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# Some recent results on interval observers design of uncertain systems

**Abstract:** Based on the theory of positive systems, the goal of interval observers is to compute sets of admissible values of the state vector at each instant of time for systems subject to bounded uncertainties (noises, disturbances and parameters). The size of the estimated sets, which should be minimised, are proportional to the model uncertainties. An interval estimation can be seen as a conventional point estimation (the centre of the interval) with an estimation error given by the interval radius. The reliable uncertainties propagation performed in this context can be useful in several fields such as robust control, diagnosis and fault-tolerant control. This paper presents some recent results on interval observers for several dynamical systems classes such as continuous-time and switched systems.

## 1 Introduction

The problem of state vector estimation is very challenging and can be encountered in many applications [1, 2, 3]. For linear time-invariant models there are plenty solutions, among them the most popular are Luenberger observer and Kalman filter for deterministic and stochastic settings, respectively. Among other popular solutions for estimations it is worth to mention high-gain observers [4] or high order sliding mode observers/differentiators [5]. In nonlinear case, observer or controller design is based on transformation of the system into a canonical form (frequently close to a linear canonical representation [1, 2, 3]), since such a transformation may depend on uncertain parameters or may be unknown due to the model complexity, then application of the transformation can be an obstruction in practice. The reader can refer to a survey on application of different observers for control and estimation of nonlinear systems [6]. In this context, the class of Linear Parameter-Varying (LPV) systems became very popular in applications: a wide class of nonlinear systems can be presented in the LPV form (in this case

the system equations are extended). A partial linearity of LPV models allows a rich spectrum of methods developed for linear systems to be applied [7, 8, 9, 10].

In several fields such as networked control systems, electrical devices/circuits and congestion modeling [11] switched systems appear as interesting modeling tools, that can be more accurate than LPV models, to take into account some complex physical behaviours. These systems are one of the most important classes of Hybrid Dynamical Systems (HDS). They consist of a family of continuous or discrete-time subsystems and a switching rule orchestrating among them. At each time only one subsystem is active. Several works treated switched systems such as [12], [11].

Apart of model complexity, another difficulty for an estimator design consists in the model uncertainty (unknown parameters or/and external disturbances). In the presence of uncertainty, design of a conventional estimator, converging to the ideal value of the state, cannot be realized. Application of sliding-mode tools [13] or other disturbance cancellation approaches may resolve this issue in some cases, however, in general, in the presence of uncertainty, the state estimation error is never approaching zero (it can be bounded or asymptotically bounded, and different versions of practical stability are used for analysis). In this case an interval estimation may still remain feasible: an observer can be constructed that, using input-output information, evaluates the set of admissible values (interval) for the state at each instant of time. The interval length has to be minimized by tuning the observer parameters, and it is proportional to the size of the model uncertainty. Despite such a formulation looks like a simplification of the state estimation problem (instead of the state just an interval is estimated), in fact it is an improvement since the interval mean can be used as the state pointwise estimate, while the interval width gives the admissible deviations from that value. Thus, an interval estimator provides a simultaneous accuracy evaluation for bounded uncertainty, which may not have a known statistics [14, 15, 16, 17, 18]. This idea has been initially proposed in [19] and but recently extended to several classes of dynamical systems.

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This paper constitutes an overview of recent results on interval observers which form a subclass of set-membership estimators and whose design is based on the monotone systems theory [16, 18, 20, 21, 22]. Only the case of continuous-times systems is considered, the reader can refer for instance to [23, 24, 25, 26, 27, 28] for the case of discrete-time systems.

In such a way the main restriction for the interval observer design consists in providing cooperativity of the interval estimation error dynamics by a proper design. Such a complexity has been overcome in [29, 21, 30] for Linear Time-Invariant (LTI) systems and extended to Linear Time-Varying (LTV), LPV and particular classes of nonlinear systems. In those studies, it has been shown that under some mild conditions, by applying a similarity transformation, a Hurwitz matrix could be transformed to a Hurwitz and Metzler one. In the following the main ideas of interval observer design are explained for different classes of systems.

## 2 Preliminaries

Real and integer numbers sets are denoted by  $\mathbb{R}$  and  $\mathbb{Z}$  respectively,  $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$  and  $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ . Euclidean norm for a vector  $x \in \mathbb{R}^n$  will be denoted as  $|x|$ , and for a measurable and locally essentially bounded input  $u : \mathbb{R}_+ \rightarrow \mathbb{R}$  ( $\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$ ) the symbol  $\|u\|_{[t_0, t_1]}$  denotes its  $L_\infty$  norm  $\|u\|_{[t_0, t_1]} = \text{ess sup}\{|u(t)|, t \in [t_0, t_1]\}$ , if  $t_1 = +\infty$  then we will simply write  $\|u\|$ . We will denote as  $\mathcal{L}_\infty$  the set of all inputs  $u$  with the property  $\|u\| < \infty$ . Denote the sequence of integers  $1, \dots, k$  as  $\overline{1, k}$ . The symbols  $I_n$ ,  $E_{n \times m}$  and  $E_p$  denote the identity matrix with dimension  $n \times n$ , the matrix with all elements equal 1 with dimensions  $n \times m$  and  $p \times 1$ , respectively. For a matrix  $A \in \mathbb{R}^{n \times n}$  the vector of its eigenvalues is denoted as  $\lambda(A)$ ,  $\|A\|_{\max} = \max_{i=\overline{1, n}, j=\overline{1, n}} |A_{i,j}|$  (the elementwise maximum norm, it is not sub-multiplicative) and  $\|A\|_2 = \sqrt{\max_{i=\overline{1, n}} \lambda_i(A^T A)}$  (the induced  $L_2$  matrix norm).

### 2.1 Interval relations

For two vectors  $x_1, x_2 \in \mathbb{R}^n$  or matrices  $A_1, A_2 \in \mathbb{R}^{n \times n}$ , the relations  $x_1 \leq x_2$  and  $A_1 \leq A_2$  are understood elementwise. The relation  $P \prec 0$  ( $P \succ 0$ ) means that the matrix  $P \in \mathbb{R}^{n \times n}$  is negative (positive) definite. Given a matrix  $A \in \mathbb{R}^{m \times n}$ , define  $A^+ = \max\{0, A\}$ ,  $A^- = A^+ - A$

(similarly for vectors) and denote the matrix of absolute values of all elements by  $|A| = A^+ + A^-$ .

