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ABSTRACT

In this work we provide an extension to analytic signal method for multidimensional signals. First, expressions for separate phase-shifted components are given. Second, we show that phase-shifted components could be obtained from hypercomplex Fourier transform in the algebra of commutative hypercomplex numbers. Finally we apply multidimensional analytic signal method to signals defined on graphs.

***Index Terms*—** Analytic signal, hypercomplex and Clifford Fourier transforms, Hilbert transform, analytic signal on graph, diffusion maps

1. INTRODUCTION

Analytic signal method [1], [2] has found many applications in science and engineering. Appropriate generalizations of analytic signals to two or more dimensions and its connection with complex and hypercomplex analysis were studied in the recent decades. This was motivated by the development of signal processing methods for image analysis [3, 4, 5, 6] and analysis of multivariate signals [7, 8, 9, 10, 11, 12, 13, 14, 15].

The main issue of the present work is how to compute the phase-shifted versions of a real-valued function in many dimensions, so that one can compute a multidimensional analytic signal satisfying the classical properties of 1-D analytic signals, especially offering the possibility to estimate the envelope of an oscillating signal. We show here that the commutative Scheffers hypercomplex algebra is convenient to compute a multidimensional analytic signal, using the corresponding hypercomplex Fourier transform.

Presented approach generalizes the standard results known for 1-D signals from four different and complementary perspectives: 1) by using Fourier integral formula; 2) by projection and reconstruction on phase-shifted harmonics; 3) using multidimensional Hilbert transform, and 4) by considering the positive frequency restriction of hypercomplex Fourier transform.

Second, we propose a method to construct analytic signal on graph. The motivation comes from graph-based machine learning methods [16, 17], graph signal processing [18] and graph characterization methods [19, 20].

In the context of graph signal processing, analytic signals on graphs were considered in [21, 22]; however their treatment of Hilbert transform for graphs is essentially one dimensional and relies on the asymmetry of the diffusion operator on the graph, like the adjacency matrix, therefore it does not extend to undirected graphs.

In Section 2, we introduce the concept and properties of multidimensional analytic signal and explain how to calculate it using Scheffers hypercomplex algebra. The proofs of the theorems are omitted and are detailed in [23]. Then, in Section 3, we provide numerical illustrations for graph analytic signals, which relies on the mapping of the original signal to a multidimensional signal using diffusion maps [24].

2. ANALYTIC SIGNAL IN MANY DIMENSIONS

For convenience we introduce the following notation. For a given binary $\{0,1\}$-string of length d, $\boldsymbol{j} \in \{0,1\}^d$, we define the functions $\alpha_{\boldsymbol{j}} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ by

$$\alpha_{\boldsymbol{j}}(\boldsymbol{x}, \boldsymbol{\omega}) := \prod_{l=1}^{d} \cos\left(\omega_l x_l - j_l \frac{\pi}{2}\right). \tag{1}$$

Definition 2.1. The phase-shifted version $f_{\boldsymbol{j}}(\boldsymbol{x})$, in the direction $\boldsymbol{j} \in \{0,1\}^d$, of a real-valued function $f(\boldsymbol{x})$ is given by

$$f_{\boldsymbol{j}}(\boldsymbol{x}) = \frac{1}{\pi^d} \int\limits_{\boldsymbol{0}}^{\infty} \int\limits_{-\infty}^{\infty} f(\boldsymbol{x'}) \alpha_{\boldsymbol{j}}\left(\boldsymbol{x} - \boldsymbol{x'}, \boldsymbol{\omega}\right) \, \mathrm{d}\boldsymbol{x'} \, \mathrm{d}\boldsymbol{\omega}. \tag{2}$$

Definition 2.2. We call

- *instantaneous amplitude* or *envelope* $a(\boldsymbol{x})$, the following combination of all phase-shifted versions of the signal:

$$a(\boldsymbol{x}) := \sqrt{\sum_{\boldsymbol{j} \in \{0,1\}^d} f_{\boldsymbol{j}}(\boldsymbol{x})^2} \tag{3}$$

- *instantaneous phase* $\phi_{\boldsymbol{j}}(\boldsymbol{x})$ in the direction $\boldsymbol{j} \in \{0,1\}^d$, the angle:

$$\phi_{\boldsymbol{j}}(\boldsymbol{x}) := \arctan\left(\frac{f_{\boldsymbol{j}}(\boldsymbol{x})}{f(\boldsymbol{x})}\right). \tag{4}$$

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The quantity $a(\boldsymbol{x})$ represents a norm [25] defined in the Banach space of commutative hypercomplex numbers that we introduce shortly. The instantaneous frequency may be defined naturally [23] by taking partial derivatives for each direction $\boldsymbol{j} \in \{0,1\}^d$. The definition of a Hilbert transform in a $d-$dimensional space follows from the definition in 1-D.

