On Implicit Finite-Time and Fixed-Time ISS Lyapunov Functions

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F. Lopez-Ramirez[1], D. Efimov[1,2], A. Polyakov[1,2] and W. Perruquetti[1,2,3]

***Abstract*— This article presents a theoretical framework to study finite-time and fixed-time input-to-state stability of non-linear systems using the implicit Lyapunov function formulation. This approach allows to determine stability, robustness and convergence type of a given system without relying in an explicit Lyapunov function.**

I. INTRODUCTION

In control systems theory, convergence time and robustness against noise and perturbations are arguably the most important features to take into consideration to study control systems behavior and to design control and estimation laws. Qualitatively speaking, convergence rates may be divided in asymptotic and non-asymptotic. The former involving the types of convergence where zero is reached in an infinite amount of time (*i.e.* $x(t) \to 0$ as $t \to \infty$). Non-asymptotic convergence on the contrary, reaches exactly zero in a finite time [1], and the so-called *fixed-time convergence* does it with a uniform upper bound on the settling-time estimate [2], [3], [4], meaning that there exists a maximum convergence time to zero, irrespective of the system's initial condition.

Regarding robustness analysis of non-linear systems, one of the most popular frameworks is the input-to-state stability (ISS) theory, which allows to characterize the conditions that assure that, for any bounded and measurable input, the system's states will remain bounded [5], [6].

Both properties, convergence time and ISS robustness, have been entirely developed using Lyapunov analysis, meaning that there exist tools such as finite-time, fixed-time and ISS Lyapunov functions that assert finite-time, fixed-time and ISS stability for a given system. In [7] the authors present a unified framework for finite-time ISS stability (FTISS) and stabilization to assert both finite-time stability in the absence of disturbances and ISS robustness when inputs (or inputs considered as disturbances) are present.

The implicit Lyapunov function theory provides an implicit yet analytical expression of a Lyapuonv function V that might be expressed, for example, as the solution of an algebraic equation $Q(V,x)=0$, where $Q(V,x)$ is called an implicit Lyapunov function (ILF). This approach not only broadens the spectrum of possible Lyapunov functions but in many cases allows to simplify the calculations involved since neither an explicit expression of the Lyapunov function nor its time derivative are needed [8], [9], [10]. In [9], [10], finite-time and fixed-time controllers and observers for linear systems are derived using the implicit approach. In [11], the implicit framework is used to obtain implicit control Lyapunov functions (ICLF).

The aim of this article is to derive sufficient conditions for an implicit Lyapunov function to be ISS, FTISS and fixed-time ISS (FXISS), joining the properties of finite-time and fixed-time ISS Lyapunov functions with those of the implicit Lyapunov function approach.

The remainder of the article is organized as follows: at the end of this section the notation used throughout the article is presented. Section II contains the required definitions and preliminary theory; before presenting the implicit framework, we complete in Section III-A the results given in [7] for the explicit case. In Section III-B the definitions and theorems that constitute the explicit framework are presented and Section IV includes some academic examples. The article finishes in Section V with some conclusions and final remarks.

A. Notation

- $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$.
- The notation $DV(x)f(x)$ stands for the directional derivative of a continuously differentiable function V with respect to the vector field f evaluated at point x.
- For a continuous function $\rho(x,y,...)$, $\partial_x \rho(x,y,...)$ represents the partial derivative of ρ with respect to x.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$, we use $||d||_{[t_0,t_1)} = \text{ess sup}_{t\in[t_0,t_1)} \|d(t)\|$ to define the norm of $d(t)$ in the interval $[t_0,t_1)$, then $||d||_\infty = ||d||_{[0,+\infty)}$ and the set of essentially bounded and measurable functions $d(t)$ with the property $||d||_\infty < +\infty$ is further denoted as $\mathcal{L}_\infty$; $\mathcal{L}_D = \{d \in \mathcal{L}_\infty : ||d||_\infty \leq D\}$ for any $D > 0$.

II. PRELIMINARIES

Consider the following nonlinear system

$$\dot{x}(t) = f(x(t), d(t)),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the input, $d \in \mathcal{L}_\infty$; $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is continuous and ensures forward existence of the system solutions, at least locally, and

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$f(0,0)=0$. For an initial condition $x_0 \in \mathbb{R}^n$ and an input $d \in \mathcal{L}_\infty$, define the corresponding solution by $X(t,x_0,d)$ for any $t \geq 0$ for which the solution exists. Since (1) might not have unique solutions and in this work we are interested in the strong stability notions only (satisfying for all solutions), then with a slight inexactness in the notation we will assume that if a property is satisfied for all initial conditions in a set, then it implies that it also holds for all solutions issued from those initial conditions.