**Lemma 1.** [22] *Let  $x \in \mathbb{R}^n$  be a vector variable,  $\underline{x} \leq x \leq \overline{x}$  for some  $\underline{x}, \overline{x} \in \mathbb{R}^n$ .*

(1) *If  $A \in \mathbb{R}^{m \times n}$  is a constant matrix, then*

$$A^+ \underline{x} - A^- \overline{x} \leq Ax \leq A^+ \overline{x} - A^- \underline{x}. \quad (1)$$

(2) *If  $A \in \mathbb{R}^{m \times n}$  is a matrix variable and  $\underline{A} \leq A \leq \overline{A}$  for some  $\underline{A}, \overline{A} \in \mathbb{R}^{m \times n}$ , then*

$$\begin{aligned} \underline{A}^+ \underline{x}^+ - \overline{A}^+ \underline{x}^- - \underline{A}^- \overline{x}^+ + \overline{A}^- \overline{x}^- &\leq Ax \\ &\leq \overline{A}^+ \overline{x}^+ - \underline{A}^+ \overline{x}^- - \overline{A}^- \underline{x}^+ + \underline{A}^- \underline{x}^-. \end{aligned} \quad (2)$$

Furthermore, if  $-\overline{A} = \underline{A} \leq 0 \leq \overline{A}$ , then the inequality (2) can be simplified:  $-\overline{A}(\overline{x}^+ + \underline{x}^-) \leq Ax \leq \overline{A}(\overline{x}^+ + \underline{x}^-)$ .

### 2.2 Nonnegative continuous-time linear systems

A matrix  $A \in \mathbb{R}^{n \times n}$  is called Hurwitz if all its eigenvalues have negative real parts, it is called Metzler if all its elements outside the main diagonal are nonnegative. Any solution of the linear system

$$\begin{aligned} \dot{x} &= Ax + B\omega(t), \quad \omega : \mathbb{R}_+ \rightarrow \mathbb{R}_+^q, \quad \omega \in \mathcal{L}_\infty^q, \\ y &= Cx + D\omega(t), \end{aligned} \quad (3)$$

with  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}^p$  and a Metzler matrix  $A \in \mathbb{R}^{n \times n}$ , is elementwise nonnegative for all  $t \geq 0$  provided that  $x(0) \geq 0$  and  $B \in \mathbb{R}_+^{n \times q}$  [31, 32]. The output solution  $y(t)$  is nonnegative if  $C \in \mathbb{R}_+^{p \times n}$  and  $D \in \mathbb{R}_+^{p \times q}$ . Such dynamical systems are called cooperative (monotone) or nonnegative if only initial conditions in  $\mathbb{R}_+^n$  are considered [31, 32]. The stability of a Metzler matrix  $A \in \mathbb{R}^{n \times n}$  can be checked verifying a Linear Programming (LP) problem

$$A^T \lambda < 0$$

for some  $\lambda \in \mathbb{R}_+^n \setminus \{0\}$ , or Lyapunov matrix equation

$$A^T P + PA \prec 0$$

for a diagonal matrix  $P \in \mathbb{R}^{n \times n}$ ,  $P > 0$ . The  $L_1$  and  $L_\infty$  gains for nonnegative systems (3), *i.e.* gains of transfer function from input to output in different norms, have been studied in [33, 34], for this kind of systems these gains are interrelated.

**Lemma 2.** [33, 34] *Let the system (3) be nonnegative (*i.e.*  $A$  is Metzler,  $B \geq 0$ ,  $C \geq 0$  and  $D \geq 0$ ), then it is*

asymptotically stable if and only if there exist  $\lambda \in \mathbb{R}_+^n \setminus \{0\}$  and a scalar  $\gamma > 0$  such that the following LP problem is feasible:

$$\begin{bmatrix} A^T\lambda + C^TE_p \\ B^T\lambda - \gamma E_q + D^TE_p \end{bmatrix} < 0.$$

Moreover, in this case  $L_1$  gain of the operator  $\omega \rightarrow y$  is lower than  $\gamma$ .

**Lemma 3.** [33, 34] *Let the system (3) be cooperative (i.e.  $A$  is Metzler,  $B \geq 0$ ,  $C \geq 0$  and  $D \geq 0$ ), then it is asymptotically stable if and only if there exist  $\lambda \in \mathbb{R}_+^n \setminus \{0\}$  and a scalar  $\gamma > 0$  such that the following LP problem is feasible:*

$$\begin{bmatrix} A\lambda + BE_q \\ C\lambda - \gamma E_p + DE_q \end{bmatrix} < 0.$$

Moreover, in this case  $L_\infty$  gain of the transfer  $\omega \rightarrow y$  is lower than  $\gamma$ .

The conventional results and definitions on  $L_2/L_\infty$  stability for linear systems can be found in [35].

### 2.3 Nonnegative systems with delays

Consider a linear system with constant delays

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^N A_i x(t - \tau_i) + \omega(t), \quad (4)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $x_t \in \mathcal{C}_\tau^n$  for  $\tau = \max_{1 \leq i \leq N} \tau_i$  where  $\tau_i \in \mathbb{R}_+$  are the delays ( $\mathcal{C}_\tau = C([- \tau, 0], \mathbb{R})$  is the set of continuous maps from  $[- \tau, 0]$  into  $\mathbb{R}$ ,  $\mathcal{C}_{\tau+} = \{y \in \mathcal{C}_\tau : y(s) \in \mathbb{R}_+, s \in [- \tau, 0]\}$ ); a piecewise continuous function  $\omega \in \mathcal{L}_\infty^n$  is the input; the constant matrices  $A_i$ ,  $i = \overline{0, N}$  have appropriate dimensions. The system (4) is called cooperative or nonnegative [36] if it admits  $x(t) \in \mathbb{R}_+^n$  for all  $t \geq t_0$  provided that  $x_{t_0} \in \mathcal{C}_{\tau+}^n$  and  $\omega : \mathbb{R} \rightarrow \mathbb{R}_+^n$ .

**Lemma 4.** [37, 36] *The system (4) is nonnegative for all  $\tau \in \mathbb{R}_+$  iff  $A_0$  is Metzler and  $A_i$ ,  $i = \overline{1, N}$  are nonnegative matrices.*

## 3 Interval observers for LTI systems

Let us start the line of design of interval observers with the simplest cases of time-invariant linear models.

