a,b[\times U(s))} \min \left\{ \mathrm{TV}\left(u_1(s)\right), \, \mathrm{TV}\left(u_2(s)\right) \right\} \mathrm{d}s \\
&\quad \left. +2 \int_0^t \left\| (f_2 - f_1)(s,a,\cdot) \right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s + 2 \int_0^t \left\| (f_2 - f_1)(s,b,\cdot) \right\|_{\mathbf{L}^\infty(U(s))} \mathrm{d}s \right).
\end{aligned}
$$

□

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References

[1] C. Bardos, A. Y. le Roux, and J.-C. Nédélec. First order quasilinear equations with boundary conditions. *Comm. Partial Differential Equations*, 4(9):1017–1034, 1979.

[2] F. Bouchut and B. Perthame. Kružkov's estimates for scalar conservation laws revisited. *Trans. Amer. Math. Soc.*, 350(7):2847–2870, 1998.

[3] R. M. Colombo and E. Rossi. Rigorous estimates on balance laws in bounded domains. *Acta Math. Sci. Ser. B Engl. Ed.*, 35(4):906–944, 2015.

[4] R. M. Colombo and E. Rossi. IBVPs for scalar conservation laws with time discontinuous fluxes. *Math. Methods Appl. Sci.*, 41(4):1463–1479, 2018.

[5] R. M. Colombo and E. Rossi. Nonlocal conservation laws in bounded domains. *SIAM J. Math. Anal.*, 50(4):4041–4065, 2018.

[6] R. M. Colombo and E. Rossi. Stability of the 1D IBVP for a non autonomous scalar conservation law. *Proc. Roy. Soc. Edinburgh Sect. A*, To appear. https://arxiv.org/abs/1601.05948.

[7] C. De Filippis and P. Goatin. The initial-boundary value problem for general non-local scalar conservation laws in one space dimension. *Nonlinear Anal.*, 161:131–156, 2017.

[8] K. H. Karlsen and N. H. Risebro. On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. *Discrete Contin. Dyn. Syst.*, 9(5):1081–1104, 2003.

[9] S. N. Kružhkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[10] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.

[11] S. Martin. First order quasilinear equations with boundary conditions in the L^∞ framework. *J. Differential Equations*, 236(2):375–406, 2007.

[12] F. Otto. Initial-boundary value problem for a scalar conservation law. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(8):729–734, 1996.

[13] E. Rossi. Definitions of solutions to the IBVP for multi-dimensional scalar balance laws. *J. Hyperbolic Differ. Equ.*, 15(2):349–374, 2018.

[14] J. Vovelle. Convergence of finite volume monotone schemes for scalar conservation laws on bounded domains. *Numer. Math.*, 90(3):563–596, 2002.