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Abstract In this work we present and analyse a time discretisation strategy for linear wave equations based on domain decomposition that aims at using locally in space the most adapted time discretisation among a family of implicit or explicit centered second order schemes. They correspond respectively to local implicit schemes and to local time stepping. In the case of vanishing initial data, we show that, if some regularity properties of the source term are satisfied and if the time step verified a stability condition, then the family of proposed time discretisations provides, in the energy norm, second order space-time convergence. Finally, we provide extensive 1D numerical convergence results that confirm the obtained theoretical results and we compare our approach to other existing local time stepping strategies for wave equations.

Keywords Wave equations · Time discretisation · Domain decomposition · Local implicit scheme · Local time stepping

1 Introduction

The appropriate time integration of systems of ordinary differential equations (ODEs) resulting from the finite element discretisation in space of partial differential equations is of crucial importance to construct efficient numerical solvers. For linear wave equations problems it is well-known that fully explicit time discretisations perform better than implicit ones in non-stiff situations, i.e. when wave propagation occurs in homogeneous media and simple geometries that are quasi-uniformly meshed. However if a strong heterogeneity (high wave speed, low density) is considered, or if the mesh size and quality degenerate locally in space, then explicit methods reach their bottlenecks: the time step of the simulation must be adapted to the local pertubation of the discretisation's parameters. Local time discretisation is a well covered topic that aims at overcoming these bottlenecks and two main strategies can be distinguished

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- **Local implicit time discretisation**, see for instance [1–6]. The strategy is to treat by an implicit time integration scheme the ODEs acting on the degrees of freedom corresponding to the region where the pertubations occur. By doing so, the time step restriction (CFL) is decorrelated from the perturbations. The price to pay is that a (hopefully small) linear problem must be solved at each iteration.
- **Local Time Stepping (LTS)**, see for instance [7–11]. The strategy is to use a first time marching scheme in the whole domain and a second one in the perturbed region. The chosen type of time discretisations used in both steps is often the same but time steps differ: a smaller time step is used locally. One can distinguish non-conservative strategies (see for instance [11]) from conservative strategies. The latter are based upon Leap-Frog schemes and can be separated into two categories depending on how sub-domains are coupled.
  - **Implicit LTS.** A domain decomposition strategy is introduced at the continuous level together with some coupling conditions at the interface of the subdomains (typically by introducing a Lagrange multiplier to enforce in a weak sense those conditions). This idea can be traced back to the work of Collino et. al. [12, 31] and has been pursued and improved in [9, 10, 25–27]. Such strategy is referred as implicit since the treatment of the transmission conditions is done implicitly at the fully discrete level.
  - **Fully explicit LTS.** The decomposition of the domain is done at the discrete level through the use of a discrete restriction operator on the region – and its surrounding – where perturbations occur. The resulting scheme does not introduce transmission conditions in the classical sense but is fully explicit. It has first been proposed in [8] and various extensions have been proposed: Maxwell's equations (see [29]) and multi-level LTS (see [30]). Recently, in [7] a proof of space-time convergence is given. It shows that, for the scalar wave equation, a second order space-time convergence holds in the L^2 norm in space.

In this work we construct and analyse local time discretisations that gather in an original framework both local implicit time discretisation and conservative implicit LTS, moreover:

- We show that the proposed time discretisations provide, under some regularity and stability conditions, second order space-time convergence, in the energy norm (for scalar wave equations, it provides convergence for the H^1 norm in space).
- We provide extensive numerical convergence experiments for a 1D scalar wave propagation problem. The results show that our approach provides better space-time convergence properties, in the H^1 norm, than existing LTS approach. In particular we study some situations where the LTS of [8] converges in $\Delta t^{3/2}$ in $L^\infty(0, T; H^1(\Omega))$ whereas our approach always provides second order convergence (all computational sources concerning this last aspect are available as supplementary materials at the web link [33]).

The outline of the article is the following:

- In Section 2 we give all the necessary notations and assumptions related to the discretisation in space of linear conservative wave type problems.
- Section 3 is devoted to the introduction of a class of time discretisations – parameterised by two polynomial functions $\mathcal{P}_p$ and $\mathcal{P}_k$ – for which we show stability and second order space-time convergence results under some assumptions on the parameters (i.e. the coefficients of the polynomials $\mathcal{P}_p$ and $\mathcal{P}_k$) and some CFL conditions.
- In Section 4 we first present two preliminary applications of our discretisation framework. By adequately choosing the polynomial functions $\mathcal{P}_p$ and $\mathcal{P}_k$ we construct a

local implicit time discretisation (Section 4.1) and, in Section 4.2, a first local time-stepping scheme (with a *ratio* 2, see Section 5.2 for an accurate definition of the term *ratio*). Finally, in Section 4.3 we propose a strategy to construct general local time-stepping schemes. This strategy is based on the use of Chebychev polynomials (more precisely on Leap-Frog Chebychev method as introduced in [20]).
o Space-time numerical convergence results in 1D are given in Section 5 and illustrate the developed theory.
o Finally, in section 6 we compare our approach algorithmically to the Implicit LTS of [9]: we first explain why the proposed schemes can be seen as a generalisation of the ones proposed in [9]. Moreover, we compare our approach numerically with the Fully explicit LTS of [8] (the source code used to obtain convergence curves of this section are available as supplementary materials at the web link [33]).

2 Semi-discrete wave propagation problem

We are interested in the simulation of coupled linear wave propagation problems. The most simple example one could think of is given by the following problem: being given a bounded connected open domain Ω partitioned as two disjoint connected domains Ω_c and Ω_f, find $u_c(t) \in H^1(\Omega_c)$ and $u_f(t) \in H^1(\Omega_f)$, for all $t \in [0,T]$ such that

$$\begin{cases} \partial_t^2 u_c - \nabla \cdot \mu_c \nabla u_c = f_c & \text{in } \Omega_c, \\ \partial_t^2 u_f - \nabla \cdot \mu_f \nabla u_f = f_f & \text{in } \Omega_f, \end{cases} \tag{1}$$

and, to ease the analysis, we choose homogeneous boundary conditions on non-empty part of the domain's boundary

$$u_c = 0 \text{ on } \Gamma_c \subset \partial\Omega_c, \quad u_f = 0 \text{ on } \Gamma_f \subset \partial\Omega_f,$$

and some transmission conditions on the complementary boundary Σ

$$u_c = u_f, \quad \mu_c \nabla u_c \cdot n = \mu_f \nabla u_f \cdot n \quad \text{ on } \Sigma = \partial\Omega_c \cap \partial\Omega_f,$$

where n is the outward unitary normal of Ω_c. The scalar functions $\mu_c \in L^\infty(\Omega_c)$ and $\mu_f \in L^\infty(\Omega_f)$ are positive and bounded by below. Such problems find applications in the wave scattering by obstacles and is of interest for modeling non destructive experiments for instance.

2.1 Continuous abstract formulation

In the following q stands for either c or f.

In this section we formulate the coupled wave propagation in a more abstract framework. To do so we use notations from [14], chapter XVIII, and [22]. We assume given separable Hilbert spaces (H_q, V_q). The space H_q is equipped with the scalar product $(\cdot,\cdot)_q$, the norm in H_q is denoted $|\cdot|_q$ whereas the norms on V_q is denoted $\|\cdot\|_q$. Moreover we assume that V_q is dense and continuously embedded in H_q. We assume given a continuous hermitian bilinear form $a_q : V_q \times V_q \to \mathbb{R}$ that satisfies

$$c_q^2 \|v\|_q^2 \leq a_q(v,v), \quad \forall v \in V_q, \tag{2}$$

where c_q is a real positive scalar. We assume also being given another Hilbert space L equipped with the norm $\|\cdot\|_L$ as well as a continuous bilinear form $b_q(v,\lambda)$ on $V_q \times L$. We consider the following abstract wave propagation problems:
Let $f_c \in C^0([0,T],H_c)$ and $f_f \in C^0([0,T],H_f)$ be given, find $(u_c(t),u_f(t),\lambda(t)) \in V_c \times V_f \times L$ solution, for all $t \in [0,T]$, to the coupled system of equations

$$\begin{cases} \dfrac{d^2}{dt^2}(u_c,v_c)_c + a_c(u_c,v_c) + b_c(v_c,\lambda) = (f_c,v_c)_c & \forall\, v_c \in V_c, \\ \dfrac{d^2}{dt^2}(u_f,v_f)_f + a_f(u_f,v_f) - b_f(v_f,\lambda) = (f_f,v_f)_f & \forall\, v_f \in V_f, \\ b_c(u_c,\mu) = b_f(u_f,\mu) & \forall\, \mu \in L. \end{cases} \tag{3}$$

System (3) can be rewritten in a more compact form using the following notation: we use bold letters to define unknowns in $\boldsymbol{V} = V_c \times V_f$, e.g., $\boldsymbol{u} = (u_c,u_f)$ and we introduce the bilinear forms

$$a(\boldsymbol{u},\boldsymbol{v}) := a_c(u_c,v_c) + a_f(u_f,v_f),\ (\boldsymbol{u},\boldsymbol{v}) := (u_c,v_c)_c + (u_f,v_f)_f$$

as well as $b(\boldsymbol{v},\lambda) := b_c(v_c,\lambda) - b_f(v_f,\lambda_h)$. Then (3) can be recast as: find $(\boldsymbol{u}(t),\lambda(t)) \in \boldsymbol{V} \times L$ solution to

$$\begin{cases} \dfrac{d^2}{dt^2}(\boldsymbol{u},\boldsymbol{v}) + a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},\lambda) = (\boldsymbol{f},\boldsymbol{v}) & \boldsymbol{v} \in \boldsymbol{V}, \\ b(\boldsymbol{u},\mu) = 0 & \mu \in L. \end{cases} \tag{4}$$

We complete (4) with initial conditions

$$\boldsymbol{u}(0) = \boldsymbol{0} \quad \text{in } \boldsymbol{V}, \quad \frac{d}{dt}\boldsymbol{u}(0) = \boldsymbol{0} \quad \text{in } \boldsymbol{H}, \tag{5}$$

where $\boldsymbol{H} = H_c \times H_f$. Existence and uniqueness results for this problem rely on the assumption that a so-called inf-sup condition holds. More precisely we assume that there exists $k > 0$ such that

$$\inf_{\lambda \in L} \sup_{\boldsymbol{v} \in \boldsymbol{V}} \frac{b(\boldsymbol{v},\lambda)}{\|\lambda\|_L\, \|\boldsymbol{v}\|} \geq k.$$

where $\|\boldsymbol{v}\|^2 = \|v_c\|_c^2 + \|v_f\|_f^2$ (similarly we denote by $|\cdot|$ the composite norm in $\boldsymbol{H}$). By standard mathematical techniques (energy analysis and/or Laplace transform) see for instance [32] (and also [19]) one can prove the following lemma.

Lemma 1 *Let $k \in \mathbb{N}^*$, assume $\boldsymbol{f} \in W^{k,1}(0,T;\boldsymbol{H})$ and $\boldsymbol{f}^{(m)}(0) = \boldsymbol{0}$ for $m \in \mathbb{N}$ such that $m < k$, then, there exists a unique solution*

$$(\boldsymbol{u},\lambda) \ \in\ C^{k+1}([0,T];\boldsymbol{H}) \cap C^k([0,T];\boldsymbol{V}) \ \times\ C^{k-1}([0,T];L) \tag{6}$$

of problem (4, 5), moreover, there exists $C > 0$ depending only on c_c and c_f such that,

$$\sum_{m=0}^{k} \sup_{t\in[0,T]} \big|\frac{d^{m+1}}{dt^{m+1}}\boldsymbol{u}(t)\,\big| + \sum_{m=0}^{k} \sup_{t\in[0,T]} \|\frac{d^m}{dt^m}\boldsymbol{u}(t)\,\| \leq C \sum_{m=0}^{k} \int_0^T \big|\frac{d^m}{dt^m}\boldsymbol{f}(t)\big|\,\mathrm{d}t. \tag{7}$$

Remark 1 The scalar wave equation problem (1)-(5) enters the abstract framework presented above by choosing

$$H_q = L^2(\Omega_q), \quad V_q = \{v \in H^1(\Omega_q) \mid v = 0 \text{ on } \Gamma_q\}, \quad L = H^{-1/2}(\Sigma),$$

where H_q is equipped with the standard L^2 scalar product, and for all u and v in V_q and for all λ in $H^{-1/2}(\Sigma)$

$$a_q(u,v) = (\mu_q \nabla u, \nabla v)_q, \quad b_q(v,\lambda) = \langle v|_\Sigma, \lambda\rangle_{H^{1/2}(\Sigma),H^{-1/2}(\Sigma)}.$$

The elastodynamics equations also enter the abstract framework by writing standard transmission problems (continuity of displacements and stresses) and using vectorial forms of all the space and scalar product introduced.

Remark 2 Note that we consider in this work only zero initial data and vanishing source terms to simplify the analysis. In particular, if initial data would be considered some compatibility condition must be satisfied so that the second equation (4) holds at the initial time, and, if high regularity properties are required these compatibility condition becomes complicated. We refer to [28] for the fully discrete analysis of Leap-frog Chebychev scheme with initial data.

2.2 Discretisation in space and stability estimates

We introduce the family of finite dimensional Hilbert spaces $\{V_{q,h}\}_{h>0}$ with $V_{q,h} \subset V_q$ and $L_h \subset L$. As usual, the subscript h is devoted to tend to 0 and represents an approximation parameter of $V_{q,h}$ to V_q and L_h to L. For each h we define the operator $A_{q,h}$ as $A_{q,h} : V_{q,h} \mapsto V_{q,h}$ and

$$A_{q,h} : u_h \to A_{q,h}u_h \qquad \text{such that} \quad (A_{q,h}u_h, v_h)_q = a_q(u_h, v_h), \quad \forall v_h \in V_{q,h}.$$

Inequality (2) implies that the operator $A_{q,h}$ is self-adjoint and positive definite. Its spectrum, denoted $\mathrm{Sp}(A_{q,h})$, is composed by a finite number of positive eigenvalues . The spectral radius is defined as the maximum eigenvalue in the set $\mathrm{Sp}(A_{q,h})$, i.e.,

$$\rho_{q,h} := \max \mathrm{Sp}(A_{q,h}).$$

We also introduce a bilinear form $b_{q,h} : V_{q,h} \times L_h$ that represents an approximation of the bilinear form b_q and define the operator $B_{q,h} : V_{q,h} \mapsto L_h$ and $B^t_{q,h} : L_h \mapsto V_{q,h}$ as

$$(B_{q,h}v_{q,h}, \lambda_h)_L = (B^t_{q,h}\lambda_h, v_{q,h})_q := b_q(v_{q,h}, \lambda_h), \quad \forall v_{q,h} \in V_{q,h} \text{ and } \forall \lambda_h \in L_h.$$

As done previously we define $\boldsymbol{V}_h = V_{c,h} \times V_{f,h}$ and represents by bold letters unknowns in $\boldsymbol{V}_h$. The semi-discrete equation we consider reads:
Let $\boldsymbol{f}_h \in C^0([0,T], V_{q,h})$ be given: find $(\boldsymbol{u}_h(t), \lambda_h(t)) \in \boldsymbol{V}_h \times L_h$ and solution, for all $t \in [0,T]$, of

$$\begin{cases} \dfrac{d^2}{dt^2}u_{c,h} + A_{c,h}u_{c,h} + B^t_{c,h}\lambda_h = f_{c,h} & \text{in } V_{c,h}, \quad (a) \\ \dfrac{d^2}{dt^2}u_{f,h} + A_{f,h}u_{f,h} - B^t_{f,h}\lambda_h = f_{f,h} & \text{in } V_{f,h}, \quad (b) \\ B_{c,h}u_{c,h} = B_{f,h}u_{f,h} & \text{in } L_h, \quad (c) \end{cases} \tag{8}$$

together with the initial conditions

$$\boldsymbol{u}_h(0) = \frac{d}{dt}\boldsymbol{u}_h(0) = 0 \quad \text{in } \boldsymbol{V}_h. \tag{9}$$

