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Abstract

We study the long time behavior of the solution to some McKean-Vlasov stochastic differential equation (SDE) driven by a Poisson process. In neuroscience, this SDE models the asymptotic dynamic of the membrane potential of a spiking neuron in a large network. We prove that for a small enough interaction parameter, any solution converges to the unique (in this case) invariant measure. To this aim, we first obtain global bounds on the jump rate and derive a Volterra type integral equation satisfied by this rate. We then replace temporary the interaction part of the equation by a deterministic external quantity (we call it the external current). For constant current, we obtain the convergence to the invariant measure. Using a perturbation method, we extend this result to more general external currents. Finally, we prove the result for the non-linear McKean-Vlasov equation.

Keywords McKean-Vlasov SDE · Long time behavior · Mean-field interaction · Volterra integral equation · Piecewise deterministic Markov process
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Contents

1 Introduction

We study a model of a network of neurons. For each $N \in \mathbb{N}$, we consider a Piecewise-Deterministic Markov Process (PDMP) $\mathbf{X}^N_t = (X^{i,N}_t, i = 1, ..., N) \in \mathbb{R}^N_+$. For $i \in \{1, ..., N\}$, $X^{i,N}_t$ models the (membrane) potential of a neuron (say neuron i) in the network. It emits spikes at random times. The spiking rate of neuron i at time t is $f(X^{i,N}_t)$: it only depends on the potential of neuron i. When the neuron i emits a spike, say at time τ, its potential is reset ($X^{i,N}_{\tau+} = 0$) and the potential of the other neurons increases by an amount $\frac{J}{N}$, where the connection strength $J \geq 0$ is fixed:

$$\forall j \neq i, \quad X^{j,N}_{\tau_+} = X^{j,N}_{\tau_-} + \frac{J}{N}.$$

Between two spikes, the potentials evolve according to the one dimensional equation

$$\frac{dX^{i,N}_t}{dt} = b(X^{i,N}_t).$$

The functions b and f are assumed to be smooth. Equivalently, the model can be described using a system of SDEs driven by Poisson measures. This process is indeed a PDMP, in particular Markov (see [10]). Let $(\mathbf{N}^i(du, dz))_{i=1,...,N}$ be a family of N independent Poisson measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dudz$. Let $(X^{i,N}_0)_{i=1,...,N}$ be a family of N random variables on $\mathbb{R}_+$, *i.i.d.* of law ν and independent of the Poisson measures. Then $(X^{i,N})$ is a *càdlàg* process solution of the SDE:

$$\begin{cases} X^{i,N}_t = X^{i,N}_0 + \int_0^t b(X^{i,N}_u)du + \frac{J}{N}\sum_{j\neq i}\int_0^t\int_{\mathbb{R}_+} \mathbb{1}_{\{z \leq f(X^{j,N}_{u-})\}}\mathbf{N}^j(du, dz) \\ \qquad - \int_0^t\int_{\mathbb{R}_+} X^{i,N}_{u-}\mathbb{1}_{\{z\leq f(X^{i,N}_{u-})\}}\mathbf{N}^i(du, dz). \end{cases} \tag{1}$$

When the number of neurons N goes to infinity, it has been proved (see [11, 18]) for specific linear functions b that $X^{1,N}_t$ - i.e. the first coordinate of the solution to (1) - converges in law to the solution of the McKean-Vlasov SDE:

$$X_t = X_0 + \int_0^t b(X_u)du + J\int_0^t \mathbb{E}\, f(X_u)du - \int_0^t\int_{\mathbb{R}_+} X_{u-}\mathbb{1}_{\{z\leq f(X_{u-})\}}\mathbf{N}(du, dz), \tag{2}$$

where, $\mathcal{L}(X_0) := \mathcal{L}(X^{1,N}_0) = \nu$ and $\mathbf{N}$ is a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dudz$. The measure $\mathbf{N}$ and X_0 are independent.

Equation (2) is a mean-field equation and is the current object of our interest. Note that the drift part of (2) makes appear the law of the solution in the term $\mathbb{E}\, f(X_u)$: the equation is non-linear in the sense of McKean-Vlasov. We study, here, existence and uniqueness of the solution of (2) and its long time behavior.

Assume the law ν of the initial condition X_0 has a density p_0 with respect to the Lebesgue measure (i.e. $\nu(dx) = p_0(x)dx$). If $p_0 \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ with $p_0(0) = \frac{r_0}{b(0)+Jr_0}$, where $r_0 = \int_0^\infty f(x)p_0(x)dx$, then the law of

X_t has also a density $p(t,.) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ which satisfies the Fokker-Planck equation:

$$\begin{cases} \dfrac{\partial}{\partial t} p(t,x) = -\dfrac{\partial}{\partial x}[(b(x) + Jr_t)p(t,x)] - f(x)p(t,x), \quad x > 0 \\ p(0,\cdot) = p_0, \quad p(t,0) = \dfrac{r_t}{b(0) + Jr_t}, \quad r_t = \displaystyle\int_0^\infty f(x)p(t,x)dx. \end{cases} \tag{3}$$

Note that if the density of the initial condition does not satisfy $p_0(0) = \frac{r_0}{b(0)+Jr_0}$ (or does not have a density at all) then $\mathcal{L}(X_t)$ is only a weak solution of (3).

The model (2) has been introduced in [11] and has been the object of further developments in [18] (see also [19, Chap. 9], for earlier works and biological considerations). The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ can be considered of the type $f(x) = (\frac{x}{\vartheta})^\xi$ for large ξ and some soft threshold ϑ. In this situation, if the potential of the neuron is equal to x then the neuron has a small probability to spike between t and $t + dt$ if $x < \vartheta$ and a large probability if $x > \vartheta$. Such a choice of f mimics the standard Integrate-And-Fire model with a fixed (deterministic) threshold around ϑ.

Results on the existence of solution to (2), in a slightly different context, has been obtained in [11]: the authors explored the case where the initial condition ν is compactly supported. This property is preserved at any time $t > 0$. So, the behavior of the solution with a rate function f locally Lipschitz continuous is similar to the case with a function f Lipschitz continuous. When the initial condition is not compactly supported, the situation is more delicate. In [18], the authors proved existence and path-wise uniqueness of the solution to (2) (in a slightly different setting than ours) using an ad-hoc distance.

Note that the global existence results obtained for this model differ from those obtained for the "standard" Integrate-and-Fire model with a fixed deterministic threshold. This situation, studied for instance in [4, 6, 12, 13], corresponds (formally) in our case to $f(x) = +\infty \mathbb{1}_{\{x \geq \vartheta\}}$, $\vartheta > 0$ being the fixed threshold. In these papers, a diffusion part is included in the modeling. In [4], the authors proved that a blow-up phenomena appears when the law of the initial condition is concentrated close to the threshold ϑ: the jump rate of the solution diverges to infinity in finite time. Here, the situation is completely different: the jump rate is uniformly bounded in time (see Theorem 6). In [4], the authors have obtained results on the stability of the solution (see also [5] for a variant).

Very little is known about the long time behavior of the solutions to (2). One can study it by considering the long time behavior of the finite particles system (1) and then apply the propagation of chaos to extend the results to the McKean-Vlasov equation (2). For instance, this strategy has been developed in [30, 2] for diffusive problems. The long time behavior of the particles system (1) has been studied in [16, 22] (again in a slightly different setting but the methods could be adapted to our case): they proved that the particles system is Harris-ergodic and consequently converges weakly to its unique invariant measure. However, transferring the long time behavior of the particles system to the McKean-Vlasov equation is only possible if the propagation of chaos holds *uniformly in time*. In [11, 18], the propagation of chaos is only proved on compact time interval $[0, T]$ and their estimates diverge as T goes to infinity. Equation (2) may have multiple invariant measures: there is no hope in general to prove such uniform propagation of chaos.

Coupling methods are also used to study the long time behavior of SDEs. In [1], the authors have studied the TCP (a linear SDE with jumps) which is close to (2). The main differences is that the size of the jumps is $-x/2$ in the TCP and $-x$ in our setting, x being the position of the process just before the jump. We failed to adapt their methods when the interactions are non-zero ($J > 0$).