A. Stability Rates

Following [12], [13], [4], let $\Gamma \subseteq \mathbb{R}^n$ be an open connected set containing the origin.

Definition 1. The origin of the system (1), for $d \in \mathcal{L}_D$ and $D > 0$, is said to be

uniformly Lyapunov stable if for any $x_0 \in \Gamma$ and $d \in \mathcal{L}_D$ the solution $X(t,x_0,d)$ is defined for all $t \geq 0$, and for any $\epsilon > 0$ there is $\delta > 0$ such that for any $x_0 \in \Gamma$, if $\|x_0\| \leq \delta$ then $\|X(t,x_0,d)\| \leq \epsilon$ for all $t \geq 0$;

uniformly asymptotically stable if it is uniformly Lyapunov stable and for any $\kappa > 0$ and $\epsilon > 0$ there exists $T(\kappa,\epsilon) \geq 0$ such that for any $x_0 \in \Gamma$ and $d \in \mathcal{L}_D$, if $\|x_0\| \leq \kappa$ then $\|X(t,x_0,d)\| \leq \epsilon$ for all $t \geq T(\kappa,\epsilon)$;

uniformly finite-time stable if it is uniformly Lyapunov stable and ***uniformly finite-time converging from*** Γ, *i.e.* for any $x_0 \in \Gamma$ and all $d \in \mathcal{L}_D$ there exists some constant $T \in \mathbb{R}_{\geq 0}$ such that $X(t,x_0,d) = 0$ for all $t \geq T$. The function $T(x_0) = \inf\{T \geq 0 : X(t,x_0,d) = 0 \ \forall t \geq T, \ \forall d \in \mathcal{L}_D\}$ is called the ***uniform settling-time function*** of the system (1);

fixed-time stable if it is uniformly finite-time stable and $\sup_{x_0 \in \Gamma} T_0(x_0) < +\infty$.

The set Γ is called the ***domain of attraction*** and throughout the paper will be considered to be $\Gamma = \mathbb{R}^n$.

B. Comparison Functions

A continuous function $\vartheta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a ***class-$\mathcal{K}$ function*** if it is strictly increasing with $\vartheta(0) = 0$; ϑ is a ***class-$\mathcal{K}_\infty$ function*** if it is a class-$\mathcal{K}$ function and $\vartheta(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to the class-$\mathcal{KL}$ if $\beta(\cdot,t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_{\geq 0}$ and $\lim_{t\to+\infty} \beta(s,t) = 0$ for each fixed $s \in \mathbb{R}_{\geq 0}$.

C. Generalized Comparison Functions

Following [7], we define the following generalization of comparison functions.

Definition 2 ([7]). A function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be a ***generalized class-$\mathcal{K}$ function*** ($\mathcal{GK}$ function) if it is continuous and satisfies

$$\begin{cases} \varphi(s_1) > \varphi(s_2), & \text{if } \varphi(s_1) > 0, s_1 > s_2 \\ \varphi(s_1) = \varphi(s_2), & \text{if } \varphi(s_1) = 0, s_1 > s_2. \end{cases}$$

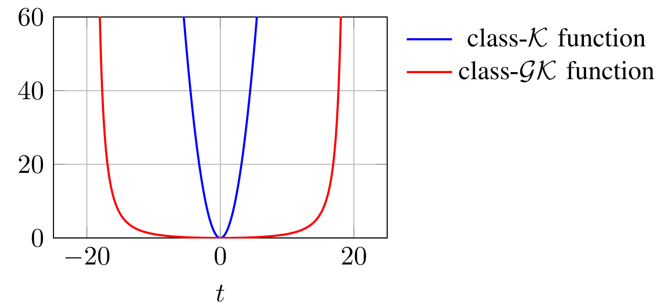

Fig. 1: Example of a class-$\mathcal{K}$ and a class-$\mathcal{GK}$ functions. Remark that a class-$\mathcal{K}$ function is only zero at zero while a class-$\mathcal{GK}$ function can be zero in a vicinity of zero.