In the rest of the work we assume that the following discrete inf-sup condition holds

$$\inf_{\lambda_h \in L_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}, \lambda)}{\|\lambda_h\|_L \, \|\boldsymbol{v}_h\|} \geq k. \tag{10}$$

where k is independent of h. Existence and uniqueness results for the semi-direct problem are then direct consequences of Lemma 1: we use the same scalar products and norms introduced, and in the paragraph above we choose $\boldsymbol{V} = \boldsymbol{V}_h$ and $\boldsymbol{H} = \boldsymbol{V}_h$ as well as $L = L_h$. Moreover, we define the discrete source terms $\boldsymbol{f}_h(t)$ at every time t as the orthogonal projection of $\boldsymbol{f}(t)$ in $\boldsymbol{V}_h$ with respect to the scalar product of $\boldsymbol{H}$, i.e., $\boldsymbol{f}_h(t) \in \boldsymbol{V}_h$ and

$$(\boldsymbol{f}_h(t), \boldsymbol{v}_h) = (\boldsymbol{f}(t), \boldsymbol{v}_h), \quad \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{11}$$

From this definition we deduce that if $\boldsymbol{f} \in W^{k,1}(0,T;\boldsymbol{H})$ then $\boldsymbol{f}_h \in W^{k,1}(0,T;\boldsymbol{V}_h)$ and

$$\sum_{m=0}^{k} \int_0^T |\frac{d^m}{dt^m}\boldsymbol{f}_h(t)| \, \mathrm{d}t \leq \sum_{m=0}^{k} \int_0^T |\frac{d^m}{dt^m}(t)| \, \mathrm{d}t.$$

Finally, by application of Lemma 1, there exists a unique solution $(\boldsymbol{u}_h, \lambda_h)$ to (8), it satisfies $\boldsymbol{u}_h \in C^{k+1}([0,T];\boldsymbol{V}_h)$ and $\lambda_h \in C^{k-1}([0,T];L_h)$ as well as the estimate

$$\sum_{m=0}^{k} \sup_{t \in [0,T]} |\frac{d^{m+1}}{dt^{m+1}}\boldsymbol{u}_h(t)\,| + \sum_{m=0}^{k} \sup_{t \in [0,T]} \|\frac{d^m}{dt^m}\boldsymbol{u}_h(t)\,\| \leq C \sum_{m=0}^{k} \int_0^T |\frac{d^m}{dt^m}\boldsymbol{f}(t)| \, \mathrm{d}t. \tag{12}$$

where $C > 0$ depends only on c_c and c_f. To state ta semi-discrete convergence result we introduce the discrepancy error

$$\boldsymbol{e}_h(t) = \boldsymbol{u}(t) - \boldsymbol{u}_h(t).$$

We have the following theorem.

Theorem 1 ***Convergence of the semi-discrete problem.***
Assume that $\boldsymbol{f}$ satisfies the hypotheses of Lemma 1 with $k = 3$. If we define the best approximation errors

$$\begin{aligned} \delta_h := \sup_{t \in [0,T]} \Bigg(\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} |\boldsymbol{v}_h - \boldsymbol{f}(t)| + \sum_{m=0}^{2} \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{v}_h - \frac{d^m}{dt^m}\boldsymbol{u}(t)\| \\ + \sum_{m=0}^{2} \inf_{\mu_h \in L_h} \|\mu_h - \frac{d^m}{dt^m}\lambda(t)\| \Bigg), \end{aligned} \tag{13}$$

then, there exists a C independent of T and h such that

$$\sup_{t \in [0,T]} \|\boldsymbol{e}_h(t)\| \leq C\,(1+T)\,\delta_h.$$

Proof First observe that from (11) we have

$$|\boldsymbol{f}_h(t) - \boldsymbol{f}(t)| \leq 2 \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} |\boldsymbol{v}_h - \boldsymbol{f}(t)|.$$

Then, the proof is rather standard, see [32], and is almost a carbon copy of the one given in [19] (article in review) and we just sketch the steps for the sake of completeness. From Lemma 1 we know that there exists a solution of (3) such that (6) holds with $k = 3$. Then, System (8) can be recast as: find $(\boldsymbol{u}_h(t), \lambda_h(t)) \in \boldsymbol{V}_h \times L_h$ solution to

$$\begin{cases} \dfrac{d^2}{dt^2}(\boldsymbol{u}_h, \boldsymbol{v}_h) + a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \lambda_h) = (\boldsymbol{f}_h, \boldsymbol{v}_h) & \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{u}_h, \mu_h) = 0 & \mu_h \in \boldsymbol{L}_h, \end{cases} \tag{14}$$

We introduce the elliptic projection $\hat{\boldsymbol{u}}_h(t)$ of $\boldsymbol{u}(t)$ defined as: for $(\boldsymbol{u}(t), \lambda(t)) \in \boldsymbol{V} \times L$ find $(\hat{\boldsymbol{u}}_h(t), \hat{\lambda}(t)) \in \boldsymbol{V}_h \times L_h$ such that

$$\begin{cases} a(\hat{\boldsymbol{u}}_h - \boldsymbol{u}, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \hat{\lambda}_h - \lambda) = 0 & \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\hat{\boldsymbol{u}}_h, \mu_h) = 0 & \mu_h \in \boldsymbol{L}_h. \end{cases} \tag{15}$$

Notice that $\hat{\boldsymbol{u}}_h$ can be seen as the solution of a static problem of mixed type and depends on the time t only because the terms $\boldsymbol{u}$ and λ do so. Therefore, $\hat{\boldsymbol{u}}_h$ inherits directly from $\boldsymbol{u}$ and λ its regularity in the parameter t.

Since the inf-sup condition (10) is satisfied and because of the coercivity and continuity of $a(\cdot, \cdot)$ (see (2)) one can use standard result on mixed problem – see [22] – to show that for every $t \in [0, T]$, there exists C independent of h such that

$$\|\frac{d^2}{dt^2}\hat{\boldsymbol{u}}_h - \frac{d^2}{dt^2}\boldsymbol{u}\| \leq C\,\Big(\inf_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \|\boldsymbol{v}_h - \frac{d^2}{dt^2}\boldsymbol{u}\| + \inf_{\mu_h \in L_h} \|\mu_h - \frac{d^2}{dt^2}\lambda\|_L \Big). \tag{16}$$

The strategy is now to obtain an estimation of $\|\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h\|$. One can show, using (14), (4) and (15) that

$$\begin{cases} \dfrac{d^2}{dt^2}(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h, \boldsymbol{v}_h) + a(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, \lambda_h - \hat{\lambda}_h) & \\ \qquad\qquad = -\dfrac{d^2}{dt^2}(\hat{\boldsymbol{u}}_h - \boldsymbol{u}, \boldsymbol{v}_h) + (\boldsymbol{f}_h - \boldsymbol{f}, \boldsymbol{v}_h) & \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ b(\boldsymbol{u}_h - \hat{\boldsymbol{u}}_h, \mu_h) = 0 & \mu_h \in \boldsymbol{L}_h, \end{cases}$$

By standard energy estimates, one can show that there exists C independent of T and h such that

$$\sup_{t \in [0,T]} \|\boldsymbol{u}_h(t) - \hat{\boldsymbol{u}}_h(t)\| \leq C\,T \Big(\sup_{t \in [0,T]} \|\frac{d^2}{dt^2}\boldsymbol{u}_h(t) - \frac{d^2}{dt^2}\hat{\boldsymbol{u}}(t)\| + \sup_{t \in [0,T]} |\boldsymbol{f}_h(t) - \boldsymbol{f}(t)| \Big).$$

Note that to obtain the inequality above we have used the property that $\boldsymbol{u}_h^{(m)}(0) = \boldsymbol{0}$ and $\hat{\boldsymbol{u}}_h^{(m)}(0) = \boldsymbol{0}$ for $m = 0$ and $m = 1$. Finally, the result of the theorem is then a consequence of (16), the definition of $\boldsymbol{f}_h$ and the triangle inequality (used after writing $\boldsymbol{e}_h = (\boldsymbol{u} - \hat{\boldsymbol{u}}_h) + (\hat{\boldsymbol{u}}_h - \boldsymbol{u}_h)$).

Remark 3 For simplicity of analysis we have assumed that bilinear forms are evaluated exactly. However the results presented in this work could be extended to take into account the use of quadrature formulae for the computation of space integrals. Moreover, numerical convergence results will be presented using the mass-lumping strategy that is obtained using specific quadrature formulae (see [15] or [16]).

3 Time discretisation

The schemes we construct here can be seen as perturbations of the standard centered two-steps discretisation of system (8). The perturbations are defined by two polynomials $\mathcal{P}_k(x)$ and $\mathcal{P}_p(x)$ to be determined. In this section we first construct time discretisation with the minimum assumptions concerning the properties that should be satisfied by the polynomials and then state a space-time convergence result. In Section 4 some examples are given that show how efficient local time discretisation can be constructed from adequate definition of $\mathcal{P}_k(x)$ and $\mathcal{P}_p(x)$.

3.1 Introduction of local time discretisations

We define the sequences $\{\boldsymbol{u}_h^n = (u_{c,h}^n, u_{f,h}^n)\}$ and $\{\lambda_h^n\}$ as the approximations of $\boldsymbol{u}_h(t)$ and $\lambda_h(t)$ at time $t = n\Delta t$ for a given time step $0 < \Delta t < 1$, and $n \in \{1, 2, \dots, N\}$. We define the final time of computation as $T = N\Delta t$. These sequences are constructed by solving the following problem:
Let $\boldsymbol{f}_h \in C^0([0,T], \boldsymbol{V}_h)$ be given, find $(\{\boldsymbol{u}_h^n\}, \{\lambda_h^n\})$ solution to

$$\begin{cases} \dfrac{u_{c,h}^{n+1} - 2u_{c,h}^n + u_{c,h}^{n-1}}{\Delta t^2} + A_{c,h} u_{c,h}^n + B_{c,h}^t \lambda_h^n = f_{c,h}(t^n) & \text{in } V_{c,h}, \ (a) \\ \mathcal{P}_k(\Delta t^2 A_{f,h}) \dfrac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} & \\ \qquad + \mathcal{P}_p(\Delta t^2 A_{f,h}) \big(A_{f,h} \{u_{f,h}^n\}_{1/4} - B_{f,h}^t \lambda_h^n - f_{f,h}(t^n) \big) = 0 & \text{in } V_{f,h}, \ (b) \\ B_{c,h} u_{c,h}^n = B_{f,h} u_{f,h}^n & \text{in } L_h, \quad (c) \end{cases} \tag{17}$$

where

$$\{u_{f,h}^n\}_{1/4} = \frac{u_{f,h}^{n+1} + 2u_{f,h}^n + u_{f,h}^{n-1}}{4},$$

with the initial conditions

$$\boldsymbol{u}_h^0 = \boldsymbol{u}_h^1 = \boldsymbol{0} \quad \text{in } \boldsymbol{V}_h. \tag{18}$$

Remark 4 For the sake of simplicity we have assumed that the solution $\boldsymbol{u}$ and source term $\boldsymbol{f}(t)$ vanishes at $t = 0$ hence we show later that the initial condition (18) introduces no significant consistency error.

The scheme (3) is consistent only if some conditions are satisfied on the polynomials $\mathcal{P}_k(x)$ and $\mathcal{P}_p(x)$. Since we want to construct perturbations of the standard centered scheme it seems natural to do the the following hypothesis.

Assumption 1

$$\mathcal{P}_k(0) = \mathcal{P}_p(0) = 1.$$

For stability the time step Δt can not be chosen arbitrarily. A so called CFL-condition has to be satisfied to obtain a stable scheme. In our case it corresponds to the assumption that follows.

Assumption 2 *The following CFL-condition holds: there exists* $0 < \alpha \leq 1$ *such that*

$$\Delta t = \alpha \frac{2}{\sqrt{\rho_{c,h}}}$$

and

$$\mathcal{P}_k(x) \geq 0, \quad \mathcal{P}_p(x) > 0, \quad \forall x \in [0, \Delta t^2 \rho_{f,h}]. \tag{19}$$

Note that since

$$\Delta t^2 \rho_{f,h} = 4\,\alpha^2\,\frac{\rho_{f,h}}{\rho_{c,h}} \tag{20}$$

and because of Assumption 1 we known that there exists Δt small enough or equivalently α small enough, such that (19) is satisfied for any fixed h. As shown later, these conditions ensure the positivity of a preserved discrete energy. We describe now more in detail an algorithm that computes the solution to (17). At each iteration, one needs to compute the Lagrange multiplier λ_h^n first, then compute $u_{f,h}^{n+1}$ and $u_{c,h}^{n+1}$. More precisely, using the property that

$$\{u_{f,h}^n\}_{1/4} = u_{f,h}^n + \frac{\Delta t^2}{4}\,\frac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} \tag{21}$$

we can re-write equation (17)(b) in the following form

$$\frac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} + D_{f,h}^{-1}\,\mathcal{P}_p(\Delta t^2 A_{f,h})\left(A_{f,h}u_{f,h}^n - B_{f,h}^t\lambda_h^n - f_{f,h}(t^n)\right) = 0, \tag{22}$$

with

$$D_{f,h} := \mathcal{P}_k(\Delta t^2 A_{f,h}) + \frac{\Delta t^2}{4}\mathcal{P}_p(\Delta t^2 A_{f,h})A_{f,h}.$$

Note that $D_{f,h}$ is a positive symmetric operator – hence invertible – if equation (19) holds. We now use a Schur complement technique: applying the operator $B_{f,h}$ to equation (22), applying the operator $B_{c,h}$ to (17)(a), we obtain by subtraction and thanks to (17)(c) the following system for λ_h^n

$$\begin{aligned}\Big(B_{c,h}B_{c,h}^t + B_{f,h}D_{f,h}^{-1}\,\mathcal{P}_p(\Delta t^2 A_{f,h})B_{f,h}^t\Big)\lambda_h^n &= B_{c,h}f_{c,h}(t^n) - B_{f,h}f_h(t^n)\\ &\quad - B_{f,h}A_{c,h}\,u_{c,h}^n + B_{f,h}D_{f,h}^{-1}\,\mathcal{P}_p(\Delta t^2 A_{f,h})\,A_{f,h}\,u_{f,h}^n.\end{aligned} \tag{23}$$

The well-posedness of the above problem is a consequence of the surjectivity of either $B_{c,h}$ or $B_{f,h}$ which is a consequence of the inf-sup condition (10). Assuming that $u_{f,h}^n$ and $u_{c,h}^n$ are known, then λ_h^n can be computed using (23), it follows that $u_{f,h}^{n+1}$ and $u_{c,h}^{n+1}$ can be computed using respectively (22) and (17)(a).

Remark 5 A drastic simplification occurs when

$$\mathcal{P}_k(x) = 1 - x\,\frac{\mathcal{P}_p(x)}{4}. \tag{24}$$

In that case $D_{f,h}$ is the identity operator on $V_{f,h}$ and the volumic unknown $u_{f,h}^{n+1}$ can be explicitly updated.

