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Juliette Chabassier, Sébastien Imperiale

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Abstract In this work we present and analyse a time discretisation strategy for linear wave propagation that aims at using locally in space the most adapted time discretisation among a family of implicit or explicit centered second order schemes. The domain of interest being decomposed into several regions, different time discretisations can be chosen depending on the local properties of the spatial discretisations (mesh size and quality) or the physical parameters (high wave speed, low density). We show that, under some conditions on the time step, the family of time discretisations obtained combined with standard finite elements methods in space ensures a second order space-time convergence.

Keywords Wave equations · Time discretisation · Local time stepping

1 Introduction

The appropriate time integration of systems of ordinary differential equations (ODEs) resulting from the finite element discretisation in space of partial differential equations is of crucial importance to construct efficient numerical solvers. For linear wave equations problems it is well-known that fully explicit time discretisations perform better than implicit ones in non-stiff situations, i.e. when wave propagation occurs in homogeneous media and simple geometries uniformly or regularly meshed. However if a strong heterogeneity (high wave speed, low density) is present, or if the mesh size and quality degenerate locally in space, then explicit methods reach their bottlenecks: the time step of the simulation must be adapted to the local pertubation of the discretisation's parameters. Local time discretisation is a standard topic that aims at overcoming these bottlenecks and two main strategies can be distinguished

- o **Local implicit time discretisation**, see for instance [1–6]. The strategy is to treat by an implicit time integration scheme the ODEs acting on the degree of freedoms

J. Chabassier
Inria Bordeaux Sud Ouest – Magique 3D team
Université de Pau et des Pays de l'Adou
E-mail: juliette.chabassier@inria.fr

S. Imperiale
Inria, Université Paris-Saclay, France,
LMS, Ecole Polytechnique, Université Paris-Saclay, CNRS, France
E-mail: sebastien.imperiale@inria.fr

corresponding to the region where the pertubations occur. By doing so, the time step restriction (CFL) is decorrolated from the pertubations. The price to pay is that a (hopefully small) linear problem must be solved at each iteration.

o **Local time stepping**, see for instance [7–11]. The strategy is to use a first time marching scheme in the whole domain and a second one in the pertubated region. The chosen type of time discretisations used in both steps is often the same but time steps differ: a smaller time step is used locally. One can distinguish non-conservative strategies (see for instance [11]) and conservative strategies. The latter are based upon Leap-Frog schemes and can be traced back to the work of Collino et. al. [12]. They can be separated into two categories depending on how transmissions between sub-domains are treated: either explicitly (a reference work on this topic is [8]) or implicitly (see [9, 10]).

In this work we construct and analyse local time discretisations that gather in an original framework both local implicit time discretisation and conservative local time stepping. We show that the local time stepping that we propose is closely related to the one developed in [10, 9], however we offer a new perspective on the problem which enable us to develop a full space-time convergence analysis, showing that second order convergence is achieved under some explicit CFL conditions.

The outline of the article is the following: In section 2 we give all the necessary notations and assumptions related to the discretisation in spaces of linear conservative wave type problems. Section 3 is devoted to the introduction of a class of time discretisations – parameterised by two polynomial functions $\mathcal{P}_p$ and $\mathcal{P}_k$ – for which we show stability and second order space-time convergence results under some assumptions on the parameters (i.e. the coefficients of the polynomials $\mathcal{P}_p$ and $\mathcal{P}_k$) and some CFL conditions. In Section 4 we present two preliminaries applications of our discretisation framework. By choosing adequately the polynomial functions $\mathcal{P}_p$ and $\mathcal{P}_k$ we construct a local implicit time discretisation and a first local time-stepping scheme (with a *ratio* 2, see Section 7.2 for an accurate definition of the term *ratio*). Section 5 is a parenthesis in the manuscript since it is devoted to show that the local time-stepping scheme constructed in Section 4 is actually equivalent to the local time stepping proposed by J. Rodriguez and P. Joly in [9, 10]. In Section 6 we propose a strategy to construct general local time-stepping schemes. This strategy is based on the use of Chebychev polynomial (more precisely on Leap-Frog Chebychev method as introduced in [20]). Space-time numerical convergence results in 1D are given Section 7 and illustrate the developed theory.

2 Semi-discrete wave propagation problem

We are interested in the simulation of coupled linear wave propagation problems. The most simple example one could think of is given by the following problem: being given a connex open domain $\varOmega$ partitioned as two disjoint domains $\varOmega_c$ and $\varOmega_f$, find $u_c(t) \in H^1(\varOmega_c)$ and $u_f(t) \in H^1(\varOmega_f)$, for all $t \in [0, T]$ such that

$$\begin{cases} \partial_t^2 u_c - \mu_c\, \Delta u_c = f & \text{in } \varOmega_c, \\ \partial_t^2 u_f - \mu_f\, \Delta u_f = 0 & \text{in } \varOmega_f, \end{cases} \tag{1}$$

and homogeneous boundary conditions on part of the domain's boundary

$$u_c = 0 \text{ on } \varGamma_c \subset \partial\Omega_c, \quad u_f = 0 \text{ on } \varGamma_f \subset \partial\Omega_f,$$

and some transmission conditions on the complementary boundary $\varSigma$

$$u_c = u_f, \quad \mu_c \nabla u_c \cdot n = \mu_f \nabla u_f \cdot n \quad \text{on } \varSigma = \partial\Omega_c \subset \partial\Omega_f.$$

The scalar numbers μ_c and μ_f are some material parameters. The source term is assumed to be only supported in Ω_c and for simplicity we only consider zero initial data although the analysis we provide in this work can be naturally extended if the initial data have a compact support in Ω_c. Such problems find applications in the wave scattering by obstacles and is of interest for modeling non destructive experiments for instance.

2.1 Continuous abstract formulation

In the following q stands for either c or f.

In this section we formulate the coupled wave propagation in a more abstract framework. To do so we use notations from [14], chapter XVIII, and [22]. We assume given separable Hilbert spaces (H_q, V_q). The space H_q is equipped with the scalar product $(\cdot,\cdot)_q$, the norm in H_q is denoted $|\cdot|_q$ whereas the norms on V_q is denoted $\|\cdot\|_q$. Moreover we assume that V_q is dense and continuously embedded in H_q. We assume given a continuous hermitian bilinear form $a_q : V_q \times V_q \to \mathbb{R}$ that satisfies

$$c_q^2 \|v\|_q^2 \leq a_q(v,v), \quad \forall v \in V^q, \tag{2}$$

where c_q is a real positive scalar. We assume also being given another Hilbert space L equipped with the norm $\|\cdot\|_L$ as well as a continuous bilinear form $b_q(v,\lambda)$ on $V_q \times L$. We consider the following abstract wave propagation problems:
Let $f \in C^0([0,T], H_c)$ be given, find $(u_c(t), u_f(t), \lambda(t)) \in V_c \times V_f \times L$ solution, for all $t \in [0,T]$, to the coupled system of equations

$$\begin{cases} \dfrac{d^2}{dt^2}(u_c, v_c) + a_c(u_c, v_c) + b_c(v_c, \lambda) = f & \forall\, v_c \in V_c, \\ \dfrac{d^2}{dt^2}(u_f, v_f) + a_f(u_f, v_f) - b_f(v_f, \lambda) & \forall\, v_f \in V_f, \\ b_c(u_c, \mu) = b_f(u_f, \mu) & \forall\, \mu \in L. \end{cases} \tag{3}$$

We complete the equations by null initial conditions

$$u_q(0) = 0 \quad \text{in } V_q, \quad \partial_t u_q(0) = 0 \quad \text{in } H_q, \qquad q \in \{c, f\}. \tag{4}$$

Existence and uniqueness results for this problem rely on the assumption that a so-called inf-sup condition holds. More precisely we assume that there exists $k > 0$ such that

$$\inf_{\lambda \in L} \sup_{(v_c, v_f) \in V_c \times V_f} \frac{b_c(v_c,\lambda) - b_f(v_f,\lambda)}{\|\lambda\|_L \sqrt{\|v_c\|_c^2 + \|v_f\|_f^2}} \geq k.$$

It is standard to show existence and uniqueness results for this problem (see for instance [19]). One can prove the following lemma

Lemma 1 *Assume $f \in C^1([0,T], H_c)$ and has a compact support in $(0,T)$ then there exists a unique solution*

$$(u_c, u_f, \lambda) \in \prod_{q \in \{c,f\}} \left(C^2([0,T]; H_q) \cap C^1([0,T]; V_q) \right) \times C^0([0,T]; L)$$

of the problem (3, 4).

Remark 1 To solve system (1)-(4) one has to choose

$$H_q = L^2(\Omega_q), \quad V_q = \{v \in H^1(\Omega_q) \mid v = 0 \text{ on } \Gamma_q\}, \quad L = H^{-1/2}(\Sigma),$$

where H_q is equipped with the standard L^2 scalar product, and for all u and v in V_q and for all λ in $H^{-1/2}(\Sigma)$

$$a_q(u, v) = (\mu_q \nabla u, \nabla v)_q, \quad b_q(v, \lambda) = \langle v, \lambda \rangle_{H^{1/2}(\Sigma), H^{-1/2}(\Sigma)}.$$

2.2 discretisation in space and stability estimates

We introduce the family of finite dimensional Hilbert spaces $\{V_{q,h}\}_{h>0}$ with $V_{q,h} \subset V_q$ and $L_h \subset L$. As usual, the subscript h is devoted to tend to 0 and represents an approximation parameter of $V_{q,h}$ to V_q and L_h to L. For each h we define the operator $A_{q,h}$ as $A_{q,h} : V_{q,h} \mapsto V_{q,h}$ and

$$A_{q,h} : u_h \to A_{q,h} u_h \qquad \text{such that} \qquad (A_{q,h} u_h, v_h)_q = a_q(u_h, v_h), \quad \forall v_h \in V_{q,h},$$

Inequality (2) implies that the operator $A_{q,h}$ is self-adjoint and positive definite. Its spectrum, denoted $\mathrm{Sp}(A_{q,h})$, is the set composed by a finite number of positive eigenvalues . The spectral radius is defined as the maximum eigenvalue in the set $\mathrm{Sp}(A_{q,h})$, i.e,

$$\rho_{q,h} := \max \mathrm{Sp}(A_{q,h}).$$

We also introduce a bilinear form $b_{q,h} : V_{q,h} \times L_h$ that represents an approximation of the bilinear form b_q and define the operator $B_{q,h} : V_{q,h} \mapsto L_h$ and $B^t_{q,h} : L_h \mapsto V_{q,h}$ as

$$(B_{q,h} v_{q,h}, \lambda_h)_L = (B^t_{q,h} \lambda_h, v_{q,h})_q := b_q(v_{q,h}, \lambda_h), \quad \forall v_{q,h} \in V_{q,h} \text{ and } \forall \lambda_h \in L_h.$$

The semi-discrete equation we consider reads:
Let $f_h \in C^0([0,T], V_{c,h})$ be given: find $(u_{c,h}(t), u_{f,h}(t), \lambda_h(t)) \in V_{c,h} \times V_{f,h} \times L_h$ and solution, for all $t \in [0,T]$, of

$$\begin{cases} \dfrac{d^2}{dt^2} u_{c,h} + A_{c,h} u_{c,h} + B^t_{c,h} \lambda_h = f_h & \text{in } V_{c,h}, \quad (a) \\ \dfrac{d^2}{dt^2} u_{f,h} + A_{f,h} u_{f,h} - B^t_{f,h} \lambda_h = 0 & \text{in } V_{f,h}, \quad (b) \\ B_{c,h} u_{c,h} = B_{f,h} u_{f,h} & \text{in } L_h \quad\;\; (c) \end{cases} \tag{5}$$

together with the initial conditions

$$u_{q,h}(0) = \frac{d}{dt} u_{q,h}(0) = 0 \quad \text{in } V_{q,h}, \qquad q \in \{c, f\}. \tag{6}$$

In the rest of the paper we assume that the following discrete inf-sup condition holds

$$\inf_{\lambda_h \in L_h} \sup_{(v_{c,h}, v_{f,h}) \in V_{c,h} \times V_{f,h}} \frac{b_{c,h}(v_{c,h}, \lambda_h) - b_{f,h}(v_{h,f}, \lambda_h)}{\|\lambda_h\|_{L_h} \sqrt{\|v_{c,h}\|_c^2 + \|v_{f,h}\|_f^2}} \geq k, \tag{7}$$

where k should be independent of h. Existence and uniqueness results for the semi-direct problem are then direct consequences of lemma 1 (in the paragraph above choose $V_q = V_{q,h} \subset H_q$ and $L = L_h$): There exists a unique solution $(u_{c,h}, u_{f,h}, \lambda_h)$ to (5), it satisfies

$$(u_c, u_f, \lambda) \in \prod_{q \in \{c,f\}} C^2([0,T]; V_{q,h}) \times C^0([0,T]; L_h).$$

Moreover standard energy analysis of the semi-discrete problem (5) can be used to show a corollary of the solution's existence result that we state below.