Note that a class-$\mathcal{K}$ function is a generalized class-$\mathcal{K}$ function while the converse is not true (see Figure 1). A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a ***generalized class-$\mathcal{KL}$ function*** ($\mathcal{GKL}$ function) if for each fixed $t \geq 0$ the function $\beta(\cdot,t)$ is a class-$\mathcal{GK}$ function, and for each fixed $s \geq 0$ the function $\beta(s,\cdot)$ is continuous, strictly decreasing and there exists some $\tilde{t}(s) \in [0,+\infty)$ such that $\beta(s,t) \to 0$ as $t \to \tilde{t}(s)$.

D. Input-to-state stability

Definition 3 ([6]). The system (1) is called ***input-to-state stable (ISS)*** if for any input $d \in \mathcal{L}_\infty$ and any $x_0 \in \mathbb{R}^n$ there exist some functions $\beta \in \mathcal{KL}$ and $\vartheta \in \mathcal{K}$ such that

$$\|X(t,x_0,d)\| \leq \beta(\|x_0\|,t) + \vartheta(||d||_{[0,t)}) \quad \forall t \geq 0.$$

The function ϑ is called the ***nonlinear gain***.

E. Finite-Time and Fixed-Time Input-to-State Stability

Definition 4 ([7]). System (1) is said to be ***finite-time ISS*** if for all $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty$, each solution $X(t,x_0,d)$ is defined for $t \geq 0$ and satisfies

$$\|X(t,x_0,d)\| \leq \beta(\|x_0\|,t) + \vartheta(\|d\|_\infty),$$

where ϑ is a class-$\mathcal{K}$ function and β is a class-$\mathcal{GKL}$ function with $\beta(r,t) = 0$ when $t \geq \tilde{T}(r)$ with $\tilde{T}(r)$ continuous with respect to r and $\tilde{T}(0) = 0$.

As noted in [7, Remark 3], when $d = 0$ the system (1) becomes finite-time stable with settling-time function $T(x_0) \leq \tilde{T}(x)$. Therefore, the existence of $T(x_0)$ implies that of $\tilde{T}(r)$ in Definition 4 and viceversa.

Definition 5. System (1) is said to be ***fixed-time ISS*** if it is finite-time ISS and $\sup_{r \in \mathbb{R}_{\geq 0}} \tilde{T}(r) < +\infty$.

F. ISS Lyapunov functions

Definition 6 ([5]). A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called ***ISS Lyapunov function*** for system (1) if for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$ there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\chi, \gamma \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{2}$$

$$\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -\gamma(\|x\|). \tag{3}$$

Remark that, without loss of generality, one can assume that $\gamma \in \mathcal{K}_\infty$ [14, Remark 4.1].

As the next lemma states, there is an alternative definition of an ISS Lyapunov function that provides a *dissipativity like* characterization.

Lemma 1 ([5]). *A smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS Lyapunov function for* (1) *if and only if there exist $\alpha_1, \alpha_2, \delta, \gamma \in \mathcal{K}_\infty$ such that* (2) *holds and*

$$DV(x)f(x,d) \leq \delta(\|d\|) - \zeta(\|x\|). \tag{4}$$

The following theorem is the main result in ISS theory, it relates the existence of an ISS Lyapunov function with the ISS property of a given system.

Theorem 1 ([5]). *The system (1) is ISS if and only if it admits an ISS Lyapunov function.*

G. Implicit Lyapunov Functions

The implicit Lyapunov function formulation combines the implicit function theorem with Lyapunov's direct method in order to determine the conditions that a function $Q(V,x)$ has to satisfy in order to implicitly define, through the solution of the equation $Q(V,x) = 0$, a Lyapunov function V.

Theorem 2 ([9]). *If there exists a continuous function*

$$Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R},$$
$$(V,x) \to Q(V,x)$$

satisfying the conditions

C1 *Q is continuously differentiable on $\mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$;*

C2 *for any $x \in \mathbb{R}^n \backslash \{0\}$ there exists $V \in \mathbb{R}_+ : Q(V,x) = 0$;*

C3 *for $\Omega = \{(V,x) \in \mathbb{R}^{n+1} : Q(V,x) = 0\}$ we have*
$$\lim_{\substack{x\to 0\\(V,x)\in\Omega}} V = 0, \quad \lim_{\substack{V\to 0^+\\(V,x)\in\Omega}} \|x\| = 0, \quad \lim_{\substack{\|x\|\to\infty\\(V,x)\in\Omega}} V = +\infty;$$

C4 *$-\infty < \partial_V Q(V,x) < 0$ for $V \in \mathbb{R}_+$ and $x \in \mathbb{R}^n \backslash \{0\}$;*

C5 *$\partial_x Q(V,x) f(x,0) < 0$ for all $(V,x) \in \Omega$,*

then the origin of (1) *with $d = 0$ is globally uniformly asymptotically stable and the function $Q(V,x) = 0$ implicitly defines a Lyapunov function $V(x)$ for* (1).