In a similar spirit to ours, in [3] Butkovsky studied the long time behavior of some McKean-Vlasov diffusion SDE of the form:

$$\forall t \geq 0, \ X_t = X_0 + \int_0^t \left[b_1(X_u) + \epsilon \int_{\mathbb{R}_+} b_2(X_u, y)\, \mu_u(dy) \right] du + W_t, \ \mu_u = \mathcal{L}(X_u), \tag{4}$$

where $(W_t)_{t\geq 0}$ is a Brownian motion. The author proved that if the parameter ϵ is small enough, (4) has an unique invariant measure which is globally stable. First, sufficient conditions are given for the uniform ergodicity on non-linear Markov chain. Then, the stability of the solution to (4) is studied. The case $\epsilon > 0$ (and small) is treated as a perturbation of the case $\epsilon = 0$ using a Girsanov transform. It could be interesting to see how this method could be adapted to SDE driven by Poisson measures, but we did not pursue this path.

Another approach consists in studying the non-linear PDE (3). Such non-linear transport equations with boundary conditions have been studied in the context of population dynamics (see for instance [21, 28, 31, 26]). In [21], the authors have characterized the stationary solutions of the PDE and found a criterion of local stability for stationary solutions. They derived a Volterra integral equation and used it to obtain the stability criteria. More recently, [24, 25, 23] have re-explored these models for neuroscience applications (see [8, 7] for a rigorous derivation of some of these PDEs using Hawkes processes).

Our PDE (3) differs from theirs in the sense that we have a non-linear transport term (theirs is constant and equal to one) and our boundary condition more complex. The long time behavior of the PDE (3) has been successfully studied in [18] and in [14] in the case where $b \equiv 0$. In this case, one can simplify the PDE (3) with a simpler boundary condition

$$p(t,0) = \frac{1}{J}.$$

The authors proved that if the density of the initial condition satisfies this boundary condition and regularity assumptions, then $p(t,.)$ converges to the density of the invariant measure as t goes to infinity. The convergence holds in L^1 norm or in stronger norms (see [14]). For $b \neq 0$, the boundary condition is more delicate and their methods cannot be easily applied.

Actually the long time behavior of the solutions to (2) may be remarkably intricate. For instance, if $b(x) = \mu - x$ and $f(x) = x^2$, where $\mu > 0$ is a constant, equation (2) may have 1, 2, or 3 non trivial invariant measures, depending on the value of the interactions J. In [15], the authors have numerically illustrated this phenomenon in a setting close to our. Furthermore, the jump rate $t \mapsto \mathbb{E}\, f(X_t)$ may have stable oscillations.

Our main result gives the long time behavior of the solution to (2) in the weakly connected regime (Theorem 12). If the connection strength J is small enough, we prove that (2) has a unique invariant measure which is globally stable. We give the explicit expression of this non-trivial invariant measure and starting from any initial condition X_0, we prove the convergence in law of X_t to this invariant measure as t goes to infinity. We argue that this result is very general: it does not depend on the explicit shape of the functions f or b. For stronger connection strengths J, such a result cannot hold true in general as equation (2) may have multiple invariant measures. To study the local stability of these invariant measures, one would have to use ad-hoc methods which rely on the explicit shape of f and b. We let such investigations for future work.

Note that we prove convergence in law, which is weaker than convergence in L^1. On the other hand, we require very few on the initial condition, in particular, we do not assume the existence of a density for the initial condition in Theorem 12. We also provide a new proof for the existence and uniqueness of the solution to (2), based on a Picard iteration scheme (see Theorem (6)). As in [18], we do not require the initial condition to be compactly supported. One of the main difficulty to study (2) (or its PDE version (3)) is that there is no simple autonomous equation for the jump rate $t \mapsto \mathbb{E}\, f(X_t)$. To overcome this difficulty, we introduce a "linearized" version of (2) for which we can derive a closed equation of the jump rate.

Fix a $s \geq 0$ and let $(a_u)_{u \geq s}$ be a continuous deterministic non-negative function, called the external current. It replaces the interaction $J\mathbb{E}f(X_u)$ in (2). We consider the linear non-homogeneous SDE:

$$\forall t \geq s,\ Y_t^{s,\nu} = Y_s^{s,\nu} + \int_s^t b(Y_u^{s,\nu})du + \int_s^t a_u du - \int_s^t \int_{\mathbb{R}_+} Y_{u-}^{s,\nu} \mathbb{1}_{\{z \leq f(Y_{u-}^{s,\nu})\}} \mathbf{N}(du, dz), \tag{5}$$

where $\mathcal{L}(Y_s^{s,\nu}) = \nu$. Under quite general assumptions on b and f, this SDE has a unique path-wise solution (see Lemma 14). We denote the jump rate of this SDE by:

$$\forall t \geq s,\ r_{(a.)}^{\nu}(t,s) := \mathbb{E}\, f(Y_t^{s,\nu}). \tag{6}$$

Moreover, taking $s = 0$ and $Y_0^{0,\nu} = X_0$, it holds that $(Y_t^{0,\nu})_{t \geq 0}$ is a solution to (2) if it satisfies the closure condition:

$$\forall t \geq 0,\ a_t = J r_{(a.)}^{\nu}(t,0). \tag{7}$$

Conversely, any solution to (2) is a solution to (5) with $a_t = J\,\mathbb{E}\, f(X_t)$. We prove that $r_{(a.)}^{\nu}(t,s) = \mathbb{E}\, f(Y_t^{s,\nu})$ satisfies a Volterra integral equation:

$$r_{(a.)}^{\nu}(t,s) = K_{(a.)}^{\nu}(t,s) + \int_s^t K_{(a.)}(t,u) r_{(a.)}^{\nu}(u,s)du, \tag{8}$$

where the kernels $K^{\nu}_{(a.)}$ and $K_{(a.)}$ are explicit (see (13) and (14)).

Our main tool is this Volterra equation: we use it with a Picard iteration scheme to "recover" the non-linear equation (2). The McKean-Vlasov equation (2), its "linearized" non-homegeneous version (5), the Fokker-Planck PDE (3) and the Volterra equation (8) are different ways to investigate this mean-field problem, each of these interpretations having their own strength and weakness. Here, we use mainly the Volterra equation (8) and the non-homegeneous SDE (5). To prove that equation (2) has a unique path-wise solution, we rely on the Volterra equation (8) and show that the following mapping:

$$(a_t)_{t\geq 0} \mapsto J r^{\nu}_{(a.)}(.,0) := [t \mapsto J\,\mathbb{E}\,f(Y^{0,\nu}_t)], \tag{9}$$

is contracting on $\mathcal{C}([0,T],\mathbb{R}_+)$ for all $T>0$. It then follows that the fixed point of this mapping satisfies the closure condition (7) and can be used to define a solution to (2). Conversely any solution to (2) defines a fixed point of this mapping and one proves strong uniqueness for (2).

Finally, we obtain our main result, the long time behavior of the solution to (2). Let us detail the structure of the proof. First, we give in Theorem 10 the long time behavior of the solution to the linear equation (5) with a constant current ($a_t \equiv a$). Any solution converges in law to a unique invariant measure ν^{∞}_a (Theorem 28). In that case, the Volterra equation (8) is of convolution type and it is possible to study finely its solution using Laplace transform techniques. Second, we prove, for small J, the uniqueness of a constant current a^* such that

$$a^* \equiv J\mathbb{E}f(Y^{0,\nu^{\infty}_{a^*}}_t).$$

Third, we extend the previous convergence result to non-constant currents (a_t) satisfying

$$|a_t - a^*| \leq C e^{-\lambda t}. \tag{10}$$

Using a perturbation method, we prove that

$$Y^{0,\nu}_t \xrightarrow[t\to\infty]{\mathcal{L}} \nu^{\infty}_{a^*}.$$

Fourth, in Theorem 12, we give the long time behavior of the solution to the non-linear equation (2) for small J. Here, we use a fixed point argument.

The layout of the paper is as follows. In Section 2, we introduce the notations and the main assumptions. Our main results are given in Section 3. In Section 4, we study the non-homogeneous linear equation (5) and derive the Volterra equation satisfied by the jump rate. In Section 5, we characterize the invariant measures of (2). In Section 7 we study the long time behavior of the solution to (5) with a constant current a. In Section 8, we introduce the perturbation method. Finally Sections 6 and 9 are devoted to the proofs of our main results (Theorems 9 and 12).