Corollary 1 ***Stability of the semi-discrete problem.*** *For any integer $n \geq 1$, assume that*

$$f_h \in C^n([0,T], V_{q,h})$$

and has compact support in $(0,T)$, then if there exists $C_f^{(n)}$ independent of h such that

$$\sup_{t\in[0,T]} \sum_{k=0}^{n} |\frac{d^k}{dt^k} f_h|_c \leq C_f^{(n)} \tag{8}$$

then there exists a constant $C_u^{(n)}$ independent of h such that for all $h > 0$ we have

$$\sup_{t\in[0,T]} \sum_{q\in\{c,f\}} \sum_{k=0}^{n} \left(\Big| A_{q,h}^{\frac{1}{2}} \frac{d^k}{dt^k} u_{q,h}(t) \Big|_q + \Big| \frac{d^{k+1}}{dt^{k+1}} u_{q,h}(t) \Big|_q \right) \leq C_u^{(n)}.$$

Remark 2 Notice that by definition, $C_f^{(n)} \leq C_f^{(n+1)}$ and $C_u^{(n)} \leq C_u^{(n+1)}$.

We introduce the discrepancy error

$$e_{q,h}(t) = u_q(t) - u_{q,h}(t).$$

The aim of this work is to study space/time convergence of some specific time discretisations, therefore we assume that the space discretisation has the expected convergence behavior. More precisely it is expected that $e_{q,h}$ vanishes when $h \to 0$. We make the following assumption

Assumption 1 ***Convergence of the semi-discrete problem.*** *There exists a positive increasing function of h denoted $\delta_V(h)$ such that*

$$\sup_{t\in[0,T]} \left(\|e_{c,h}(t)\|_c + \|e_{f,h}(t)\|_f \right) \leq \delta_V(h).$$

Remark 3 For the simplicity of presentation we assume that bilinear forms are evaluated exactly. However, since we are studying time discretisation, we believe that the results presented in this work can be extended without difficulties to take into account the use of quadrature formulae for the computation of space integral terms.

3 Time discretisation

The schemes we construct here can be seen as perturbations of standard centered two-steps discretisation of system (5). The perturbations are embedded in the definition of two given polynomials $\mathcal{P}_k(x)$ and $\mathcal{P}_p(x)$. In this section we first construct time discretisation with the minimum assumptions concerning the properties that should be satisfied by polynomials and then state a space-time convergence result. In Sections 4 and 6 some examples are given that show how efficient local time discretisation can be constructed from adequate definition of $\mathcal{P}_k(x)$ and $\mathcal{P}_p(x)$.

3.1 Introduction of local time discretisations

We define the sequences $\{u^n_{c,h}\}$, $\{u^n_{f,h}\}$ and $\{\lambda^n_h\}$ as the approximations of $u_{c,h}(t)$, $u_{f,h}(t)$ and $\lambda_h(t)$ at time $t = n\Delta t$ for a given time step $\Delta t > 0$, $n \in \mathbb{N}$. These sequences are constructed by solving the following problem:
Let $f_h \in C^3_0([0,T], V_{c,h})$ be given, find $\big(\{u^n_{c,h}\}, \{u^n_{f,h}\}, \{\lambda^n_h\}\big)$ solution to

$$\begin{cases} \dfrac{u^{n+1}_{c,h} - 2u^n_{c,h} + u^{n-1}_{c,h}}{\Delta t^2} + A_{c,h}u^n_{c,h} + B^t_{c,h}\lambda^n_h = f_h(t^n) \ \text{ in } V_{c,h}, & (a) \\ \mathcal{P}_k(\Delta t^2 A_{f,h}) \dfrac{u^{n+1}_{f,h} - 2u^n_{f,h} + u^{n-1}_{f,h}}{\Delta t^2} & \\ \qquad + \mathcal{P}_p(\Delta t^2 A_{f,h})\big(A_{f,h}\{u^n_{f,h}\}_{1/4} - B^t_{f,h}\lambda^n_h\big) = 0 \ \text{ in } V_{f,h}, & (b) \\ B_{c,h}u^n_{c,h} = B_{f,h}u^n_{f,h} \qquad \text{ in } L_h, & (c) \end{cases} \tag{9}$$

where

$$\{u^n_{f,h}\}_{1/4} = \frac{u^{n+1}_{f,h} + 2u^n_{f,h} + u^{n-1}_{f,h}}{4},$$

with the initial conditions

$$u^0_{q,h} = u^1_{q,h} = 0 \quad \text{in } V_{q,h}, \qquad q \in \{c, f\}. \tag{10}$$

For the sake of simplicity but without loss of generality we have assumed that the source term $f(t)$ and its approximation $f_h(t)$ have compact support in $(0,T)$ and that the solution of the continuous problem and semi-discrete problem vanishes identically in a neighborhood of $t = 0$. Hence the initial condition (10) introduces no consistency error for Δt sufficiently small.

The scheme (3) is consistent only if some conditions are satisfied on the polynomials $\mathcal{P}_k(x)$ and $\mathcal{P}_p(x)$. Since we want to construct perturbations of the standard centered scheme it seems reasonable to do the the following hypothesis.

Assumption 2 *We assume that*

$$\mathcal{P}_k(0) = \mathcal{P}_p(0) = 1.$$

For stability the time step Δt can not be chosen arbitrarily. A so called CFL-condition has to be satisfied to obtain a stable scheme. In our case it corresponds to the assumption that follows:

Assumption 3 *The following CFL-condition is satisfied: there exists* $0 < \alpha \leq 1$ *such that*

$$\Delta t = \alpha \frac{2}{\sqrt{\rho_{c,h}}}$$

and

$$\mathcal{P}_k(x) \geq 0, \quad \mathcal{P}_p(x) > 0, \quad \forall x \in [0, \Delta t^2 \rho_{f,h}]. \tag{11}$$

Note that since

$$\Delta t^2 \rho_{f,h} = 4\,\alpha^2\,\frac{\rho_{f,h}}{\rho_{c,h}} \tag{12}$$

and because of assumption (2) we known that there exists Δt small enough or equivalently α small enough, such that (11) is satisfied for a given h. As shown later, these conditions ensure the positivity of a preserved discrete energy. At each iteration, one needs to compute the Lagrange multiplier λ_h^n then compute $u_{f,h}^{n+1}$ and $u_{c,h}^{n+1}$. We describe now more in detail an algorithm that computes the solution to (9). Using the property that

$$\{u_{f,h}^n\}_{1/4} = u_{f,h}^n + \frac{\Delta t^2}{4} \frac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} \tag{13}$$

we can re-write equation (9)(b) in the following form

$$\frac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} + D_{f,h}^{-1}\, \mathcal{P}_p(\Delta t^2 A_{f,h}) \left(A_{f,h} u_{f,h}^n - B_{f,h}^t \lambda_h^n\right) = 0. \tag{14}$$

with

$$D_{f,h} := \mathcal{P}_k(\Delta t^2 A_{f,h}) + \frac{\Delta t^2}{4} \mathcal{P}_p(\Delta t^2 A_{f,h}) A_{f,h}.$$

Note that $D_{f,h}$ is a positive symmetric operator – hence invertible – if equation (11) holds. We now use a Schur complement: applying the operator $B_{f,h}$ to equation (14), applying the operator $B_{c,h}$ to (9)(a), we obtain by subtraction and thanks to (9)(c) the following system for λ_h^n

$$\Big(B_{c,h} B_{c,h}^t + B_{f,h} D_{f,h}^{-1}\, \mathcal{P}_p(\Delta t^2 A_{f,h}) B_{f,h}^t \Big) \lambda_h^n$$
$$= B_{f,h} f_h^n - B_{f,h} A_{c,h}\, u_{c,h}^n + B_{f,h} D_{f,h}^{-1}\, \mathcal{P}_p(\Delta t^2 A_{f,h})\, u_{f,h}^n.$$

The well-posedness of the above problem is a consequence of the surjectivity of either $B_{c,h}$ or $B_{f,h}$ which is a consequence of the inf-sup condition (7). Assuming that $u_{f,h}^n$ and $u_{c,h}^n$ are known, then λ_h^n can be computed, it follows that $u_{f,h}^{n+1}$ and $u_{c,h}^{n+1}$ can be computed using respectively (14) and (9)(a).

3.2 Space-time convergence analysis

We define the error terms $e_{c,h}^n$, $e_{f,h}^n$, ℓ_h^n as

$$e_{c,h}^n = u_{c,h}(t^n) - u_{c,h}^n, \quad e_{f,h}^n = u_{f,h}(t^n) - u_{f,h}^n \quad \text{and} \quad \ell_h^n = \lambda_h(t^n) - \lambda_h^n.$$

In this section we show that the terms $e_{c,h}^n$ and $e_{f,h}^n$ tend to 0 as h and Δt go to 0. More precisely we show that under the conditions of Assumptions 3 we obtain second order convergence in time for the norm in $V_{q,h}$. The section is organized as follows

- Definition of the consistency errors: we write an equation for the sequence ($e_{c,h}^n, e_{f,h}^n$ and ℓ_h^n) that is similar to (9) with source terms that correspond to consistency errors that we will then specify.
- Energy identity for the error equation: we proceed by energy analysis and write an energy identity satisfied by the error terms $e_{c,h}^n$ and $e_{f,h}^n$. The introduced energy is positive under the CFL-condition of Assumption 3.
- Main result: we prove convergence of $\{e_{c,h}^n\}$ and $\{e_{f,h}^n\}$ in energy norm. To do so we use discrete by-parts integration and discrete energy analysis including the use of a discrete Gronwall's lemma.
- Corollaries: we obtain convergence for the norms $\|\cdot\|_q$ and deduce a complete space-time convergence result.

Definition of the consistency errors. Using equations (5)(c) and (9) we obtain

$$\begin{cases} \dfrac{e_{c,h}^{n+1} - 2e_{c,h}^{n} + e_{c,h}^{n-1}}{\Delta t^2} + A_{c,h} e_{c,h}^n + B_{c,h}^t \ell_h^n = r_{c,h}^n & \text{in } V_{c,h}, \\ \mathcal{P}_k(\Delta t^2 A_{f,h}) \dfrac{e_{f,h}^{n+1} - 2e_{f,h}^{n} + e_{f,h}^{n-1}}{\Delta t^2} & \\ \qquad + \mathcal{P}_p(\Delta t^2 A_{f,h}) \big(A_{f,h} \{e_{f,h}^n\}_{1/4} - B_{f,h}^t \ell_h^n \big) = r_{f,h}^n & \text{in } V_{f,h}, \\ B_{c,h} e_{c,h}^n = B_{f,h} e_{f,h}^n & \text{in } L_h \end{cases} \tag{15}$$

with the consistency errors given by

$$r_{c,h}^n = \frac{u_{c,h}(t^{n+1}) - 2u_{c,h}(t^n) + u_{c,h}(t^{n-1})}{\Delta t^2} + A_{c,h} u_{c,h}(t^n) + B_{c,h}^t \lambda_h(t^n) - f_h(t^n) \tag{16}$$

and

$$r_{f,h}^n = \mathcal{P}_k(\Delta t^2 A_{f,h}) \frac{u_{f,h}(t^{n+1}) - 2u_{f,h}(t^n) + u_{f,h}(t^{n-1})}{\Delta t^2} + \mathcal{P}_p(\Delta t^2 A_{f,h}) \big(A_{f,h} \{u_{f,h}(t^n)\}_{1/4} - B_{f,h}^t \lambda_h(t^n) \big). \tag{17}$$

Standard Taylor expansions allow us to simplify equations (16) and (17). First, using equation (5)(a), there exist intermediate times $(t^{n,\heartsuit}, t^{n,\spadesuit}, t^{n,\clubsuit})$ with

$$t^{n-1} \leq t^{n,\heartsuit}, t^{n,\spadesuit}, t^{n,\clubsuit} \leq t^{n+1}$$

such that

$$r_{c,h}^n = \frac{\Delta t^2}{12} \frac{d^4}{dt^4} u_{c,h}(t^{n,\heartsuit}) \tag{18}$$

and

$$r_{f,h}^n = \mathcal{P}_k(\Delta t^2 A_{f,h}) \frac{d^2}{dt^2} u_{f,h}(t^n) + \mathcal{P}_p(\Delta t^2 A_{f,h}) \big(A_{f,h} u_{f,h}(t^n) - B_{f,h}^t \lambda_h(t^n) \big) + \frac{\Delta t^2}{12} \mathcal{P}_k(\Delta t^2 A_{f,h}) \frac{d^4}{dt^4} u_{f,h}(t^{n,\spadesuit}) + \frac{\Delta t^2}{4} \mathcal{P}_p(\Delta t^2 A_{f,h}) A_{f,h} \frac{d^2}{dt^2} u_{f,h}(t^{n,\clubsuit}).$$