Consider the following system

$$\dot{x}(t) = Ax(t) + d(t), \quad y(t) = Cx(t) + v(t), \quad t \in \mathbb{R}_+ \quad (5)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $y(t) \in \mathbb{R}^p$  is the output;  $d(t) \in \mathbb{R}^n$  is the disturbance,  $d \in \mathcal{L}_\infty^n$ ;  $v(t) \in \mathbb{R}^p$  is the measurement noise,  $v \in \mathcal{L}_\infty^p$ ; the matrices  $A$ ,  $C$  have appropriate dimensions. This model has three sources of uncertainty: initial conditions for  $x(0)$ , instant values of  $d$  and  $v$ . It is assumed that all these uncertain factors belong to known intervals.

**Assumption 1.** *Let  $x(0) \in [\underline{x}_0, \bar{x}_0]$  for some known  $\underline{x}_0, \bar{x}_0 \in \mathbb{R}^n$ , let also two functions  $\underline{d}, \bar{d} \in \mathcal{L}_\infty^n$  and a constant  $V > 0$  be given such that*

$$\underline{d}(t) \leq d(t) \leq \bar{d}(t), \quad |v(t)| \leq V \quad \forall t \geq 0.$$

Using the available information, the goal is to calculate two bounds  $\underline{x}, \bar{x} \in \mathcal{L}_\infty^n$  verifying

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t) \quad \forall t \geq 0. \quad (6)$$

An interval observer, composed of two conventional ones, is a solution to this problem:

$$\begin{cases} \dot{\underline{x}}(t) &= A\underline{x}(t) + L[y(t) - C\underline{x}(t)] - |L|E_pV + \underline{d}(t), \\ \dot{\bar{x}}(t) &= A\bar{x}(t) + L[y(t) - C\bar{x}(t)] + |L|E_pV + \bar{d}(t), \\ \underline{x}(0) &= \underline{x}_0, \quad \bar{x}(0) = \bar{x}_0, \end{cases} \quad (7)$$

where  $L \in \mathbb{R}^{n \times p}$  is the observer gain to be designed. The conditions to satisfy for  $L$  are given below.

**Theorem 1.** [19] *Let Assumption 1 hold, the matrix  $A - LC$  be Metzler and  $x \in \mathcal{L}_\infty^n$ , then the solutions of (5) and (7) satisfy the relations (6). In addition,  $\underline{x}, \bar{x} \in \mathcal{L}_\infty^n$  if  $A - LC$  is Hurwitz.*

In the following only the proof of this theorem is given as an illustration.

*Proof.* Define two estimation errors

$$\underline{e}(t) = x(t) - \underline{x}(t), \quad \bar{e}(t) = \bar{x}(t) - x(t),$$

which yield differential equations:

$$\begin{aligned} \dot{\underline{e}}(t) &= [A - LC]\underline{e}(t) + Lv(t) + |L|E_pV + d(t) - \underline{d}(t), \\ \dot{\bar{e}}(t) &= [A - LC]\bar{e}(t) - Lv(t) + |L|E_pV + \bar{d}(t) - d(t). \end{aligned}$$

By Assumption 1, for all  $t \geq 0$

$$|L|E_pV \pm Lv(t) \geq 0, \quad d(t) - \underline{d}(t) \geq 0, \quad \bar{d}(t) - d(t) \geq 0.$$

If  $A - LC$  is a Metzler matrix, since all inputs of  $\underline{e}(t)$ ,  $\bar{e}(t)$  are positive and  $\underline{e}(0) \geq 0$ ,  $\bar{e}(0) \geq 0$ , then  $\underline{e}(t) \geq 0$ ,  $\bar{e}(t) \geq 0$  for all  $t \geq 0$  [31, 32]. The property (6) follows from these relations. If  $A - LC$  is Hurwitz, since all inputs of  $\underline{e}(t)$ ,  $\bar{e}(t)$  are bounded, then  $\underline{e}, \bar{e} \in \mathcal{L}_\infty^n$  and boundedness of  $\underline{x}, \bar{x}$  is implied by boundedness of  $x$ .  $\square$

From this proof we can conclude that the idea to design an interval observer is to guarantee nonnegativity of the estimation error dynamics. The observer gain  $L$  has to be designed such that the matrix  $A - LC$  is Metzler and Hurwitz. In addition, in order to optimize the width of estimated interval  $[\underline{x}(t), \bar{x}(t)]$  the problem of  $L_1$  or  $L_\infty$  optimization of that gain value can be posed. Using lemmas 2 and 3 this problem can be formulated as a LP computational procedure. The reader can also refer to [38] for another LP problem formulation and solution for a more generic LPV system, and to [39] for a  $H_\infty$  optimization procedure. In the case of a  $L_1$  optimization, it is necessary to find  $\lambda \in \mathbb{R}^n$ ,  $w \in \mathbb{R}^p$  and a diagonal matrix  $M \in \mathbb{R}^{n \times n}$  such that

$$\begin{aligned} \begin{bmatrix} A^T \lambda - C^T w + E_n \\ \lambda - \gamma E_n \end{bmatrix} &< 0, \\ \lambda &> 0, \quad M \geq 0, \\ A^T \lambda - C^T w + M \lambda &\geq 0, \end{aligned} \quad (8)$$

then  $w = L^T \lambda$  [40].

Formulation (8) provides an effective computational tool to design interval observers, but it is only sufficient condition and in some cases this LP problem may have no solution, but it does not imply that it is not possible to design an interval observer. Roughly speaking in this case it is not possible to find  $L$  such that  $A - LC$  is simultaneously Metzler and Hurwitz. However, it is well known that Hurwitz property of a matrix is preserved under a similarity transformation of coordinates, then to overcome the issue it is possible to design the gain  $L$  such that the matrix  $A - LC$  is Hurwitz and next to find a nonsingular matrix  $S \in \mathbb{R}^{n \times n}$  such that in the new coordinates  $z = Sx$  the state matrix  $D = S(A - LC)S^{-1}$  is Metzler (it is Hurwitz by construction). The conditions of existence of such a real transformation matrix  $S$  are given in the following lemma.