2 Notations, definitions and main assumptions

Let us introduce some notations and definitions that will be used all along. For $s \geq 0$, let $Y^{s,\nu}_s$ be a random variable independent of a Poisson process $\mathbf{N}(du,dz)$ on $\mathbb{R}_+\times\mathbb{R}_+$ of intensity measure $dudz$. We consider the canonical filtration $(\mathcal{F}^s_t)_{t\geq s}$ associated to the Poisson measure $\mathbf{N}$ and to the initial condition $Y^{s,\nu}_s$, that is the completion of:

$$\sigma\{Y^{s,\nu}_s, \mathbf{N}([s,r]\times A):\ s\leq r\leq t,\ A\in\mathcal{B}(\mathbb{R}_+)\}.$$

Definition 1. *Let $s \geq 0$ and consider $(a.) : [s,\infty[\to\mathbb{R}_+$ a measurable locally integrable function ($\forall t\geq s$, $\int_s^t a_u du < \infty$).*

- *A process $(Y^{s,\nu}_t)_{t\geq s}$ is said to be a solution of the non-homegeneous linear equation (5) with a current $(a_t)_{t\geq s}$ if the law of $Y^{s,\nu}_s$ is ν, $(Y^{s,\nu}_t)_{t\geq s}$ is $(\mathcal{F}^s_t)_{t\geq s}$-adapted, càdlàg, a.s. $\forall t\geq s$, $\int_s^t f(Y^{s,\nu}_u)du < \infty$ and (5) holds a.s.*
- *An $(\mathcal{F}^0_t)_{t\geq 0}$-adapted càdlàg process $(X_t)_{t\geq 0}$ is said to solve the non-linear SDE (2) if $t\mapsto\mathbb{E}\,f(X_t)$ is measurable locally integrable and if $(X_t)_{t\geq 0}$ is a solution of (5) with $s=0$, $Y^{0,\nu}_0 = X_0$ and $\forall t\geq 0$, $a_t := \mathbb{E}\,f(X_t)$.*

Let $t \geq s \geq 0$. $Y_t^{s,\nu}$ denotes a solution of the linear non-homogeneous SDE (5) driven by $(a_t)_{t\geq s} \in \mathcal{C}([s,\infty[,\mathbb{R}_+)$ starting with law ν at time s. We denote its associated jump rate by: $r^\nu_{(a.)}(t,s) := \mathbb{E}\, f(Y_t^{s,\nu})$. For any measurable function g, we write $\nu(g) := \int_0^\infty g(x)\nu(dx) = \mathbb{E}\, g(Y_s^{s,\nu})$ whenever this integral makes sense.

Note that $(Y_t^{s,\nu})_{t\geq s}$ is a non-homogeneous Markov process with values in $\mathbb{R}_+$ which can also be described by its infinitesimal generator:

$$\forall \phi \in \mathcal{C}^1_b(\mathbb{R}_+),\ \forall t \geq s,\ \mathcal{L}_t\phi(x) := \phi'(x)[b(x) + a_t] + [\phi(0) - \phi(x)]f(x). \tag{11}$$

Between its random jumps, the SDE (5) is reduced to a non-homogeneous ODE. Let us introduce its flow $\varphi^{(a.)}_{t,s}(x)$, which by definition is the solution of:

$$\forall t \geq s,\ \frac{d}{dt}\varphi^{(a.)}_{t,s}(x) = b(\varphi^{(a.)}_{t,s}(x)) + a_t \tag{12}$$
$$\varphi^{(a.)}_{s,s}(x) = x.$$

In this article, we show that the jump rate $r^\nu_{(a.)}$ of the non-homogeneous SDE (5) satisfies the Volterra equation (8) where the kernels $K^\nu_{(a.)}$ and $K_{(a.)}$ are given by:

$$\forall x \geq 0,\ \forall t \geq s \geq 0,\ K^x_{(a.)}(t,s) := f(\varphi^{(a.)}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}(x))du\right), \tag{13}$$

$$K^\nu_{(a.)}(t,s) := \int_0^\infty K^x_{(a.)}(t,s)\nu(dx),\quad K_{(a.)}(t,s) := K^0_{(a.)}(t,s). \tag{14}$$

Given two "kernels" α and β, it is convenient to follow the notation of [20] and define:

$$\forall t \geq s,\ (\alpha * \beta)(t,s) := \int_s^t \alpha(t,u)\beta(u,s)du.$$

The Volterra equation (8) becomes:

$$r^\nu_{(a.)} = K^\nu_{(a.)} + K_{(a.)} * r^\nu_{(a.)}. \tag{15}$$

Similarly to (13) and (14), we define the kernels:

$$\forall x \geq 0,\ \forall t \geq s,\ H^x_{(a.)}(t,s) := \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}(x))du\right),\quad H^\nu_{(a.)}(t,s) := \int_0^\infty H^x_{(a.)}(t,s)\nu(dx). \tag{16}$$

To shorten notations, we shall write: $r_{(a.)}(t,s) := r^{\delta_0}_{(a.)}(t,s)$, $K_{(a.)}(t,s) := K^{\delta_0}_{(a.)}(t,s)$, $H_{(a.)}(t,s) := H^{\delta_0}_{(a.)}(t,s)$ and $\varphi^{(a.)}_{t,s} := \varphi^{(a.)}_{t,s}(0)$. From the definition, one can check directly the following relation:

$$1 * K^\nu_{(a.)} = 1 - H^\nu_{(a.)}. \tag{17}$$

When the input current $(a_t)_{t\geq 0}$ is constant and equal to a, equation (5) is homogeneous and we write:

$$\forall t \geq 0,\ Y^\nu_t := Y^{0,\nu}_t,\ r^\nu_a(t) := r^\nu_{(a.)}(t,0),\ K^\nu_a(t) := K^\nu_{(a.)}(t,0),\ H^\nu_a(t) := H^\nu_{(a.)}(t,0),\ \varphi^a_t(x) := \varphi^{(a.)}_{t,0}(x).$$

Note that in that case, the operation $*$ corresponds to the classical convolution operation. In particular this operation is commutative in the homogeneous setting and equation (15) is a *convolution Volterra equation*.

Assumptions 2. *We assume that $b \in \mathcal{C}(\mathbb{R}_+,\mathbb{R})$ satisfies $b(0) > 0$ and is such that for every $(a_t)_{t\geq 0}$, $(d_t)_{t\geq 0} \in \mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)$ we have:*

2.1 $\forall x \geq 0$, $\forall s \geq 0$, the ODE (12) has a unique solution $t \mapsto \varphi^{(a.)}_{t,s}(x)$ defined on $[s,\infty[$. This is the flow associated to the drift b and to the external current $(a_t)_{t\geq 0}$.

2.2 the flow satisfies the following comparison principle:

$$[\forall t \geq 0,\ a_t \geq d_t] \implies [\forall x \geq y \geq 0,\ \forall t \geq s \geq 0,\ \varphi^{(a.)}_{t,s}(x) \geq \varphi^{(d.)}_{t,s}(y)].$$

2.3 The drift is bounded from above:

$$\exists C_b > 0 :\ \forall x \in \mathbb{R}_+,\ b(x) \leq C_b.$$

In particular the flow grows at most linearly with respect to the initial condition:

$$\forall a \geq 0, \quad \forall x \geq 0, \forall t \geq 0,\ \varphi^a_t(x) \leq x + C^a_b t, \quad \textit{where } C^a_b := C_b + a.$$

2.4 There exists a constant C_φ (independent of (a_t) and (d_t)) such that:

$$\forall x \geq 0,\ \forall s \leq t,\ |\varphi^{(a.)}_{t,s}(x) - \varphi^{(d.)}_{t,s}(x)| \leq C_\varphi \int_s^t |a_u - d_u| du.$$

2.5 Finally we assume that the function $(t, s, x) \mapsto \varphi^{(a.)}_{t,s}(x)$ is continuous on $\{(t, s) : 0 \leq s \leq t < \infty\} \times \mathbb{R}_+$.