Then using equation (5)(b) one can simplify further $r_{f,h}^n$, we have

$$r_{f,h}^n = \big(\mathcal{P}_k(\Delta t^2 A_{f,h}) - \mathcal{P}_p(\Delta t^2 A_{f,h}) \big) \frac{d^2}{dt^2} u_{f,h}(t^n) + \frac{\Delta t^2}{12} \mathcal{P}_k(\Delta t^2 A_{f,h}) \frac{d^4}{dt^4} u_{f,h}(t^{n,\spadesuit}) + \frac{\Delta t^2}{4} \mathcal{P}_p(\Delta t^2 A_{f,h}) A_{f,h} \frac{d^2}{dt^2} u_{f,h}(t^{n,\clubsuit}).$$

If Assumption 2 holds then there exists a polynomial function $\mathcal{Q}$ such that

$$\big(\mathcal{P}_k(\Delta t^2 A_{f,h}) - \mathcal{P}_p(\Delta t^2 A_{f,h}) \big) \frac{d^2}{dt^2} u_{f,h}(t^n) = \Delta t^2 \mathcal{Q}(\Delta t^2 A_{f,h}) A_{f,h} \frac{d^2}{dt^2} u_{f,h}(t^n)$$

where $\mathcal{Q}(x)$ is given by

$$\mathcal{Q}(x) := \frac{\mathcal{P}_k(x) - \mathcal{P}_p(x)}{x}.$$

The consistency error $r^n_{f,h}$ has then the final expression

$$r^n_{f,h} = \frac{\Delta t^2}{12}\mathcal{P}_k(\Delta t^2 A_{f,h})\frac{d^4}{dt^4}u_{f,h}(t^{n,\spadesuit}) + \frac{\Delta t^2}{4}\mathcal{P}_p(\Delta t^2 A_{f,h})A_{f,h}\frac{d^2}{dt^2}u_{f,h}(t^{n,\clubsuit}) + \Delta t^2\mathcal{Q}(\Delta t^2 A_{f,h})A_{f,h}\frac{d^2}{dt^2}u_{f,h}(t^n). \quad (19)$$

Energy identity for the error equation. The initial conditions for equation (15) are deduced from (10) and the fact that the solution of the semi-discrete problem vanishes identically in a neighborhood of $t = 0$ (since the source term is compactly supported in $(0, T)$). We have, for Δt sufficiently small

$$e^0_{q,h} = e^1_{q,h} = 0 \quad \text{in } V_{q,h}. \quad (20)$$

To obtain an energy identity on the error equation (15)-(18)-(19) we use standard discrete energy techniques. The steps are the following:

i) compute the scalar product $(\cdot,\cdot)_c$ of the first equation of (15) with

$$\frac{e^{n+1}_{c,h} - e^{n-1}_{c,h}}{2\Delta t},$$

ii) compute the scalar product $(\cdot,\cdot)_f$ of the second equation of (15) with

$$\mathcal{P}_p(\Delta t^2 A_{f,h})^{-1}\frac{e^{n+1}_{f,h} - e^{n-1}_{f,h}}{2\Delta t},$$

iii) sum the two obtained equations and use the third equation of (15) to get rid of the term involving ℓ^n.

After standard algebraic manipulations ([6]) one can show that for $n \geq 1$, if Assumption 3 holds,

$$\frac{\mathcal{E}^{n+1/2}_{c,h} - \mathcal{E}^{n-1/2}_{c,h}}{\Delta t} + \frac{\mathcal{E}^{n+1/2}_{f,h} - \mathcal{E}^{n-1/2}_{h,h}}{\Delta t} = \left(r^n_{c,h}, \frac{e^{n+1}_{c,h} - e^{n-1}_{c,h}}{2\Delta t}\right)_c + \left(r^n_{f,h}, \mathcal{P}_p(\Delta t^2 A_{f,h})^{-1}\,\frac{e^{n+1}_{f,h} - e^{n-1}_{f,h}}{2\Delta t}\right)_f, \quad (21)$$

with

$$\mathcal{E}^{n+1/2}_{c,h} = \frac{1}{2}\left(\left(I_{c,h} - \frac{\Delta t^2}{4}A_{c,h}\right)\frac{e^{n+1}_{c,h} - e^n_{c,h}}{\Delta t}, \frac{e^{n+1}_{c,h} - e^n_{c,h}}{\Delta t}\right)_c + \frac{1}{2}\left|A^{1/2}_{c,h}\frac{e^{n+1}_{c,h} + e^n_{c,h}}{2}\right|^2_c \quad (22)$$

where $I_{c,h}$ is the identity operator in H_c, and with

$$\mathcal{E}^{n+1/2}_{f,h} = \frac{1}{2}\left|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}}\,\frac{e^{n+1}_{f,h} - e^n_{f,h}}{\Delta t}\right|^2_f + \frac{1}{2}\left|A^{1/2}_{c,h}\frac{e^{n+1}_{f,h} + e^n_{f,h}}{2}\right|^2_f, \quad (23)$$

where we have introduce the following notation, for all $x \in \mathbb{R}^+$

$$\mathcal{R}(x) := \mathcal{P}_p(x)^{-1}\mathcal{P}_k(x) \text{ and } \mathcal{R}(\Delta t^2 A_{f,h}) = \mathcal{P}_p(\Delta t^2 A_{f,h})^{-1}\mathcal{P}_k(\Delta t^2 A_{f,h}). \qquad (24)$$

Note that $\mathcal{R}(\Delta t^2 A_{f,h})$ is well defined and is a positive symmetric operator if condition (11) of Assumption 3 holds.

Corollary 2 *Suppose Assumptions 3 holds. Then for all $n \geq 0$*

$$\mathcal{E}_{c,h}^{n+1/2} \geq 0 \quad \text{and} \quad \mathcal{E}_{f,h}^{n+1/2} \geq 0.$$

Main result. For the statement of the proof we introduce the positive scalar $C_{\mathcal{R}}$ as

$$C_{\mathcal{R}} := \sup_{x \in [0, \Delta t^2 \rho_{f,h}]} |\mathcal{R}(x)|^{\frac{1}{2}}. \qquad (25)$$

We have, for all u_h in $V_{f,h}$ the following inequality $|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}} u_h|_f \leq C_{\mathcal{R}} |u_h|_f$. Moreover we define $C_{\mathcal{Q}}$ as

$$C_{\mathcal{Q}} := \sup_{x \in [0, \Delta t^2 \rho_{f,h}]} |{\mathcal{P}_p}^{-1}(x)\mathcal{Q}(x)|, \qquad (26)$$

we have $|{\mathcal{P}_p}^{-1}(\Delta t^2 A_{f,h})\mathcal{Q}(\Delta t^2 A_{f,h}) u_h|_f \leq C_{\mathcal{Q}} |u_h|_f$ for all u_h in $V_{f,h}$.

Theorem 1 *Let $\tau > 0$. There exists a generic constant $C > 0$ such that, if Assumptions 2 and 3 hold with $\Delta t < \tau$ then for all $f_h \in C^3([0,T], V_{c,h})$ satisfying (8) with compact support in (τ, T) we have*

$$(\mathcal{E}_{f,h}^{n-1/2})^{\frac{1}{2}} + (\mathcal{E}_{c,h}^{n-1/2})^{\frac{1}{2}} \leq (1+T)\, C\, C_u^{(3)}\, (1 + C_{\mathcal{R}} + C_{\mathcal{Q}}) \Delta t^2 \quad \text{for all} \quad n\,\Delta t \leq T. \quad (27)$$

Proof Note that by "generic constant" we mean that C is a scalar independent of the discretisation parameters, of the polynomial $(\mathcal{P}_p, \mathcal{P}_k)$, of the solution of the continuous problem (3) and of T. In the following proof it is allowed to change from one line to the other.

After summing over $n = 1$ to $n = N-1$ equation (21) and taking into account equations (18) and (19) and the initial condition (20) (i.e. $\mathcal{E}_{c,h}^{1/2} = \mathcal{E}_{f,h}^{1/2} = 0$), we find

$$\mathcal{E}_{c,h}^{N-1/2} + \mathcal{E}_{f,h}^{N-1/2} \leq C\, \Delta t^2 \big(\Xi_c^N + \Xi_f^N + \Pi_f^N + \Lambda_f^N \big)$$

where

$$\begin{cases} \Xi_c^N = \Delta t \displaystyle\sum_{n=1}^{N-1} \left(\partial_t^4 u_{c,h}(t^{n,\heartsuit}), \frac{e_{c,h}^{n+1} - e_{c,h}^{n-1}}{2\Delta t} \right)_c, \\ \Xi_f^N = \Delta t \displaystyle\sum_{n=1}^{N-1} \left(\mathcal{R}(\Delta t^2 A_{f,h}) \partial_t^4 u_{f,h}(t^{n,\spadesuit}), \frac{e_{f,h}^{n+1} - e_{f,h}^{n-1}}{2\Delta t} \right)_f, \\ \Pi_f^N = \Delta t \displaystyle\sum_{n=1}^{N-1} \left(A_{f,h} \partial_t^2 u_{f,h}(t^{n,\clubsuit}), \frac{e_{f,h}^{n+1} - e_{f,h}^{n-1}}{2\Delta t} \right)_f, \\ \Lambda_f^N = \Delta t \displaystyle\sum_{n=1}^{N-1} \left(\mathcal{P}_p^{-1}(\Delta t^2 A_{f,h}) \mathcal{Q}(\Delta t^2 A_{f,h}) A_{f,h} \partial_t^2 u_{f,h}(t^n), \frac{e_{f,h}^{n+1} - e_{f,h}^{n-1}}{2\Delta t} \right)_f. \end{cases}$$

The proof then proceeds in five steps. One step for the estimation of each of the four above terms and a final step that collects all the obtained estimations in order to obtain (27) using a discrete Gronwall's lemma.

Step 1: Estimation of Ξ_c^N. Following the proof given in [13] (proof 2.4 of Lemma 2.3 and appendix) it is possible to show that

$$\Big|\frac{e_{c,h}^{n+1}-e_{c,h}^{n-1}}{2\Delta t}\Big|_c \leq 2\left(\mathcal{E}_{c,h}^{n+1/2}\right)^{\frac{1}{2}} + 2\left(\mathcal{E}_{c,h}^{n-1/2}\right)^{\frac{1}{2}}. \tag{28}$$

It has to be noted that this inequality holds uniformly of the time step (in the limit given by Assumption 3) and in particular it is valid if $\Delta t = 2/\sqrt{\rho_{c,h}}$. This result is not trivial. It is proven using a decomposition into low and high frequency components of the solution $u_{c,h}$. Then using Cauchy-Schwarz inequality, estimate (28), as well as standard algebraic manipulations one gets

$$\Xi_c^N \leq \Delta t \sum_{n=1}^{N-1} \Big|\frac{d^4}{dt^4}u_{c,h}(t^{n,\spadesuit})\Big|_c \Big|\frac{e_{c,h}^{n+1}-e_{c,h}^{n-1}}{2\Delta t}\Big|_c \tag{29}$$

$$\leq C \sup_{t\in[0,T]} \Big|\frac{d^4}{dt^4}u_{c,h}(t)\Big|_c \;\; \Delta t \sum_{n=1}^{N-1} (\mathcal{E}_{c,h}^{n+1/2})^{\frac{1}{2}}. \tag{30}$$

$$\leq C\, C_u^{(3)} \;\; \Delta t \sum_{n=1}^{N-1} (\mathcal{E}_{c,h}^{n+1/2})^{\frac{1}{2}}. \tag{31}$$

Step 2: Estimation of Ξ_f^N. Writing $e_{f,h}^{n+1} - e_{f,h}^{n-1} = e_{f,h}^{n+1} - e_{f,h}^{n} + e_{f,h}^{n} - e_{f,h}^{n-1}$ and using the symmetry of $\mathcal{R}(\Delta t^2 A_{f,h})$ one can show, with the Cauchy-Schwarz and triangular inequalities, that

$$\Xi_f^N \leq \frac{\Delta t}{2} \sum_{n=1}^{N-1} \Big|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}} \frac{d^4}{dt^4}u_{f,h}(t^{n,\spadesuit})\Big|_f \left(\left|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}} \frac{e_{f,h}^{n+1}-e_{f,h}^{n}}{\Delta t}\right|_f + \left|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}} \frac{e_{f,h}^{n+1}-e_{f,h}^{n}}{\Delta t}\right|_f\right) \tag{32}$$

then, since by definition of the energy (23) we have

$$\left|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}} \frac{e_{f,h}^{n+1}-e_{f,h}^{n}}{\Delta t}\right|_f \leq \sqrt{2}\,(\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}$$

we can simplify (32), and we obtain

$$\begin{aligned}
\Xi_f^N &\leq \Delta t\, \frac{\sqrt{2}}{2} \sum_{n=1}^{N-1} \Big|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}} \frac{d^4}{dt^4}u_{f,h}(t^{n,\spadesuit})\Big|_f \left((\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}} + \left(\mathcal{E}_{f,h}^{n-1/2}\right)^{\frac{1}{2}}\right)\\
&\leq C\, C_{\mathcal{R}} \sup_{t\in[0,T]} \Big|\frac{d^4}{dt^4}u_{f,h}(t)\Big|_f \;\; \Delta t \sum_{n=1}^{N-1} (\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}\\
&\leq C\, C_{\mathcal{R}}\, C_u^{(3)} \;\; \Delta t \sum_{n=1}^{N-1} (\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}
\end{aligned} \tag{33}$$

where $C_{\mathcal{R}}$ is the positive scalar constant defined by (25).