**Lemma 5.** [21] *Given the matrices  $A \in \mathbb{R}^{n \times n}$ ,  $D \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{p \times n}$ . If there is a matrix  $L \in \mathbb{R}^{n \times p}$  such that the matrices  $A - LC$  and  $D$  have the same eigenvalues, then there is a matrix  $S \in \mathbb{R}^{n \times n}$  such that  $D = S(A - LC)S^{-1}$  provided that the pairs  $(A - LC, \chi_1)$  and  $(D, \chi_2)$  are observable for some  $\chi_1 \in \mathbb{R}^{1 \times n}$ ,  $\chi_2 \in \mathbb{R}^{1 \times n}$ .*

Note that (5) can be rewritten as follows:

$$\dot{x}(t) = (A - LC)x(t) + Ly(t) - Lv(t) + d(t).$$

Under conditions of this lemma and in the new coordinates  $z = Sx$  the system (5) takes the form:

$$\dot{z}(t) = Dz(t) + SLy(t) + \delta(t), \quad \delta(t) = S[d(t) - Lv(t)]. \quad (9)$$

And using Lemma 1 we obtain that  $\delta(t) \leq \delta(t) \leq \bar{\delta}(t)$ , where  $\underline{\delta}(t) = S^+ \underline{d}(t) - S^- \bar{d}(t) - |SL|E_p V$  and  $\bar{\delta}(t) = S^+ \bar{d}(t) - S^- \underline{d}(t) + |SL|E_p V$ . For the system (9) all conditions of Theorem 1 are satisfied and an interval observer similar to (7) can be designed:

$$\begin{aligned} \dot{z}(t) &= Dz(t) + SLy(t) + \delta(t), \\ \dot{\bar{z}}(t) &= D\bar{z}(t) + SLy(t) + \bar{\delta}(t), \\ z(0) &= S^+ \underline{x}_0 - S^- \bar{x}_0, \quad \bar{z}(0) = S^+ \bar{x}_0 - S^- \underline{x}_0, \\ \underline{x}(t) &= (S^{-1})^+ z(t) - (S^{-1})^- \bar{z}(t), \\ \bar{x}(t) &= (S^{-1})^+ \bar{z}(t) - (S^{-1})^- z(t), \end{aligned} \quad (10)$$

where the relations (1) are used to calculate the initial conditions for  $z, \bar{z}$  and the estimates  $\underline{x}, \bar{x}$ . It is easy to show that in (10) the inclusion (6) is satisfied and  $\underline{x}, \bar{x} \in \mathcal{L}_\infty^n$ .

If conditions of Lemma 5 are not satisfied, then it is possible also to find a time-varying change of coordinates [29, 30], for instance, based on the Jordan canonical form, as in the next lemma.

**Lemma 6.** [29] *Let  $A - LC$  be Hurwitz, then there exists an invertible matrix function  $P : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ , of class  $C^\infty$  elementwise,  $\|P(t)\|_2 < +\infty$  for all  $t \in \mathbb{R}$ , such that for all  $t \in \mathbb{R}$*

$$\dot{P}(t) = DP(t) - P(t)(A - LC),$$

where  $D \in \mathbb{R}^{n \times n}$  is a Hurwitz and Metzler matrix.

## 4 Interval observers for time-delay systems

Consider the system (4) with an output  $y \in \mathbb{R}^p$  subject to a bounded noise  $v \in \mathcal{L}_\infty^p$ :

$$y = Cx, \quad \psi = y + v(t), \quad (11)$$

where  $C \in \mathbb{R}^{p \times n}$ . Below the relation  $a \leq b$  for  $a, b \in \mathcal{C}_\tau^n$  is understood in the sense that  $a(\theta) \leq b(\theta)$  for all  $\theta \in [-\tau, 0]$ .

**Assumption 2.** *Let  $x \in \mathcal{L}_\infty^n$  with  $\underline{x}_0 \leq x_{t_0} \leq \bar{x}_0$  for some  $\underline{x}_0, \bar{x}_0 \in \mathcal{C}_\tau^n$ ;  $\|v\| \leq V$  for a given  $V > 0$ ; and  $\underline{\omega}(t) \leq \omega(t) \leq \bar{\omega}(t)$  for all  $t \geq t_0$  for some known  $\underline{\omega}, \bar{\omega} \in \mathcal{L}_\infty^n$ .*

In this assumption it is supposed that the state of the system (4) is bounded with an unknown upper bound, but with a specified admissible set for initial conditions  $[\underline{x}_0, \bar{x}_0]$ . Uncertainty of the system is collected in the external input  $\omega$  with known bounds on the incertitude  $\underline{\omega}, \bar{\omega}$ , the

measurement noise  $v$  and the interval of initial conditions  $[\underline{x}_0, \bar{x}_0]$ .

As we see above, interval observers have an enlarged dimension (the examples given above have  $2n$  variables to estimate  $n$  states). Thus, design of reduced order interval observers is of a big importance for applications. A reduced order interval observer for time-delay system (4), (11) has been proposed in [41, 42], those ideas are explained below.

For the system (4), (11) there exists a nonsingular matrix  $S \in \mathbb{R}^{n \times n}$  such that  $x = S[y^T z^T]^T$  for an auxiliary variable  $z \in \mathbb{R}^{n-p}$  (define  $S^{-1} = [C^T Z^T]^T$  for a matrix  $Z \in \mathbb{R}^{(n-p) \times n}$ ), then

$$\dot{y}(t) = R_1 y(t) + R_2 z(t) + \sum_{i=1}^N [D_{1i} y(t - \tau_i) + D_{2i} z(t - \tau_i)] + C \omega(t), \quad (12)$$

$$\dot{z}(t) = R_3 y(t) + R_4 z(t) + \sum_{i=1}^N [D_{3i} y(t - \tau_i) + D_{4i} z(t - \tau_i)] + Z \omega(t),$$

for some matrices  $R_k, D_{ki}, k = \overline{1, 4}, i = \overline{1, N}$  of appropriate dimensions. Introducing a new variable  $w = z - Ky = Ux$  for a matrix  $K \in \mathbb{R}^{(n-p) \times p}$  with  $U = Z - KC$ , from (12) the following equation is obtained

$$\begin{aligned} \dot{w}(t) &= G_0 \psi(t) + M_0 w(t) + \sum_{i=1}^N [G_i \psi(t - \tau_i) + M_i w(t - \tau_i)] \\ &+ \beta(t), \quad \beta(t) = U \omega(t) - G_0 v(t) - \sum_{i=1}^N G_i v(t - \tau_i), \end{aligned} \quad (13)$$

where  $\psi(t)$  is defined in (11),  $G_0 = R_3 - KR_1 + (R_4 - KR_2)K$ ,  $M_0 = R_4 - KR_2$ , and  $G_i = D_{3i} - KD_{1i} + \{D_{4i} - KD_{2i}\}K$ ,  $M_i = D_{4i} - KD_{2i}$  for  $i = \overline{1, N}$ . Under Assumption 2 and using the relations (1) the following inequalities follow:

$$\begin{aligned} \underline{\beta}(t) &\leq \beta(t) \leq \bar{\beta}(t), \\ \underline{\beta}(t) &= U^+ \underline{\omega}(t) - U^- \bar{\omega}(t) - \sum_{i=0}^N |G_i| E_p V, \\ \bar{\beta}(t) &= U^+ \bar{\omega}(t) - U^- \underline{\omega}(t) + \sum_{i=0}^N |G_i| E_p V. \end{aligned}$$