Conditions **C1** and **C4** are required to satisfy the implicit function theorem. Conditions **C2** and **C3** ensure that $Q(V,x) = 0$ defines implicitly a unique, continuously differentiable, radially unbounded and positive definite function V. The last condition implies that V satisfies the differential inequality of Lyapunov's direct method.

III. Finite-Time and Fixed-Time ISS Lyapunov Functions

A. Explicit Approach

In [7], some sufficient conditions for finite-time ISS with continuous settling-time function are presented. However converse results are not obtained. The following results show that if some assumptions on the Lipschitz continuity of the system and of the settling-time function are added, then a converse result exists.

Assumption 1. Let on any compact set $K \subset (\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^m$ the function $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be locally Lipschitz continuous and, in addition, there exists some continuous $L : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\|f(x,d) - f(x,0)\| \leq L(\|x\|)\|d\|$$

for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$.

Definition 7. Consider a positive definite and radially unbounded C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. V is called a ***finite-time ISS Lyapunov function*** for system (1) if there exist some $\chi \in \mathcal{K}$, $c_1 > 0$ and $a \in (0,1)$ such that for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$

i) $\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -c_1 V(x)^a$.

V is called a ***fixed-time ISS Lyapunov function*** for system (1) if there exist some $\chi \in \mathcal{K}$, $c_1, c_2 > 0$, $a \in (0,1)$ and $b > 1$ such that for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$

ii) $\|x\| \geq \chi(\|d\|) \Rightarrow DV(x)f(x,d) \leq -c_1 V(x)^a - c_2 V(x)^b$.

Theorem 3. *The system* (1) *is FTISS (respectively, FXISS) if it admits a finite-time (respectively, fixed-time) ISS Lyapunov function. Conversely, if* (1) *is FTISS with a Lipschitz continuous settling-time function and Assumption 1 is satisfied, then there exists a finite-time ISS Lyapunov function for it.*

This last theorem along with Theorem 2 will be the stepping stones for the implicit ISS formulation.

B. Explicit Approach

Definition 8. A continuous function $Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is called an ***implicit ISS Lyapunov function*** for system (1) if it satisfies all conditions of Theorem 2 for $d = 0$ and

C5$^{\text{iss}}$ $\|x\| \geq \chi(\|d\|) \Rightarrow \dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq \gamma(\|x\|)$

for all $(V,x) \in \Omega$ and all $d \in \mathbb{R}^m$, with $\chi, \gamma \in \mathcal{K}$.

Theorem 4. *System* (1) *is ISS if and only if there exists an implicit ISS Lyapunov function $Q(V,x)$ for it.*

Using the arguments of Lemma 1, it is possible to show that the following corollary is an equivalent definition of an implicit finite-time ISS Lyapunov function

Corollary 1. *Suppose that there exists a continuous function $Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ that satisfies conditions* **C1**-**C4** *of Theorem 2 for $d = 0$ and*

C5$^{\text{iss}*}$ $\dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq \zeta(\|x\|) - \delta(\|d\|)$

for all $(V,x) \in \Omega$ and all $d \in \mathbb{R}^m$, where $\delta, \zeta \in \mathcal{K}_\infty$. Then (1) *is ISS and $Q(V,x)$ is an implicit ISS Lyapunov function for* (1).