Step 3: Estimation of Π_f^N. The difficulty here is that one can not expect in general to have a uniform bound on $A_{f,h} d_t^2 u_{f,h}(t)$ in the norm in H_f. As a matter of fact, the stability estimates of Corollary 1 are rather sharp with respect to the involved norms. So the standard strategy is to do first a discrete by part integration in space, i.e.,

$$(A_{f,h} u_h, v_h)_q = (A_{f,h}^{\frac{1}{2}} u_h, A_{f,h}^{\frac{1}{2}} v_h)_q, \quad \forall (u_h, v_h) \in V_{q,h} \times V_{q,h}, \tag{34}$$

then, a discrete by part integration in time. The objective is to "exchange space and time derivatives" between the error term and the solution of the semi-discrete problem. The by part integration in time is done using the following algebraic rule: for all sequences of real numbers $\{v^n\}$ and $\{w^n\}$ we have

$$\sum_{n=1}^{N-1} v^n (w^{n+1} - w^{n-1})$$
$$= - \sum_{n=1}^{N-2} (v^{n+1} - v^n)(w^{n+1} + w^n) + v^{N-1}(w^N + w^{N-1}) - v^1(w^1 + w^0)$$

We apply the above equality to the term Π_f^N and use property (34) as mentioned. Using the fact that $e_{f,h}^0 = e_{f,h}^1 = 0$ we obtain

$$\Pi_f^N = -\Delta t \sum_{n=1}^{N-2} \left(\frac{A_{f,h}^{\frac{1}{2}}}{\Delta t} \Big(\frac{d^2}{dt^2} u_{f,h}(t^{n+1,\clubsuit}) - \frac{d^2}{dt^2} u_{f,h}(t^{n,\clubsuit}) \Big),\ A_{f,h}^{\frac{1}{2}} \frac{e_{f,h}^{n+1} + e_{f,h}^n}{2} \right)_f$$
$$+ \left(A_{f,h}^{\frac{1}{2}} \frac{d^2}{dt^2} u_{f,h}(t^{N-1,\clubsuit}),\ A_{f,h}^{\frac{1}{2}} \frac{e_{f,h}^N + e_{f,h}^{N-1}}{2} \right)_f. \tag{35}$$

Moreover using the mean value theorem we find that

$$\left| \frac{A_{f,h}^{\frac{1}{2}}}{\Delta t} \Big(\frac{d^2}{dt^2} u_{f,h}(t^{n+1,\clubsuit}) - \frac{d^2}{dt^2} u_{f,h}(t^{n,\clubsuit}) \Big) \right|_f \leq \sup_{t \in [0,T]} \left| A_{f,h}^{\frac{1}{2}} \frac{d^3}{dt^3} u_{f,h}(t) \right|_{f,h} \leq C_u^{(3)}, \tag{36}$$

and by the definition of the energy (23) one gets

$$\left| A_{f,h}^{\frac{1}{2}} \frac{e_{f,h}^{n+1} + e_{f,h}^n}{2} \right|_{f,h} \leq \sqrt{2} \, (\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}$$

Injecting the estimate above as well as estimate (36) into (35) one obtains after using Cauchy-Schwarz inequality

$$\Pi_f^N \leq C\, C_u^{(2)} (\mathcal{E}_{f,h}^{N-1/2})^{\frac{1}{2}} + C\, C_u^{(3)} \ \Delta t \sum_{n=1}^{N-1} (\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}. \tag{37}$$

Step 4: Estimation of Λ_f^N As similar strategy than for the estimation of Π_f^N can be applied. For that it is essential to observe the following property

$$\mathcal{P}_p^{-1}(\Delta t^2 A_{f,h})\mathcal{Q}(\Delta t^2 A_{f,h})A_{f,h} = A_{f,h}^{\frac{1}{2}}\mathcal{P}_p^{-1}(\Delta t^2 A_{f,h})\mathcal{Q}(\Delta t^2 A_{f,h})\, A_{f,h}^{\frac{1}{2}}$$

that can be proven by diagonalisation of the operators involved on the family of eigenvectors of $A_{f,h}$. Then the same proof as in step 3 can be used. We obtain

$$\Lambda_f^N \leq C\, C_{\mathcal{Q}}\, C_u^{(2)} (\mathcal{E}_{f,h}^{N-1/2})^{\frac{1}{2}} + C\, C_{\mathcal{Q}}\, C_u^{(3)}\ \Delta t \sum_{n=1}^{N-1} (\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}, \tag{38}$$

where $C_{\mathcal{Q}}$ is given by (26).

Step 5: Final energy estimate and Gronwall's lemma application. Combining inequalities (29), (33), (37) and (38) obtained above, we find

$$\begin{aligned}\mathcal{E}_{c,h}^{N-1/2} + \mathcal{E}_{f,h}^{N-1/2} &\leq C\, C_u^{(2)}\, \Delta t^2\ (1 + C_{\mathcal{Q}})\, (\mathcal{E}_{f,h}^{N-1/2})^{\frac{1}{2}} \\ &+ C\, C_u^{(3)}\, \Delta t^3 \sum_{n=1}^{N-1} \Big((\mathcal{E}_{c,h}^{n+1/2})^{\frac{1}{2}} + \big(1 + C_{\mathcal{R}} + C_{\mathcal{Q}}\big)(\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}} \Big). \end{aligned} \tag{39}$$

Then using Young's inequality we write that

$$C\, C_u^{(2)}\, \Delta t^2\, (1 + C_{\mathcal{Q}})\, (\mathcal{E}_{f,h}^{N-1/2})^{\frac{1}{2}} \leq \frac{C^2 \left(C_u^{(2)}\right)^2 \Delta t^4}{2}\, (1 + C_{\mathcal{Q}})^2 + \frac{1}{2}\mathcal{E}_{f,h}^{N-1/2},$$

and using the above estimation into (39) we can write that

$$\begin{aligned}&\mathcal{E}_{c,h}^{N-1/2} + \mathcal{E}_{f,h}^{N-1/2} \\ &\leq C \left(C_u^{(2)}\right)^2 (1 + C_{\mathcal{Q}})^2\, \Delta t^4\ + C\, C_u^{(3)}(1 + C_{\mathcal{R}} + C_{\mathcal{Q}})\, \Delta t^3 \sum_{n=1}^{N-1} \left(\mathcal{E}_{c,h}^{n+1/2} + \mathcal{E}_{f,h}^{n+1/2}\right)^{\frac{1}{2}}.\end{aligned}$$

To conclude let us state the following discrete Gronwall's lemma: for any real sequences $\{v_n\}$ and any positive scalar numbers A and B we have

$$v^{N-1} \leq A + B \sum_{n=1}^{N-1} (v^n)^{\frac{1}{2}} \quad \Rightarrow \quad (v^{N-1})^{\frac{1}{2}} \leq A^{\frac{1}{2}} + (N-1)B.$$

Applying this result with $v^n = \mathcal{E}_{c,h}^{n+1/2} + \mathcal{E}_{f,h}^{n+1/2}$ and observing that $(N-1)\Delta t < T$ and $C_u^{(2)} \leq C_u^{(3)}$, we obtain the result of the theorem. □

Corollaries. We give now two consequences of Theorem 1: first we state a convergence result for the norm in V_c and V_f and second we state the full space time convergence result.

Corollary 3 *Let $\tau > 0$. There exists a generic constant $C > 0$ such that, if Assumptions 2 and 3 hold with $\Delta t < \tau$ then for all $f_h \in C^4([0,T], V_{c,h})$ satisfying (8) with compact support in (τ, T) we have*

$$\|e_{c,h}^n\|_c + \|e_{f,h}^n\|_f \leq (1+T)\,(c_c^{-1} + c_f^{-1})\, C\, C_u^{(4)}\, (1 + C_{\mathcal{R}} + C_{\mathcal{Q}})\Delta t^2 \ \ \textit{for all}\ \ n\,\Delta t\, \leq\, T.$$

where c_c and c_f are the coercivity coefficients defined in equation (2).

Proof We introduce the unknowns

$$v_{q,h}(t) := \frac{u_{q,h}(t) - u_{q,h}(t-\Delta t)}{\Delta t}, \quad v^n_{q,h} := \frac{u^n_{q,h} - u^{n-1}_{q,h}}{\Delta t}, \quad d^n_{q,h} := \frac{e^n_{q,h} - e^{n-1}_{q,h}}{\Delta t}$$

and

$$\eta_h(t) := \frac{\lambda_h(t) - \lambda_h(t-\Delta t)}{\Delta t}, \quad \eta^n_h := \frac{\lambda^n_h - \lambda^{n-1}_h}{\Delta t}.$$

Then one can see that $(v_{c,h}, v_{f,h}, \eta_h)$ and $(v^n_{c,h}, v^n_{f,h}, \eta^n_h)$ satisfy respectively the semi-discrete problem (5) and the discrete problem (9) with a source term given by

$$g_h(t) := \frac{f_h(t) - f_h(t-\Delta t)}{\Delta t}$$

instead of $f_h(t)$. Note that we have assumed that $f_h(t)$ is extended by 0 for negative values of time. Since $f_h(t)$ is compactly supported in (τ, T) then $g_h(t)$ belongs to $C^3([0,T], V_{c,h})$ and is compactly supported in (τ, T). By application of the mean value theorem we deduce the following stability result

$$\sup_{t\in[0,T]} \sum_{k=0}^{3} |\frac{d^k}{dt^k} g_h|_c \leq C_f^{(4)}$$

and

$$\sup_{t\in[0,T]} \sum_{q\in\{c,f\}} \sum_{k=0}^{3} \left(\Big| A^{\frac{1}{2}}_{q,h} \frac{d^k}{dt^k} v_{q,h}(t) \Big|_q + \Big| \frac{d^{k+1}}{dt^{k+1}} v_{q,h}(t) \Big|_q \right) \leq C_u^{(4)}.$$

It is then possible to use Theorem 1 (with f_h replaced by g_h, $u^n_{q,h}$ replaced by $v^n_{q,h}$, etc...) in order to show that

$$\left(\mathcal{E}^{n-1/2}_{f,h}\right)^{1/2} + \left(\mathcal{E}^{n-1/2}_{c,h}\right)^{1/2} \leq (1+T)\, C\, C_u^{(4)}\, (1 + C_{\mathcal{R}} + C_{\mathcal{Q}})\, \Delta t^2 \quad \text{for all} \quad n\,\Delta t \leq T,$$

where the energy terms are defined by (22) and (23) with $d^n_{q,h}$ instead of $e^n_{q,h}$. In particular we have

$$\left| A^{1/2}_{q,h} \frac{d^{n+1}_{q,h} + d^n_{q,h}}{2} \right|_q \leq (1+T)\, C\, C_u^{(4)}\, (1 + C_{\mathcal{R}} + C_{\mathcal{Q}})\, \Delta t^2$$

$$\Rightarrow \quad \left| A^{1/2}_{q,h} \frac{e^{n+1}_{q,h} - e^{n-1}_{q,h}}{2\Delta t} \right|_q \leq (1+T)\, C\, C_u^{(4)}\, (1 + C_{\mathcal{R}} + C_{\mathcal{Q}})\, \Delta t^2.$$

Then from the triangular inequality and the coercivity property (2) we get

$$\|e^{n+1}_{q,h}\|_q \leq \|e^{n-1}_{q,h}\|_q + 2\Delta t\, (1+T)\, \frac{C\, C_u^{(4)}}{c_q}\, (1 + C_{\mathcal{R}} + C_{\mathcal{Q}})\, \Delta t^2. \tag{40}$$

We obtain the results of the corollary using (40) recursively and that $e^1_{q,h} = e^0_{q,h} = 0$. □

Corollary 4 *Let Assumptions 1, 2 hold. Moreover let Assumption 3 hold for some α independent of h and assume that there exists β independent of h such that*

$$4\,\alpha^2\,\frac{\rho_{f,h}}{\rho_{c,h}} \leq \beta \tag{41}$$

Then, for all $f_h \in C^4([0,T], V_{c,h})$ compactly supported in $(0,T)$ and satisfying (8), there exists a constant C independent of h and T such that for all h and Δt sufficiently small

$$\|u_c(t^n) - u_{c,h}^n\|_c + \|u_f(t^n) - u_{f,h}^n\|_f \leq \delta_V(h) + (1+T)\,C\,\Delta t^2 \quad \text{for all} \;\; n\,\Delta t \,\leq\, T.$$

Proof We write

$$\|u_q(t^n) - u_{q,h}^n\|_q \leq \|u_q(t^n) - u_{q,h}(t^n)\|_q + \|u_{q,h}(t^n) - u_{q,h}^n\|_q. \tag{42}$$

The first term is bounded by $\delta_V(h)$ by Assumption 1. Note that we require Δt small enough so that $f_h(\Delta t) = 0$ hence we can bound the second term of (42) by

$$(1+T)\,(c_c^{-1} + c_f^{-1})\,C\,C_u^{(4)}\,(1 + C_{\mathcal{R}} + C_{\mathcal{Q}})\Delta t^2$$

by application of Corollary 3. The statement of the corollary is obtained by observing that c_q and $C_u^{(4)}$ do not depend on h and that thanks to (41) the coefficients $C_{\mathcal{R}}$ and $C_{\mathcal{Q}}$ are independent of h as well.