Then the next interval reduced-order observer can be proposed for (4):

$$\dot{\underline{w}}(t) = G_0 \psi(t) + M_0 \underline{w}(t) + \sum_{i=1}^N [G_i \psi(t - \tau_i) + M_i \underline{w}(t - \tau_i)] + \underline{\beta}(t), \quad (14)$$

$$\dot{\bar{w}}(t) = G_0 \psi(t) + M_0 \bar{w}(t) + \sum_{i=1}^N [G_i \psi(t - \tau_i) + M_i \bar{w}(t - \tau_i)] + \bar{\beta}(t),$$

$$\underline{w}_0 = U^+ \underline{x}_0 - U^- \bar{x}_0, \quad \bar{w}_0 = U^+ \bar{x}_0 - U^- \underline{x}_0.$$

The applicability conditions for (14) are given below.

**Theorem 2.** [41] *Let Assumption 2 be satisfied and the matrices  $M_0, M_i, i = \overline{1, N}$  form an asymptotically stable cooperative system. Then  $\underline{x}, \bar{x} \in \mathcal{L}_\infty^n$  and*

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t) \quad \forall t \geq 0,$$

where

$$\begin{aligned} \underline{x}(t) &= S^+ [y(t)^T z(t)^T]^T - S^- [\bar{y}(t)^T \bar{z}(t)^T]^T, \\ \bar{x}(t) &= S^+ [\bar{y}(t)^T \bar{z}(t)^T]^T - S^- [y(t)^T z(t)^T]^T, \end{aligned} \quad (15)$$

$$y(t) = \psi(t) - V, \quad \bar{y}(t) = \psi(t) + V, \quad (16)$$

$$z(t) = \underline{w}(t) + K^+ \underline{y} - K^- \bar{y}, \quad \bar{z}(t) = \bar{w}(t) + K^+ \bar{y} - K^- \underline{y}.$$

The main condition of Theorem 2 is rather straightforward: the matrices  $M_0, M_i, i = \overline{1, N}$  have to form a stable cooperative system. It is a standard LMI problem to find a matrix  $K$  such that the system composed by  $M_0, M_i, i = \overline{1, N}$  is stable, but to find a matrix  $K$  making the system stable and cooperative simultaneously could be more complicated. However, the advantage of Theorem 2 is that its main condition can be reformulated using LMIs following the idea of [43].

**Proposition 1.** [41] *Let there exist  $\varsigma \in \mathbb{R}_+, p \in \mathbb{R}_+^{n-p}, q \in \mathbb{R}_+^{n-p}$  and  $B \in \mathbb{R}^{(n-p) \times p}$  such that the following LMIs are satisfied:*

$$p^T \Pi_0 - E_{n-p}^T B \Pi_1 + q^T \leq 0, \quad p > 0, \quad q > 0;$$

$$\text{diag}[p] R_4 - B R_2 + \varsigma I_{n-p} \geq 0, \quad \varsigma > 0;$$

$$\text{diag}[p] D_{4i} - B D_{2i} \geq 0, \quad i = \overline{1, N};$$

$$\Pi_0 = R_4 + \sum_{i=1}^N D_{4i}, \quad \Pi_1 = R_2 + \sum_{i=1}^N D_{2i},$$

then  $K = \text{diag}[p]^{-1} B$  and the matrices  $M_0 = R_4 - K R_2, M_i = D_{4i} - K D_{2i}, i = \overline{1, N}$  represent a stable cooperative system in (14).

If these LMIs are not satisfied, the assumption that the matrix  $M_0$  is Metzler and the matrices  $M_i$ ,  $i = \overline{1, N}$  are nonnegative can be relaxed using Lemma 5[41]. This approach can also be applied to the system (4) with time-varying and uncertain delays [41]. The cases of delayed measurements are studied in [44, 42]. An extension to descriptor delay systems is presented in [45].

## 5 The case of LPV models

Consider a LPV system:

$$\dot{x} = [A_0 + \Delta A(\rho(t))]x + d(t), \quad y = Cx + v(t), \quad (17)$$

where  $x \in \mathbb{R}^n$  is the state,  $y \in \mathbb{R}^p$  is the output available for measurements,  $\rho(t) \in \Pi \subset \mathbb{R}^r$  is the vector of scheduling parameters,  $\rho \in \mathcal{L}_{\infty}^r$ . The values of the scheduling vector  $\rho$  are not available for measurements, and only the set of admissible values  $\Pi$  is known. The matrices  $A_0 \in \mathbb{R}^{n \times n}$  and  $C \in \mathbb{R}^{p \times n}$  are known, the matrix function  $\Delta A : \Pi \rightarrow \mathbb{R}^{n \times n}$  is piecewise continuous and also known for a given value of  $\rho$ . The signals  $d : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  and  $v : \mathbb{R}_+ \rightarrow \mathbb{R}^p$  are the external input and measurement noise respectively.

**Assumption 3.**  $x \in \mathcal{L}_{\infty}^n$  and  $x(0) \in [\underline{x}_0, \overline{x}_0]$  for some known  $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$ ;  $\underline{d}(t) \leq d(t) \leq \overline{d}(t)$  and  $|v(t)| \leq V$  for all  $t \geq 0$  and some known  $\underline{d}, \overline{d} \in \mathcal{L}_{\infty}^n$  and  $V > 0$ ;  $\underline{\Delta A} \leq \Delta A(\rho) \leq \overline{\Delta A}$  for all  $\rho \in \Pi$  and some known  $\underline{\Delta A}, \overline{\Delta A} \in \mathbb{R}^{n \times n}$ .