Remark 6 With the choice $\mathcal{P}_p(x) = 1$ and $\mathcal{P}_k(x) = 1 - x/4$ we obtain a standard coupled explicit leap-frog schemes. It is not difficult to prove that the corresponding stability condition reads

$$\Delta t^2 \le \min\big(\frac{4}{\rho_{c,h}}, \frac{4}{\rho_{f,h}}\big). \tag{25}$$

Condition (25) is penalizing since it depends in the same way in $\rho_{c,h}$ and $\rho_{f,h}$ but the latter can be large compared to $\rho_{c,h}$.

3.2 Space-time convergence analysis

We define the error terms $\boldsymbol{e}_h^n = (e_{c,h}^n, e_{f,h}^n)$ and ℓ_h^n as

$$\boldsymbol{e}_h^n = \boldsymbol{u}_h(t^n) - \boldsymbol{u}_h^n \quad \text{and} \quad \ell_h^n = \lambda_h(t^n) - \lambda_h^n.$$

In this section we show that the terms $\boldsymbol{e}_h^n$ tends to 0 as h and Δt go to 0. More precisely we show that under the conditions of assumptions 1, 2 and 3 (given below) we obtain an estimation in the norm $L^\infty(0,T;\boldsymbol{V})$ of the error in $O(\Delta t^2) + O(\delta_h)$. The section is organized as follows

- o Definition of the consistency errors: we write an equation for the sequence ($e_{c,h}^n, e_{f,h}^n$ and ℓ_h^n) that is similar to (17) with source terms that correspond to consistency errors that we will then specify.
- o Energy identity for the error equation: we proceed by energy analysis and write an energy identity satisfied by the error terms $e_{c,h}^n$ and $e_{f,h}^n$. The introduced energy is positive under the CFL-condition of Assumption 2.
- o Stability result: we prove a stability result for $\{\boldsymbol{e}_h^n\}$ in energy norm. To do so we use discrete by-parts integration and discrete energy analysis including the use of a discrete Gronwall's lemma.
- o Space-time convergence results: using Theorem 1, we deduce space-time convergence results in the norm $L^\infty(0,T;\boldsymbol{V})$,

3.2.1 Definition of the consistency errors.

We assume now that $\boldsymbol{f} \in W^{3,1}(0,T;\boldsymbol{H})$ and $\boldsymbol{f}^{(m)}(0) = 0$ for $m \in \{0,1,2\}$ and hence the solution given by Lemma 1 is sufficiently smooth so that all the manipulations and expression used below make sense, in particular,

$$\boldsymbol{u}_h \in C^4([0,T];\boldsymbol{V}_h).$$

Using equations (8) and (17) we obtain, for $n \in \{1,2,\dots,N-1\}$,

$$\begin{cases} \dfrac{e_{c,h}^{n+1} - 2e_{c,h}^n + e_{c,h}^{n-1}}{\Delta t^2} + A_{c,h} e_{c,h}^n + B_{c,h}^t \ell_h^n = r_{c,h}^n & \text{in } V_{c,h}, \\[2ex] \mathcal{P}_k(\Delta t^2 A_{f,h}) \dfrac{e_{f,h}^{n+1} - 2e_{f,h}^n + e_{f,h}^{n-1}}{\Delta t^2} & \\ \qquad + \mathcal{P}_p(\Delta t^2 A_{f,h}) \big(A_{f,h}\{e_{f,h}^n\}_{1/4} - B_{f,h}^t \ell_h^n\big) = r_{f,h}^n & \text{in } V_{f,h}, \\[2ex] B_{c,h} e_{c,h}^n = B_{f,h} e_{f,h}^n & \text{in } L_h \end{cases} \tag{26}$$

with the consistency errors given by

$$r_{c,h}^n = \frac{u_{c,h}(t^{n+1}) - 2u_{c,h}(t^n) + u_{c,h}(t^{n-1})}{\Delta t^2} + A_{c,h} u_{c,h}(t^n) + B_{c,h}^t \lambda_h(t^n) - f_{c,h}(t^n) \tag{27}$$

and

$$\begin{aligned} r_{f,h}^n = \mathcal{P}_k(\Delta t^2 A_{f,h}) & \frac{u_{f,h}(t^{n+1}) - 2u_{f,h}(t^n) + u_{f,h}(t^{n-1})}{\Delta t^2} \\ & + \mathcal{P}_p(\Delta t^2 A_{f,h}) \big(A_{f,h}\{u_{f,h}(t^n)\}_{1/4} - B_{f,h}^t \lambda_h(t^n) - f_{f,h}(t^n)\big). \end{aligned} \tag{28}$$

Standard Taylor expansions allow us to simplify equations (27) and (28). First, using equation (8)(a), there exist intermediate times $(t^{n,\heartsuit}, t^{n,\spadesuit}, t^{n,\clubsuit})$ with

$$t^{n-1} \leq t^{n,\heartsuit}, t^{n,\spadesuit}, t^{n,\clubsuit} \leq t^{n+1}$$

such that

$$r_{c,h}^n = \frac{\Delta t^2}{12} \frac{d^4}{dt^4} u_{c,h}(t^{n,\heartsuit}) \tag{29}$$

and

$$\begin{aligned} r_{f,h}^n = {} & \mathcal{P}_k(\Delta t^2 A_{f,h}) \frac{d^2}{dt^2} u_{f,h}(t^n) + \mathcal{P}_p(\Delta t^2 A_{f,h}) \big(A_{f,h} u_{f,h}(t^n) - B_{f,h}^t \lambda_h(t^n) - f_{f,h}(t^n)\big) \\ & + \frac{\Delta t^2}{12} \mathcal{P}_k(\Delta t^2 A_{f,h}) \frac{d^4}{dt^4} u_{f,h}(t^{n,\spadesuit}) + \frac{\Delta t^2}{4} \mathcal{P}_p(\Delta t^2 A_{f,h}) A_{f,h} \frac{d^2}{dt^2} u_{f,h}(t^{n,\clubsuit}). \end{aligned}$$

Then using equation (8)(b) one can simplify further $r_{f,h}^n$, we have

$$\begin{aligned} r_{f,h}^n = {} & \big(\mathcal{P}_k(\Delta t^2 A_{f,h}) - \mathcal{P}_p(\Delta t^2 A_{f,h})\big) \frac{d^2}{dt^2} u_{f,h}(t^n) \\ & + \frac{\Delta t^2}{12} \mathcal{P}_k(\Delta t^2 A_{f,h}) \frac{d^4}{dt^4} u_{f,h}(t^{n,\spadesuit}) + \frac{\Delta t^2}{4} \mathcal{P}_p(\Delta t^2 A_{f,h}) A_{f,h} \frac{d^2}{dt^2} u_{f,h}(t^{n,\clubsuit}). \end{aligned}$$

If Assumption 1 holds then there exists a polynomial function $\mathcal{Q}$ such that

$$\begin{aligned} & \big(\mathcal{P}_k(\Delta t^2 A_{f,h}) - \mathcal{P}_p(\Delta t^2 A_{f,h})\big) \frac{d^2}{dt^2} u_{f,h}(t^n) \\ & \qquad = \Delta t^2 \mathcal{P}_p(\Delta t^2 A_{f,h}) \mathcal{Q}(\Delta t^2 A_{f,h}) A_{f,h} \frac{d^2}{dt^2} u_{f,h}(t^n) \end{aligned}$$

where $\mathcal{Q}(x)$ is given by

$$\mathcal{Q}(x) := \mathcal{P}_p^{-1}(x) \frac{\mathcal{P}_k(x) - \mathcal{P}_p(x)}{x}. \tag{30}$$

The consistency error $r_{f,h}^n$ has then the final expression

$$\begin{aligned} r_{f,h}^n = {} & \frac{\Delta t^2}{12} \mathcal{P}_k(\Delta t^2 A_{f,h}) \frac{d^4}{dt^4} u_{f,h}(t^{n,\spadesuit}) + \frac{\Delta t^2}{4} \mathcal{P}_p(\Delta t^2 A_{f,h}) A_{f,h} \frac{d^2}{dt^2} u_{f,h}(t^{n,\clubsuit}) \\ & + \Delta t^2 \mathcal{P}_p(\Delta t^2 A_{f,h}) \mathcal{Q}(\Delta t^2 A_{f,h}) A_{f,h} \frac{d^2}{dt^2} u_{f,h}(t^n). \end{aligned} \tag{31}$$

3.2.2 Energy identity for the error equation

To obtain an energy identity on the error equation (26)-(29)-(31) we use standard discrete energy techniques. The main ingredients of the strategy is to observe that, if Assumption 2 holds then

$$I_{c,h} - \frac{\Delta t^2}{4} A_{c,h}$$

is a non-negative symmetric operator. Moreover, with the same assumption, if we introduce the following notation,

$$\mathcal{R}(x) := \mathcal{P}_p(x)^{-1} \mathcal{P}_k(x) \tag{32}$$

then $\mathcal{R}(\Delta t^2 A_{f,h})$ is well defined and is a non-negative symmetric operator. Note that from (32) and (30) we deduce that

$$\mathcal{Q}(x) := \frac{\mathcal{R}(x) - 1}{x}. \tag{33}$$

After standard algebraic manipulations (similar to the computations done in [6]) one can show the following lemma.

Lemma 2 *Let Assumption 2 holds. Then, for $n \in \{1, 2, \dots, N-1\}$,*

$$\frac{\mathcal{E}_{c,h}^{n+1/2} - \mathcal{E}_{c,h}^{n-1/2}}{\Delta t} + \frac{\mathcal{E}_{f,h}^{n+1/2} - \mathcal{E}_{h,h}^{n-1/2}}{\Delta t}$$
$$= \left(r_{c,h}^n, \frac{e_{c,h}^{n+1} - e_{c,h}^{n-1}}{2\Delta t} \right)_c + \left(r_{f,h}^n, \mathcal{P}_p(\Delta t^2 A_{f,h})^{-1} \frac{e_{f,h}^{n+1} - e_{f,h}^{n-1}}{2\Delta t} \right)_f, \tag{34}$$

with

$$\mathcal{E}_{c,h}^{n+1/2} = \frac{1}{2} \left| \left(I_{c,h} - \frac{\Delta t^2}{4} A_{c,h} \right)^{\frac{1}{2}} \frac{e_{c,h}^{n+1} - e_{c,h}^n}{\Delta t} \right|_c^2 + \frac{1}{2} \left| A_{c,h}^{1/2} \frac{e_{c,h}^{n+1} + e_{c,h}^n}{2} \right|_c^2, \tag{35}$$

where $I_{c,h}$ is the identity operator in H_c, and with

$$\mathcal{E}_{f,h}^{n+1/2} = \frac{1}{2} \left| \mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}} \frac{e_{f,h}^{n+1} - e_{f,h}^n}{\Delta t} \right|_f^2 + \frac{1}{2} \left| A_{f,h}^{1/2} \frac{e_{f,h}^{n+1} + e_{f,h}^n}{2} \right|_f^2. \tag{36}$$

Proof We omit the detail of the proof of Lemma 2 but instead list the main steps:

- compute the scalar product $(\cdot,\cdot)_c$ of the first equation of (26) with

$$\frac{e_{c,h}^{n+1} - e_{c,h}^{n-1}}{2\Delta t},$$

- compute the scalar product $(\cdot,\cdot)_f$ of the second equation of (26) with

$$\mathcal{P}_p(\Delta t^2 A_{f,h})^{-1} \frac{e_{f,h}^{n+1} - e_{f,h}^{n-1}}{2\Delta t},$$

- sum the two obtained equations and use the third equation of (26) to get rid of the term involving ℓ^n.
- observe that $\mathcal{E}_{c,h}^{n+1/2}$ and $\mathcal{E}_{f,h}^{n+1/2}$ are positive quadratic energy functional if Assumption 2 holds.

□

3.2.3 Stability results.

To obtain meaningful results we need more assumptions on how the spectral radius of $A_{f,h}$ behaves with respect to h compare to the spectral radius of $A_{c,h}$. More precisely we assume the following property

Assumption 3 *There exists β independent of h such that*

$$\frac{\rho_{f,h}}{\rho_{c,h}} \leq \beta^2. \tag{37}$$

Let us now suppose that Assumption 2 holds. We introduce the positive scalar $C_{\mathcal{R}}$, independent of h, as

$$C_{\mathcal{R}} := \sup_{x\in[0,4\alpha^2\beta^2]} |\mathcal{R}(x)|, \tag{38}$$

where $\mathcal{R}(x)$ is given by (32). Since $\Delta t^2 \rho_{f,h} \leq 4\alpha^2\beta^2$ one can see that for all v_h in $V_{f,h}$ the following inequality holds

$$|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}} v_h|_f \leq C_{\mathcal{R}}^{\frac{1}{2}} |v_h|_f.$$

Moreover we define $C_{\mathcal{Q}}$, independent of h, as

$$C_{\mathcal{Q}} := \sup_{x\in[0,4\alpha^2\beta^2]} |\mathcal{Q}(x)|, \tag{39}$$

where $\mathcal{Q}(x)$ is given by (33). Again, since $\Delta t^2 \rho_{f,h} \leq 4\alpha^2\beta^2$ one can see that for all v_h in $V_{f,h}$ the following inequality holds

$$|\mathcal{Q}(\Delta t^2 A_{f,h}) v_h|_f \leq C_{\mathcal{Q}} |v_h|_f.$$

Theorem 2 *Let assumptions 1, 2 and 3 hold and assume that the source term satisfied the hypothesis of Theorem 1. Then, there exists a scalar C depending on c_c, c_f and $\boldsymbol{f}$ only, such that we have for $n \in \{1, \ldots, N\}$,*

$$(\mathcal{E}_{f,h}^{n-1/2})^{\frac{1}{2}} + (\mathcal{E}_{c,h}^{n-1/2})^{\frac{1}{2}} \leq C\,(1 + C_{\mathcal{R}}^{\frac{1}{2}} + C_{\mathcal{Q}}) \left((\mathcal{E}_{f,h}^{1/2})^{\frac{1}{2}} + (\mathcal{E}_{c,h}^{1/2})^{\frac{1}{2}} + (1+T)\,\Delta t^2 \right). \tag{40}$$

Proof In what follows the scalar C is a scalar C – depending only on c_c, c_f and $\boldsymbol{f}$ – is allowed to change from one line to the other. After summing over $n = 1$ to $n = N-1$ equation (34) and taking into account equations (29) and (31), we find

$$\mathcal{E}_{c,h}^{N-1/2} + \mathcal{E}_{f,h}^{N-1/2} \leq \mathcal{E}_{c,h}^{1/2} + \mathcal{E}_{f,h}^{1/2} + C\,\Delta t^2 \big(\Xi_c^N + \Xi_f^N + \Pi_f^N + \Lambda_f^N \big)$$

where

$$\begin{cases}
\Xi_c^N = \Delta t \displaystyle\sum_{n=1}^{N-1} \left(\frac{d^4}{dt^4} u_{c,h}(t^{n,\heartsuit}), \frac{e_{c,h}^{n+1} - e_{c,h}^{n-1}}{2\Delta t} \right)_c, \\[2ex]
\Xi_f^N = \Delta t \displaystyle\sum_{n=1}^{N-1} \left(\mathcal{R}(\Delta t^2 A_{f,h}) \frac{d^4}{dt^4} u_{f,h}(t^{n,\spadesuit}), \frac{e_{f,h}^{n+1} - e_{f,h}^{n-1}}{2\Delta t} \right)_f, \\[2ex]
\Pi_f^N = \Delta t \displaystyle\sum_{n=1}^{N-1} \left(A_{f,h} \frac{d^2}{dt^2} u_{f,h}(t^{n,\clubsuit}), \frac{e_{f,h}^{n+1} - e_{f,h}^{n-1}}{2\Delta t} \right)_f, \\[2ex]
\Lambda_f^N = \Delta t \displaystyle\sum_{n=1}^{N-1} \left(\mathcal{Q}(\Delta t^2 A_{f,h}) A_{f,h} \frac{d^2}{dt^2} u_{f,h}(t^n), \frac{e_{f,h}^{n+1} - e_{f,h}^{n-1}}{2\Delta t} \right)_f.
\end{cases}$$

The proof then proceeds in five steps. One step for the estimation of each of the four above terms and a final step that collects all the obtained estimations in order to obtain (40) using a discrete Gronwall's lemma.