Remark 3. *1. Assumption $b(0) > 0$ implies that for all $x \in \mathbb{R}_+$ and s, t, we have $\varphi^{(a.)}_{t,s}(x) \in \mathbb{R}_+$.*

2. If b is locally Lipschitz and satisfies assumption 2.3 , then by the Gronwall Lemma, one can prove that 2.1, 2.2 and 2.5 are satisfied.

We make the following technical assumptions on f:

Assumptions 4. *We assume that $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is $\mathcal{C}^1$ convex increasing with $f(0) = 0$, $\forall x > 0$ $f(x) > 0$ and satisfies:*

4.1 there exists a constant C_f such that

$$\forall x, y \geq 0,\ f(x + y) \leq C_f(1 + f(x) + f(y)) \textit{ and } f'(x + y) \leq C_f(1 + f'(x) + f'(y)).$$

4.2 $\forall \theta \geq 0 :\ \sup_{x \geq 0}\{\theta f'(x) - f(x)\} < \infty.$

Define $\psi(\theta) := \sup_{x \geq 0}\{\theta f'(x) - \frac{1}{2} f^2(x)\} < \infty$. We also assume that:

$$\lim_{\theta \to +\infty} \frac{\psi(\theta)}{\theta^2} = 0.$$

4.3 Finally we assume that there is a constant $C_{b,f} > 0$ such that

$$\forall x \geq 0,\ |b(x)| \leq C_{b,f}(1 + f(x)).$$

Note that $f(x) := x^p$ with $p \geq 1$ satisfies the Assumptions 4.1 and 4.2. In that case, we have $\forall x, y \in \mathbb{R}_+,\ f(x+y) \leq 2^{p-1}(f(x)+f(y))$. A similar estimate holds for f'. Moreover $\psi(\theta) = \frac{1}{2}\theta^{\frac{2p}{p+1}}.(p-1)^{\frac{p-1}{p+1}}(1+p)$, so Assumption 4.2 holds.

Assumption 4.2 ensures that f do not grow too fast in the sense that $\forall \epsilon > 0$, there is a constant $C_\epsilon > 0$, such that: $\forall x \geq 0$ $f(x) \leq C_\epsilon e^{\epsilon x}$. We make the following assumption on the initial condition:

Assumptions 5. *We assume that the law of the initial condition is a (probability) measure ν satisfying $\nu(f^2) < \infty$.*

Examples

We present examples of drift b and of function f which satisfy our assumptions. $f(x) := x^p$, $b(x) := \mu - \lambda x$ where $p \geq 1$, $\mu > 0$ and $\lambda \geq 0$ are constant. In that case, the flow is given by:

$$\varphi_{t,s}^{(a.)}(x) = xe^{-\lambda(t-s)} + \frac{\mu}{\lambda}[1 - e^{-\lambda(t-s)}] + \int_s^t e^{-\lambda(t-u)} a_u du.$$

Note that the case $\lambda > 0$ and $\lambda = 0$ are qualitatively different: when $\lambda = 0$ one can show that the non-linear process (2) has a unique invariant measure for all $J \geq 0$. For $\lambda > 0$, the process may have one, two or three invariant measures depending on the value of μ, λ and J and all of these invariant measures are compactly supported.

3 Main results

Let us give our main theorems.

Theorem 6. *Under Assumptions 2, 4 and 5, the non-linear SDE* (2) *has a unique path-wise (strong) solution* $(X_t)_{t\geq 0}$ *in the sense of Definition 1. Furthermore, there is a finite constant* $\bar{r} > 0$ *(only depending on* b, f *and* J*) such that:*

$$\sup_{t\geq 0} \mathbb{E} f(X_t) \leq \max(\bar{r}, \mathbb{E} f(X_0)), \quad \limsup_{t\to\infty} \mathbb{E} f(X_t) \leq \bar{r}.$$

$\bar{r}$ *can be chosen to be an increasing function of* J.

Assumptions 7. *We assume that for any constant current* $a \geq 0$*, the flow converges to a limit as* t *goes to infinity (eventually equal to* $+\infty$*):*

$$\forall a \geq 0, \ \forall x \geq 0, \ \lim_{t\to+\infty} \varphi_t^a(x) := \sigma_a^x \in \mathbb{R}_+^* \cup \{\infty\}. \tag{18}$$

We assume that $\inf_{a,x\geq 0} \sigma_a^x > 0$*. Moreover if we define:*

$$\sigma_a := \inf\{x \geq 0 : \ b(x) + a = 0\},$$

we assume that $\sigma_a^0 = \sigma_a$.

Theorem 8. *Under Assumptions 2, 4 and 7, the invariant measures of the non-linear SDE* (2) *are:*

$$\nu_{a^*}^\infty(dx) := \frac{\gamma(a^*)}{b(x) + a^*} \exp\left(-\int_0^x \frac{f(y)}{b(y) + a^*} dy\right) \mathbb{1}_{\{x\in[0,\sigma_{a^*}]\}} dx, \tag{19}$$

where:

1. $\gamma(a)$ *is the normalizing factor, given by:*

$$\gamma(a) := \left[\int_0^{\sigma_a} \frac{1}{b(x) + a} \exp\left(-\int_0^x \frac{f(y)}{b(y) + a} dy\right) dx\right]^{-1}. \tag{20}$$

2. *the* $a^* \geq 0$ *are the solutions of the following scalar equation:*

$$\frac{a}{\gamma(a)} = J, \ a \in [0, \infty[\text{ is the unknown.} \tag{21}$$

Note that we have $\nu_{a^*}^\infty(f) = \gamma(a^*)$.

Moreover, define $J_m := \sup\{J_0 \geq 0 : \ \forall J \in [0, J_0]$ *equation* (21) *has a unique solution*$\}$*. Then* $J_m > 0$*. Consequently, for all* $0 \leq J < J_m$ *the non-linear process* (2) *has a unique invariant measure.*

We prove, in the following theorem, that controlling the behavior of the jump rate $\mathbb{E} f(X_t)$ can be sufficient to deduce the asymptotic law of (X_t), solution of (2).

Theorem 9. *Grant Assumptions 2, 4, 5 and 7. Let $(X_t)_{t\geq 0}$ be the solution of the non-linear equation* (2) *and define: $a_t := J\,\mathbb{E}\, f(X_t)$. Assume that there exists constants $\lambda, C > 0$ and $a^* \geq 0$ (that may depends on b, f, ν, and J) such that:*

$$\forall t \geq 0,\ |a_t - a^*| \leq Ce^{-\lambda t},$$

and that a^* satisfies equation (21)*: $\frac{a^*}{\gamma(a^*)} = J$. Then*

$$X_t \xrightarrow[t\to\infty]{\mathcal{L}} \nu^{\infty}_{a^*}.$$

Moreover, if $\phi : \mathbb{R}_+ \to \mathbb{R}$ is any bounded Lipschitz-continuous function, it holds that:

$$\forall 0 < \lambda' < \min(\lambda, f(\sigma_0)),\ \exists D > 0:\ |\mathbb{E}\,\phi(X_t) - \nu^{\infty}_{a^*}(\phi)| \leq De^{-\lambda' t},$$

where the constant D only depends on $b, f, J, C, \nu, \lambda'$ and ϕ through its infinite norm and its Lipschitz constant.

When the interactions are small enough, we are able to prove the convergence of the jump rate, and consequently the convergence to the invariant measure. The next theorem states the result for J equal to zero. In this case, we use Laplace transform techniques to characterize the convergence.

Theorem 10. *Grant Assumptions 2, 4, 5, 7. Let $(Y^{\nu}_t)_{t\geq 0}$ be the solution of* (5)*, driven by a constant current $(a_t) \equiv a$, $a \geq 0$; starting at time $s = 0$ with law ν.*

One can find a constant $\lambda^*_a \in]0, f(\sigma_a)]$ (only depending on b, f and a) such that for any $0 < \lambda < \lambda^*_a$ it holds:

$$\forall t \geq 0,\ |\mathbb{E}\, f(Y^{\nu}_t) - \gamma(a)| \leq De^{-\lambda t} \int_0^{\infty} [1 + f(x)] |\nu - \nu^{\infty}_a|(dx),$$

where D is a constant only depending on f, b, a and λ. Moreover, one has

$$Y^{\nu}_t \xrightarrow[t\to\infty]{\mathcal{L}} \nu^{\infty}_a.$$

Remark 11. *In the above theorem, λ^*_a is explicitly known in term of f, b and a (see its expression* (31)*) and is optimal in the sense that: $\limsup_{t\to\infty} |\mathbb{E}\, f(Y^{\nu}_t) - \gamma(a)| e^{\lambda^*_a t} > 0$. Note also that in the above theorem, the dependence to the initial distribution is stated explicitly through its distance to the invariant measure.*

We now state our main result: the convergence to the unique invariant measure for weak enough interactions.