In the system (17), the uncertainties are due to the initial state, the noises, the disturbances and also to the fact that the matrix  $\Delta A(\rho)$  belongs into the interval  $[\underline{\Delta A}, \overline{\Delta A}]$  for all  $t \geq 0$ . The interval  $[\underline{\Delta A}, \overline{\Delta A}]$  is easy to compute for a given set  $\Pi$  and known  $\Delta A(\rho)$ .

The objective is to design an interval observer for the system (17). Before introduction of interval observer equations note that for a matrix  $L \in \mathbb{R}^{n \times p}$  the system (17) can be rewritten as follows:

$$\dot{x} = [A_0 - LC]x + \Delta A(\rho(t))x + L[y - v(t)] + d(t),$$

and according to Lemma 1 and Assumption 3 we have for all  $\rho \in \Pi$ :

$$\underline{\Delta A}^+ \underline{x}^+ - \overline{\Delta A}^+ \underline{x}^- - \underline{\Delta A}^- \overline{x}^+ + \overline{\Delta A}^- \overline{x}^- \leq \Delta A(\rho)x \quad (18)$$

$$\leq \overline{\Delta A}^+ \overline{x}^+ - \underline{\Delta A}^+ \overline{x}^- - \overline{\Delta A}^- \underline{x}^+ + \underline{\Delta A}^- \underline{x}^-$$

provided that  $\underline{x} \leq x \leq \overline{x}$  for some  $\underline{x}, \overline{x} \in \mathbb{R}^n$ .

The following interval observer structure is proposed [46, 40] for the LPV system (17):

$$\begin{aligned} \dot{\underline{x}} &= [A_0 - \underline{L}C]\underline{x} + [\underline{\Delta A}^+ \underline{x}^+ - \overline{\Delta A}^+ \underline{x}^- \\ &\quad - \underline{\Delta A}^- \overline{x}^+ + \overline{\Delta A}^- \overline{x}^-] + \underline{L}y - |\underline{L}|VE_p + \underline{d}(t), \\ \dot{\overline{x}} &= [A_0 - \overline{L}C]\overline{x} + [\overline{\Delta A}^+ \overline{x}^+ - \underline{\Delta A}^+ \overline{x}^- \\ &\quad - \overline{\Delta A}^- \underline{x}^+ + \underline{\Delta A}^- \underline{x}^-] + \overline{L}y + |\overline{L}|VE_p + \overline{d}(t), \\ \underline{x}(0) &= \underline{x}_0, \quad \overline{x}(0) = \overline{x}_0. \end{aligned} \quad (19)$$

Note that due to the presence of  $\underline{x}^+, \underline{x}^-, \overline{x}^+$  and  $\overline{x}^-$ , the interval observer (19) is a globally Lipschitz *nonlinear* system. In addition, in (19) the dynamics of  $\underline{x}$  and  $\overline{x}$  are coupled.

**Theorem 3.** [40] *Let assumption 3 be satisfied and the matrices  $A_0 - \underline{L}C$ ,  $A_0 - \overline{L}C$  be Metzler. Then the relations  $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$  hold for all  $t \geq t_0$ . Furthermore, if there exist  $P \in \mathbb{R}^{2n \times 2n}$ ,  $P = P^T \succ 0$  and  $\gamma > 0$  such that the following Riccati matrix inequality is verified*

$$G^T P + PG + 2\gamma^{-2}P^2 + \gamma^2\eta^2 I_{2n} + Z^T Z \prec 0,$$

where  $\eta = 2n\|\overline{\Delta A} - \underline{\Delta A}\|_{max}$ ,  $Z \in \mathbb{R}^{s \times 2n}$ ,  $0 < s \leq 2n$  and

$$G = \begin{bmatrix} A_0 - \underline{L}C + \underline{\Delta A}^+ & -\underline{\Delta A}^- \\ -\underline{\Delta A}^- & A_0 - \overline{L}C + \overline{\Delta A}^+ \end{bmatrix},$$

then  $\underline{x}, \overline{x} \in \mathcal{L}_{\infty}^n$ . In addition, the operator  $\begin{bmatrix} \underline{b} \\ \overline{b} \end{bmatrix} \rightarrow Z \begin{bmatrix} \underline{x} \\ \overline{x} \end{bmatrix}$  in (19) has an  $L_2$  gain less than  $\gamma$ .

Note that if there are no gains such that  $A_0 - \underline{L}C$  and  $A_0 - \overline{L}C$  are Metzler, then a transformation of coordinates  $T$  can be used for (19) in order to relax such requirement. The Riccati matrix inequality from Theorem 3 can be reformulated in terms of LMIs with respect to  $\underline{L}, \overline{L}$  and  $P$  [40].

**Example 1.** Consider a nonlinear system [40]:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \epsilon \cos t & 1 + \epsilon \sin x_3 & \epsilon \sin x_2 \\ \epsilon \sin x_3 & -0.5 + \epsilon \sin t & 1 + \epsilon \cos 2t \\ \epsilon \sin x_2 & 0.3 + \epsilon \cos 2t & -1 + \epsilon \sin t \end{bmatrix} x \\ &\quad + \begin{bmatrix} 6 \cos x_1 \\ \sin t + 0.1 \sin x_3 \\ -\cos 3t + 0.1 \sin 2x_2 \end{bmatrix}, \quad y = x_1 + v(t), \end{aligned}$$

where  $\epsilon = 0.01$  and  $\varepsilon = 0.001$ . We assume that  $V = 0.1$ , and for simulation we selected  $v(t) = V(\sin 5t + \cos 3t)/2$ . For initial conditions  $|x_i(0)| \leq 5$  the system has bounded