Definition 2.3. The Hilbert transform of a function $f : \mathbb{R}^d \to \mathbb{R}$ in the direction $\boldsymbol{j} \in \{0,1\}^d$ is defined by

$$\left(H_{\boldsymbol{j}} f\right)(\boldsymbol{x}) = \text{p.v.}\frac{1}{\pi^{|\boldsymbol{j}|}} \int\limits_{\mathbb{R}^{|\boldsymbol{j}|}} \frac{f(\boldsymbol{y})}{(\boldsymbol{x}-\boldsymbol{y})^{\boldsymbol{j}}}\ \mathrm{d}\boldsymbol{y}^{\boldsymbol{j}}, \tag{5}$$

where $|\boldsymbol{j}|$ gives the number of 1-s in $\boldsymbol{j}$, $(\boldsymbol{x}-\boldsymbol{y})^{\boldsymbol{j}} = \prod_{i=1}^d (x_i - y_i)^{j_i}$ and $\mathrm{d}\boldsymbol{y}^{\boldsymbol{j}} = \mathrm{d}y_1^{j_1} \dots \mathrm{d}y_d^{j_d}$, i.e. integration is performed only over the variables indicated by the vector $\boldsymbol{j}$.

Thus defined, the multidimensional Hilbert transform binds to the phase shift of Def. 2.1; like in the 1-D case, where the Hilbert transform coincides with its quadrature version. It is stated in the following

Theorem 2.4. *For a continuous and absolutely integrable $f(\boldsymbol{x})$, we have*

$$f_{\boldsymbol{j}}(\boldsymbol{x}) = \left(H_{\boldsymbol{j}} f\right)(\boldsymbol{x}). \tag{6}$$

Continuing the parallel with 1-D, it is of interest to express the phase-shifted function $f_{\boldsymbol{j}}(\boldsymbol{x})$ as a combination of the harmonics $\alpha_{\boldsymbol{j}}(\boldsymbol{x},\boldsymbol{\omega})$ and the corresponding projection coefficients $\alpha^{\boldsymbol{j}}(\boldsymbol{\omega})$. Indeed, as there exist efficient algorithms implementing sine and cosine transforms, it is plausible that knowing the coefficients $\alpha^{\boldsymbol{j}}(\boldsymbol{\omega})$ will ease the numerical calculation of the phase-shifted functions $f_{\boldsymbol{j}}$.

To this end, we define the projection coefficients as

$$\alpha^{\boldsymbol{j}}(\boldsymbol{\omega}) := \int\limits_{-\infty}^{\infty} f(\boldsymbol{x})\alpha_{\boldsymbol{j}}(\boldsymbol{x},\boldsymbol{\omega})\ \mathrm{d}\boldsymbol{x}, \tag{7}$$

and introduce the brackets $\langle a, b\rangle_+$ for two functions $a(\boldsymbol{\omega}) = a(\omega_1,\dots,\omega_d)$ and $b(\boldsymbol{x},\boldsymbol{\omega}) = b(x_1,\dots,x_d,\omega_1,\dots,\omega_d)$ as following

$$\langle a, b\rangle_+(\boldsymbol{x}) := \frac{1}{\pi^d}\int\limits_{\boldsymbol{0}}^{\infty} a(\boldsymbol{\omega}) b(\boldsymbol{x},\boldsymbol{\omega})\ \mathrm{d}\boldsymbol{\omega}. \tag{8}$$