C. Implicit Finite-Time and Fixed-Time ISS Lyapunov Functions

Definition 9. Consider a continuous function $Q : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ that satisfies all conditions of Theorem 2 for $d = 0$. Q is called an ***implicit finite-time ISS Lyapunov function*** for (1) if there exist some $\chi \in \mathcal{K}$, $c_1 > 0$ and $a \in (0,1)$ such that for all $(V,x) \in \Omega$ and all $d \in \mathbb{R}^m$

C5$^{\text{ft}}$ $\|x\| \geq \chi(\|d\|) \Rightarrow \dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq cV^a.$

Q is called an ***implcit fixed-time ISS Lyapunov Function*** for (1) if there exist some $\chi \in \mathcal{K}$, $c_1, c_2 > 0$, $a \in (0,1)$ and $b > 1$ such that for all $(V,x) \in \Omega$ and all $d \in \mathbb{R}^m$

C5$^{\text{fx}}$ $\|x\| \geq \chi(\|d\|) \Rightarrow \dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq c_1 V^a + c_2 V^b.$

Theorem 5. *System* (1) *is FTISS (respectively FXISS) if there exists an implicit finite-time (respectively fixed-time) ISS Lyapunov function for it. Conversely, if* (1) *is FTISS with a Lipschitz continuous settling-time function and Assumption 1 is satisfied, then there exists an implicit finite-time ISS Lyapunov function for it.*

With a mild modification of the poof of Theorem 5, we obtain the next corollary, which presents dissipativity-like conditions.

Corollary 2. *Consider a continuous function* $Q : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *that satisfies conditions* **C1**-**C4** *for* $d = 0$*. If there exist some* $\delta \in \mathcal{K}_\infty$*,* $c_1 > 0$ *and* $a \in (0,1)$ *such that*

C5$^{\text{ft}*}$ $\dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq c_1 V^a - \delta(\|d\|)$

for all $(V,x) \in \Omega$ *and all* $d \in \mathbb{R}^m$*, then* (1) *is FTISS and* $Q(V,x)$ *is an implicit finite-time ISS Lyapunov function for* (1)*. If there exist some* $\delta \in \mathcal{K}_\infty$*,* $c_1, c_2 > 0$*,* $a \in (0,1)$ *and* $b > 1$ *such that*

C5$^{\text{fx}*}$ $\dfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq c_1 V^a + c_2 V^b - \delta(\|d\|)$

for all $(V,x) \in \Omega$ *and all* $d \in \mathbb{R}^m$*, then* (1) *is FXISS and* $Q(V,x)$ *is an implicit fixed-time ISS Lyapunov function for* (1).

IV. Examples

Example 1. Consider the system

$$\dot{x} = -x^3 + x^2 d_1 - x d_2 + d_1 d_2 \tag{5}$$

and the following implicit ISS Lyapunov function candidate:

$$Q(V,x) = \frac{x^2}{2V} - 1. \tag{6}$$

We have that $\frac{\partial Q(V,x)}{\partial V} = -\frac{x^2}{2V^2}$, and that $\frac{\partial Q(V,x)}{\partial x} = \frac{x}{V}$, hence

$$-\frac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) = -\frac{Vx}{2x^2}(-x^3 + x^2 d_1 - x d_2 + d_1 d_2),$$

if $3|d_1| \leq |x|$ and $3|d_2| \leq x^2$ and since $Q = 0 \Rightarrow 1 = \frac{x^2}{2V}$ we have

$$\tfrac{\partial_x Q(V,x)}{\partial_V Q(V,x)} f(x,d) \geq \tfrac{2}{9} x^4.$$

Then, according to Theorem 4 the function $Q(V,x)$ defined by (6) is an implicit ISS Lyapunov function for (5) with $\gamma(|x|) = \frac{2}{9}x^4$ and $\chi(|d|) = \nu^{-1}$, $\nu(r) = \min\{\frac{r}{3}, \frac{r^2}{3}\}$ and we conclude that the origin of (5) is ISS. Note that although in this example it is possible to obtain an explicit expression for V, using the implicit framework this is not necessary.

Example 2. Consider the double integrator

$$\dot{x} = A_0 x + b u(x) + d, \tag{7}$$

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ d \in \mathbb{R}^2,$$

with the following control law

$$u(x) = -k_1 \lceil x_1 \rfloor^{\frac{\nu}{2-\nu}} - k_2 \lceil x_2 \rfloor^{\nu}, \tag{8}$$

where $\lceil \cdot \rfloor^\rho = |\cdot|^\rho \operatorname{sign}(\cdot)$ and $\nu \in (0,1)$.