Step 1: Estimation of Ξ_c^N. Following the proof given in [13] (proof 2.4 of Lemma 2.3 and appendix) it is possible to show that

$$\Big|\frac{e_{c,h}^{n+1}-e_{c,h}^{n-1}}{2\Delta t}\Big|_c \leq 2\left(\mathcal{E}_{c,h}^{n+1/2}\right)^{\frac{1}{2}} + 2\left(\mathcal{E}_{c,h}^{n-1/2}\right)^{\frac{1}{2}}. \tag{41}$$

It has to be noted that this inequality holds uniformly with respect to the time step (in the limit given by Assumption 2) and in particular it is valid if $\Delta t = 2/\sqrt{\rho_{c,h}}$. This result is not trivial: it is proven using a decomposition into low and high frequency components of the solution $u_{c,h}$. Then using Cauchy-Schwarz inequality, estimate (41), as well as standard algebraic manipulations one gets

$$\Xi_c^N \leq \Delta t \sum_{n=1}^{N-1} \Big|\frac{d^4}{dt^4}u_{c,h}(t^{n,\spadesuit})\Big|_c \Big|\frac{e_{c,h}^{n+1}-e_{c,h}^{n-1}}{2\Delta t}\Big|_c \tag{42}$$

$$\leq C \sup_{t\in[0,T]} \Big|\frac{d^4}{dt^4}u_{c,h}(t)\Big|_c \ \Delta t \sum_{n=0}^{N-1} (\mathcal{E}_{c,h}^{n+1/2})^{\frac{1}{2}}. \tag{43}$$

Using the stability estimate (12) we obtain

$$\Xi_c^N \leq C\,\Delta t \sum_{n=0}^{N-1} (\mathcal{E}_{c,h}^{n+1/2})^{\frac{1}{2}}. \tag{44}$$

Step 2: Estimation of Ξ_f^N. Writing $e_{f,h}^{n+1}-e_{f,h}^{n-1} = e_{f,h}^{n+1}-e_{f,h}^{n}+e_{f,h}^{n}-e_{f,h}^{n-1}$ and using the symmetry of $\mathcal{R}(\Delta t^2 A_{f,h})$ one can show, with the Cauchy-Schwarz and triangular inequalities, that

$$\begin{aligned}\Xi_f^N \leq \frac{\Delta t}{2}\sum_{n=1}^{N-1}\Big|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}}\frac{d^4}{dt^4}u_{f,h}(t^{n,\spadesuit})\Big|_f \Bigg(&\left|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}}\frac{e_{f,h}^{n+1}-e_{f,h}^{n}}{\Delta t}\right|_f \\ &+\left|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}}\frac{e_{f,h}^{n}-e_{f,h}^{n-1}}{\Delta t}\right|_f\Bigg)\end{aligned} \tag{45}$$

then, since by definition of the energy (36) we have

$$\left|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}}\frac{e_{f,h}^{n+1}-e_{f,h}^{n}}{\Delta t}\right|_f \leq \sqrt{2}\,(\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}$$

we can simplify (45), and we obtain

$$\begin{aligned}\Xi_f^N &\leq \Delta t\,\frac{\sqrt{2}}{2}\sum_{n=1}^{N-1}\Big|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}}\frac{d^4}{dt^4}u_{f,h}(t^{n,\spadesuit})\Big|_f\big((\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}+\big(\mathcal{E}_{f,h}^{n-1/2}\big)^{\frac{1}{2}}\big)\\ &\leq C\,C_{\mathcal{R}}^{\frac{1}{2}}\sup_{t\in[0,T]}\Big|\frac{d^4}{dt^4}u_{f,h}(t)\Big|_f\ \Delta t\sum_{n=1}^{N-1}(\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}.\end{aligned} \tag{46}$$

Using the stability estimate (12) we obtain

$$\Xi_f^N \leq C\,C_{\mathcal{R}}^{\frac{1}{2}}\,\Delta t\sum_{n=0}^{N-1}(\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}. \tag{47}$$

Step 3: Estimation of Π_f^N. The difficulty here is that one can not expect in general to have a uniform bound on $A_{f,h} d_t^2 u_{f,h}(t)$ in the norm in H_f. As a matter of fact, the stability estimates (12) is sharp with respect to the involved norms. The standard strategy is to use the following equality

$$(A_{f,h} u_h, v_h)_q = (A_{f,h}^{\frac{1}{2}} u_h, A_{f,h}^{\frac{1}{2}} v_h)_q, \quad \forall (u_h, v_h) \in V_{q,h} \times V_{q,h}, \tag{48}$$

then, a discrete by part integration in time. The objective is to "exchange space and time derivatives" between the error term and the solution of the semi-discrete problem. The by-part integration in time is done using the following algebraic rule: for all sequences of real numbers $\{v^n\}$ and $\{w^n\}$ we have

$$\begin{aligned}&\sum_{n=1}^{N-1} v^n (w^{n+1} - w^{n-1}) \\ &\qquad = -\sum_{n=1}^{N-2} (v^{n+1} - v^n)(w^{n+1} + w^n) + v^{N-1}(w^N + w^{N-1}) - v^1 (w^1 + w^0).\end{aligned}$$

We apply the above equality to the term Π_f^N and use property (48) as mentioned. We obtain

$$\begin{aligned}\Pi_f^N = -\Delta t \sum_{n=1}^{N-2} &\left(\frac{A_{f,h}^{\frac{1}{2}}}{\Delta t} \Big(\frac{d^2}{dt^2} u_{f,h}(t^{n+1,\clubsuit}) - \frac{d^2}{dt^2} u_{f,h}(t^{n,\clubsuit}) \Big),\, A_{f,h}^{\frac{1}{2}} \frac{e_{f,h}^{n+1} + e_{f,h}^n}{2} \right)_f \\ &+ \left(A_{f,h}^{\frac{1}{2}} \frac{d^2}{dt^2} u_{f,h}(t^{N-1,\clubsuit}),\, A_{f,h}^{\frac{1}{2}} \frac{e_{f,h}^N + e_{f,h}^{N-1}}{2} \right)_f \\ &- \left(A_{f,h}^{\frac{1}{2}} \partial_t^2 u_{f,h}(t^{1,\clubsuit}), A_{f,h}^{\frac{1}{2}} \frac{e_{f,h}^1 + e_{f,h}^0}{2} \right)_f .\end{aligned} \tag{49}$$

Moreover using the mean value theorem we find that

$$\left| \frac{A_{f,h}^{\frac{1}{2}}}{\Delta t} \Big(\frac{d^2}{dt^2} u_{f,h}(t^{n+1,\clubsuit}) - \frac{d^2}{dt^2} u_{f,h}(t^{n,\clubsuit}) \Big) \right|_f \le \sup_{t \in [0,T]} \left| A_{f,h}^{\frac{1}{2}} \frac{d^3}{dt^3} u_{f,h}(t) \right|_f \le C, \tag{50}$$

and by the definition of the energy (36) one gets

$$\left| A_{f,h}^{\frac{1}{2}} \frac{e_{f,h}^{n+1} + e_{f,h}^n}{2} \right|_f \le \sqrt{2}\, (\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}.$$

Injecting the estimate above as well as estimate (50) into (49) one obtains after using Cauchy-Schwarz inequality

$$\Pi_f^N \le C \left((\mathcal{E}_{f,h}^{N-1/2})^{\frac{1}{2}} + (\mathcal{E}_{f,h}^{1/2})^{\frac{1}{2}} \right) + C\, \Delta t \sum_{n=1}^{N-1} (\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}. \tag{51}$$

Step 4: Estimation of Λ_f^N A similar strategy than for the estimation of Π_f^N can be applied. For that it is essential to observe the following property

$$\mathcal{Q}(\Delta t^2 A_{f,h}) A_{f,h} = A_{f,h}^{\frac{1}{2}} \mathcal{Q}(\Delta t^2 A_{f,h})\, A_{f,h}^{\frac{1}{2}},$$

that can be proven by diagonalisation of the operators involved on the family of eigenvectors of $A_{f,h}$. Then the same proof as in step 3 can be used. We obtain

$$\Lambda_f^N \leq C\, C_{\mathcal{Q}} \left((\mathcal{E}_{f,h}^{N-1/2})^{\frac{1}{2}} + (\mathcal{E}_{f,h}^{1/2})^{\frac{1}{2}} \right) + C\, C_{\mathcal{Q}}\ \Delta t \sum_{n=1}^{N-1} (\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}}, \tag{52}$$

where $C_{\mathcal{Q}}$ is given by (39).

Step 5: Final energy estimate and Gronwall's lemma application. Combining inequalities (44), (47), (51) and (52) obtained above, we find

$$\begin{aligned} \mathcal{E}_{c,h}^{N-1/2} + \mathcal{E}_{f,h}^{N-1/2} &\leq C \left(\mathcal{E}_{c,h}^{1/2} + \mathcal{E}_{f,h}^{1/2} \right) \\ &\quad + C\, \Delta t^2\ (1 + C_{\mathcal{Q}}) \left((\mathcal{E}_{f,h}^{N-1/2})^{\frac{1}{2}} + (\mathcal{E}_{f,h}^{1/2})^{\frac{1}{2}} \right) \\ &\quad + C\, \Delta t^3 \sum_{n=0}^{N-1} \left((\mathcal{E}_{c,h}^{n+1/2})^{\frac{1}{2}} + \left(1 + C_{\mathcal{R}}^{\frac{1}{2}} + C_{\mathcal{Q}}\right) (\mathcal{E}_{f,h}^{n+1/2})^{\frac{1}{2}} \right). \end{aligned} \tag{53}$$

Then using Young's inequality we write that

$$C\, \Delta t^2\, (1 + C_{\mathcal{Q}})\, (\mathcal{E}_{f,h}^{m-1/2})^{\frac{1}{2}} \leq \frac{C^2\, \Delta t^4}{2}\, (1 + C_{\mathcal{Q}})^2 + \frac{1}{2} \mathcal{E}_{f,h}^{m-1/2},$$

for $m = 1$ and $m = N$, and, using the above estimation into (53) we obtain

$$\begin{aligned} \mathcal{E}_{c,h}^{N-1/2} + \mathcal{E}_{f,h}^{N-1/2} &\leq C \left(\mathcal{E}_{c,h}^{1/2} + \mathcal{E}_{f,h}^{1/2} \right) \\ &\quad + C\, (1 + C_{\mathcal{Q}})^2\, \Delta t^4\ + C\, (1 + C_{\mathcal{R}}^{\frac{1}{2}} + C_{\mathcal{Q}})\, \Delta t^3 \sum_{n=1}^{N-1} \left(\mathcal{E}_{c,h}^{n+1/2} + \mathcal{E}_{f,h}^{n+1/2} \right)^{\frac{1}{2}}. \end{aligned} \tag{54}$$

To conclude let us state the following discrete Gronwall's lemma: for any real positive sequences $\{v^n\}$ and any positive scalar numbers A and B we have, for all $m \geq 2$,

$$v^{m-1} \leq A + B \sum_{n=0}^{m-1} (v^n)^{\frac{1}{2}} \quad \Rightarrow \quad (v^{m-1})^{\frac{1}{2}} \leq A^{\frac{1}{2}} + (m-1)B + B(v^0)^{\frac{1}{2}}.$$

Applying this result with $v^n = \mathcal{E}_{c,h}^{n+1/2} + \mathcal{E}_{f,h}^{n+1/2}$ in (54) we obtain, after some algebraic manipulations, the result of the theorem (note that we use the property $(N-1)\Delta t < T$ and the assumption $\Delta t < 1$). $\square$

Estimate (40) shows that it is important to obtain reasonable bounds on the coefficients $C_{\mathcal{R}}$ and $C_{\mathcal{Q}}$. In particular, if $\mathcal{P}(x)$ has some roots then these coefficients may blow up. This is the main difficulty that is addressed in Section 4.3.2 when constructing polynomials for explicit local time discretisation.

3.2.4 Space-time convergence results.

Corollary 1 *If the assumptions of theorems 1 and 2 hold, then, there exists C independent of Δt, T and h such that, for all $n \in \{1, \dots, N-1\}$*

$$\|\{\boldsymbol{u}_h^n\}_{1/4} - u(t^n)\| \leq C\,(1+T)\left(\Delta t^2 + \sup_{t\in[0,T]} \delta_h(t)\right). \tag{55}$$

Proof First, we observe that, for all $n \in \{1, \dots, N-1\}$,

$$\begin{aligned}\|\{\boldsymbol{e}_h^n\}_{1/4}\| &\leq \frac{1}{2}\|\frac{\boldsymbol{e}_h^{n+1}+\boldsymbol{e}_h^n}{2}\| + \frac{1}{2}\|\frac{\boldsymbol{e}_h^n+\boldsymbol{e}_h^{n-1}}{2}\| \\ &\leq \frac{c_c^{-1}}{\sqrt{2}}\big((\mathcal{E}_{c,h}^{n+1/2})^{\frac{1}{2}} + (\mathcal{E}_{c,h}^{n-1/2})^{\frac{1}{2}}\big) + \frac{c_f^{-1}}{\sqrt{2}}\big((\mathcal{E}_{f,h}^{n+1/2}))^{\frac{1}{2}} + (\mathcal{E}_{f,h}^{n-1/2})^{\frac{1}{2}}\big)\end{aligned} \tag{56}$$

where c_c and c_f are the coercivity coefficients defined in equation (2). Applying Theorem 2 we can show that there exists C independent of T and h such that,

$$\|\{\boldsymbol{e}_h^n\}_{1/4}\| \leq C\,\left((\mathcal{E}_{f,h}^{1/2})^{\frac{1}{2}} + (\mathcal{E}_{c,h}^{1/2})^{\frac{1}{2}} + (1+T)\,\Delta t^2\right).$$

Moreover, thanks to the initial condition (18) we have

$$\mathcal{E}_{f,h}^{1/2} = \frac{1}{2}\left|\mathcal{R}(\Delta t^2 A_{f,h})^{\frac{1}{2}}\frac{u_{f,h}(\Delta t) - u_{f,h}(0)}{\Delta t}\right|_f^2 + \frac{1}{2}\left|A_{f,h}^{1/2}\frac{u_{f,h}(\Delta t)+u_{f,h}(0)}{2}\right|_f^2. \tag{57}$$

However, on the one hand, $u_{f,h}(t) \in C^3([0,T], \boldsymbol{V}_h)$ and the uniform estimate (7) holds; on the other hand, we have

$$\frac{d^m}{dt^m}u_{f,h}(0) = 0, \quad m \in \{0,1,2\},$$

because of the initial condition (9) and the fact that $\boldsymbol{f}_h(0) = \boldsymbol{0}$; this two arguments implies, using a Taylor expansion, that the initial error term (57) is uniformly bounded by Δt^2. With the same arguments we can estimate $\mathcal{E}_{c,h}^{1/2}$ and we obtain the existence of another scalar C independent of T and h such that

$$\|\{\boldsymbol{e}_h^n\}_{1/4}\| \leq C(1+T)\,\Delta t^2. \tag{58}$$

The statement of the corollary is obtained using an adequate decomposition of the difference $\{\boldsymbol{u}_h^n\}_{1/4} - u(t^n)$ and triangle inequalities. More precisely we have

$$\|\{\boldsymbol{u}_h^n\}_{1/4} - u(t^n)\| \leq \|\{\boldsymbol{e}_h^n\}_{1/4}\| + \|\{\boldsymbol{u}_h(t^n)\}_{1/4} - u_h(t^n)\| + \|\boldsymbol{e}_h(t^n)\|,$$

where the first term can be estimated by (58), the second term is uniformly bounded by Δt^2 (with respect to T and h) since $\boldsymbol{u}_h \in C^2([0,T], \boldsymbol{V}_h)$ and (12) holds with $k=3$, and the last term can be estimated using Theorem 1. □

Corollary 1 means that the "good" quantity that approximates $u(t^n)$ is $\widetilde{\boldsymbol{u}}_h^n = \{\boldsymbol{u}_h^n\}_{1/4}$. By linearity $\widetilde{\boldsymbol{u}}_h^n$ can be computed solving (17) for $n \in \{0, \dots, N-2\}$ with source term $\widetilde{\boldsymbol{f}}_h^n = \{\boldsymbol{f}_h(t^n)\}_{1/4}$ instead of $\boldsymbol{f}_h(t^n)$ (we extend $\boldsymbol{f}_h(t)$ by $\boldsymbol{0}$ for negative time) and with initial data

$$\widetilde{\boldsymbol{u}}_h^0 = \widetilde{\boldsymbol{u}}_h^{-1} = \boldsymbol{0}.$$

This means that a small change in the computations of the source terms in (17) allows to recover the expected estimate $\|\widetilde{\boldsymbol{u}}_h^n - u(t^n)\| = O\big(\Delta t^2 + \delta_h\big)$. Finally, we give below a direct estimate of $\|\boldsymbol{u}_h^n - u(t^n)\|$ that holds if the source term is more regular.