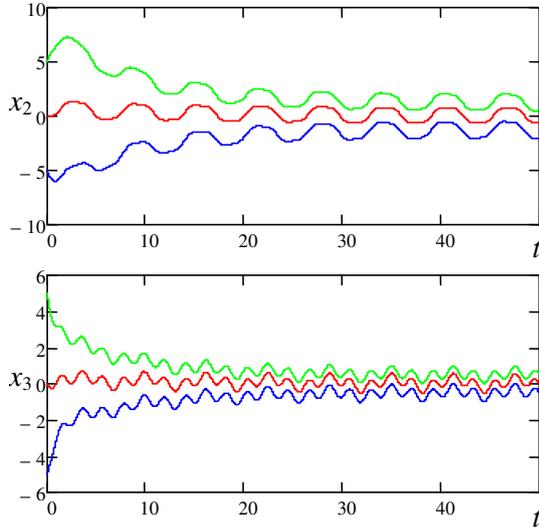


Fig. 1: The results of simulations for an LPV system [40]

solutions. This system can be presented in the form of (17) for

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.5 & 1 \\ 0 & 0.3 & -1 \end{bmatrix}, \quad \overline{\Delta A} = \begin{bmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \end{bmatrix} = -\underline{\Delta A},$$

$$\underline{b}(t, y) = \begin{bmatrix} 6\underline{f}(y) \\ \sin t - 0.1 \\ -\cos 3t - 0.1 \end{bmatrix}, \quad \overline{b}(t, y) = \begin{bmatrix} 6\overline{f}(y) \\ \sin t + 0.1 \\ -\cos 3t + 0.1 \end{bmatrix},$$

$$\underline{f}(y) = \begin{cases} \cos y \cos V & \text{if } \cos y \geq 0 \\ \cos y & \text{if } \cos y < 0 \end{cases} - |\sin y| \sin V,$$

$$\overline{f}(y) = \begin{cases} \cos y & \text{if } \cos y \geq 0 \\ \cos y \cos V & \text{if } \cos y < 0 \end{cases} + |\sin y| \sin V$$

and a properly selected  $\rho$ , clearly assumption 3 is satisfied. Solution of LMIs formulated for Theorem 3 gives the  $L_2$  optimal solution [40]:

$$\underline{L} = \begin{bmatrix} 82.923 \\ -3e - 4 \\ -4e - 4 \end{bmatrix}, \quad \overline{L} = \begin{bmatrix} 97.16 \\ -2e - 5 \\ -1e - 5 \end{bmatrix}$$

for  $\gamma = 31.4$ . The results of interval simulations for the variables  $x_2$  and  $x_3$  are given in Fig. 1 (the variable  $x_1$  is omitted since it is available for measurements).

## 6 Case of LTV systems

LTV models can be considered as a special case of LPV ones when the vector of scheduling parameters is available for measurements. Interval observers have been proposed

for LTV systems in [47, 48], where different variants of transformations of time-varying systems to nonnegative forms are presented. Consider a LTV system described by:

$$\begin{cases} \dot{x}(t) = A(t)x(t) + d(t), \\ y(t) = C(t)x(t) + v(t), \\ x(t_0) \in \mathbb{R}^n, t \geq t_0 \geq 0, \end{cases} \quad (20)$$

where  $x(t) \in \mathbb{R}^n$ ,  $d(t) \in \mathbb{R}^n$ ,  $y(t) \in \mathbb{R}^p$  and  $v(t) \in \mathbb{R}^p$  are respectively the state vector, an unknown but bounded input, the output vector and a bounded noise. Let  $x(t_0) \in [\underline{x}_0, \overline{x}_0]$  for some known  $\underline{x}_0, \overline{x}_0 \in \mathbb{R}^n$ ;  $d(t) \in [\underline{d}(t), \overline{d}(t)]$  for all  $t \geq t_0$ , where  $\underline{d}, \overline{d} \in \mathcal{L}_\infty^n$ ;  $v(t) \in [\underline{v}(t), \overline{v}(t)]$  for all  $t \geq t_0$ , where  $\underline{v}, \overline{v} \in \mathcal{L}_\infty^p$ .

In [48], an effective technique has been proposed to build an interval observer for systems described by (20).

**Assumption 4.** *There exist bounded matrix functions  $L : \mathbb{R} \rightarrow \mathbb{R}^{n \times p}$ ,  $M : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ ,  $M(\cdot) = M(\cdot)^T \succ 0$  such that for all  $t \geq t_0$ ,*

$$\dot{M}(t) + D(t)^T M(t) + M(t)D(t) \prec 0, \quad D(t) = A(t) - L(t)C(t).$$

Assumption 4 is a conventional requirement for LTV systems [49]. Under this assumption, the observer gain  $L(t)$  and the matrix function  $M(t)$  are such that the stability of the LTV system  $\dot{x}(t) = D(t)x(t)$  can be proven by taking  $V(t) = x(t)^T M(t)x(t)$  as a Lyapunov function. It determines the output stabilization conditions of the system (20) which can be rewritten as:

$$\begin{cases} \dot{x}(t) = D(t)x(t) + \tilde{d}(t), \\ y(t) = C(t)x(t) + v(t) \end{cases} \quad (21)$$

with  $\tilde{d}(t) = d(t) - L(t)v(t) + L(t)y(t)$ . When the gain  $L(t)$  is computed such that the closed loop matrix  $D(t)$  is stable and Metzler, the observer design is similar to the ones for LTI and LPV systems. In [48], such a restrictive condition has been avoided. The methodology is based on the D-similarities approach developed in [50, 51, 52], where any time-varying matrix can be transformed under a Metzler form.

**Proposition 2.** *There exists a time-varying transformation  $z = T(t)x$  transforming  $D(t)$  into a Metzler matrix  $\Gamma(t)$ :*

$$\Gamma(t) = T(t) (D(t)T^{-1}(t) - d(T^{-1}(t))/dt) \quad (22)$$

where  $T(t) = P(t)L^{-1}(t)$  with

$$\Gamma(t) = \begin{bmatrix} \lambda_1(t) & 1 & \cdots & 0 \\ 0 & \lambda_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_n(t) \end{bmatrix}. \quad (23)$$

The elements  $\lambda_i(t)$  are called Essential D-eigenvalues of  $D(t)$  (or ED-eigenvalues, see Definition 3.1 in [50]).