In [15], instead of using an explicit Lyapunov function, finite-time stability of this system, for $x \in \mathbb{R}^n$ and $d = 0$, is proven by first considering the asymptotic stability of (7)-(8) when $\nu = 1$. In this case, the closed loop system becomes linear and it is possible to propose a quadratic Lyapunov function

$$V(x) = x^T P x, \tag{9}$$

$\partial_x V(x) f(x,0) < 0$ for all $x \in \mathbb{R}^2 \backslash \{0\}$ and for properly selected $k = (k_1, k_2)$. Next the authors show that $\{x : V(x) \leq 1\}$ is strictly positively invariant under (7)-(8), for ν sufficiently close to 1, and using homogeneity arguments (see [15, Theorem 6.1]), global finite-time stability and ISS are obtained. In this example we will show that it is possible to construct an implicit FT ISS Lyapunov function for (7)-(8), provided that ν is sufficiently close to 1.

Let us propose the following implicit Lyapunov function candidate:

$$Q(V,x) = x^T D_r(V^{-1}) P D_r(V^{-1}) x - 1, \tag{10}$$

where $P > 0$ and $D_r(V^{-1}) = \begin{pmatrix} V^{-r_1} & 0 \\ 0 & V^{-r_2} \end{pmatrix}$, $r_1 = \frac{2-\nu}{2}, r_2 = \frac{1}{2}$. The function $Q(V,x)$ is differentiable for any $(V,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$ and since $P > 0$ then

$$\frac{\lambda_{\min}(P)\|x\|^2}{V} \leq Q(V,x) + 1 \leq \frac{\lambda_{\max}(P)\|x\|^2}{V^{2-\nu}}, \tag{11}$$

and there exist some $V^-, V^+ \in \mathbb{R}_+$ such that $Q(V^-,x) < 0 < Q(V^+,x)$ and some $V \in \mathbb{R}_+$ such that $Q(V,x) = 0$. Hence conditions **C1**-**C3** are fulfilled. Remark that for $\nu = 1$, the identity $Q(V,x) = 0$ defines the quadratic Lyapunov function (9).

The derivative of Q w.r.t. V is given by

$$\partial_V Q = -V^{-1} x^T D_r(V^{-1})(H_r P + P H_r) D_r(V^{-1}) x,$$

where $H_r = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$. Since $H_r = \frac{1}{2} I_2 + \frac{1-\nu}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ where I_2 is the identity matrix, $H_r \to \frac{1}{2} I_2$ as $\nu \to 1$ and

$$0 < P H_r + H_r P,$$

so that $\partial_V Q(V,x) < 0$ for all $(V,x) \in \mathbb{R}_+ \times \mathbb{R}^n \backslash \{0\}$ and condition **C4** is satisfied. Assuming additionally that $P H_r + H_r P \leq P$ and taking into account (from (10)) that $Q(V,x) = 0 \Rightarrow x^T D_r(\frac{1}{V}) P D_r(\frac{1}{V}) x = 1$ we obtain

$$-V^{-1} \leq \partial_V Q(V,x) < 0. \tag{12}$$

Similarly, the derivative of Q along the trajectories of (7)-(8), denoted as $\partial_x Q f$, is given by

$$\partial_x Q f = 2x^T D_r(\tfrac{1}{V}) P D_r(\tfrac{1}{V})(A_0 x + b u(x) + d).$$

Let us assume that the following condition holds for some $\mu > 0$:

$$A_0 S + S A_0^T + bq + b^T q^T + S + \mu I_2 \leq 0, \tag{13}$$

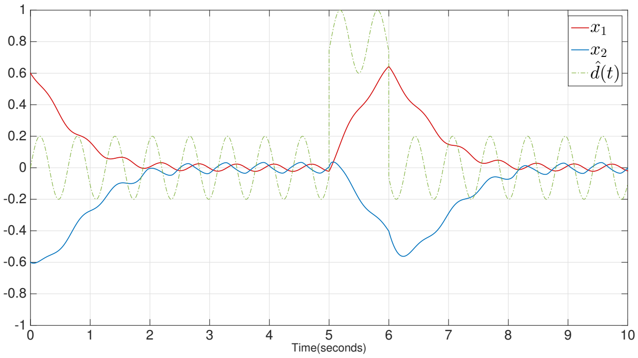

Fig. 2: Simulation of system (7) for $\nu = 0.8$ and initial conditions $x(0) = (0.6, -0.6)$.