Corollary 2 *If the assumptions of theorems 1 and 2 hold and if moreover,*

$$\boldsymbol{f} \in W^{4,1}(0,T;\boldsymbol{H}) \text{ and } \frac{d^m}{dt^m}\boldsymbol{f}(0) = \boldsymbol{0} \text{ for } m \in \{0,1,2,3\},$$

then, there exists C independent of Δt, T and h such that, for all $n \in \{1,\dots,N-1\}$,

$$\|\boldsymbol{u}_h^n - u(t^n)\| \leq C\,T\,(1+T)\left(\Delta t^2 + \sup_{t\in[0,T]} \delta_h(t)\right). \tag{59}$$

Proof We introduce the unknowns

$$\boldsymbol{v}_h(t) := \frac{\boldsymbol{u}_h(t) - \boldsymbol{u}_h(t-\Delta t)}{\Delta t}, \quad \boldsymbol{d}_h^n := \frac{\boldsymbol{e}_h^n - \boldsymbol{e}_h^{n-1}}{\Delta t} \text{ and } \eta_h^n := \frac{\ell_h^n - \ell_h^{n-1}}{\Delta t},$$

where $\boldsymbol{d}_h^0$ is defined by setting $\boldsymbol{e}_h^{-1} := \boldsymbol{0}$ and where $\boldsymbol{u}_h(t)$ is extended by 0 for negative value of time. Note that this extension do not deteriorate the smoothness of $\boldsymbol{u}_h$, indeed, thanks to the assumption on $\boldsymbol{f}$, one can show that $\boldsymbol{u}_h^{(m)}(0)$ vanishes for $m \in \{0,\dots,5\}$.

Then one can see that $(\boldsymbol{d}_h^n, \eta_h^n)$ satisfies the error equation (26) with a right-hand side corresponding to the finite difference of the consistency errors, i.e.,

$$\boldsymbol{s}_h^n := \frac{\boldsymbol{r}_h^n - \boldsymbol{r}_h^{n-1}}{\Delta t},$$

where $\boldsymbol{s}_h^0$ is defined by setting $\boldsymbol{r}_h^0 := \boldsymbol{0}$. A verbatim copy of Theorem 2 can be written replacing $\boldsymbol{r}_h^n$ by $\boldsymbol{s}_h^n$. We obtain that, if there exists C depending only on c_c, c_f and $\boldsymbol{f}$ such that

$$\sup_{t\in[0,T]} \left|\frac{d^4}{dt^4}\boldsymbol{v}_h(t)\right| + \sup_{t\in[0,T]} \|\frac{d^3}{dt^3}\boldsymbol{v}_h(t)\| \leq C, \tag{60}$$

then, there exists another C (with the same dependence) such that for $n \in \{1,\dots,N\}$,

$$\|\frac{\boldsymbol{d}_h^n + \boldsymbol{d}_h^{n-1}}{2}\| \leq C\,(1 + C_{\mathcal{R}}^{\frac{1}{2}} + C_{\mathcal{Q}})\,\left((\mathcal{D}_{f,h}^{1/2})^{\frac{1}{2}} + (\mathcal{D}_{c,h}^{1/2})^{\frac{1}{2}} + (1+T)\,\Delta t^2\right), \tag{61}$$

where $\mathcal{D}_{f,h}^{1/2}$ and $\mathcal{D}_{c,h}^{1/2}$ corresponds to energy terms associated to the initial data $\boldsymbol{d}_h^0$ and $\boldsymbol{d}_h^1$. With the same arguments used in Corollary 1 one can show that these terms are bounded uniformly (with respect to Δt, T and h) in Δt^2. It is easy to show that (60) holds thanks to the assumptions on the source term and the stability estimate of Lemma 1. Then, (61) holds and we deduce that

$$\|\boldsymbol{e}_h^n\| \leq \|\boldsymbol{e}_h^{n-2}\| + 2\,C\,\Delta t(1+T)\,(1 + C_{\mathcal{R}}^{\frac{1}{2}} + C_{\mathcal{Q}})\Delta t^2.$$

We proceed as in the proof of Corollary 1 to show that (59) holds.

4 Derivation of local implicit or explicit time discretisations

In this section we derive three specific local time discretisations that enter the framework described in Section 3. The schemes we present are of increasing complexity and are constructed assuming $\rho_{c,h}$ and $\rho_{f,h}$ known.

4.1 Local implicit scheme

Local implicit strategies for wave equations have been developed and analysed by several authors, see for instance [1–3, 5]. Moreover in [6] a second order and a fourth order local implicit time discretisation based on domain decomposition have been constructed. The second order method of [6] fits naturally into the family of discrete problems (17) that we have constructed. It is obtained by choosing

$$\mathcal{P}_k(x) = 1 \quad \text{and} \quad \mathcal{P}_p(x) = 1.$$

With this choice, Assumption 2 holds for any $0 < \alpha \leq 1$. Moreover we have

$$\mathcal{Q}(x) \equiv 0 \quad \text{and} \quad \mathcal{R}(x) \equiv 1.$$

The complete scheme reads

$$\begin{cases} \dfrac{u_{c,h}^{n+1} - 2u_{c,h}^n + u_{c,h}^{n-1}}{\Delta t^2} + A_{c,h}u_{c,h}^n + B_{c,h}^t \lambda_h^n = f_{c,h}(t^n) & \text{in } V_{c,h}, \\ \dfrac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} + A_{f,h}\{u_{f,h}^n\}_{1/4} - B_{f,h}^t \lambda_h^n = f_{f,h}(t^n) & \text{in } V_{f,h}, \\ B_{c,h}u_{c,h}^n = B_{f,h}u_{f,h}^n & \text{in } L_h. \end{cases} \tag{62}$$

Notice that this means that the first equation of (8) is discretised with an explicit leaf-frog scheme, while the second is discretised with an unconditionally stable implicit θ-scheme with $\theta = 1/4$. It has been shown in [6] that at each time iteration, one needs to solve the following problem

$$\begin{pmatrix} \frac{1}{\Delta t^2} I_{f,h} + \frac{1}{4} A_{f,h} & -B_{f,h}^t \\ B_{f,h} & B_{c,h}^t B_{c,h} \end{pmatrix} \begin{pmatrix} u_{f,h}^{n+1} \\ \lambda_h^n \end{pmatrix} = \begin{pmatrix} \hat{f}_{f,h}^n \\ \hat{g}_h^n \end{pmatrix}, \tag{63}$$

where $I_{f,h}$ is the identity operator in $V_{f,h}$ and where $\hat{f}_h^n$ and $\hat{g}_h^n$ are some source terms that depend on previous iterates and of $\boldsymbol{f}_h(t^n)$. The invertibility of the above system is guaranteed if the discrete inf-sup condition (10) holds as explained in Section 3.1. Since $\mathcal{Q}(x)$ and $\mathcal{R}(x)$ are independent of x then $C_{\mathcal{Q}} = 0$ and $C_{\mathcal{R}} = 1$ are obviously independent of β that can be arbitrarily high (hence the ratio $\rho_{f,h}/\rho_{c,h}$ can be arbitrarily high). Finally, the application of Corollary 1 proves the space-time convergence of (62).

Note that, when applied to the wave equation (1), solving System (63) corresponds to solving the wave equation in Ω_f with an implicit scheme augmented by some operator acting on boundaries that accounts for the transmission of fluxes between Ω_c and Ω_f as well as the equality between u_c and u_f on Σ. This scheme is particularly adapted if a very strong and very local heterogeneity is considered in the propagating medium. In that case (63) is not well conditioned but the algebraic system has a small size and can be solved efficiently.

4.2 Stabilised explicit scheme

Our objective is to construct a time discretisation that allows to treat situations for which we have

$$\rho_{f,h} \leq 4\rho_{c,h}, \tag{64}$$

that is to say $\beta = 2$ in Assumption 3. Note that we expect $\rho_{f,h} \simeq 4\rho_{c,h}$ for the scheme to be meaningful and efficient. For instance, in the case of standard finite elements on a uniform mesh for the scalar wave equation (1), if the mesh size used to discretise Ω_f is two times smaller that the mesh size used to discretise Ω_c, we have $\rho_{f,h} = 4\rho_{c,h}$.

The scheme is constructed by choosing

$$\mathcal{P}_p(x) = 1 - \frac{x}{16} \tag{65}$$

and set

$$\mathcal{P}_k(x) = 1 - x\,\frac{\mathcal{P}_p(x)}{4} = 1 - \frac{x}{4} + \frac{x^2}{64} \quad \Big(= \big(\frac{x}{8} - 1\big)^2\Big). \tag{66}$$

With this choice, the scheme (17) is explicit (see Remark 5). The complete scheme reads

$$\begin{cases} \dfrac{u_{c,h}^{n+1} - 2u_{c,h}^n + u_{c,h}^{n-1}}{\Delta t^2} + A_{c,h}u_{c,h}^n + B_{c,h}^t\lambda_h^n = f_{c,h}(t^n) & \text{in } V_{c,h}, \\ \dfrac{u_{f,h}^{n+1} - 2u_{f,h}^n + u_{f,h}^{n-1}}{\Delta t^2} & \\ \qquad + \big(I_{f,h} - \dfrac{\Delta t^2}{16}A_{f,h}\big)(A_{f,h}u_{f,h}^n - B_{f,h}^t\lambda_h^n - f_{f,h}(t^n)) = 0 & \text{in } V_{f,h}, \\ B_{c,h}u_{c,h}^n = B_{f,h}u_{f,h}^n & \text{in } L_h. \end{cases} \tag{67}$$

Observe that at each time iteration, computing λ_h^n requires to solve:

$$\begin{aligned} \Big(B_{c,h}\,B_{c,h}^t + B_{f,h}\,\big(I_{f,h} - \frac{\Delta t^2}{16}A_{f,h}\big)\,B_{f,h}^t \Big)\lambda_h^n &= B_{c,h}f_{c,h}(t^n) - B_{f,h}f_h(t^n) \\ &- B_{c,h}\,A_{c,h}u_{c,h}^n + B_{c,h}\,\big(I_{f,h} - \frac{\Delta t^2}{16}A_{f,h}\big)\,A_{f,h}u_{f,h}^n. \end{aligned} \tag{68}$$

Then λ_h^n is used to compute $u_{c,h}^{n+1}$ and $u_{f,h}^{n+1}$ explicitly using the first two equations of (67). The well-posedness property of (68) is a consequence of the discrete inf-sup condition (10). To apply Corollary 1 one needs to check that Assumption 2 holds. Since we have assumed $\beta = 2$ (i.e. $\rho_{f,h} \leq 4\rho_{c,h}$), we need to check (19), which reads, using (20),

$$\mathcal{P}_k(x) \geq 0, \quad \mathcal{P}_p(x) > 0, \quad \forall x \in [0, 16\,\alpha^2].$$

From the definition of $\mathcal{P}_k(x)$ given by (66) (see Figure 1) it is clear that $\mathcal{P}_k(x) \geq 0$ for all positive α less or equal one (it has a double root at $x = 8$). However from the definition of (65) we see that $\mathcal{P}_p(x) > 0$ only if α is strictly less than one, moreover we have

$$Q(x) = \frac{-\frac{3}{16} + \frac{x}{64}}{1 - \frac{x}{16}}, \quad R(x) = \frac{1 - \frac{x}{4} + \frac{x^2}{64}}{1 - \frac{x}{16}}. \tag{69}$$

and therefore

$$C_{\mathcal{Q}} = \max_{x\in[0,16\alpha^2]} |\mathcal{Q}(x)| \underset{\alpha\to 1}{\sim} \frac{1}{16(1-\alpha^2)}, \quad C_{\mathcal{R}} = \max_{x\in[0,16\alpha^2]} |\mathcal{R}(x)| \underset{\alpha\to 1}{\sim} \frac{1}{(1-\alpha^2)}.$$

This estimate illustrates that the value $\alpha = 1$ is forbidden to apply Theorem 2. However we will see that in practice a value really close to 1 gives satisfactory results (to back up this claim, several space-time convergence curves for different values of α are presented in Section 5.2). To conclude, we have constructed a time discretisation that is stable and convergent if $\rho_{f,h} \leq 4\rho_{c,h}$ and Δt is chosen below the optimal value $2/\sqrt{\rho_{c,h}}$.

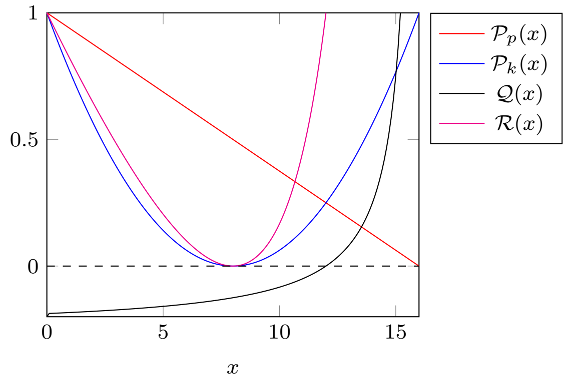


Fig. 1 Representation of $\mathcal{P}_p(x)$ defined by (65), $\mathcal{P}_k(x)$ defined by (66) and $\mathcal{Q}(x)$ and $\mathcal{R}(x)$ given by (69).