Theorem 2.5. *For a given function $f : \mathbb{R}^d \to \mathbb{R}$, the phase-shifted version $f_{\boldsymbol{j}}$ in the direction $\boldsymbol{j} \in \{0,1\}^d$ reads*

$$f_{\boldsymbol{j}}(\boldsymbol{x}) = \sum_{\boldsymbol{i}\in\{0,1\}^d} (-1)^{|(\boldsymbol{i}\oplus\boldsymbol{j})\ominus\boldsymbol{i}|} \left\langle \alpha^{\boldsymbol{i}\oplus\boldsymbol{j}}(\boldsymbol{\omega}), \alpha_{\boldsymbol{i}}(\boldsymbol{x},\boldsymbol{\omega})\right\rangle_+(\boldsymbol{x}), \tag{9}$$

where $\oplus$ is a binary exclusive OR operation acting element-wise on its arguments and $\ominus$ is defined as following: $1\ominus 0 = 1$ and the result is 0 otherwise. To state it simply: we have "$-$" sign before the term in the sum when the corresponding upper index inside the bracket is 1 and down index is 0 and "$+$" sign in any other case.

As a brief demonstration to this theorem we provide an example in 1-D.

Example 2.6. In 1-D case, the analytic signal $f_a : \mathbb{R} \to \mathbb{C}$ of a function $f : \mathbb{R} \to \mathbb{R}$ is given by $f_a(x) = f_0(x) + i f_1(x)$, where $f_0(x) = f(x)$ and $f_1(x) = (Hf)(x)$, its Hilbert transform. Theorem 2.5 in this case reads

$$\begin{aligned} f_0(x) &= \left\langle \alpha^0(\omega), \alpha_0(x,\omega)\right\rangle_+ + \left\langle \alpha^1(\omega), \alpha_1(x,\omega)\right\rangle_+ \\ f_1(x) &= \left\langle \alpha^0(\omega), \alpha_1(x,\omega)\right\rangle_+ - \left\langle \alpha^1(\omega), \alpha_0(x,\omega)\right\rangle_+ . \end{aligned} \tag{10}$$

It follows that component f_0 is obtained by combination of inverse cosine transform of the forward cosine transform plus inverse sine transform of the forward sine transform. While the component f_1 is computed by inverse cosine transform of the forward sine transform minus inverse sine transform of the forward cosine transform of the original function f.

2.1. Scheffers algebra-valued Fourier transform

One corollary property of analytic 1-D signals defined in the algebra of complex numbers, is that their Fourier spectrum has support only over positive frequencies. During the last few decades a number of works were dedicated to the generalization of the Fourier transform, in order to define analytic signal in several dimensions. These generalizations were mainly based on non-commutative algebras like quaternions, Caley-Dickson and Clifford algebras, see e.g. [10, 26, 27, 28, 11, 3, 29, 12, 30, 31, 32, 33, 34]. In this paper we show that a commutative and associative hypercomplex algebra yields a nice approach to define multidimensional analytic signal. Moreover in [23] we demonstrate that one cannot obtain the phase-shifted components $f_{\boldsymbol{j}}$ from Fourier transform defined in any non-commutative algebra in dimension higher than two, not contradicting the fact that for $d = 2$ the symmetric quaternionic Fourier transform [9, 3] does the job.

Theory of analytic functions, extended to the hypercomplex functions of hypercomplex variables, was first introduced in 1893 by G.W. Scheffers [35] and then extended by [36, 25, 37]. In particular, the existence of Cauchy formula in commutative hypercomplex algebras [38], implies existence of the corresponding Hilbert transform (5) in hypercomplex space.

Definition 2.7. The elliptic *Scheffers* algebra $\mathbb{S}_d$ is a unital, commutative and associative algebra over the field $\mathbb{R}$ of dimension 2^d. It has generators $\{1, e_1, \dots, e_d\}$ satisfying the conditions $e_i^2 = -1,\ e_i e_j = e_j e_i,\ i,j = 1,\dots,d$. The basis of the algebra $\mathbb{S}_d$ consists of the elements of the form $e_0 = 1,\ e_\beta = e_{\beta_1} e_{\beta_2} \dots e_{\beta_s},\ \beta = (\beta_1,\dots,\beta_s), \beta_1 < \beta_2 < \dots < \beta_s,\ 1 \le s \le d$. Each element $w \in \mathbb{S}_d$ has the form

$$w = \sum_\beta w_\beta e_\beta = w_0 e_0 + \sum_{s=1}^{d} \sum_{\beta_1<\dots<\beta_s} w_\beta e_\beta,\ (w_0, w_\beta) \in \mathbb{R}. \tag{11}$$

Definition 2.8. The spaces $\mathbb{S}_+(i)$ and $\mathbb{S}_+^d,\ i = 1, ..., d$, are defined by

$$\mathbb{S}_+(i) = \{a + be_i | a, b \in \mathbb{R}; b > 0\}, \tag{12}$$

$$\mathbb{S}_+^d = \bigoplus_{i=1}^{d} \mathbb{S}_+(i). \tag{13}$$

Hypercomplex analytic signal is defined to be any holomorphic function $f : \mathbb{S}_+^d \to \mathbb{S}_d$ on the boundary $\partial\mathbb{S}_+^d \simeq \mathbb{R}^d$.