Theorem 12. *Grant Assumptions 2, 4, 5, 7. There exists strictly positive constants J^* and λ (both only depending on b and f) satisfying:*

$$0 < J^* < J_m,\ 0 < \lambda < f(\sigma_0),$$

(J_m is defined in Theorem 8) such that for any $0 \leq J \leq J^*$, there is a constant $D > 0$ satisfying:

$$\forall t \geq 0,\ |\mathbb{E}\, f(X_t) - \gamma(a^*)| \leq De^{-\lambda t},$$

where $(X_t)_{t\geq 0}$ is the solution of the non-linear SDE (2) *starting with law ν and a^* is the unique solution of* (21)*. The constant D only depends on b, f, $\mathbb{E}\, f(X_0)$, J and λ. In particular Theorem 9 applies and*

$$X_t \xrightarrow[t\to\infty]{\mathcal{L}} \nu^{\infty}_{a^*}.$$

Note that in Theorem 12, the unique invariant measure is globally stable: for weak enough interactions, starting from any initial condition, the system converges to its steady state.

Remark 13. *The theorems above have been stated under restrictive assumptions on b and f in order to keep the proofs as readable as possible. However, the reader should be convinced by now that we do not need the rate function f to be non-decreasing and convex on $\mathbb{R}_+$.*

- *For instance, assume only $f \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$, $\forall x > 0, f(x) > 0$ and f satisfies 4.2. Assume moreover that there is a constant $K_f \geq 0$ such that f is convex increasing on $[K_f, \infty[$ and such that assumptions 4.1 and 4.3 are satisfied for all $x, y \geq K_f$. Then the conclusions of the theorems above should hold.*
- *More generally, grant Assumptions 2 and 4.2 and assume $f \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$. Define*

$$\Theta_f(x) := \sup_{y \in [0,x]} f(y), \quad \Theta_{f'}(x) := \sup_{y \in [0,x]} |f'(y)|.$$

Assume further that:

13.1 There exist $C_f, C_{f'} > 0$ such that

$$\forall x, y \geq 0, \; \Theta_f(x+y) \leq C_f(1 + \Theta_f(x) + \Theta_f(y)) \text{ and } \Theta_{f'}(x+y) \leq C_{f'}(1 + \Theta_{f'}(x) + \Theta_{f'}(y)).$$

13.2 There exists $C_{b,f} > 0$ such that

$$\forall x \geq 0, \; |b(x)| \leq C_{b,f}(1 + \Theta_f(x)).$$

13.3

$$\sup_{x \geq 0} \left\{ f'(x)b(x) - \tfrac{1}{2} f^2(x) \right\} < \infty.$$

13.4 The initial condition ν satisfies $\nu(\Theta_f^2) < +\infty$ and $\nu(\Theta_{f'}^2) < +\infty$.

13.5 Finally, assume that

$$\inf_{a, x \geq 0} f(\sigma_a^x) > 0,$$

where σ_a^x is defined by (18).

These Assumptions 13.1, 13.2, 13.4 and 13.5 replace respectively 4.1, 4.3, 5 and 7. Under these assumptions, the conclusions of the theorems above should hold. Note that with this set of hypothesis, the assumption $b(0) > 0$ can be relaxed to $b(0) \geq 0$.

4 Study of the non-linear SDE (2) and of its linearized version (5)

4.1 On the non-homogeneous linear SDE (5)

Fix $s \geq 0$ and let $(a_t) : [s, \infty[\to \mathbb{R}_+$ be a continuous function. We consider the non-homogeneous linear SDE (5). We always assume that ν, the law of the initial condition $Y_s^{s,\nu}$, satisfies Assumptions 5.

Lemma 14. *Grant Assumptions 2, 4 and 5. Then the SDE (5) has a unique path-wise solution on $[s, \infty[$ in the sense of Definition 1.*

Proof. We give a direct proof by considering the jumps of $Y_t^{s,\nu}$ and by solving the equation between the jumps.

- **Step 1**: we grant Assumptions 2, 5 and assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is measurable and bounded: there exists a constant $K < \infty$ such that:

$$\forall x \geq 0, \; f(x) \leq K.$$

In this case, the solution of (5) can be constructed in the following way. Define by induction:

$$\tau_0 := \inf\{t \geq s : \int_s^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \leq f(\varphi_{u,s}^{(a.)}(Y_s^{s,\nu}))\}} \mathbf{N}(du, dz) > 0\},$$

$$\forall n \geq 0, \; \tau_{n+1} := \inf\{t \geq \tau_n : \int_{\tau_n}^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \leq f(\varphi_{u,\tau_n}^{(a.)})\}} \mathbf{N}(du, dz) > 0\}.$$

Using that $f \leq K$, it follows that $a.s. \lim_{n\to\infty} \tau_n = +\infty$. We define:

$$Y_t^{s,\nu} = \varphi_{t,s}^{(a.)}(Y_s^{s,\nu})\mathbb{1}_{t\in[s,\tau_0[} + \sum_{n\geq 1} \varphi_{t,\tau_n}^{(a.)}\mathbb{1}_{t\in[\tau_n,\tau_{n+1}[},$$

and we can directly verify that $t \mapsto Y_t^{s,\nu}$ is almost surely a solution of (5).

Uniqueness of equation (5) follows immediately from the uniqueness of the ODE (Assumption 2.1): two solutions have to be equal almost surely before the first jump, from which we deduce that the two solutions have to jump at the exact same time: by induction on the number of jumps, the two trajectories are almost surely equal.

- **Step 2**: We now come back to the general case where f is not assumed to be bounded and we adapt the strategy of [18], proof of proposition 2. We grant Assumptions 2, 4 and 5.

  We use Step 1 with $f^K(x) := f(\min(x, K))$ for some $K > 0$. f^K being bounded, we have a unique path-wise solution $Y_t^{s,\nu,K}$ of (5). We introduce $\zeta_K := \inf\{t \geq 0 : \ |Y_t^{s,\nu,K}| \geq K\}$, it holds by path-wise uniqueness that $a.s.$ $Y_t^{s,\nu,K} = Y_t^{s,\nu,K+1}$ for all $t \in [0, \zeta_K]$ and all $K \in \mathbb{N}$ and that $a.s.$ $\zeta_K < \zeta_{K+1}$. We define $\zeta := \sup_K \zeta_K$ and deduce the existence and uniqueness of a solution $t \mapsto Y_t^{s,\nu}$ of (5) on $[0, \zeta[$ and that $\limsup_{t\to\zeta} Y_t^{s,\nu} = \infty$ on the event $\{\zeta < \infty\}$. But any solution of (5) satisfies $\forall t \geq s,\ a.s.\ Y_t^{s,\nu} \leq \varphi_{t,s}^{(a.)}(Y_s^{s,\nu}) < \infty$ and so it holds that $\zeta = +\infty$ a.s.