4.3 Local time discretisation using the Leap-Frog Chebychev method

4.3.1 Principle

In the same spirit than Section 4.2, we construct now a method that can be characterised as a conservative local time stepping technique with an implicit treatment of transmission terms. As in Section 4.2, the unknown $u^n_{f,h}$ should be explicitly update, hence following Remark 5, we assume that $\mathcal{P}_k(x)$ satisfies

$$\mathcal{P}_k(x) = 1 - x\,\frac{\mathcal{P}_p(x)}{4}. \tag{70}$$

The complete scheme reads

$$\begin{cases} \dfrac{u^{n+1}_{c,h} - 2u^n_{c,h} + u^{n-1}_{c,h}}{\Delta t^2} + A_{c,h}u^n_{c,h} + B^t_{c,h}\lambda^n_h = f_h(t^n) & \text{in } V_{c,h}, \\ \dfrac{u^{n+1}_{f,h} - 2u^n_{f,h} + u^{n-1}_{f,h}}{\Delta t^2} & \\ \qquad +\mathcal{P}_p(\Delta t^2 A_{f,h})\big(A_{f,h}u^n_{f,h} - B^t_{f,h}\lambda^n_h - f_{f,h}(t^n)\big) = 0 & \text{in } V_{f,h}, \\ B_{c,h}u^n_{c,h} = B_{f,h}u^n_{f,h} & \text{in } L_h. \end{cases} \tag{71}$$

We do not yet specify the polynomial $\mathcal{P}_p(x)$, but from Assumption 1 (consistency assumption) we must have $\mathcal{P}_p(0) = \mathcal{P}_k(0) = 1$ and remark that by definition (70) we have

$$\mathcal{P}_p(0) = 1 \quad \Rightarrow \quad \mathcal{P}_k(0) = 1.$$

Our objective is then to construct a sequence of polynomials $\mathcal{P}_{p,\ell}$ that satisfy the properties

$$\mathcal{P}_{p,\ell}(0) = 1, \quad 1 - x\,\frac{\mathcal{P}_{p,\ell}(x)}{4} \geq 0, \quad \mathcal{P}_{p,\ell}(x) > 0, \quad \forall x \in \big[0, 4\,\beta_\ell^2\big] \tag{72}$$

for a monotonically increasing sequence $\beta_\ell > 2$. To satisfy Assumption 2 (stability assumption) one should check that

$$\Delta t^2\,\rho_{f,h} \leq \beta_\ell^2 \quad \underset{(20)}{\Rightarrow} \quad \frac{\rho_{f,h}}{\rho_{c,h}} \leq \beta_\ell^2.$$

Which means that Assumption 3 should be verified with $\beta = \beta_\ell$. Observe that this is an improvement compare to the condition (64) associated to the scheme (67).

In Section 4.3.2 below we present a procedure to construct the sequence of polynomials that satisfy the property (72) for some increasing sequence of β_ℓ. In Section 4.3.3 we apply the algorithm of Section 4.3.2 and construct a family of polynomials for which we have $\beta_2 \simeq 3$, $\beta_3 \simeq 4$ and $\beta_4 \simeq 5$.

Remark 7 The stability condition of the scheme (71) with $\mathcal{P}_p \equiv \mathcal{P}_{p,\ell}$ can be rewritten

$$\Delta t^2 \leq \min\big(\frac{4\alpha^2}{\rho_{c,h}}, \frac{4\beta_\ell^2}{\rho_{f,h}}\big). \tag{73}$$

This is clearly in improvement compared to the stability condition (25). Finally, in practice, to ensure that (73) holds one can choose $\alpha \leq 1$, set $\Delta t = 2\alpha/\sqrt{\rho_{c,h}}$ and set $\mathcal{P}_p(x) \equiv \mathcal{P}_{p,\ell}(x)$ where ℓ is the smallest integer such that $\alpha^2\,\rho_{f,h} \leq \beta_\ell^2\,\rho_{c,h}$.

4.3.2 Construction of a parametrized polynomials sequence

To construct the sequence of polynomials that satisfy property (72) for a monotonically increasing sequence β_ℓ, we start from the polynomials introduced in [20] that correspond to shifted and stretched Chebychev's polynomials. They are given by

$$\widetilde{\mathcal{P}}_{p,\ell}(x) = \frac{2}{x}\left[1 - \mathcal{T}_{\ell+1}\Big(1 - \frac{2x}{4(\ell+1)^2}\Big)\right] \tag{74}$$

where $\mathcal{T}_\ell(x)$ is the ℓth Chebychev polynomial. The first polynomials being given by

$$\mathcal{T}_3(x) = 4x^3 - 3x, \quad \mathcal{T}_4(x) = 8x^4 - 8x^2 + 1, \quad \mathcal{T}_5(x) = 16x^5 - 20x^3 + 5x,$$

hence for $\ell = 2$ and $\ell = 3$, we have

$$\widetilde{\mathcal{P}}_{p,2}(x) = 1 - \frac{6}{3^4}x + \frac{1}{3^6}x^2, \quad \widetilde{\mathcal{P}}_{p,3}(x) = 1 - \frac{20}{4^4}x + \frac{8}{4^6}x^2 - \frac{1}{4^8}x^3,$$

and for $\ell = 4$,

$$\widetilde{\mathcal{P}}_{p,4}(x) = 1 - \frac{50}{5^4}x + \frac{35}{5^6}x^2 - \frac{10}{5^8}x^3 + \frac{1}{5^{10}}x^4.$$

It is proven in [20] that the polynomials $\widetilde{\mathcal{P}}_{p,\ell}(x)$ satisfy

$$\widetilde{\mathcal{P}}_{p,\ell}(0) = 1, \quad 1 - \frac{x\,\widetilde{\mathcal{P}}_{p,\ell}(x)}{4} \geq 0, \quad \widetilde{\mathcal{P}}_{p,\ell}(x) \geq 0 \quad \forall x \in [0, 4(\ell+1)^2]. \tag{75}$$

The polynomials satisfy the good requirements that we have stated in order to construct the local time stepping process, i.e. (72), except for the fact that the $\widetilde{\mathcal{P}}_{p,\ell}(x)$ do vanish for some $x < 4(\ell+1)^2$. An idea used in [23] and [24] in the context of stabilisation of the Runge-Kutta method is to transform $\widetilde{\mathcal{P}}_{p,\ell}(x)$ to obtain the required behavior (i.e. $\widetilde{\mathcal{P}}_{p,\ell}(x) > 0$). Note that a similar idea is used concurrently in the context of non-linear wave propagation phenomena in [28]. We define the family of functions $\mathcal{P}^\varepsilon_{p,\ell}(x)$ parametrized by (a, b, ε) such that, for ε positive and sufficiently small,

$$\mathcal{P}^\varepsilon_{p,\ell}(x) = \frac{1}{x}\left((1 - \frac{\varepsilon}{4})(ax+b)\,\widetilde{\mathcal{P}}_{p,\ell}(ax+b) + \varepsilon\right). \tag{76}$$

One can see that if $a = 1$, $b = 0$ and $\varepsilon = 0$ one recovers $\mathcal{P}^\varepsilon_{p,\ell} = \widetilde{\mathcal{P}}_{p,\ell}$ and for a given ε small enough we propose a procedure (see Figure 2) that compute $a \equiv a^\varepsilon$ and $b \equiv b^\varepsilon$

such that $\mathcal{P}^{\varepsilon}_{p,\ell}$ is a well-defined polynomial and consistency as well as stability are ensured. Namely, one should check that

(i) $\mathcal{P}^{\varepsilon}_{p,\ell}$ is a polynomial, (ii) $\mathcal{P}^{\varepsilon}_{p,\ell}(0) = 1$, (iii) Eq. (72) holds.

Step (i). From the definition (76) one can check that $\mathcal{P}^{\varepsilon}_{p,\ell}$ is a polynomial if the a-priori blow-up at $x = 0$ is compensated, this means, that one should have,

$$\big(1 - \frac{\varepsilon}{4}\big) b^{\varepsilon}\, \widetilde{\mathcal{P}}_{p,\ell}(b^{\varepsilon}) + \varepsilon = 0 \quad \Rightarrow \quad b^{\varepsilon}\, \widetilde{\mathcal{P}}_{p,\ell}(b^{\varepsilon}) = -\frac{\varepsilon}{1 - \frac{\varepsilon}{4}} \tag{77}$$

It can be observed that b^{ε} is a root of polynomial of order $\ell + 1$. However since the polynomial $x\widetilde{\mathcal{P}}_{p,\ell}(x)$ behave like the linear function x is a neighborhood of $x = 0$ for ε positive and sufficiently small there exists a real negative solution to (77). Hence, we choose b^{ε} as the negative solution of (77) with the smallest absolute value. Note that we have $b^{\varepsilon} \to 0$ when $\varepsilon \to 0$.

Step (ii). To satisfy Assumption 1, i.e. the consistency assumption, one must check that $\mathcal{P}^{\varepsilon}_{p,\ell}(0) = 1$.. To do so we first differentiate (76) with respect to x, we obtain

$$\mathcal{P}^{\varepsilon}_{p,\ell}(x) + x(\mathcal{P}^{\varepsilon}_{p,\ell})'(x) = \big(1 - \frac{\varepsilon}{4}\big)\, a^{\varepsilon}\, \big(\widetilde{\mathcal{P}}_{p,\ell}(a^{\varepsilon}x + b^{\varepsilon}) + (a^{\varepsilon}x + b^{\varepsilon})\, \widetilde{\mathcal{P}}'_{p,\ell}(a^{\varepsilon}x + b^{\varepsilon})\big)$$

and therefore

$$\mathcal{P}^{\varepsilon}_{p,\ell}(0) = 1 \quad \Rightarrow \quad a^{\varepsilon} = \frac{1}{\big(1 - \frac{\varepsilon}{4}\big)\big(\widetilde{\mathcal{P}}_{p,\ell}(b^{\varepsilon}) + b^{\varepsilon}\, \widetilde{\mathcal{P}}'_{p,\ell}(b^{\varepsilon})\big)}. \tag{78}$$

Note that for ε positive and small enough the coefficient a^{ε} is well defined and close to one and we we have $a^{\varepsilon} \to 1$ when $\varepsilon \to 0$.

Step (iii). From the definition (76) and the property (75) one can see that (72) holds for some $\beta^{\varepsilon}_{\ell} \leq (\ell + 1)$. Thanks to the definition of a^{ε} and b^{ε}we have

$$\mathcal{P}^{\varepsilon}_{p,\ell}(x) \underset{\varepsilon \to 0}{\to} \widetilde{\mathcal{P}}_{p,\ell}(x)$$

and therefore $\beta^{\varepsilon}_{\ell} \to \ell + 1$ when $\varepsilon \to 0$.

4.3.3 Numerical construction

In this subsection we apply the algorithm explained above for $\ell = 3$ to $\ell = 5$. Since the proposed limit is parametrized by ε we choose $\varepsilon \in \{0.1, 0.5, 1\}$. Table 1 gives the values of $(a^{\varepsilon}, b^{\varepsilon})$ that are computed. We also provide a value of $\beta_{\ell} \equiv \beta^{\varepsilon}_{\ell}$, obtained numerically, that guarantees the corresponding bounds on $C_{\mathcal{Q}} \equiv C^{\varepsilon}_{\mathcal{Q}}$ and $C_{\mathcal{R}} \equiv C^{\varepsilon}_{\mathcal{R}}$ (defined in (38) and (39)) provided in Table 1. In Figure 3 we have plotted the obtained polynomials.

The process we have presented can be used for arbitrarily large ℓ however ε should be chosen small enough (and presumably smaller and smaller as ℓ increase) and in that case the constant $C^{\varepsilon}_{\mathcal{Q}}$ and $C^{\varepsilon}_{\mathcal{R}}$ will degenerate. Numerical results presented Section 5.2 and Section 5.3 confirm the fact that it is necessary to have $C^{\varepsilon}_{\mathcal{Q}}$ and $C^{\varepsilon}_{\mathcal{R}}$ bounded to obtain second order convergence in the norm $\|\cdot\|_q$.

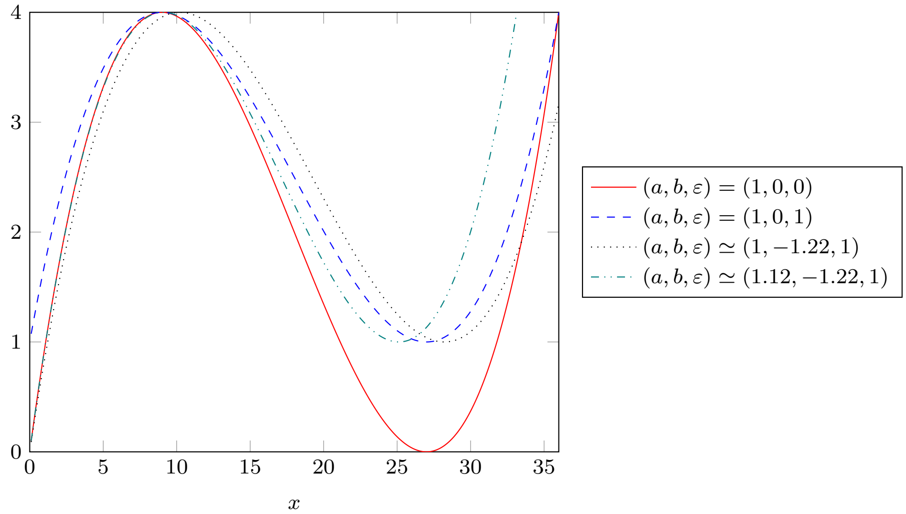
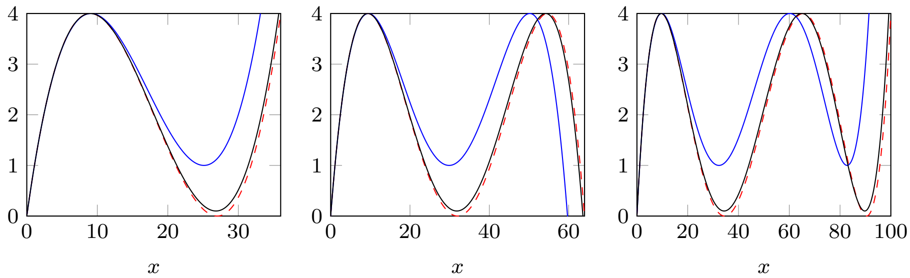


Fig. 2 Representation of the polynomial $x\mathcal{P}^{\varepsilon}_{p,\ell}(x)$ given by (76) for $\ell = 2$ and for different values of the parameters (a, b, ε)

ℓ	ε	b^{ε}	a^{ε}	$\beta^{\varepsilon}_{\ell}$	$C^{\varepsilon}_{\mathcal{Q}}$	$(C^{\varepsilon}_{\mathcal{R}})^{\frac{1}{2}}$
2	1	-1.220497601922388	1.123332443935161	2.87	1	5
2	0.5	-0.548885078878804	1.055702443069509	2.94	2	7
2	0.1	-0.101795082372209	1.010360937184039	2.98	10	17
3	1	-1.214605698792632	1.112468647367209	3.80	1	5
3	0.5	-0.547676655750322	1.051055803796928	3.90	2	8
3	0.1	-0.101753177452728	1.009529277032937	3.97	10	14
4	1	-1.211812534393700	1.107473444638217	4.77	1	8
4	0.5	-0.547112834621174	1.048910062073242	4.89	2	13
4	0.1	-0.101733760636154	1.009144480323238	4.97	10	30

Table 1 Computed values of a and b for given values of ε. We give an estimation of the constants $C_{\mathcal{Q}}$ and $C_{\mathcal{R}}$ obtained if the polynomial $\mathcal{P}^{\varepsilon}_{p,\ell}(x)$ is considered on the interval $[0, 4\,(\beta^{\varepsilon}_{\ell})^2]$.



(a) $\ell = 2$ (b) $\ell = 3$ (c) $\ell = 4$

Fig. 3 Plain line: Representation of $x\mathcal{P}^{\varepsilon}_{p,\ell}(x)$ for different values of ℓ and for $\varepsilon = 0.1$ (in black) or $\varepsilon = 1$ (in blue). Dashed line: Representation of $x\widetilde{\mathcal{P}}_{p,\ell}(x)$ for different values of ℓ.