The Scheffers algebra valued Fourier transform of the function $f : \mathbb{R}^d \to \mathbb{R}$ is defined as follows

$$\hat{f}(\boldsymbol{\omega}) = \int_{-\infty}^{\infty} f(x_1, ..., x_d) \prod_{i=1}^{d} e^{-e_i\omega_i x_i}\ \mathrm{d}x_1 \dots \mathrm{d}x_d, \tag{14}$$

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} f(\omega_1, ..., \omega_d) \prod_{i=1}^{d} e^{e_i\omega_i x_i}\ \mathrm{d}\omega_1 \dots \mathrm{d}\omega_d. \tag{15}$$

Theorem 2.9. *The hypercomplex valued analytic signal $f_h : \mathbb{R}^d \to \mathbb{S}_d$ of the real valued function $f : \mathbb{R}^d \to \mathbb{R}$ defined as*

$$\begin{aligned} f_h(\boldsymbol{x}) := &\left\langle \hat{f}(\boldsymbol{\omega}), \alpha_{0\dots0}(\boldsymbol{x}, \boldsymbol{\omega}) \right\rangle_+ (\boldsymbol{x}) \\ &+ \sum_{i=1}^{d} e_i \left\langle \hat{f}(\boldsymbol{\omega}), \alpha_{0\dots1(i)\dots0}(\boldsymbol{x}, \boldsymbol{\omega}) \right\rangle_+ (\boldsymbol{x}) \\ &+ \sum_{i<j} e_i e_j \left\langle \hat{f}(\boldsymbol{\omega}), \alpha_{0\dots1(i,j)\dots0}(\boldsymbol{x}, \boldsymbol{\omega}) \right\rangle_+ (\boldsymbol{x}) + \dots \end{aligned} \tag{16}$$

has as components the corresponding phase-shifted functions $f_{\boldsymbol{j}}(\boldsymbol{x})$, i.e.

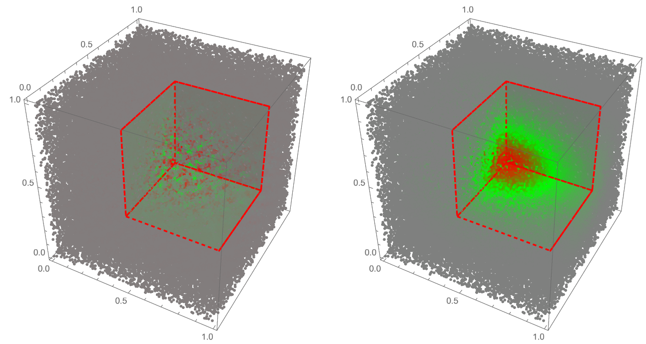
$$f_h(\boldsymbol{x}) = f(\boldsymbol{x}) + \sum_{i=1}^{d} e_i f_{0\dots1(i)\dots0}(\boldsymbol{x}) \tag{17}$$

$$+ \sum_{i<j} e_i e_j f_{0\dots1(i,j)\dots0}(\boldsymbol{x}) + \dots \tag{18}$$

Observation 2.10. As it is the case in 1-D, the hypercomplex Fourier transform of $f_h(\boldsymbol{x})$ is supported only over non-negative frequencies, as we have:

$$\hat{f}_h(\boldsymbol{\omega}) = \prod_{i=1}^{d} (1 + \mathrm{sign}(\omega_i))\, \hat{f}(\omega_1, \dots, \omega_d). \tag{19}$$

In practical situations, Bedrosian theorem [39] facilitates the computation of the Hilbert transform of product of functions with non-overlapping spectra. In [23] we extend this result to multidimensional analytic signals.