□

Lemma 15. *Grant Assumptions 2, 4 and 5. Let $(Y_t^{s,\nu})_{t\geq s}$ be the solution of (5). The functions $t \mapsto \mathbb{E} f(Y_t^{s,\nu})$, $t \mapsto \mathbb{E} f'(Y_t^{s,\nu})$, $t \mapsto \mathbb{E} f'(Y_t^{s,\nu})|b(Y_t^{s,\nu})|$ and $t \mapsto \mathbb{E} f^2(Y_t^{s,\nu})$ are locally bounded on $[s,\infty[$. Moreover, $t \mapsto \mathbb{E} f(Y_t^{s,\nu}) =: r_{(a.)}^{\nu}(t, s)$ is continuous on $[s,\infty[$.*

Proof. Consider the interval $[s, T]$ for some $T > 0$. Let $A := \sup_{t\in[s,T]} a_t$. It is clear that:

$$\forall t \in [s, T], a.s.\ Y_t^{s,\nu} \leq Y_s^{s,\nu} + \int_s^t [b(Y_u^{s,\nu}) + a_u]du \leq Y_s^{s,\nu} + C_T,$$

with $C_T := (C_b + A)(T - s)$. We used here Assumption 2.3. Using that f^2 is non-decreasing and Assumption 4.1, it yields:

$$a.s.\ f^2(Y_t^{s,\nu}) \leq f^2(Y_s^{s,\nu} + C_T) \leq C_f^2(1 + f(C_T) + f(Y_s^{s,\nu}))^2.$$

Using Assumption 5, we deduce that $t \mapsto \mathbb{E} f^2(Y_t^{s,\nu})$ is bounded on $[s, T]$. By the Cauchy–Schwarz inequality, this implies that $t \mapsto \mathbb{E} f(Y_t^{s,\nu})$ is also bounded on $[s, T]$. Finally, using the Assumption 4.2 (with $\theta = 1$), there is a constant C such that: $\forall x \geq 0\ f'(x) \leq C + f(x)$. Assumption 4.3 yields:

$$\forall x \geq 0\ f'(x)|b(x)| \leq C_{b,f}(1 + f(x))(C + f(x)),$$

and so this proves that $t \mapsto \mathbb{E} f'(Y_t^{s,\nu})|b(Y_t^{s,\nu})|$ is also bounded on $[s, T]$. We now apply the Itô's Formula (see for instance Theorem 32 of [27, Chap. II]) to $Y_t^{s,\nu}$. It yields:

$$f(Y_{t+\epsilon}^{s,\nu}) = f(Y_t^{s,\nu}) + \int_t^{t+\epsilon} f'(Y_u^{s,\nu})[b(Y_u^{s,\nu}) + a_u]du - \int_t^{t+\epsilon}\int_0^{\infty} f(Y_{u-}^{s,\nu})\mathbb{1}_{\{z\leq f(Y_{u-}^{s,\nu})\}}\mathbf{N}(du, dz).$$

Taking the expectation, it follows that:

$$\mathbb{E} f(Y_{t+\epsilon}^{s,\nu}) - \mathbb{E} f(Y_t^{s,\nu}) = \int_t^{t+\epsilon} \mathbb{E} f'(Y_u^{s,\nu})[b(Y_u^{s,\nu}) + a_u]du - \int_t^{t+\epsilon} \mathbb{E} f^2(Y_u^{s,\nu})du,$$

from which we deduce that $t \mapsto \mathbb{E} f(Y_t^{s,\nu})$ is locally Lipschitz and consequently continuous. □

4.2 The Volterra equation

Along this section, we grant Assumptions 2, 4 and 5. Let $s \geq 0$ and $(a_t)_{t\geq s} \in \mathcal{C}([s,\infty[,\mathbb{R}_+)$ be fixed. We consider $(Y^{s,\nu}_t)_{t\geq s}$ the unique path-wise solution of equation (5) driven by the current $(a_t)_{t\geq s}$. Following [18], we define:

$$\tau_{s,t} := \sup\{u \in [s,t] : \Delta Y^{s,\nu}_u \neq 0\},$$

the time of the last jump before t, with the convention that $\tau_{s,t} = s$ if there is no jump during $[s,t]$. It follows directly from (5) that:

$$\forall t \geq s,\ a.s.\ Y^{s,\nu}_t = \varphi^{(a.)}_{t,s}(Y^{s,\nu}_s)\mathbb{1}_{\{\tau_{s,t}=s\}} + \varphi^{(a.)}_{t,\tau_{s,t}}\mathbb{1}_{\{\tau_{s,t}>s\}}.$$

We also define:

$$\forall t \geq s,\ J_t := \int_s^t \int_0^\infty \mathbb{1}_{\{z\leq f(Y^{s,\nu}_{u-})\}}\mathbf{N}(du,dz),$$

the number of jumps between s and t.

Lemma 16.

$$\forall t \geq u \geq s,\ \mathbb{P}(J_t = J_u|\mathcal{F}_u) = H^{Y^{s,\nu}_u}_{(a.)}(t,u)\ a.s.$$

where $H^x_{(a.)}$ is defined by (16)*, that is:*

$$\forall t \geq u, \forall x \in \mathbb{R}_+,\ H^x_{(a.)}(t,u) := \exp\left(-\int_u^t f(\varphi^{(a.)}_{\theta,u}(x))d\theta\right).$$

Proof. $\{J_t = J_u\} = \{\int_u^t\int_0^\infty \mathbb{1}_{\{z\leq f(Y^{s,\nu}_{\theta-})\}}\mathbf{N}(d\theta,dz) = 0\}$. $\mathcal{F}_u$ and $\sigma\{\mathbf{N}([u,\theta]\times A) : \theta \in [u,t], A \in \mathcal{B}(\mathbb{R}_+)\}$ are independent. It follows from the Markov property satisfied by $(Y^{s,\nu}_t)_{t\geq s}$ that:

$$a.s.\ \mathbb{P}(J_t = J_u|\mathcal{F}_u) = \Phi(Y^{s,\nu}_u)$$

where: $\Phi(x) := \mathbb{P}(\int_u^t\int_0^\infty \mathbb{1}_{\{z\leq f(\varphi^{(a.)}_{\theta,u}(x))\}}\mathbf{N}(d\theta,dz) = 0) = H^x_{(a.)}(t,u)$. □

Lemma 17 (See also [18], proposition 25)**.** *For all $t > s$, the law of $\tau_{s,t}$ is given by:*

$$\mathcal{L}(\tau_{s,t})(du) = H^\nu_{(a.)}(t,s)\delta_s(du) + r^\nu_{(a.)}(u,s)H_{(a.)}(t,u)\mathbb{1}_{\{s<u<t\}}du.$$

Proof. First, from Lemma 16 it follows that:

$$\mathbb{P}(\tau_{s,t} = s) = \mathbb{P}(J_t = J_s) = \mathbb{E}(H^{Y^{s,\nu}_s}_{(a.)}(t,s)) = H^\nu_{(a.)}(t,s).$$

Let now $u \in]s,t]$ and $h > 0$ such that: $s < u-h < u \leq t$. We have:

$$\mathbb{P}(\tau_{s,t} \in]u-h,u]) = \mathbb{P}(J_u > J_{u-h}, J_t = J_u) = \mathbb{P}(J_u > J_{u-h}\mathbb{P}(J_t = J_u|\mathcal{F}_u)) = \mathbb{E}(\mathbb{1}_{\{J_u>J_{u-h}\}}H^{Y^{s,\nu}_u}_{(a.)}(t,u)).$$

Let $A := \sup_{u\in[s,t]} a_u$. On the event $\{J_u > J_{u-h}\}$, the process jumps at least once during $]u-h,u]$ and so: $Y^{s,\nu}_u \in [0,\varphi^{(a.)}_{u,u-h}] \subset [0,\varphi^A_h]$. We used the fact that $\sup_{t\geq 0} a_t \leq A$ and the comparison principle 2.2. It follows that:

$$|\mathbb{P}(\tau_{s,t} \in]u-h,u]) - \mathbb{E}(\mathbb{1}_{\{J_u>J_{u-h}\}}H_{(a.)}(t,u))| \leq \sup_{x\in[0,\varphi^A_h]} |H^x_{(a.)}(t,u) - H_{(a.)}(t,u)|\mathbb{P}(J_u > J_{u-h}).$$

From the next Lemma 18, we have:

$$\lim_{h\downarrow 0}\frac{1}{h}\mathbb{P}(J_u > J_{u-h}) = r^\nu_{(a.)}(u,s).$$

Using Assumption 2.5, $x \mapsto H^x_{(a.)}(t,u)$ is continuous at $x=0$. From the continuity of $h \mapsto \varphi^A_h$ at $h=0$, it yields:

$$\lim_{h\downarrow 0} \frac{1}{h} |\mathbb{P}(\tau_{s,t} \in]u-h,u]) - \mathbb{E}(\mathbb{1}_{\{J_u > J_{u-h}\}} H_{(a.)}(t,u))| = 0.$$

Combining the two results, we obtain the stated formula:

$$\lim_{h\downarrow 0} \frac{1}{h} \mathbb{P}(\tau_{s,t} \in]u-h,u]) = r^{\nu}_{(a.)}(u,s) H_{(a.)}(t,u).$$

Moreover, from Lemma 18, we have:

$$\forall u \in]s,t],\ \forall h: s < u-h < u,\ \mathbb{P}(\tau_{s,t} \in]u-h,u]) = \mathbb{P}(J_u > J_{u-h}, J_t = J_u) \le \mathbb{P}(J_u > J_{u-h}) \le C_{s,t} h.$$

This estimate and the previous equalities imply the result. □

Lemma 18 (See also [18], lemme 23)**.** *1. There exists a constant $C_{s,t}$ such that:*

$$\forall u \in]s,t], \forall h: \ s < u-h < u,\ \mathbb{P}(J_u > J_{u-h}) \le C_{s,t} h.$$

2. $\forall u \in]s,t],\ \lim_{h\downarrow 0} \frac{1}{h} \mathbb{P}(J_u > J_{u-h}) = r^{\nu}_{(a.)}(u,s)$.