**Theorem 4.** [48] *Let assumption 4 be satisfied for (20) and  $\|y(t)\| \leq Y$  for all  $t \geq t_0$  for a known constant  $Y > 0$ . Given the matrix  $T$  defined in Proposition 2, assume that  $\exists M_1 \in \mathbb{R}_+$  such that  $\|T(t)\| + \|T^{-1}(t)\| \leq M_1$  for all  $t \geq t_0$ . Then,*

$$\begin{aligned}\dot{\underline{z}}(t) &= \Gamma(t)\underline{z}(t) + \underline{d}_{obs}(t) + \underline{\Psi}_{obs}(t) + T_{obs}(t)y(t) \\ \dot{\bar{z}}(t) &= \Gamma(t)\bar{z}(t) + \bar{d}_{obs}(t) + \bar{\Psi}_{obs}(t) + T_{obs}(t)y(t)\end{aligned}$$

is an interval observer for (21) and

$$\underline{z}(t) \leq T(t)x(t) \leq \bar{z}(t), \quad \forall t \geq t_0,$$

where  $\underline{d}_{obs}(t) = T^+(t)\underline{d}(t) - T^-(t)\bar{d}(t)$ ,  $T_{obs}(t) = T(t)L(t)$ ,  $\underline{\Psi}_{obs}(t) = T_{obs}^-(t)\underline{v}(t) - T_{obs}^+(t)\bar{v}(t)$ ,  $\bar{d}_{obs}(t) = T^+(t)\bar{d}(t) - T^-(t)\underline{d}(t)$ ,  $\bar{\Psi}_{obs}(t) = T_{obs}^-(t)\bar{v}(t) - T_{obs}^+(t)\underline{v}(t)$ .

## 7 Case of switched systems

Consider a continuous-time switched system described by:

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + w(t) \\ y_m(t) = C_{\sigma(t)}x(t) + v(t) \end{cases}, \quad \sigma(t) \in \mathcal{I} \quad (24)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y_m \in \mathbb{R}^p$ ,  $w \in \mathbb{R}^n$ ,  $v \in \mathbb{R}^p$  are the state vector, the input, the output, the disturbance and the measurement noise respectively.  $q = \sigma(t)$  is the index of the active subsystem and assumed to be known.  $A_q$ ,  $B_q$  and  $C_q$  are constant matrices of appropriate dimensions.

The aim is to derive two signals  $\underline{x}(t)$  and  $\bar{x}(t)$  such that

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t), \quad \forall t \geq 0$$

holds despite the disturbances and uncertainties provided that

$$\underline{x}_0 \leq x_0 \leq \bar{x}_0$$

is satisfied.

As in the previous sections, assume that the measurement noise and the state disturbances are assumed to be unknown but bounded with *a priori* known bounds such that  $-\bar{w}(t) \leq w(t) \leq \bar{w}(t)$ , and  $|v(t)| \leq \bar{V}E_p, \forall t \geq 0$ , where  $\bar{w} : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ ,  $\bar{w}(t) \in \mathcal{L}_\infty^n$  and  $\bar{V}$  is a positive scalar.

As in [53, 54], assume that there exist gains  $L_q$  such that the matrices  $(A_q - L_qC_q)$  are Metzler for all  $q \in \mathcal{I}$ , where the matrices  $L_q$  ( $q \in \mathcal{I}$ ) denote the observer gains associated with each subsystem  $q$ .

A candidate interval observer structure to compute  $\bar{x}$  and  $\underline{x}$  is described by:

$$\begin{cases} \dot{\bar{x}} = (A_q - L_qC_q)\bar{x} + B_qu + \bar{w} + L_qy_m + |L_q|\bar{V}E_p \\ \dot{\underline{x}} = (A_q - L_qC_q)\underline{x} + B_qu - \bar{w} + L_qy_m - |L_q|\bar{V}E_p \end{cases} \quad (25)$$

The following theorem gives the conditions for achieving the desired design goal.

**Theorem 5.** [55] *Given  $\underline{x}_0, \bar{x}_0 \in \mathbb{R}^n$ . Let Assumption 1 be satisfied and  $x \in \mathcal{L}_\infty^n$ . If there exist  $\lambda, \eta \in \mathbb{R}^n$ ,  $\lambda > 0$ ,  $\eta > 0$ , matrices  $W_q \in \mathbb{R}^{n \times n}$  and positive scalars  $\alpha_q$ ,  $\forall q \in \mathcal{I}$  such that*

$$\begin{cases} A_q^T S + S A_q - C_q^T W_q^T - W_q C_q + \alpha_q S < 0 \\ A_q^T S - C_q^T W_q^T + \text{diag}(\eta) \geq 0 \\ S = \text{diag}(\lambda) \end{cases} \quad (26)$$

then, the solutions of (25) and (24) satisfy  $\underline{x}, \bar{x} \in \mathcal{L}_\infty^n$  and the inclusion

$$\underline{x}(t) \leq x(t) \leq \bar{x}(t)$$

provided that  $\underline{x}_0 \leq x_0 \leq \bar{x}_0$ , where  $L_q = S^{-1}W_q$ .

*Remark 1.* The system (25) is initialized with the initial conditions  $\underline{x}_0$  and  $\bar{x}_0$  for the first active subsystem. At the switching time instant  $t_{i+1}$ , the output of the previous active subsystem ( $q = \sigma(t_i)$ ) is used to initialize (25) with the subsystem ( $q = \sigma(t_{i+1})$ ). ■

In the LMI (26) only the stability of the interval observer (25) is ensured. As in the case of non-switched systems, this LMI can be reformulated to find observer gains  $L_q$  minimizing the total interval estimation error  $\bar{x} - \underline{x}$  as in [56, 27, 39]. The main limitation of the approach detailed in this section is the existence of gains  $L_q$  such that the matrices  $(A_q - L_qC_q)$  are Metzler for all  $q \in \mathcal{I}$ . To overcome this problem, some techniques which consists in finding a change of coordinates that transforms the observation errors into cooperative forms are proposed in [55, 57, 58]. The changes of coordinates proposed for instance in [29, 59] for continuous systems can be used to transform the matrices  $(A_q - L_qC_q)$  into a Metzler form.

## 8 Conclusion

Some recent results on interval observers for continuous-time systems have been presented in this overview. Interval observers can be considered as conventional pointwise observers with an estimate and an uncertainty quantification respectively given by the midpoint and the radius of each interval. This overview can be completed in a

further paper by some results on discrete-time systems [23, 24, 25, 26, 27, 28], joint unknown and input estimation [60, 61] and applications of interval observers in the field of robust control [62, 26, 63, 64, 65], diagnosis [66, 67, 68, 69], prognosis [70] and automotive [71, 72].

In addition, sliding mode control and other estimation techniques are well known for their compensation of matched disturbances and finite-time convergence. Combination of sliding mode differentiators and interval observers is investigated in [22]. More generally, the combination of standard and interval observers can be considered as an appealing research issue for future works. Furthermore, a lot of open problems are still open in the field of interval observers design, namely for time-delay, distributed and hybrid systems. The optimization of the width of interval estimation remains also an interesting issue in the field of linear parameter varying systems.

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