where $S = P^{-1}$ and $q = kS^{-1}$. By adding and subtracting the term $2V^{\nu/2}x^T D_r(\frac{1}{V})PD_r(\frac{1}{V})bkD_r(\frac{1}{V})x$, and taking into account that $D_r(\frac{1}{V})A_0D_r^{-1}(\frac{1}{V}) = V^{(\nu-1)/2}A_0$ and that $D_r(\frac{1}{V})b = V^{-\frac{1}{2}}b$, we obtain

$$\partial_x Qf = \left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right)^T \Theta \left(\begin{smallmatrix} y \\ z \end{smallmatrix}\right) + V^{\frac{\nu-1}{2}}(2y^T Pbk\tilde{y}_\nu - y^T Py) + \tfrac{1}{\mu}V^{\frac{1-\nu}{2}}z^T z,$$

where $y = D_r(\frac{1}{V})x$, $z = D_r(\frac{1}{V})d$, $\tilde{y}_\nu = y - (\lceil y_1 \rfloor^{\frac{\nu}{2-\nu}}, \lceil y_2 \rfloor^{\nu})^T$ and

$$\Theta = \begin{pmatrix} V^{\frac{\nu-1}{2}}(P(A_0 - bk) + (A_0 - bk)^T P + P) & P \\ P & -\frac{1}{\mu}V^{\frac{1-\nu}{2}}I_2 \end{pmatrix}.$$

Since the Schur complement of $\left(\begin{smallmatrix} P^{-1} & 0 \\ 0 & \eta \end{smallmatrix}\right)\Theta\left(\begin{smallmatrix} P^{-1} & 0 \\ 0 & \eta \end{smallmatrix}\right)$ for any $\eta \in \mathbb{R}$ is equivalent to the left hand side of (13) and $Q(V, x) = 0 \Rightarrow y^T Py = 1$, we have that

$$\partial_x Qf \leq -V^{\frac{\nu-1}{2}}(2y^T Pbk\tilde{y}_\nu - 1) + \tfrac{1}{\mu}V^{\frac{1-\nu}{2}}d^T D_r^2(\tfrac{1}{V})d.$$

Since $\tilde{y}_\nu \to 0$ and $D_r^2(\frac{1}{V}) \to V^{-1}I_2$ as $\nu \to 1$, there exists some ν, sufficiently close to one, such that $\max_{y:y^T Py=1} y^T Pbk\tilde{y}_\nu < c_1 \leq 1$. Then

$$\partial_x Q < -c_2 V^{\frac{\nu-1}{2}} + \tfrac{1}{\mu}V^{-1+\frac{1-\nu}{2}}d^T d,$$

where $c_2 = 1 - c_1 > 0, c_1 > \frac{1}{\mu}$. From (12) we obtain

$$\frac{\partial_x Q}{\partial_V Q}f(x, d) \geq c_1 V^{1+\frac{\nu-1}{2}} - \tfrac{1}{\mu}V^{-1+\frac{1-\nu}{2}}d^T d,$$

and from (11) we finally derive

$$\|x\| \geq \chi(\|d\|) \Rightarrow \frac{\partial_x Q}{\partial_V Q}f(x, d) \geq (c_1 - \tfrac{1}{\mu})V^{1+\frac{\nu-1}{2}}$$

where $\chi(r) = \frac{1}{\lambda_{\min}(P)}r^{\frac{1}{\nu+1}}$ and we recover the condition $\mathbf{C5}^{\mathrm{ft}}$. Thus, we conclude that $Q(V, x)$ is a finite-time implicit Lyapunov function and form Theorem 2, the origin of (7)-(8) is finite-time ISS stable for any ν sufficiently close to 1.

V. Conclusions

A theoretical framework, with necessary and sufficient conditions, for implicit ISS and implicit FT ISS Lyapunov functions has been developed. For the implicit FX ISS case sufficient conditions were provided.

All the theorems and definitions here presented allow to assert, with a single implicitly defined function, the convergence type and the robustness, in an input-to-state sense, of a given nonlinear system. Also, with the results presented, it is possible to obtain the convergence time to zero whenever the disturbances are absent. In order to achieve these results, necessary and sufficient conditions for finite-time ISS explicit Lyapunov functions were developed for the first time.

The implicit framework, being developed from Lyapunov's direct method, also lacks a universal methodology to obtain implicit Lyapunov functions, however, as seen in Example 2, it can be used in systems where no explicit LF is known, thus broadening the Lyapunov analysis tools.

To derive nonlinear gains form a given implicit Lyapunov function is proposed as a future research topic.

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