(a) Oscillating process (b) Envelope

Fig. 1: Point cloud of cube with a deleted octant is shown. (a) Illustration of oscillating process $f(x, y, z)$ that takes place inside the cube; (b) Envelope function of the oscillating process.

3. NUMERICAL EXAMPLES

3.1. Simple example in 3D

As a direct illustration of how the construction of the envelope works, we provide an example in 3D. The oscillating function that we analyse is given by the cosine wave modulated by a gaussian window:

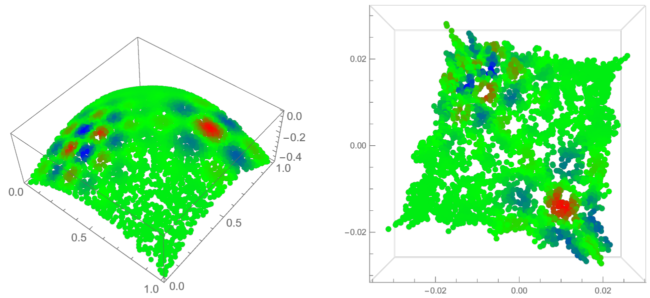
$$\begin{aligned} f(x, y, z) &= e^{-10x^2 - 20y^2 - 20z^2} \cos(50x) \cos(40y) \cos(60z) \\ &= f_x(x) f_y(y) f_z(z). \end{aligned} \tag{20}$$

This particular toy example allows us to separate the variables when calculating the phase-shifted components $f_{\boldsymbol{j}}$ from (9). In three dimensions we will have in total $|\{0, 1\}^3| = 8$ shifted functions.

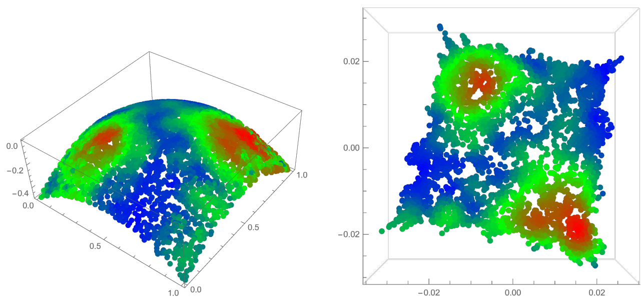
Calculation of some $f_{\boldsymbol{j}}$ using (2) reduces to the calculation of forward and inverse cosine and sine transforms. For example for the f_{100} shifted component, we will have

$$\begin{aligned} f_{100}(x, y, z) = &\frac{f_y(y) f_z(z)}{\pi} \cdot \\ &\cdot \int_0^{\infty} \int_{-\infty}^{\infty} f_x(x) \sin(\omega_1(x - x'))\ \mathrm{d}x'\, \mathrm{d}\omega_1. \end{aligned} \tag{21}$$

Taking into account that f is even in each variable, after expanding the sine of difference, we see that we have to calculate only forward cosine and inverse sine transform. This double integral can be calculated into semi-analytical form by expressing it in terms of $\mathrm{erf}(z) = \int_0^z e^{-t^2}\, \mathrm{d}t$. To illustrate the resulting envelope of signal (20), in Fig. 1 we vizualize a point cloud in a cube with a removed octant where each point is colour coded according to the signal value (the origin is shifted by 0.5). This method allows to visualize three dimensional signals over the inner faces of the deleted octant. In Fig. 1(a) the original $f(x, y, z)$ is plotted, while in Fig. 1(b) we plot the envelope function obtained from (3) computed on the 8 phase-shifted functions obtained similarly to (21).