4 Two preliminaries applications

4.1 Locally implicit scheme

Locally implicit strategies for wave equations have been developed and analysed by several authors, see for instance [1–3,5]. Moreover in [6] a second order and a fourth order locally implicit time discretisation based on domain decomposition have been constructed. The second order method of [6] fits naturally into the family of discrete problems (9) that we have constructed. It is obtained by choosing

$$\mathcal{P}_k(x) = 1 \quad \text{and} \quad \mathcal{P}_p(x) = 1.$$

With this choice, Assumption 3 holds for any $0 < \alpha \leq 1$. Moreover we have

$$\mathcal{Q}(x) \equiv 0 \quad \text{and} \quad \mathcal{R}(x) \equiv 1.$$

The complete scheme reads

$$\begin{cases} \dfrac{u_{c,h}^{n+1} - 2u_{c,h}^n + u_{c,h}^{n-1}}{\Delta t^2} + A_{c,h}u_{c,h}^n + B_{c,h}^t\lambda_h^n = f_h(t^n) & \text{in } V_{c,h}, \\ \dfrac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} + A_{f,h}\{u_{f,h}^n\}_{1/4} - B_{f,h}^t\lambda_h^n = 0 & \text{in } V_{f,h}, \\ B_{c,h}u_{c,h}^n = B_{f,h}u_{f,h}^n & \text{in } L_h \end{cases} \tag{43}$$

Notice that this means that domain Ω_c is discretized with an explicit leaf-frog scheme, while the domain Ω_f is discretized with an unconditionally stable implicit θ-scheme with $\theta = 1/4$. It has been shown in [6] that at each time iteration, one needs to solve the following problem

$$\begin{pmatrix} \frac{1}{\Delta t^2} I_{f,h} + \frac{1}{4}A_{f,h} & -B_{f,h}^t \\ B_{f,h} & B_{c,h}^t B_{c,h} \end{pmatrix} \begin{pmatrix} u_{f,h}^{n+1} \\ \lambda_h^n \end{pmatrix} = \begin{pmatrix} \tilde{f}_h^n \\ \tilde{g}_h^n \end{pmatrix}, \tag{44}$$

where $I_{f,h}$ is the identity operator in $V_{f,h}$ and where $\tilde{f}_h^n$ and $\tilde{g}_h^n$ are some source terms that depend on previous iterates and of $f_h(t^n)$. The invertibility of the above system is guaranteed if the discrete inf-sup condition (7) holds, as explained in Section 3.1. Application of Corollary 4 (with β as large as required) proves the space-time convergence of (43).

Note that, when applied to the wave equation (1), system (44) corresponds to solving the wave equation in Ω_f with an implicit scheme augmented by some operator acting on boundaries that accounts for the transmission of fluxes between Ω_c and Ω_f as well as the trace equality between u_c and u_f.

4.2 stabilised explicit scheme

Our objective is to construct a time discretisation that allows to treat situations for which we have

$$\rho_{f,h} \leq 4\rho_{c,h}.$$

Note that we expect $\rho_{f,h} \simeq 4\rho_{c,h}$ for the scheme to be meaningful and efficient. For instance in the case of standard finite elements on a uniform mesh for the scalar wave equation (1), if the mesh size used to discretise Ω_f is two times smaller that the mesh size used to discretise Ω_c, we have $\rho_{f,h} = 4\rho_{c,h}$. We choose

$$\mathcal{P}_p(x) = 1 - \frac{x}{16} \tag{45}$$

and set

$$\mathcal{P}_k(x) = 1 - x\,\frac{\mathcal{P}_p(x)}{4} = 1 - \frac{x}{4} + \frac{x^2}{64} \quad \left(= \left(\frac{x}{8} - 1\right)^2 \right). \tag{46}$$

We have

$$Q(x) = -\frac{3}{16} + \frac{x}{64}, \quad R(x) = \frac{1 - \frac{x}{4} - \frac{x^2}{64}}{1 - \frac{x}{16}}. \tag{47}$$

With this choice one can see that the second equation of (15) becomes

$$\big(I_{f,h} - \frac{\Delta t^2}{4}\mathcal{P}_p(\Delta t^2 A_{f,h})A_{f,h}\big)\frac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} + \mathcal{P}_p(\Delta t^2 A_{f,h})\big(A_{f,h}\{u_{f,h}^n\}_{1/4} - B_{f,h}^t\lambda_h^n\big) = 0. \tag{48}$$

Then using the algebraic relation (13), equation (48) can be simplified, we obtain

$$\frac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} + \mathcal{P}_p(\Delta t^2 A_{f,h})\big(A_{f,h}u_{f,h}^n - B_{f,h}^t\lambda_h^n\big) = 0.$$

The complete scheme reads

$$\begin{cases} \dfrac{u_{c,h}^{n+1} - 2u_{c,h}^n + u_{c,h}^{n-1}}{\Delta t^2} + A_{c,h}u_{c,h}^n + B_{c,h}^t\lambda_h^n = f_h(t^n) & \text{in } V_{c,h}, \\ \dfrac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} + \big(I_{f,h} - \dfrac{\Delta t^2}{16}A_{f,h}\big)(A_{f,h}u_{f,h}^n - B_{f,h}^t\lambda_h^n = 0 & \text{in } V_{f,h}, \\ B_{c,h}u_{c,h}^n = B_{f,h}u_{f,h}^n & \text{in } L_h. \end{cases} \tag{49}$$

Observe that at each time iteration, computing λ_h^n requires to solve:

$$\Big(B_{c,h}\, B_{c,h}^t + B_{f,h}\, \big(I_{f,h} - \frac{\Delta t^2}{16} A_{f,h}\big)\, B_{f,h}^t \Big) \lambda_h^n$$
$$= B_{f,h} f_h(t^n) - B_{c,h}\, A_{c,h} u_{c,h}^n + B_{c,h}\, \big(I_{f,h} - \frac{\Delta t^2}{16} A_{f,h}\big)\, A_{f,h} u_{f,h}^n. \quad (50)$$

Then λ_h^n is used to compute $u_{c,h}^{n+1}$ and $u_{f,h}^{n+1}$. The well-posedness property of (50) is a consequence of the discrete inf-sup condition (7). To apply Corollary 4 one needs to check that Assumption 3 holds. Since we have assumed that $\rho_{f,h} \leq 4\rho_{c,h}$, we need to check (11), which reads, using (12),

$$\mathcal{P}_k(x) \geq 0, \quad \mathcal{P}_p(x) > 0, \quad \forall x \in [0, 16\,\alpha^2].$$

From the definition of $\mathcal{P}_k(x)$ given by (46) (see Figure 1) it is clear that $\mathcal{P}_k(x) \geq 0$ for all positive α less or equal one (it has a double root at $x = 8$). However from the definition of (45) we see that $\mathcal{P}_p(x) > 0$ only if α is strictly less than one, moreover we have from (47)

$$C_{\mathcal{Q}} = \max_{x \in [0,16\alpha^2]} |\mathcal{P}_p^{-1}(x)\mathcal{Q}(x)| \underset{\alpha\to 1}{\sim} \frac{1}{16(1-\alpha^2)}, \quad C_{\mathcal{R}} = \max_{x \in [0,16\alpha^2]} |\mathcal{R}(x)|^{\frac{1}{2}} \underset{\alpha\to 1}{\sim} \frac{\sqrt{7}}{\sqrt{1-\alpha^2}}.$$

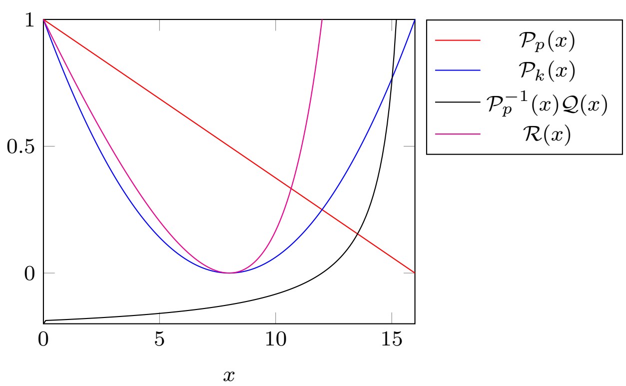


Fig. 1 Representation of $\mathcal{P}_p(x)$ defined by (45), $\mathcal{P}_k(x)$ defined by (46) and $\mathcal{P}_p^{-1}(x)\mathcal{Q}(x)$ and $\mathcal{R}(x)$ given by (47).

This estimate illustrates that the value $\alpha = 1$ is forbidden to apply Corollary 4. However we will see that in practice a value really close to one gives satisfactory results (to back up this claim, several space-time convergence curves for different values of α are presented in Section 7.2). To conclude, we have constructed a time discretisation that is stable and convergent if $\rho_{f,h} \leq 4\rho_{c,h}$ and Δt is chosen close to the optimal value $\frac{2}{\sqrt{\rho_{c,h}}}$.

5 A remark on the local time stepping of [9,10]

In equation (13.74)-(13.76) of [9], we find an algebraic formulation for conservative local time stepping. This formulation is written for the system of elastodynamics written at first order in time. However, by elimination (of the variable corresponding to the velocity), one can show that the algebraic formulation is equivalent to the following system

$$\left\{\begin{aligned} & M_{\sigma,c}\frac{\Sigma_c^{2n+1}-2\Sigma_c^{2n-1}+\Sigma_c^{2n-3}}{(2\Delta t)^2}+K_c\Sigma_c^{2n-1}-C_c^*\frac{J^{2n+1}-J^{2n-3}}{4\Delta t}=0, && \text{(51a)}\\ & M_{\sigma,f}\frac{\Sigma_f^{2n+1}-2\Sigma_f^{2n}+\Sigma_f^{2n-1}}{\Delta t^2}+K_f\Sigma_f^{2n}-C_f^*\frac{J^{2n+1}-J^{2n-1}}{2\Delta t}=0, && \text{(51b)}\\ & M_{\sigma,f}\frac{\Sigma_f^{2n}-2\Sigma_f^{2n-1}+\Sigma_f^{2n-2}}{\Delta t^2}+K_f\Sigma_f^{2n-1}-C_f^*\frac{J^{2n+1}-J^{2n-3}}{4\Delta t}=0. && \text{(51c)}\\ & C_c\Sigma_c^{2n+1}+C_f\Sigma_f^{2n+1}=0. && \text{(51d)}\end{aligned}\right.$$

In this formulation the unknowns Σ_c^n and Σ_f^n are vectors corresponding to stresses in a coarse and fine region respectively and, J^n are vectors corresponding to normal stresses, K_c and K_f are stiffness matrices (equal respectively to $B_c^* M_{v,c}^{-1} B_c$ and $B_f^* M_{v,c}^{-1} B_f$ with the notation given in [9]), $M_{\sigma,c}$ and $M_{\sigma,f}$ are mass matrices. Now we aim at eliminating intermediate steps in the evaluation for the fine region, more precisely, the sequence of even iterates $\{\Sigma_f^{2n}\}$ for $n \geq 0$. To do so, we write (51b) centered at time t^{2n} and subtract two times equation (51c) centered around t^{2n-1} and add the equation (51b) centered at time t^{2n-2}. We obtain

$$M_{\sigma,f}\frac{\Sigma_f^{2n+1}-4\Sigma_f^{2n}+6\Sigma_f^{2n-1}-4\Sigma_f^{2n-2}+\Sigma_f^{2n-3}}{\Delta t^2}+K_f\left[\Sigma_f^{2n}-2\Sigma_f^{2n-1}+\Sigma_f^{2n-2}\right]=0.$$