Proof. Again let $A := \sup_{u\in[s,t]} a_u < \infty$. Using Lemma 16, we have:

$$\begin{aligned} \mathbb{P}(J_u > J_{u-h}) &= 1 - \mathbb{P}(J_u = J_{u-h}) \\ &= 1 - \mathbb{E} \exp\left(- \int_{u-h}^{u} f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu}_{u-h})) d\theta\right). \end{aligned}$$

Now, using $\forall x \ge 0 : 1 - e^{-x} \le x$:

$$\begin{aligned} 1 - \mathbb{E} \exp\left(- \int_{u-h}^{u} f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu}_{u-h})) d\theta\right) &\le \mathbb{E} \int_{u-h}^{u} f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu}_{u-h})) d\theta \\ &\le \mathbb{E} \int_0^h f(\varphi^A_\theta(Y^{s,\nu}_{u-h})) d\theta \\ &\le \mathbb{E} \int_0^h f(Y^{s,\nu}_{u-h} + C^A_b \theta) d\theta \\ &\le C_f\, \mathbb{E} \int_{u-h}^{u} (f(Y^{s,\nu}_{u-h}) + 1 + f(C^A_b \theta)) d\theta \\ &\le C_{s,t} h, \end{aligned}$$

for some constant $C_{s,t}$. We used here our Assumption 2.3 with $a = A$, the fact that f is increasing, Assumption 4.1 and finally the fact that $u \mapsto \mathbb{E} f(Y^{s,\nu}_u)$ is locally bounded (Lemma 15). This proves point 1. Moreover:

$$\begin{aligned} |hr^{\nu}_{(a.)}(u,s) - \mathbb{P}(J_u > J_{u-h})| \le &|hr^{\nu}_{(a.)}(u,s) - hr^{\nu}_{(a.)}(u-h,s)| + \\ &|hr^{\nu}_{(a.)}(u-h,s) - \mathbb{E} \int_{u-h}^{u} f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu}_{u-h})) d\theta| + \\ &\left| \mathbb{E} \int_{u-h}^{u} f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu}_{u-h})) d\theta - \left[1 - \mathbb{E} \exp\left(- \int_{u-h}^{u} f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu}_{u-h})) d\theta\right)\right]\right| \\ := &\Delta^1_h + \Delta^2_h + \Delta^3_h. \end{aligned}$$

From the continuity of $u \mapsto r^{\nu}_{(a.)}(u,s)$ (Lemma 15) it follows that $\lim_{h\downarrow 0} \frac{\Delta^1_h}{h} = 0$. Moreover,

$$\Delta^2_h = \left| \int_{u-h}^{u} \mathbb{E} f(Y^{s,\nu}_{u-h}) d\theta - \mathbb{E} \int_{u-h}^{u} f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu}_{u-h})) d\theta \right|,$$

and from our Assumption 2 it yields:

$$\forall y \geq 0,\ \forall \theta \in [u-h, u],\ 0 \leq \varphi^{(a.)}_{\theta,u-h}(y) - y \leq \varphi^A_h(y) - y \leq C^A_b h.$$

We deduce that:

$$\Delta^2_h \leq h \int_{u-h}^{u} \mathbb{E}\, g_h(Y^{s,\nu}_{u-h}) C^A_b d\theta,$$

with $g_h(x) := \sup_{y \in [0, C^A_b h]} f'(x+y) = f'(x + C^A_b h)$. Using our Assumption 4.1, we have $f'(x+C^A_b h) \leq C_f(1 + f'(C^A_b h) + f'(x))$. It follows that $\mathbb{E}\, g_h(Y^{s,\nu}_{u-h}) \leq C_f(1 + f(C^A_b h) + \mathbb{E}\, f'(Y^{s,\nu}_{u-h}))$. The function $t \mapsto \mathbb{E}\, f'(Y^{s,\nu}_t)$ being locally bounded, we deduce that: $\limsup_{h \downarrow 0} \frac{\Delta^2_h}{h} = 0$. Finally, using that $\forall x \geq 0 : \ |x - (1 - e^{-x})| \leq x^2$, it yields:

$$\Delta^3_h \leq \mathbb{E} \left(\int_{u-h}^{u} f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu}_{u-h})) d\theta \right)^2.$$

Using the Cauchy-Schwarz inequality, we obtain:

$$\Delta^3_h \leq h\, \mathbb{E} \int_{u-h}^{u} f^2(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu}_{u-h})) d\theta \leq h^2\, \mathbb{E}\, f^2(Y^{s,\nu}_{u-h} + C^A_b h).$$

Using $\forall x \geq 0, \forall y \in [0, C^A_b t] : \ f^2(x+y) \leq C^2_f (1 + f(C^A_b) + f(x))^2$ (Assumption 4.1) and the fact that $t \mapsto \mathbb{E}\, f^2(Y^{s,\nu}_t)$ and $t \mapsto \mathbb{E}\, f(Y^{s,\nu}_t)$ are locally bounded (as seen in the Lemma 15), one can find a constant C_t such that:

$$\Delta^3_h \leq C_t h^2.$$

This shows that $\lim_{h \downarrow 0} \frac{\Delta^3_h}{h} = 0$. Combining the three results prove the second point of the lemma. $\square$

Theorem 19 (See also [18], Theorem 12)**.** *Grant assumptions 2, 4 and 5 . Let $s \geq 0$ and $(a_t)_{t \geq s} \in \mathcal{C}([s, \infty[, \mathbb{R}_+)$. Let $Y^{s,\nu}_t$ be the solution of equation* (5)*, starting from $\mathcal{L}(Y^{s,\nu}_s) = \nu$.*

Then if $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is any continuous non-negative function, it holds that:

$$\mathbb{E}\, \phi(Y^{s,\nu}_t) = \int_s^t \phi(\varphi^{(a.)}_{t,u}) H_{(a.)}(t,u) r^\nu_{(a.)}(u,s) du + \int_0^\infty \phi(\varphi^{(a.)}_{t,s}(x)) H^x_{(a.)}(t,s) \nu(dx).$$

Proof.

$$\begin{aligned}
\mathbb{E}\, \phi(Y^{s,\nu}_t) &= \mathbb{E}\, \phi(Y^{s,\nu}_t) \mathbb{1}_{\{\tau_{s,t} = s\}} + \mathbb{E}\, \phi(Y^{s,\nu}_t) \mathbb{1}_{\{\tau_{s,t} > s\}} \\
&= \mathbb{E}\, \phi(\varphi^{(a.)}_{t,s}(Y^{s,\nu}_s)) \mathbb{1}_{\{\tau_{s,t} = s\}} + \mathbb{E}\, \phi(\varphi^{(a.)}_{t,\tau_{s,t}}) \mathbb{1}_{\{\tau_{s,t} > s\}} \\
&:= \alpha_t + \beta_t
\end{aligned}$$

Using Lemma 16 it follows that:

$$\alpha_t = \mathbb{E}[\phi(\varphi^{(a.)}_{t,s}(Y^{s,\nu}_s))\, \mathbb{P}(J_t = J_s | \mathcal{F}_s)] = \mathbb{E}[\phi(\varphi^{(a.)}_{t,s}(Y^{s,\nu}_s)) H^{Y^{s,\nu}_s}_{(a.)}(t,s)] = \int_0^\infty \phi(\varphi^{(a.)}_{t,s}(x)) H^x_{(a.)}(t,s) \nu(dx).$$

Moreover, using Lemma 17, we have $\beta_t = \int_s^t \phi(\varphi^{(a.)}_{t,u}) r^\nu_{(a.)}(u,s) H_{(a.)}(t,u) du$. $\square$

In particular, taking $\phi = f$ we obtain the Volterra equation (15):

$$r^\nu_{(a.)} = K^\nu_{(a.)} + K_{(a.)} * r^\nu_{(a.)}, \quad r^\nu_{(a.)}(t,s) := \mathbb{E}\, f(Y^{s,\nu}_t).$$

Note that using Lemma 17, $\int_s^t \mathcal{L}(\tau_{s,t})(du) = 1$ gives:

$$H^\nu_{(a.)} + H_{(a.)} * r^\nu_{(a.)} = 1.$$

This last formula is interesting by itself but does not characterize the jump rate $r^\nu_{(a.)}$. We prefer to work with (15) because, as shown in the next lemma, this Volterra equation admits a unique solution.