4.3.4 Algorithm

In practice System (71) is solved by first computing λ^n_h by solving

$$\Big(B_{c,h}B^t_{c,h} + B_{f,h}\mathcal{P}^{\varepsilon}_{p,\ell}(\Delta t^2 A_{f,h})B^t_{f,h}\Big)\lambda^n_h = B_{c,h}f_{c,h}(t^n) - B_{f,h}f_h(t^n)$$
$$- B_{f,h}A_{c,h}\,u^n_{c,h} + B_{f,h}\mathcal{P}^{\varepsilon}_{p,\ell}(\Delta t^2 A_{f,h})\,A_{f,h}\,u^n_{f,h}m \quad (79)$$

and, once λ_h^n is known, $\boldsymbol{u}_h^{n+1}$ is computed explicitly from the two first equation (71). As long as wave equations are considered, the algorithmic complexity to solve this system is the same as in the local time stepping introduced in [9]. The Lagrange multiplier λ_h^n corresponds to an unknown discretised on an interface and the bandwidth of the corresponding matrix system increases with ℓ. The evaluation of the term

$$\mathcal{P}_{p,\ell}^{\varepsilon}(\Delta t^2 A_{f,h}) A_{f,h} u_{f,h}^n$$

requires $\ell + 1$ evaluations of the operator $A_{f,h}$, which is consistent with the fact that the stability condition (73) is relaxed almost $\beta_\ell^\varepsilon \simeq \ell + 1$ times. These evaluations can be done by computing explicitly the coefficient of the polynomials $\mathcal{P}_{p,\ell}^{\varepsilon}(x)$ and using the Horner algorithm. For high values of ℓ it is preferable, for numerical stability reasons, to use the second order recurrence relation of the Chebychev polynomials to evaluate $\mathcal{P}_{p,\ell}^{\varepsilon}(x)$ (such relation can be derived from the definitions (74) and (76)).

Remark 8 Note that although the size of the matrix system corresponding to equation (79) may be relatively small (it is reduced to the Lagrange multiplier unknown) the matrix may become full for large ℓ. In that case local implicit scheme may be preferred.

5 Numerical convergence results in 1D

In this section we present numerical results in 1D that illustrate the convergence behavior of the schemes we have proposed. We consider the wave equation (1), with homogeneous Neumann boundary condition, posed on the domain $\Omega = (-0.5, 0.5)$ with $\Omega_c = (-0.5, 0)$, $\Omega_f = (0, 0.5)$ and $\Sigma = \{0\}$. We consider the initial value problem with smooth initial data compactly supported in Ω_c such that the exact solution around the initial times is given by

$$u_c(x, 0) = r(x), \quad u_c(x, \Delta t) = r(x - \Delta t) \text{ with } r(x) = 1_{[x_0-\sigma, x_0+\sigma]} e^{-2/(1-(x-x_0)^2/\sigma^2)}.$$

with $x_0 = -0.25$, $\sigma = 0.05$, and

$$u_f(x, 0) = 0, \quad u_f(x, \Delta t) = 0.$$

Although the present our theoretical results was establish with vanishing initial data, the induced numerical error of considering exact initial data is null and therefore the analysis is still valid in this case. We set $T = 0.5$. We assume that $\mu_c = 1$ and we denote $\mu \equiv \mu_f \leq 1$. Note that μ is therefore a measure of the contrast between the two subdomains. The solution is smooth and is given by

$$u(x,t) = \begin{cases} u_c(x,t) & x < 0 \\ u_f(x,t) & x > 0 \end{cases} = \begin{cases} r(x-t) + \mathcal{R} r(-x-t) & x < 0 \\ \mathcal{T} r(x - c_f t) & x > 0 \end{cases} \tag{80}$$

with

$$\mathcal{R} = \frac{1-\sqrt{\mu}}{1+\sqrt{\mu}} \quad \text{and} \quad \mathcal{T} = 1 - \mathcal{R} = 2\,\frac{1}{1+\sqrt{\mu}}.$$

For the space discretisation we use standard second order Galerkin finite elements with a lumped mass matrix (see for instance [16]) on a uniform mesh of Ω_c and Ω_f and we denote h_c and h_f the respective mesh sizes and q_f the refinement rate. We have

$$h_c = h, \quad h_f = \frac{h}{q_f} \quad \text{and} \quad \frac{\rho_{f,h}}{\rho_{c,h}} = \mu\, q_f^2. \tag{81}$$

We recall that $\Delta t = 2\alpha/\sqrt{\rho_{c,h}}$ for some $0 < \alpha \leq 1$. In what follows, we plot space-time convergence curves by setting α to some given values, and computing the solution of the discrete problem for some sequence h going to zero (this implies that Δt goes to zero accordingly). Then the discrete solution $(u^n_{c,h}, u^n_{f,h})$ is compared to the analytic expression (80) and we plot

$$\frac{\sup_{n\Delta t\leq T} \|\mathcal{I}_{c,h} u_c(t^n) - u^n_{c,h}\|_c}{\sup_{n\Delta t\leq T} \|\mathcal{I}_{c,h} u_c(t^n)\|_c} + \frac{\sup_{n\Delta t\leq T} \|\mathcal{I}_{f,h} u_f(t^n) - u^n_{f,h}\|_f}{\sup_{n\Delta t\leq T} \|\mathcal{I}_{f,h} u_f(t^n)\|_f}$$

where $\mathcal{I}_{c,h}$ and $\mathcal{I}_{f,h}$ denote here the interpolation operators on the nodal finite element spaces.

5.1 Local implicit scheme

In order to assess numerically the behavior of local implicit schemes described in Section 4.1, we set $\mathcal{P}_k(x) = 1$ and $\mathcal{P}_p(x) = 1$. More specifically, this means that the left-hand side of the domain is discretized with an explicit leaf-frog scheme, while the right-hand side of the domain is discretized with an unconditionally stable implicit θ-scheme with $\theta = 1/4$. The convergence plots are represented in Figure 4. We have chosen two values for $\alpha \in \{0.9, 1\}$ and two values for the refinement ratio $q_f \in \{2, 3\}$. In the first four curves, μ is set equal to 0.5. It can be observed that all configuration provide a second order rate of convergence, which was to be expected since Corollary 2 can be applied with $C_\mathcal{Q} = 0$ and $C_\mathcal{R} = 1$. Finally, we consider a situation with a high contrast: we set $\mu = 0.001$ and we choose a refinement ratio $q_f = 6$, $\alpha = 1$. We still observe an asymptotic convergence rate of 2.

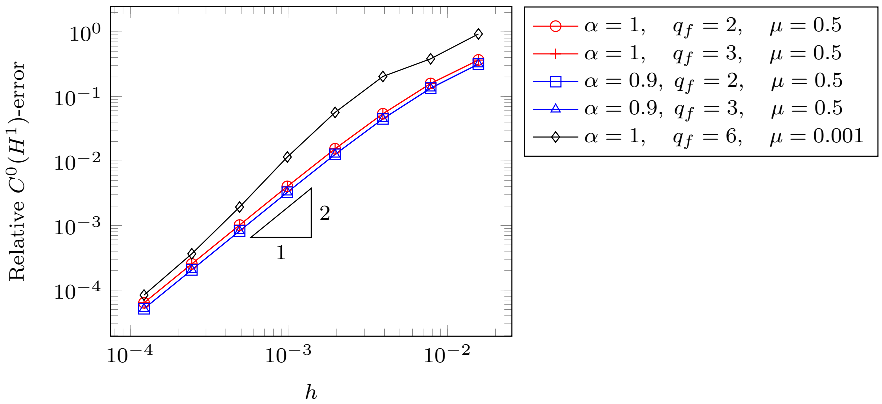


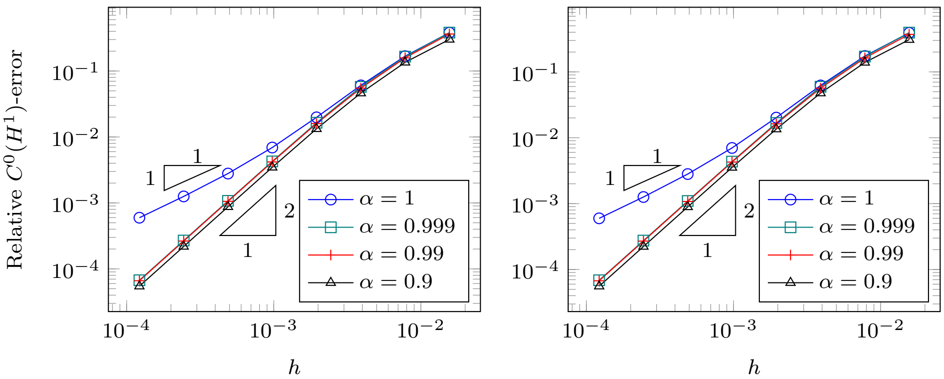
Fig. 4 Space-time convergence plot for local implicit schemes for different values of (α, μ, q_f).

5.2 Stabilised explicit scheme

In order to assess the behavior of the stabilised explicit scheme described in Section 4.2, we set

$$\mathcal{P}_p(x) = 1 - \frac{x}{16} \quad \text{and} \quad \mathcal{P}_k(x) = 1 - x\,\frac{\mathcal{P}_p(x)}{4} = 1 - \frac{x}{4} + \frac{x^2}{64}.$$

We first investigate the situation of a homogeneous medium ($\mu = 1$) where the subdomain Ω_f is refined by a factor $q_f = 2$. We make the value of α increase from $\alpha = 0.9$ to 1. As stated in Section 4.2, the value $\alpha = 1$ prevents us from applying Corollary 1 since the values of $C_{\mathcal{R}}$ and $C_{\mathcal{Q}}$ blow up when α approaches 1. The numerical results displayed in Figure 5(a) show that values of α very close to 1 (up to 0.999) give the expected convergence rate of 2, and that indeed, choosing $\alpha = 1$ does not lead to a second order space/time convergence (the convergence is of order 1). As a second example, we consider an inhomogeneous medium with $\mu = 0.25$, we choose $q_f = 4$, and we perform the same numerical tests. As observed in Figure 5(b), the same conclusions can be drawn.



(a) $\mu = 1$ and $q_f = 2$

(b) $\mu = 0.25$ and $q_f = 4$

Fig. 5 Space-time convergence plots for the stabilised explicit scheme for different values of (α, μ, q_f).

5.3 Local time stepping using the Leap-Frog Chebychev method

In order to assess the behavior of the schemes constructed in Section 4.3, we set, for a given ε, $\mathcal{P}_p(x) = \mathcal{P}^{\varepsilon}_{p,\ell}(x)$ according to the values given in Table 1 or $\mathcal{P}_p(x) = \widetilde{\mathcal{P}}_\ell(x)$ and we always choose $\mathcal{P}_k(x)$ as in (70). We consider three cases

- In a first configuration, we choose $\ell = 2$ and we consider the case of a homogeneous medium ($\mu = 1$) and $q_f = 3$ (therefore $\rho_{f,h} = 9\rho_{c,h}$). We choose $\varepsilon \in \{1, 0.5, 0.1\}$ and $\alpha = 0.9$. We always have

$$\frac{\rho_{f,h}}{\rho_{c,h}} \leq \frac{(\beta_2^\varepsilon)^2}{\alpha^2} \ (> 9) \tag{82}$$

  hence the schemes are stable and convergent. The resulting curves are displayed in Figure 6(a) where it is seen that the modified polynomials offer an expected rate of convergence of 2 while the use of $\mathcal{P}_p(x) = \widetilde{\mathcal{P}}_\ell(x)$ provide only first order convergence.

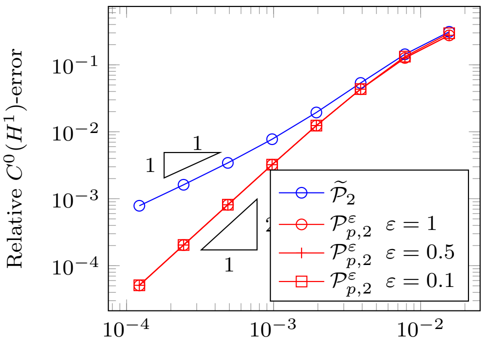
- In a second configuration, we consider the case of an inhomogeneous medium with $\mu = 0.25$ as well as a refinement factor $q_f = 6$. According to equation (81) we have

$\rho_{f,h} = 9\,\rho_{c,h}$. Again, we choose $\ell = 2$ and $\varepsilon \in \{1, 0.5, 0.1\}$. The value of $\alpha \equiv \alpha^\varepsilon$ are chosen as

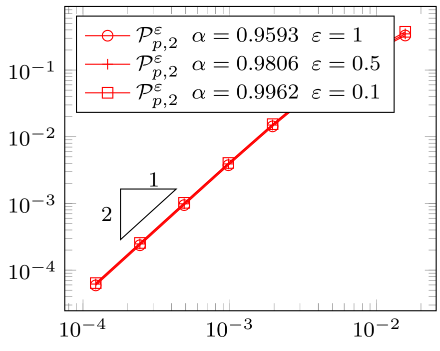
$$\alpha^\varepsilon = \beta_2^\varepsilon \sqrt{\frac{\rho_{c,h}}{\rho_{f,h}}}$$

where the the values of β_2^ε are given in Table 1. This choices of parameters ensures that stability condition (82). In Figure 6(b), we have displayed the convergence obtained with the three different values of ε. In all cases we observe second order space-time convergence.

- o Finally, we choose $\mu = 0.25$, $\varepsilon = 0.1$ and a refinement factor of respectively $q_f = 8$ and $q_f = 10$. In Figure 7 we compare the results obtained with, first, $\mathcal{P}_p(x) = \widetilde{\mathcal{P}}_\ell(x)$ and, second, $\mathcal{P}_p(x) = \mathcal{P}^\varepsilon_{p,\ell}(x)$ for respectively $\ell = 3$ and 4. We observe, in both cases, that second order space-time convergence is obtained with the second choice of polynomial while the first choice gives only a first order convergence behavior.



(a) $\mu = 1$, $q_f = 3$ and $\alpha = 0.9$



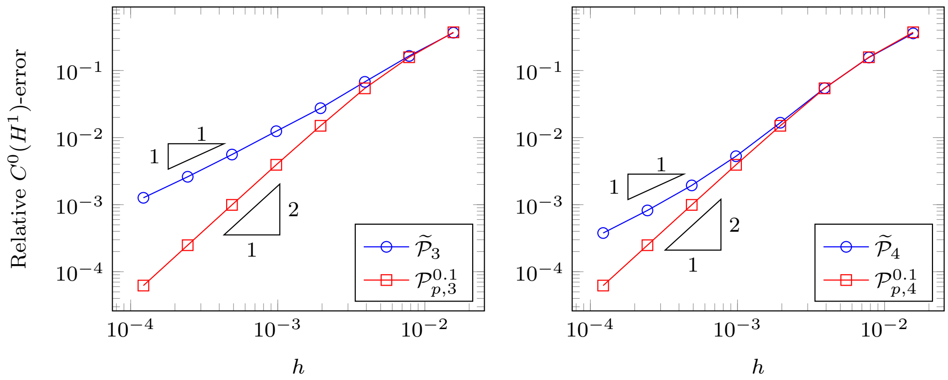
(b) $\mu = 0.25$ and $q_f = 6$

Fig. 6 Space-time convergence plots for the local time stepping explicit scheme for different values of (α, μ, q_f) and $\mathcal{P}_p(x)$.