Now we use the fact that

$$\begin{aligned}\frac{\Sigma_f^{2n+1}-4\Sigma_f^{2n}+6\Sigma_f^{2n-1}-4\Sigma_f^{2n-2}+\Sigma_f^{2n-3}}{\Delta t^2}=&\frac{\Sigma_f^{2n+1}-2\Sigma_f^{2n-1}+\Sigma_f^{2n-3}}{\Delta t^2}\\ &-\frac{4}{\Delta t^2}\left[\Sigma_f^{2n}-2\Sigma_f^{2n-1}+\Sigma_f^{2n-2}\right],\end{aligned}\qquad (52)$$

and therefore we obtain

$$M_{\sigma,f}\frac{\Sigma_f^{2n+1}-2\Sigma_f^{2n-1}+\Sigma_f^{2n-3}}{\Delta t^2}+\left[K_f-\frac{4}{\Delta t^2}M_{\sigma,f}\right]\left(\Sigma_f^{2n}-2\Sigma_f^{2n-1}+\Sigma_f^{2n-2}\right)=0.$$

Now using (51c), we replace the quantity inside the parenthesis

$$\begin{aligned}&M_{\sigma,f}\frac{\Sigma_f^{2n+1}-2\Sigma_f^{2n-1}+\Sigma_f^{2n-3}}{\Delta t^2}\\ &\quad+\Delta t^2\left[K_f-\frac{4}{\Delta t^2}M_{\sigma,f}\right]\left(-M_{\sigma,f}^{-1}K_f\Sigma_f^{2n-1}+M_{\sigma,f}^{-1}C_f^*\frac{J^{2n+1}-J^{2n-3}}{4\Delta t}\right)=0.\end{aligned}$$

Dividing by 4 the previous equation and rearranging terms we obtain

$$M_{\sigma,f}\frac{\Sigma_f^{2n+1}-2\Sigma_f^{2n-1}+\Sigma_f^{2n-3}}{(2\Delta t)^2} + \left[I_f - \frac{(2\Delta t)^2}{16}K_f M_{\sigma,f}^{-1}\right]\left[K_f\,\Sigma_f^{2n-1} - C_f^* \frac{J^{2n+1}-J^{2n-3}}{4\Delta t}\right] = 0, \quad (53)$$

where I_f is the identity matrix in the appropriate vector space. Let us denote

$$U_f^n := \Sigma_f^{2n-1}, \quad U_c^n := \Sigma_c^{2n-1}, \quad \Delta\tau := 2\Delta t \quad \text{and} \quad \Lambda^n := \frac{J^{2n+1}-J^{2n-3}}{4\Delta t}.$$

Then collecting (51a), (51d) and (53), one can show that the following system holds

$$\begin{cases} M_{\sigma,c}\dfrac{U_c^{n+1}-2U_c^n+U_c^{n-1}}{\Delta\tau^2} + K_c U_c^n - C_c^*\Lambda^n = 0, \\ M_{\sigma,f}\dfrac{U_f^{n+1}-2U_f^n+U_f^{n-1}}{\Delta\tau^2} + \left[I_f - \dfrac{\Delta\tau^2}{16}K_f M_{\sigma,f}^{-1}\right]\left[K_f\,U_f^n - C_f^*\Lambda^n\right] = 0, \\ C_c U_c^{n+1} + C_f U_f^{n+1} = 0. \end{cases}$$

This new formulation of system (51) shows that the local time stepping proposed in [9] is also equivalent to the scheme developed in Section 4.2 (compare the above equations and (49)). Therefore the local time stepping proposed in [9] can be seen as a transmission problem between two second order schemes, one of which having a relaxed stability condition by adding stabilising terms.

6 Local time stepping using the Leap-Frog Chebychev method

6.1 Principle

In the same spirit than the stabilised scheme constructed in Section 4.2 and in light of the general observation of Section 5, we construct a method that can be characterised as a conservative local time stepping technique with an implicit treatment of transmission terms. The method we construct fits the framework we have described in Section 3 (more precisely it has the form given by equation (9)). Therefore the space-time convergence result, Corollary 4, can be applied. Because we want to construct an explicit scheme, we assume that $\mathcal{P}_k(x)$ satisfies

$$\mathcal{P}_k(x) = 1 - x\,\frac{\mathcal{P}_p(x)}{4}. \quad (54)$$

Indeed, as in Section 4.2 the second equation of (9) reads

$$\frac{u_{f,h}^{n+1}-2u_{f,h}^n+u_{f,h}^{n-1}}{\Delta t^2} + \mathcal{P}_p(\Delta t^2 A_{f,h})\big(A_{f,h}u_{f,h}^n - B_{f,h}^t\lambda_h^n\big) = 0,$$

and the complete scheme reads

$$\begin{cases} \dfrac{u_{c,h}^{n+1}-2u_{c,h}^n+u_{c,h}^{n-1}}{\Delta t^2} + A_{c,h}u_{c,h}^n + B_{c,h}^t\lambda_h^n = f_h(t^n) & \text{in } V_{c,h}, \\ \dfrac{u_{f,h}^{n+1}-2u_{f,h}^n+u_{f,h}^{n-1}}{\Delta t^2} + \mathcal{P}_p(\Delta t^2 A_{f,h})\big(A_{f,h}u_{f,h}^n - B_{f,h}^t\lambda_h^n\big) = 0 & \text{in } V_{f,h}, \\ B_{c,h}u_{c,h}^n = B_{f,h}u_{f,h}^n & \text{in } L_h. \end{cases} \quad (55)$$

We do not yet specify the polynomial $\mathcal{P}_p(x)$, but the consistency Assumption 2, that is to say $\mathcal{P}_p(0) = \mathcal{P}_k(0) = 1$, should be satisfied. By definition (54), observe that

$$\mathcal{P}_p(0) = 1 \quad \Rightarrow \quad \mathcal{P}_k(0) = 1.$$

For stability Assumption 3, one should check that

$$\mathcal{P}_k(x) = 1 - x\,\frac{\mathcal{P}_p(x)}{4} \geq 0, \quad \mathcal{P}_p(x) > 0, \quad \forall x \in \big[0, \Delta t^2\,\rho_{f,h}\big] = \big[0, 4\,\alpha^2\,\frac{\rho_{f,h}}{\rho_{c,h}}\big]$$

for some $0 < \alpha \leq 1$. Our objective is to construct a sequence of polynomials $\mathcal{P}_{p,\ell}$ that satisfy the properties

$$\mathcal{P}_{p,\ell}(0) = 1, \quad 1 - x\,\frac{\mathcal{P}_{p,\ell}(x)}{4} \geq 0, \quad \mathcal{P}_{p,\ell}(x) > 0, \quad \forall x \in \big[0, \beta_\ell\big] \tag{56}$$

for a monotonically increasing sequence β_ℓ going to infinity. This would allow to use the scheme (55) with $\mathcal{P}_p(x) \equiv \mathcal{P}_{p,\ell}(x)$ and where ℓ is chosen as follows:

- First set $\Delta t = 2\alpha/\sqrt{\rho_{c,h}}$ for some $0 < \alpha \leq 1$.
- Then, choose the smallest ℓ such that $4\,\alpha^2\,\dfrac{\rho_{f,h}}{\rho_{c,h}} \leq \beta_\ell$.

In Section 6.2 we explain how one can construct in general the sequence of polynomials that satisfy the property mentioned above for some sequence of β_ℓ that have no explicit expressions. In Section 6.3 we apply the algorithm of Section 6.2 and construct a family of polynomials for which we have $\beta_2 \simeq 36$, $\beta_3 \simeq 64$ and $\beta_4 \simeq 100$.

6.2 Construction of the polynomials sequence

To construct the sequence of polynomials that satisfy property (56) for a monotonically increasing sequence β_ℓ, we start from the polynomials introduced in [20] that correspond to shifted and stretched Chebychev's polynomials. They are given by

$$\widetilde{\mathcal{P}}_{p,\ell}(x) = \frac{2}{x}\left[1 - \mathcal{T}_{\ell+1}\Big(1 - \frac{2x}{4(\ell+1)^2}\Big)\right]$$

where $\mathcal{T}^\ell(x)$ is the ℓth Chebychev polynomial. The first polynomials being given by

$$\mathcal{T}_3(x) = 4x^3 - 3x, \quad \mathcal{T}_4(x) = 8x^4 - 8x^2 + 1, \quad \mathcal{T}_5(x) = 16x^5 - 20x^3 + 5x.$$

For $\ell = 2$ and $\ell = 3$ we have

$$\widetilde{\mathcal{P}}_{p,2}(x) = 1 - \frac{6}{3^4}x + \frac{1}{3^6}x^2, \quad \widetilde{\mathcal{P}}_{p,3}(x) = 1 - \frac{20}{4^4}x + \frac{8}{4^6}x^2 - \frac{1}{4^8}x^3,$$

and for $\ell = 4$

$$\widetilde{\mathcal{P}}_{p,4}(x) = 1 - \frac{50}{5^4}x + \frac{35}{5^6}x^2 - \frac{10}{5^8}x^3 + \frac{1}{5^{10}}x^4.$$

It is proven in [20] that the polynomials $\widetilde{\mathcal{P}}_{p,\ell}(x)$ satisfy

$$\widetilde{\mathcal{P}}_{p,\ell}(0) = 1, \quad x\,\widetilde{\mathcal{P}}_{p,\ell}(x) \leq 4 \quad \text{and} \quad 0 \leq \widetilde{\mathcal{P}}_{p,\ell}(x) \quad \text{for } x \in [0, 4(\ell+1)^2].$$

The polynomials satisfy the good requirements that we have stated in order to construct the local time stepping process, except for the fact that the $\widetilde{\mathcal{P}}_{p,\ell}(x)$ do vanish for some $x < 4(\ell+1)^2$ and therefore equation (11) is not satisfied. An idea used in [23] and [24] in the context of stabilisation of the Runge-Kutta method [?] is to slightly transform

$x\,\widetilde{\mathcal{P}}_{p,\ell}(x)$ to obtain the required behavior (i.e (11) is satisfied and the stabilisation property are kept). Note that this idea is also presented concurrently in [?] with some slight changes. We define the family of functions $(\mathcal{P}_{p,\ell}(x), \mathcal{P}_{k,\ell}(x))$ parametrized by (a, b, c, d) such that for all $x \neq 0$

$$x\,\mathcal{P}_{p,\ell}(x) = c\,(ax+b)\,\widetilde{\mathcal{P}}_{p,\ell}(ax+b) + d, \quad \mathcal{P}_{k,\ell} = 1 - x\mathcal{P}_{p,\ell}/4. \tag{57}$$

Note that $\mathcal{P}_{p,\ell}(x)$ is not a polynomial for all values of (a, b, c, d). This is why we will need to impose some conditions on these parameters, in order to ensure that $\mathcal{P}_{p,\ell}(x)$ is indeed a polynomial. Now for some given and sufficiently small ε we choose

$$c = 1 - \varepsilon/4, \quad d = \varepsilon. \tag{58}$$

Then if one sets $a = 1$ and $b = 0$ we ensure that $x\,\mathcal{P}_{p,\ell}(x) \equiv c\,x\,\widetilde{\mathcal{P}}_{p,\ell}(x) + d$ is strictly greater than 0 and less or equal to 4 for $x \in [0, 4(\ell+1)^2]$ (see Figure 2). However Assumption 2 is not satisfied for $\mathcal{P}_{p,\ell}$ and $\mathcal{P}_{k,\ell}$. We now describe a process on how to choose a and b to tackle this issue. Assumption 2 requires that

$$\mathcal{P}_{k,\ell}(0) = 1 \quad \Rightarrow \quad 1 - \frac{x\,\mathcal{P}_{p,\ell}(x)}{4}|_{x=0} = 1 \quad \Rightarrow \quad \frac{1}{4}\big(c\,b\,\widetilde{\mathcal{P}}_{p,\ell}(b) + d\big) = 0$$

and therefore

$$\mathcal{P}_{k,\ell}(0) = 1 \quad \Rightarrow \quad b\,\widetilde{\mathcal{P}}_{p,\ell}(b) = -\frac{d}{c} = \frac{\varepsilon}{\varepsilon/4 - 1}. \tag{59}$$

It can be observed that a real solution b to the equation above exists for ε sufficiently small (remark that for $\varepsilon = 0$ we have $b = 0$). Moreover it is clear from Figure 2 that b should be negative hence we choose b as the negative solution of (59) with the smallest absolute value. Moreover differentiating (57) with respect to x grants

$$\mathcal{P}_{p,\ell}(x) + x\mathcal{P}'_{p,\ell}(x) = c\,a\left(\widetilde{\mathcal{P}}_{p,\ell}(ax+b) + (ax+b)\,\widetilde{\mathcal{P}}'_{p,\ell}(ax+b)\right)$$

and therefore

$$\mathcal{P}_{p,\ell}(0) = 1 \quad \Rightarrow \quad a = \frac{1}{c\big(\widetilde{\mathcal{P}}_{p,\ell}(b) + b\,\widetilde{\mathcal{P}}'_{p,\ell}(b)\big)}. \tag{60}$$

Note that by the choices (60) and (59) we guarantee that $x\,\mathcal{P}_{p,\ell}(x)$ behaves like $o(1)$ close to $x = 0$ (because $x\mathcal{P}_{p,\ell}(x)$ vanishes at $x = 0$) hence by the definition (57), the function $\mathcal{P}_{p,\ell}(x)$ is a polynomial.