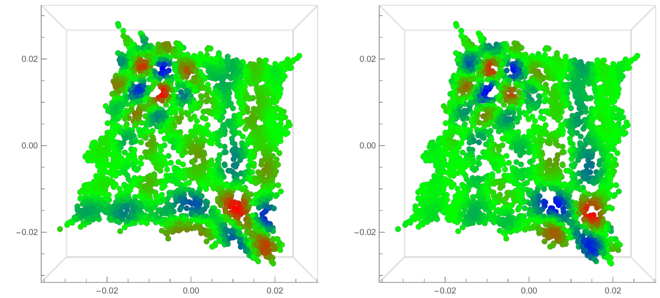


(a) signal over point cloud (b) signal in diffusion space

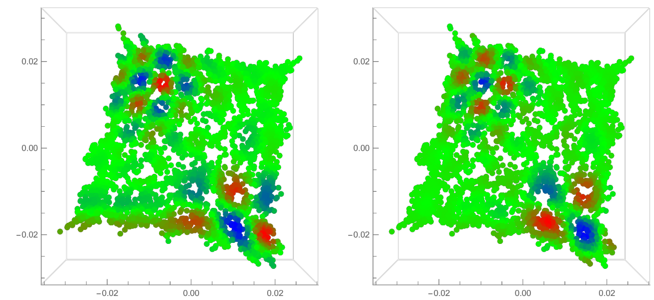


(c) envelope over point cloud (d) envelope in diffusion space

Fig. 2: Point cloud and diffusion embedding of the corresponding knn-graph. Oscillation process is given by the sum of modulated gaussians: (a) original point cloud with signal over it; (b) embedding of the point cloud by the first two eigenvectors of normalized Laplacian; (c) envelope of the original signal; (d) envelope of the original signal in diffusion space.



(a) f_{00} shifted component (b) f_{10} shifted component



(a) f_{01} shifted component (b) f_{11} shifted component

Fig. 3: Phase shifted components in diffusion space.

3.2. Analytic signal on graph

Our objective is to apply the multidimensional analytic signal method to study oscillating signals on graphs. Let us consider a graph $G = (V, E)$ consisting of the set of nodes V and the set of edges E. The topology of a graph can be captured by a variety of operators like adjacency operator or Laplacian [40]. In machine learning scenario, data is frequently assumed to be noisy samples taken from some smooth manifold. Locally, the manifold is homeomorphic to some open ball in $\mathbb{R}^d$, so that we can easily apply the theory of multidimensional analytic signal on a patch of such a manifold. A now classical approach to embed a graph (or part of it) into $\mathbb{R}^d$, is to use the diffusion maps proposed by Coifman and Lafon [24].

Let us consider then the mapping $\phi : V \to \mathbb{R}^d$, for instance coming from the diffusion map [24]. Let us suppose also, that we have some oscillating process $f : V \to \mathbb{R}$ over the vertices of a graph. Our goal is to obtain some global information on this process, here instantaneous amplitude and phase. We illustrate the technique on a simple, yet edifying example. We consider a sum of two gaussians modulated by different frequencies that lie on a curved surface. The sampled patch is then given by a set of 4096 randomly picked points on the surface as displayed in Fig. 2(a). This resulting point cloud is inherently 2-dimensional although it is embedded in 3 dimensions. Then, to obtain an associated graph, we consider the $k = 20$ nearest neighbours graph to mesh the point cloud. Using the diffusion map given by the first two eigenvectors of the normalised Laplacian, to embed the patch in a 2-D Euclidean space, the resulting embedding of the corresponding k-nn graph with the same oscillating process is shown in Fig. 2 (b).

To calculate the discrete version of hypercomplex Fourier transform $\hat{f}(\boldsymbol{\omega})$, we applied the transformation (14) to the discrete signal of Fig. 2(b). Then corresponding Scheffers algebra-valued analytic signal $f_h(\boldsymbol{x})$ was constructed, by first restricting the spectrum $\hat{f}(\boldsymbol{\omega})$ to positive frequencies according to (19) and applying inverse formula (15). All computations were performed in the diffusion map domain. The signal domain was rescaled by a factor of 10 for convenience. Discrete version of the Fourier transform (14) was calculated for the first 50 frequencies with step 1. Then formula (19) was applied prior to the inverse discrete transform. In Fig. 2 (c) and (d), the envelope function of Eq. (3) is displayed on the original point cloud and in the diffusion space, respectively. Fig. 3 shows separate phase-shifted components $f_{\boldsymbol{j}}(\boldsymbol{x})$ for the four directions $\boldsymbol{j} = 00,\ 10,\ 01$ and 11 in the diffusion space.

4. CONCLUSION

In this paper we developed basic theory for analytic signal in many dimensions and applied the discretized theory to signals supported on graphs. The presented hypercomplex analytic signal, aside being analytic function of hypercomplex variable, satisfies in many dimensions all the desirable properties that we expect from 1-D analytic signals. Future work will show how the presented theory will be used in the applications from multidimensional data analysis and machine learning for global characterization of locally oscillating processes.

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