6 Comparisons with existing approaches

6.1 Comparison with the implicit LTS of [9]

In equation (13.74)-(13.76) of [9], we find an algebraic formulation for conservative local time stepping. This formulation is written for the system of elastodynamics written at first order in time. However, by elimination (of the variable corresponding to the velocity), one can show that the algebraic formulation is equivalent to the following



(a) $\mu = 0.25$, $q_f = 8$ and $\alpha = 0.9943$ (b) $\mu = 0.25$ and $q_f = 10$, $\alpha = 0.9959$

Fig. 7 Space-time convergence plots for the local time stepping explicit scheme for different values of (α, μ, q_f) and $\mathcal{P}_p(x)$.

system

$$\begin{cases} M_{\sigma,c}\dfrac{\Sigma_c^{2n+1} - 2\Sigma_c^{2n-1} + \Sigma_c^{2n-3}}{(2\Delta t)^2} + K_c\Sigma_c^{2n-1} - C_c^*\dfrac{J^{2n+1} - J^{2n-3}}{4\Delta t} = 0, & \text{(83a)} \\[2ex] M_{\sigma,f}\dfrac{\Sigma_f^{2n+1} - 2\Sigma_f^{2n} + \Sigma_f^{2n-1}}{\Delta t^2} + K_f\Sigma_f^{2n} - C_f^*\dfrac{J^{2n+1} - J^{2n-1}}{2\Delta t} = 0, & \text{(83b)} \\[2ex] M_{\sigma,f}\dfrac{\Sigma_f^{2n} - 2\Sigma_f^{2n-1} + \Sigma_f^{2n-2}}{\Delta t^2} + K_f\Sigma_f^{2n-1} - C_f^*\dfrac{J^{2n+1} - J^{2n-3}}{4\Delta t} = 0. & \text{(83c)} \\[2ex] C_c\Sigma_c^{2n+1} + C_f\Sigma_f^{2n+1} = 0. & \text{(83d)} \end{cases}$$

In this formulation the unknowns Σ_c^n and Σ_f^n are vectors corresponding to stresses in a coarse and fine region respectively and, J^n are vectors corresponding to normal stresses, K_c and K_f are stiffness matrices (equal respectively to $B_c^* M_{v,c}^{-1} B_c$ and $B_f^* M_{v,c}^{-1} B_f$ with the notation given in [9]), $M_{\sigma,c}$ and $M_{\sigma,f}$ are mass matrices. Now we aim at eliminating intermediate steps in the evaluation for the fine region, more precisely, the sequence of even iterates $\{\Sigma_f^{2n}\}$ for $n \geq 0$. To do so, we write (83b) centered at time t^{2n} and subtract two times equation (83c) centered around t^{2n-1} and add the equation (83b) centered at time t^{2n-2}. We obtain

$$M_{\sigma,f}\frac{\Sigma_f^{2n+1} - 4\Sigma_f^{2n} + 6\Sigma_f^{2n-1} - 4\Sigma_f^{2n-2} + \Sigma_f^{2n-3}}{\Delta t^2} + K_f\left[\Sigma_f^{2n} - 2\Sigma_f^{2n-1} + \Sigma_f^{2n-2}\right] = 0.$$

Now we use the fact that

$$\frac{\Sigma_f^{2n+1} - 4\Sigma_f^{2n} + 6\Sigma_f^{2n-1} - 4\Sigma_f^{2n-2} + \Sigma_f^{2n-3}}{\Delta t^2} = \frac{\Sigma_f^{2n+1} - 2\Sigma_f^{2n-1} + \Sigma_f^{2n-3}}{\Delta t^2} - \frac{4}{\Delta t^2}\left[\Sigma_f^{2n} - 2\Sigma_f^{2n-1} + \Sigma_f^{2n-2}\right], \quad (84)$$

and therefore we obtain

$$M_{\sigma,f}\frac{\Sigma_f^{2n+1} - 2\Sigma_f^{2n-1} + \Sigma_f^{2n-3}}{\Delta t^2} + \left[K_f - \frac{4}{\Delta t^2}M_{\sigma,f}\right]\left(\Sigma_f^{2n} - 2\Sigma_f^{2n-1} + \Sigma_f^{2n-2}\right) = 0.$$

Now using (83c), we replace the quantity inside the parenthesis

$$M_{\sigma,f}\frac{\Sigma_f^{2n+1}-2\Sigma_f^{2n-1}+\Sigma_f^{2n-3}}{\Delta t^2} \\ +\Delta t^2\left[K_f-\frac{4}{\Delta t^2}M_{\sigma,f}\right]\left(-M_{\sigma,f}^{-1}K_f\Sigma_f^{2n-1}+M_{\sigma,f}^{-1}C_f^*\frac{J^{2n+1}-J^{2n-3}}{4\Delta t}\right)=0.$$

Dividing by 4 the previous equation and rearranging terms we obtain

$$M_{\sigma,f}\frac{\Sigma_f^{2n+1}-2\Sigma_f^{2n-1}+\Sigma_f^{2n-3}}{(2\Delta t)^2} \\ +\left[I_f-\frac{(2\Delta t)^2}{16}K_fM_{\sigma,f}^{-1}\right]\left[K_f\,\Sigma_f^{2n-1}-C_f^*\frac{J^{2n+1}-J^{2n-3}}{4\Delta t}\right]=0, \quad (85)$$

where I_f is the identity matrix in the appropriate vector space. Let us denote

$$U_f^n:=\Sigma_f^{2n-1},\quad U_c^n:=\Sigma_c^{2n-1},\quad \Delta\tau:=2\Delta t\quad \text{and}\quad \Lambda^n:=\frac{J^{2n+1}-J^{2n-3}}{4\Delta t}.$$

then collecting (83a), (83d) and (85), one can show that the following system holds

$$\begin{cases} M_{\sigma,c}\dfrac{U_c^{n+1}-2U_c^n+U_c^{n-1}}{\Delta\tau^2}+K_cU_c^n-C_c^*\Lambda^n=0, \\ M_{\sigma,f}\dfrac{U_f^{n+1}-2U_f^n+U_f^{n-1}}{\Delta\tau^2}+\left[I_f-\dfrac{\Delta\tau^2}{16}K_fM_{\sigma,f}^{-1}\right]\left[K_f\,U_f^n-C_f^*\Lambda^n\right]=0, \\ C_cU_c^{n+1}+C_fU_f^{n+1}=0. \end{cases}$$

This new formulation of system (83) shows that the local time stepping proposed in [9] is in fact equivalent to the scheme developed in Section 4.2 (compare the above equations with (67)). Therefore the local time stepping proposed in [9] can be seen as a transmission problem between two second order schemes, one of which having a relaxed stability condition by adding stabilising terms. The computational burden of the schemes we proposed Section 4.3 is equivalent to the one of the schemes proposed in [9]. In fact we conjecture that, the local time stepping of [9] can be recast in the formalism of Section 4.3 with $\mathcal{P}_p \equiv \widetilde{\mathcal{P}}_{p,\ell}$. It has to be noted that the schemes in [9] are not proven to be second order convergent (in space and time) for the H^1-norm which is in accordance with the convergence results of Section 5.3.

6.2 Comparison with the fully explicit Local Time Stepping of [7]

The computational code used to obtain the results of this section is available as supplementary material at the web link [33].

In [8] and [7], an explicit Local Time Stepping Algorithm is proposed and is proved to be second order convergent for the L^2-norm. It is used in the context of solving the following semi-discrete wave equation:

$$\frac{d^2}{dt^2}u_h+A_hu_h=f_h. \quad (86)$$

From Formula (12) of [7] with $p = 2$ one can derive the following scheme

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + \big(A_h - \frac{\Delta t^2}{16} A_h P_h A_h\big) u_h^n = f_h^n, \tag{EX-2b}$$

where $P_h : V_h \to V_h$ is a restriction operator on a region discretised with a fine grid (with an overlap of one element in our computations). Notice that this algorithm amounts to solving a Leap-Frog scheme for the kernel of P_h and to a modified scheme for the complement. Moreover, from Algorithm 1 (page 1000) of [7] we can deduce a variant of (EX-2b),

$$\frac{u_h^{n+1} - 2u_h^n + u_h^{n-1}}{\Delta t^2} + \big(A_h - \frac{\Delta t^2}{16} A_h P_h A_h\big) u_h^n = \tilde{f}_h^n - \frac{\Delta t^2}{16} A_h P_h f_h^n, \tag{EX-2}$$

where $\tilde{f}_h^n$ can be define in two ways, denoting $f_h^{n+\frac{1}{2}} = f_h(t^n + \Delta t/2)$ we set

$$\tilde{f}_h^n = \frac{1}{4} P_h\big(f_h^{n+\frac{1}{2}} + 2\, f_h^n + f_h^{n-\frac{1}{2}}\big) + (I_h - P_h) f_h^n \quad (a) \qquad \text{or} \qquad \tilde{f}_h^n = f_h^n \quad (b). \tag{87}$$

The choice (87)(a) gives exactly the algorithm 1 of [7] while the most simple choice (87)(b) gives similar observed convergence behavior and so will be used in what follows (note that in the provided supplementary materials both choices (87)(b) and (87)(a) are implemented).

6.3 Numerical assessment

In the following we present a numerical assessment, in a one-dimensional setting, of local time stepping procedures that have the same computational cost: the stabilised explicit scheme (67) of Section 4.2, the scheme (EX-2) and its variant (EX-2b).

The considered case is the same as in Section 5 with $\mu = 1$ (homogeneous medium). More precisely, we solve up to time $T = 0.5$, the equation

$$\partial_t^2 u - \partial_x^2 u = f, \quad x \text{ in } (-0.5, -0.5), \tag{88}$$

with homogeneous Neumann boundary condition. The discretisation parameters are $\alpha = 0.99$ and $q_f = 2$. The purpose of these tests is to quantify the relative L^2 and H^1-errors with respect to three chosen continuous analytical solutions associated with adequate source terms or initial data.

6.3.1 Propagating pulse

The first considered case is a propagating pulse as described in Section 5. In figure 8 are displayed the relative L^2 and H^1-errors between the solutions of the three numerical schemes (Scheme (67), (EX-2) and (EX-2b)) and the analytical solution, with respect to the mesh size h (note that Δt goes to zero with h because of Assumption 2).

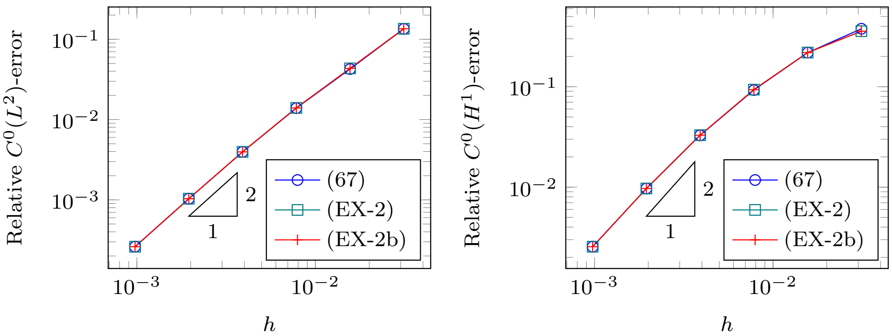


Fig. 8 Space-time convergence plots for the LTS explicit scheme. The analytical solution is a propagating pulse.

6.3.2 Quasi-static solution

We choose vanishing initial data for the wave equation (88) and choose f such that the solution is smooth and given by

$$u(x,t) = g(x)\, h\left(\frac{t-\tau}{t_0}\right),$$

with

$$g(x) = \begin{cases} x^2\left(x+\frac{1}{2}\right)^2 & x<0, \\ x^2\left(x-\frac{1}{2}\right)^2 & x>0, \end{cases} \qquad h(t) = \begin{cases} 0 & t \leq 0, \\ 1 & t \geq 1, \\ \dfrac{1}{1+e^{\frac{1}{t}+\frac{1}{t-1}}} & 0<t<1. \end{cases}$$

and $\tau = 0.1$ and $t_0 = 0.25$. Note that the solution reaches a static state after times $t \geq 0.35$. The obtained convergence curves are displayed in Figure 9. One can see that the three schemes behave similarly in terms of convergence in the L^2-norm, however the scheme (EX-2b) is less accurate in the H^1-norm. More precisely, half an order of convergence is lost.

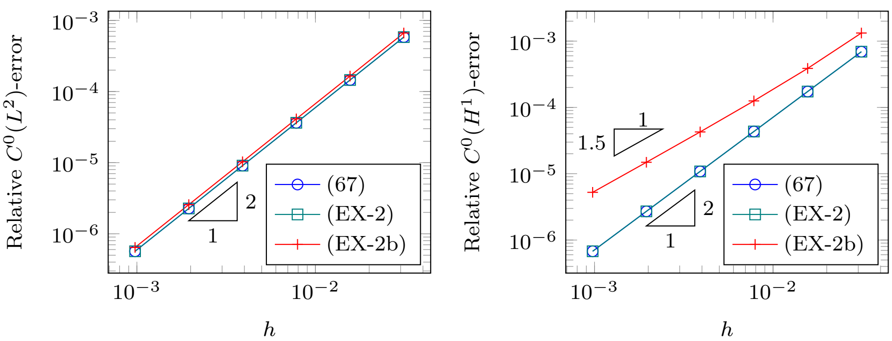


Fig. 9 Space-time convergence plots for the LTS explicit schemes. The analytical solution is static for $t \geq 0.35$.

6.3.3 Spatially constant solution

We choose vanishing initial data for the wave equation (88) and choose f such that the solution is smooth and given by

$$u(x,t) = h\left(\frac{t-\tau}{t_1}\right).$$

with $\tau = 0.1$ and $t_1 = 0.8$. The analytical solution is therefore constant in space. The obtained convergence curves are displayed in Figure 10. Again, one the one hand, one can see that the three schemes behave similarly in terms of convergence in the L^2-norm. On the other hand, it is this time the scheme (EX-2) which is less accurate in the H^1-norm (half an order of convergence is lost).

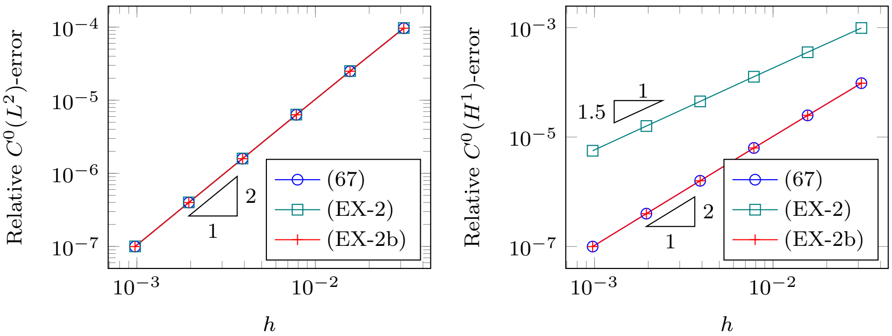


Fig. 10 Space-time convergence plots for the LTS explicit schemes. The analytical solution is constant in space.

7 Conclusions

In this work we have presented and analysed a family of second order in time discretisation strategy for linear wave equations. We have shown that they correspond to either locally implicit schemes or to local time stepping. For the analysis we have considered the case of smooth solutions vanishing at the initial time. Then, we have shown that, if a well-defined stability condition holds, then second order space-time convergence property holds in the context of abstract Galerkin approximations of the wave equation. Finally, we have presented 1D numerical convergence results that confirm the obtained theoretical results. As far as local time stepping strategies are concerned, after comparisons with existing methods we have confirmed the interest of the proposed approach since, in terms of accuracy, it yields second order convergence results in the H^1-norm (contrary to [9] and [8]) and in terms of computational cost, it is equivalent to the method proposed in [9].

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