It is not clear that for ℓ going to infinity the sequence of polynomials $\mathcal{P}_{p,\ell}(x)$ have the adequate behavior. However we believe that the approach we suggest should be used for some ℓ not too large. Indeed beyond an environment dependent threshold, it happens that the proposed local time stepping strategy becomes more costly than the use of locally implicit method with some dedicated linear algebra solver.

6.3 Numerical construction

In this subsection we apply the algorithm explained above for $\ell = 3$ to $\ell = 5$. Since the proposed limit is parametrized by ε we choose $\varepsilon \in \{0.1, 0.5, 1\}$. Table 1 gives the values of (a, b) that are computed. The values of (c, d) are given by (58) from the value of ε in Table 1. We also provide a value of β_ℓ chosen by inspection that guarantees the corresponding bounds on $C_{\mathcal{Q}}$ and $C_{\mathcal{R}}$ also provided in Table 1. In Figure 3 we have plotted the obtained polynomials.

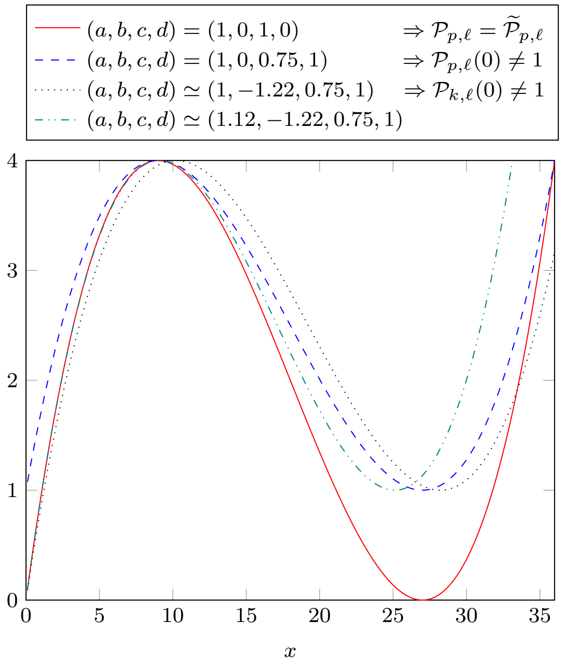


Fig. 2 Representation of $\mathcal{P}_p(x)$ for different values of the parameters (a,b,c,d)

notation	ℓ	n	ε	b	a	$\beta_{\ell,n}$	$C_{\mathcal{Q}}$	$C_{\mathcal{R}}$
$\mathcal{P}^{(1)}_{p,2}$	2	1	1	-1.220497601922388	1.123332443935161	33.13	1	5
$\mathcal{P}^{(2)}_{p,2}$	2	2	0.5	-0.548885078878804	1.055702443069509	34.62	2	7
$\mathcal{P}^{(3)}_{p,2}$	2	3	0.1	-0.101795082372209	1.010360937184039	35.73	10	17
$\mathcal{P}^{(1)}_{p,3}$	3	1	1	-1.214605698792632	1.112468647367209	57.89	1	5
$\mathcal{P}^{(2)}_{p,3}$	3	2	0.5	-0.547676655750322	1.051055803796928	60.93	2	8
$\mathcal{P}^{(3)}_{p,3}$	3	3	0.1	-0.101753177452728	1.009529277032937	63.28	10	14
$\mathcal{P}^{(1)}_{p,4}$	4	1	1	-1.211812534393700	1.107473444638217	91.38	1	8
$\mathcal{P}^{(2)}_{p,4}$	4	2	0.5	-0.547112834621174	1.048910062073242	95.85	2	13
$\mathcal{P}^{(3)}_{p,4}$	4	3	0.1	-0.101733760636154	1.009144480323238	99.19	10	30

Table 1 Computed values of a and b for given values of ε. We give an estimation of the constants $C_{\mathcal{Q}}$ and $C_{\mathcal{R}}$ obtained if the polynomial $\mathcal{P}^{(n)}_{p,\ell}(x)$ is considered on the interval $[0,\beta_\ell]$. The β^n_ℓ are chosen by inspection.

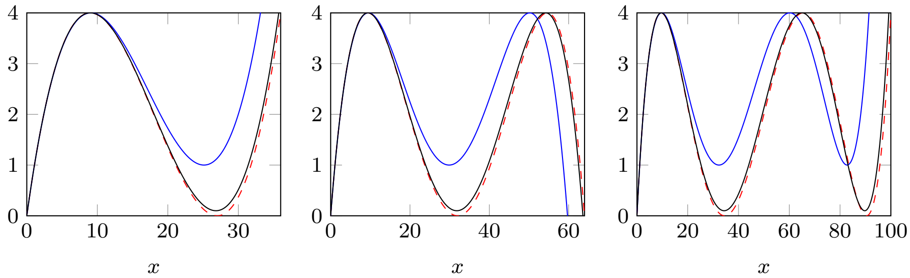
7 Numerical convergence results in 1D

In this section we present numerical results in 1D that illustrate the convergence behavior of the schemes we have proposed. We consider the wave equation (1) posed on the domain $\Omega = (-0.5, 0.5)$ with $\Omega_c = (-0.5, 0)$, $\Omega_f = (0, 0.5)$ and $\Sigma = \{0\}$. We consider the initial value problem with smooth initial data compactly supported in Ω_c such that the exact solution around the initial times is given by

$$u_c(x,0) = g(x), \quad u_c(x,\Delta t) = g(x-\Delta t) \text{ with } g(x) = 1_{[-0.3,-0.2]} e^{-2/(1-400(x+0.25)^2)}.$$

and

$$u_f(x,0) = 0, \quad u_f(x,\Delta t) = 0.$$



(a) $\ell = 3$ (b) $\ell = 4$ (c) $\ell = 5$

Fig. 3 Plain line: Representation of $x\mathcal{P}_{p,\ell}(x)$ for different values of ℓ and for $\varepsilon = 0.1$ (in black) or $\varepsilon = 1$ (in blue). Dashed line: Representation of $x\widetilde{\mathcal{P}}_{p,\ell}(x)$ for different values of ℓ.

We set $T = 0.5$. We assume that $\mu_c = 1$ and we denote $\mu \equiv \mu_f \leq 1$. Note that μ is therefore a measure of the contrast between the two subdomains. The solution is then given by

$$u(x,t) = \begin{cases} u_c(x,t) & x < 0 \\ u_f(x,t) & x > 0 \end{cases} = \begin{cases} g(x-t) + \mathcal{R}g(-x-t) & x < 0 \\ \mathcal{T}g(x - c_f t) & x > 0 \end{cases} \tag{61}$$

with

$$\mathcal{R} = \frac{1-\sqrt{\mu}}{1+\sqrt{\mu}} \text{ and } \mathcal{T} = 1 - \mathcal{R} = 2\,\frac{1}{1+\sqrt{\mu}}.$$

For the space discretisation we use standard second order Galerkin finite elements with a lumped mass matrix (see for instance [16]) on a uniform mesh of Ω_c and Ω_f and we denote $h_c = h$ and $h_f = h/q_f$ the respective mesh sizes. We have

$$h_c = h, \quad h_f = \frac{h}{q_f} \quad \text{and} \quad \frac{\rho_{f,h}}{\rho_{c,h}} = \mu\, q_f^2$$

where q_f is the refinement coefficient. We recall that $\Delta t = 2\alpha/\sqrt{\rho_{c,h}}$ for some $0 < \alpha \leq 1$. In what follows, we plot space-time convergence curves by setting α to some given values, and computing the solution of the discrete problem for some sequence h going to zero (this implies that Δt goes to zero accordingly). Then the discrete solution $(u_{c,h}^n, u_{f,h}^n)$ is compared to the analytic expression (61) and we plot

$$\frac{\sup_{n\Delta t \leq T} \|\mathcal{I}_{c,h} u_c(t^n) - u_{c,h}^n\|_c}{\sup_{n\Delta t \leq T} \|\mathcal{I}_{c,h} u_c(t^n)\|_c} + \frac{\sup_{n\Delta t \leq T} \|\mathcal{I}_{f,h} u_f(t^n) - u_{f,h}^n\|_f}{\sup_{n\Delta t \leq T} \|\mathcal{I}_{f,h} u_f(t^n)\|_f}$$

where $\mathcal{I}_{c,h}$ and $\mathcal{I}_{f,h}$ denote here the interpolation operators on the nodal finite element spaces.

7.1 Locally implicit scheme

In order to assess numerically the behavior of locally implicit schemes described in Section 4.1, we set $\mathcal{P}_k(x) = 1$ and $\mathcal{P}_p(x) = 1$. More specifically, this means that the left-hand side of the domain is discretized with an explicit leaf-frog scheme, while the right-hand side of the domain is discretized with an unconditionally stable implicit θ-scheme with $\theta = 1/4$. The convergence plots are represented in Figure 4. We have chosen two values for $\alpha \in \{0.9, 1\}$ and two values for the refinement ratio $q_f \in \{2, 3\}$. In the first four curves, μ is set equal to 0.5. It can be observed that all curves converge at second order, which was to be expected since Corollary 3 can be applied with $C_{\mathcal{Q}} = 0$ and $C_{\mathcal{R}} = 1$. Finally, we set $\mu = 0.001$ and we choose a refinement ratio $q_f = 6$, $\alpha = 1$ and we still observe an asymptotic convergence rate of 2.

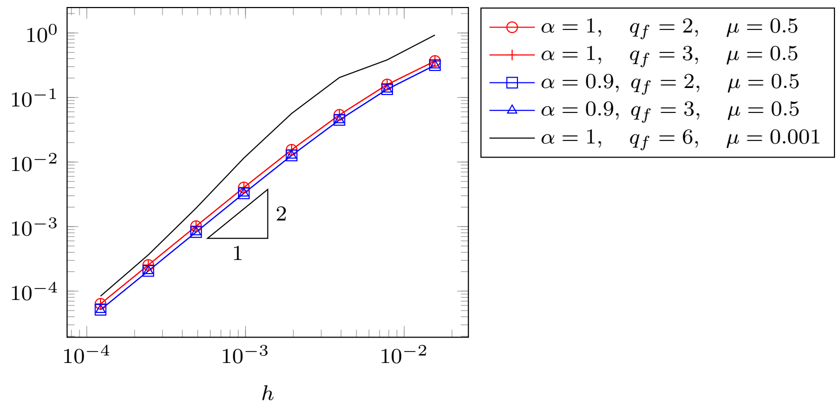


Fig. 4 Convergence curves for locally implicit schemes for different values of α, μ and q_f.

7.2 stabilised explicit scheme

In order to assess the behavior of stabilised explicit schemes described in Section 4.2, we set

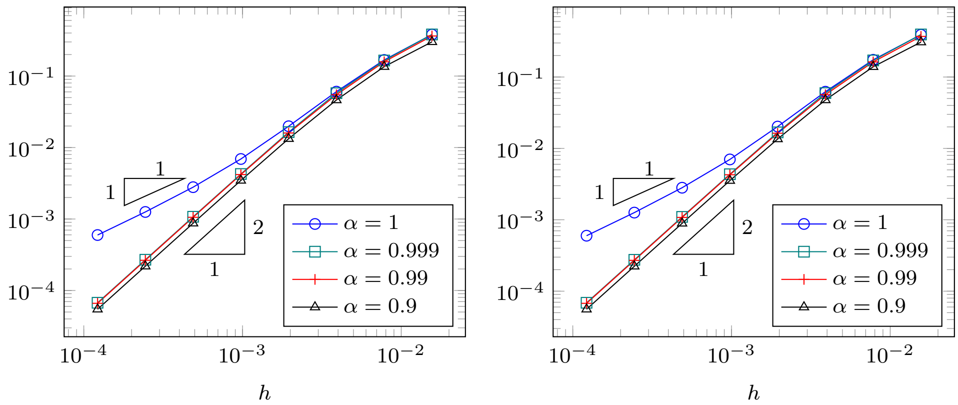
$$\mathcal{P}_p(x) = 1 - \frac{x}{16} \quad \text{and} \quad \mathcal{P}_k(x) = 1 - x\,\frac{\mathcal{P}_p(x)}{4} = 1 - \frac{x}{4} + \frac{x^2}{64}.$$

As explained in Section 5, this scheme can be seen as an alternative algebraic formulation of the local time stepping strategy described in [9,10]. We first investigate the situation of a uniform medium (with $\mu = 1$) where the right-hand side subdomain is refined by a factor 2. We make the value of α increase from $\alpha = 0.9$ to 1. As stated in Section 4.2, the value $\alpha = 1$ prevents us from applying Corollary 4 since the values of the constants blow up when α approaches 1. The numerical results displayed in Figure 5(a) show that values of α very close to 1 (up to 0.999) give the expected convergence rate of 2, and that indeed, choosing $\alpha = 1$ does not lead to a second order space/time convergence (we seem to loose one order of convergence). As a second example, we consider a inhomogeneous medium with $\mu = 0.25$, we refine the right-hand side subdomain by a factor 4, and we perform the same numerical tests. As observed in Figure 5(b), the same conclusions can be drawn.

7.3 Local time stepping using the Leap-Frog Chebychev method

Finally, in order to assess the behavior of the schemes constructed in Section 6, we set $\mathcal{P}_p(x)$ and $\mathcal{P}_k(x)$ equal to $\mathcal{P}_{p,\ell}(x)$ and $\mathcal{P}_{k,\ell}(x)$ according to the values given in Table 1. Because of the time step restriction we must find for $\mathcal{P}_{p,\ell}$ a polynomial $\mathcal{P}_{p,\ell}^{(n)}$ associated to a $\beta_{\ell,n}$ such that, from (56)

$$4\,\alpha^2\,\frac{\rho_{f,h}}{\rho_{c,h}} \leq \beta_{\ell,n} \tag{62}$$



(a) $\mu = 1$ and $q_f = 2$

(b) $\mu = 0.25$ and $q_f = 4$

Fig. 5 Convergence curves for the stabilised explicit scheme for different values of α, μ and q_f.

So, for each situation, it might possible to choose between several polynomials. First, let us consider the case of a homogeneous medium ($\mu = 1$) where the right-hand side is refined by a factor 3. We compare the space/time convergence curves obtained by using the polynomial $\widetilde{\mathcal{P}}_2$ and the different polynomials $\mathcal{P}^{(n)}_{p,2}$ for $n \in \{1, 2, 3\}$ respectively found by using $\varepsilon \in \{1, 0.5, 0.1\}$. We choose $\alpha = 0.9$ which is always compatible with condition (62) in this case. The resulting curves are displayed in Figure 6(a) where it is seen that the modified polynomials offer an expected rate of convergence of 2 while the shifted and stretched Chebychev's polynomial seems to loose one order of convergence. To further test the convergence behavior of the different schemes constructed using the polynomials $\mathcal{P}^{(n)}_{p,\ell}$, we use the values of $\beta_{\ell,n}$ given in Table 1 and choose α as

$$\alpha = \frac{1}{2}\sqrt{\beta\,\frac{\rho_{c,h}}{\rho_{f,h}}} = \frac{1}{2}\frac{1}{q_f}\frac{\sqrt{\beta}}{\sqrt{\mu}}.$$

In Figure 6(b), we have displayed the convergence for the polynomials $\mathcal{P}^{(n)}_{p,2}$ for $n \in \{1, 2, 3\}$ respectively found by using $\varepsilon \in \{1, 0.5, 0.1\}$, in the case of an inhomogeneous medium with $\mu = 0.25$ and a refinement factor of 6. The three polynomials lead to second order time/space convergence as expected, when using the highest possible value of α. Finally, we choose $\mu = 0.25$ and a refinement factor of respectively 8 and 10 and compare the shifted and stretched Chebychev's polynomials with the polynomial constructed with $\varepsilon = 0.1$ for respectively $\ell = 3$ and 4. The convergence curves are displayed in Figure 7 and both illustrate that the constructed polynomials offer the expected space/time convergence rate of 2 while the original Chebychev's polynomials seem to loose one order of convergence.

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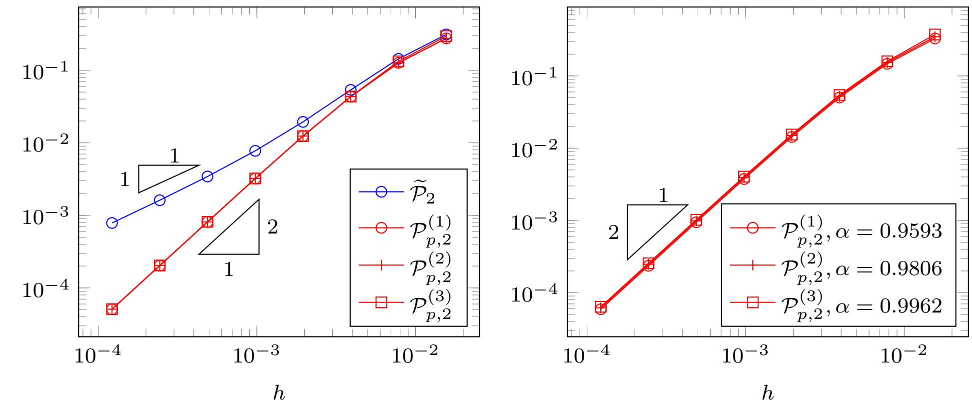
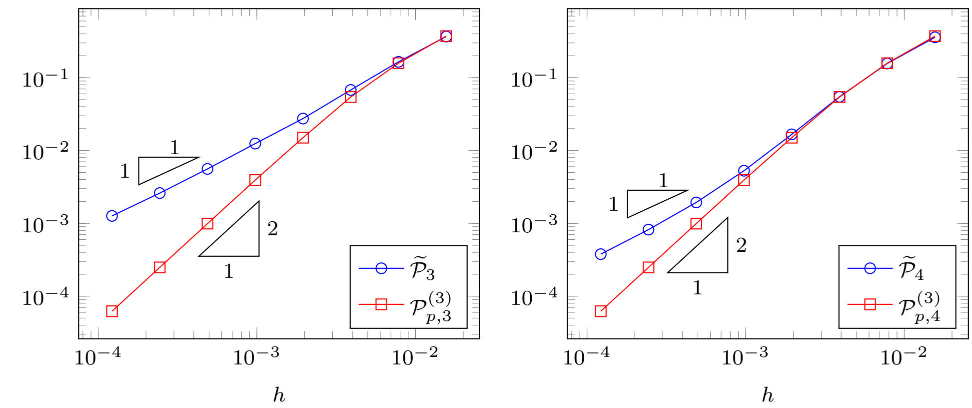


(a) $\mu = 1$, $q_f = 3$ and $\alpha = 0.9$ (b) $\mu = 0.25$ and $q_f = 6$

Fig. 6 Convergence curves for the local time stepping explicit scheme for different values of α, μ. The polynomials $\mathcal{P}^{(n)}_{p,2}$ are used for the local time stepping strategy.



(a) $\mu = 0.25$, $q_f = 8$ and $\alpha = 0.9943$ (b) $\mu = 0.25$ and $q_f = 10$, $\alpha = 0.9959$

Fig. 7 Convergence curves for the local time stepping explicit scheme for different values of α, μ. The polynomials $\mathcal{P}^{(3)}_{p,3}$ and $\mathcal{P}^{(3)}_{p,4}$ are used for the local time stepping strategy.

References

1. S. Descombes, S. Lanteri, L. Moya, Locally implicit time integration strategies in a discontinuous Galerkin method for Maxwell's equations, *Journal of Scientific Computing*, vol 56(1), pp 190–218, 2013.
2. V. Doleana, H. Fahs, L. Fezoui, S. Lanteri. Locally implicit discontinuous Galerkin method for time domain electromagnetics, *Journal of Computational Physics*, vol. 229(2), pp 512–526, 2010.
3. T. Rylander, Stability of Explicit–Implicit Hybrid Time-Stepping Schemes for Maxwell's Equations, *Journal of Computational Physics*, vol. 179 (2), pp 426–438, 2002.
4. M. Hochbruck, A. Sturm, Upwind discontinuous Galerkin space discretisation and locally implicit time integration for linear Maxwell's equations, *Mathematics of Computation*, pp. 1—33, 2018.
5. M. Hochbruck, A. Sturm, Error analysis of a second-order locally implicit method for linear Maxwell's equations, *SIAM Journal of Numerical Analysis*, vol. 54(5), pp 3167–3191, 2016.
6. J. Chabassier, S. Imperiale. Fourth order energy-preserving locally implicit time discretisation for linear wave equations. *International Journal for Numerical Methods in Engineering*, vol. 106(8), 2015.

7. M. J. Grote, M. Mehlin, S. Sauter, Convergence analysis of energy conserving explicit local time-stepping methods for the wave equation, *SIAM Journal of numerical analysis* vol. 56, no. 2, 994–1021, 2018.
8. J. Diaz, M. J. Grote Energy conserving explicit local time-stepping for second-order wave equations *SIAM Journal of Scientific Computing*, vol. 31, pp. 1985–2014, 2009.
9. G. Derveaux and P. Joly and J. Rodríguez Effective computational methods for wave propagation. Chap 13: Space time mesh refinement methods, *Chapman and Hall/CRC*, 2008.
10. J. Rodríguez A spurious-free space-time mesh refinement for elastodynamics, *International Journal For Multiscale Computational Engineering*, vol. 6(3), pp. 263–279, 2008.
11. M. Dumbser, M. Käser, E. F. Toro, An arbitrary high-order Discontinuous Galerkin method for elastic waves on unstructured meshes - V. Local time stepping and p-adaptivity, *Geophysical Journal International*, , vol. 171, pp. 695—717, 2007.
12. F. Collino, T. Fouquet, P. Joly, A conservative space-time mesh refinement method for the 1-d wave equation. Part I: Construction, *Numerische Mathematik*, vol. 95(2), pp 197–221, 2003.
13. J. Chabassier, S. Imperiale. Space/Time convergence analysis of a class of conservative schemes for linear wave equations . *Comptes Rendus Mathmatique*, 355(3), pp.282–289, 2017 .
14. R. Dautray, J-L. Lions. Mathematical Analysis and Numerical Methods for Science and Technology - Volume 5 and 6 - Evolution Problems I and II. *Springer-Verlag Berlin*, 2000.
15. G. Cohen, P. Joly, J. E. Roberts, N. Tordjman, Higher-order triangular finite elements with mass lumping for the wave equation, SIAM, *J. Numer. Anal.*, vol. 38(6), pp 2047–2078, 2001.
16. G. Cohen, Higher-order numerical methods for transient wave equations, *Springer-Verlag*, 2001.
17. Y. Maday, and A T Patera, Spectral element methods for the incompressible Navier-Stokes equations, *State-of-the-art surveys on computational mechanics*, American Society of Mechanical Engineers, 1989.
18. Y. Maday, C. Mavriplis, A. T. Patera, Nonconforming mortar element methods - Application to spectral discretisations in Domain decomposition methods, *SIAM Philadelphia*, pp 392–418, 1989.
19. J. Albella, H. Ben Dhia, , S. Imperiale, J. Rodríguez, Mathematical and numerical study of transient wave scattering by obstacles with a new class of Arlequin Method, *submitted in 2018.*
20. J. C. Gilbert, P. Joly. Higher order time stepping for second order hyperbolic problems and optimal CFL conditions. *Partial Differential Equations*, vol 16, pp. 67-93, 2008.
21. P. Joly and J. Rodríguez, Optimized higher order time discretisation of second order hyperbolic problems: construction and numerical study, *Journal of Computational and Applied Mathematics*, vol. 234(6), 2010.
22. F. Brezzi, M. Fortin, *Mixed and hybrid finite element methods*, Springer Science, vol. 15, 2012.
23. P. J. van Der Houwen, B. P. Sommeijer, On the Internal Stability of Explicit, m-Stage Runge-Kutta Methods for Large m-Values, *Journal of Applied Mathematics and Mechanics*, vol. 60(10), 1980.
24. W. Hundsdorfer, J. G. Verwer, Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations, *Springer Series in Computational Mathematics*, 2003.