Lemma 20. *Let $s \geq 0$ be fixed, $(a_t)_{t\geq s} \in \mathcal{C}([s,\infty[,\mathbb{R}_+)$. Then equation* (15) *has a unique solution $t \mapsto r^{\nu}_{(a.)}(t,s)$ on $[s,\infty[$.*

Proof. Fix $T > s$. It is sufficient to prove the existence and uniqueness result on $[s,T]$. We consider the Banach space $(\mathcal{C}([s,T],\mathbb{R}), ||\cdot||_{\infty,T})$ and define on this space the following operator: $\Gamma : r \mapsto K^{\nu}_{(a.)} + K_{(a.)} * r$. Let $A := \sup_{t\in[s,T]} a_u$, we have: $M^T_s = \sup_{s\leq u\leq t\leq T} K_{(a.)}(t,u) < \infty$. This follows from:

$$\forall s \leq u \leq t \leq T,\ K_{(a.)}(t,u) \leq f(\varphi^{(a.)}_{t,u}) \leq f(C^A_b(T-s)) < \infty.$$

It is clear (using Assumptions 4.1 and 5) that the operator $\Gamma : \mathcal{C}([s,T],\mathbb{R}) \to \mathcal{C}([s,T],\mathbb{R})$ is well defined. Given $n \in \mathbb{N}$, the iteration Γ^n is an affine operator with linear part $\Gamma^n_0 : r \mapsto (K_{(a.)})^{*(n)} * r$. We prove that Γ^n is contracting for n large enough, which is equivalent to proving that Γ^n_0 is contracting for large enough n.

By induction it is easily shown that:

$$\forall r \in \mathcal{C}([s,T],\mathbb{R}), \forall n \in \mathbb{N}\ \ ||\Gamma^n_0(r)||_{\infty,t} := \sup_{u\in[s,t]} |(\Gamma^n_0(r))(u,s)| \leq \frac{||r||_{\infty,T}(M^T_s(t-s))^n}{n!}.$$

Consequently $\forall r \in \mathcal{C}([s,T],\mathbb{R}), \forall n \in \mathbb{N}, ||\Gamma^n_0(r)||_{\infty,T} \leq \frac{(M^T_s(T-s))^n}{n!}||r||_{\infty,T}$ and Γ^n_0 is contracting for n large enough. We deduce that the operator Γ has a unique fixed point in $\mathcal{C}([s,T],\mathbb{R})$, or equivalently, that the Volterra equation (15) has a unique solution in $\mathcal{C}([s,T],\mathbb{R})$. ☐

4.3 The jump rate is uniformly bounded

Lemma 21. *Grant assumptions 2, 4 and 5 . Let $s \geq 0$ and $(a_t)_{t\geq s} \in \mathcal{C}([s,\infty[,\mathbb{R}_+)$. Let $Y^{s,\nu}_t$ be the solution of equation* (5)*, starting from $\mathcal{L}(Y^{s,\nu}_s) = \nu$. Then the functions $t \mapsto \mathbb{E} f'(Y^{s,\nu}_t)$, $t \mapsto \mathbb{E} f'(Y^{s,\nu}_t)b(Y^{s,\nu}_t)$ and $t \mapsto \mathbb{E} f^2(Y^{s,\nu}_t)$ are continuous on $[s,\infty[$.*

Proof. The proof relies on Theorem 19. Consider the interval $[s,T]$ for some fixed $T > s \geq 0$ and let $A := \sup_{t\in[s,T]} a_t$. Let $\phi \in \{f', f'b, f^2\}$. By Assumption 2.5, the function $(t,u) \mapsto \phi(\varphi^{(a.)}_{t,u})H_{(a.)}(t,u)r^{\nu}_{(a.)}(u,s)$ is uniformly continuous on $\{(t,u) : s \leq u \leq t \leq T\}$. Consequently:

$$t \mapsto \int_s^t \phi(\varphi^{(a.)}_{t,u})H_{(a.)}(t,u)r^{\nu}_{(a.)}(u,s)du \text{ is continuous on } [s,T].$$

The continuity of $t \mapsto \int_0^\infty \phi(\varphi^{(a.)}_{t,s}(x))H^x_{(a.)}(t,s)\nu(dx)$ follows from the dominated convergence theorem. For instance, for $\phi \equiv f'$, one has:

$$\forall t \in [s,T], \forall x \geq 0,\ f'(\varphi^{(a.)}_{t,s}(x)) \leq f'(\varphi^A_{t-s}(x)) \leq f'(x + C^A_b(t-s)) \leq C_f(f'(x) + 1 + f(C^A_b(T-s))),$$

from which the result follows easily using Assumption 5 and Assumption 4.2. The same method can be applied for $\phi(x) := f'(x)b(x)$ (using our Assumption 4.3) and for $\phi(x) := f^2(x)$. ☐

Theorem 22. *Grant Assumptions 2, 4 and 5. Let $s, J \geq 0$ be fixed. Given any $\kappa \geq 0$, there a constant $\bar{a} \geq \kappa$ only depending on b, f, J and κ such that:*

$$\forall (a_t)_{t\geq s} \in \mathcal{C}([s,\infty[,\mathbb{R}_+),\ \left\{\sup_{t\geq s} a_t \leq \bar{a} \text{ and } J\nu(f) \leq \bar{a}\right\} \implies \sup_{t\geq s} Jr^{\nu}_{(a.)}(t,s) \leq \bar{a}.$$

Moreover $\bar{a}$ is an increasing function of J and κ.

Proof. Assume $\sup_{t\geq s} a_t \leq \bar{a}$ for some $\bar{a} > 0$ that we specify later. Applying the Itô's formula and taking expectation, it yields:

$$\forall t \geq s,\ \mathbb{E} f(Y^{s,\nu}_t) = \mathbb{E} f(Y^{s,\nu}_s) + \int_s^t \mathbb{E} f'(Y^{s,\nu}_u)[b(Y^{s,\nu}_u) + a_u]du - \int_s^t \mathbb{E} f^2(Y^{s,\nu}_u)du.$$

From the continuity of $t \mapsto \mathbb{E}\, f'(Y_t^{s,\nu})b(Y_t^{s,\nu})$, $t \mapsto \mathbb{E}\, f'(Y_t^{s,\nu})$ and $t \mapsto \mathbb{E}\, f^2(Y_t^{s,\nu})$ (Lemma 21), we see that $t \mapsto \mathbb{E}\, f(Y_t^{s,\nu})$ is $\mathcal{C}^1$ on $[s, \infty[$ and :

$$\forall t \geq s,\ \frac{d}{dt}\mathbb{E}\, f(Y_t^{s,\nu}) = \mathbb{E}\, f'(Y_t^{s,\nu})(b(Y_t^{s,\nu}) + a_t) - \mathbb{E}\, f^2(Y_t^{s,\nu}).$$

Using our Assumption 2.3, the Cauchy-Schwarz inequality, it yields:

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\, f(Y_t^{s,\nu}) &\leq \left\{[\bar{a} + C_b]\,\mathbb{E}\, f'(Y_t^{s,\nu}) - \frac{1}{2}\mathbb{E}\, f^2(Y_t^{s,\nu})\right\} - \frac{1}{2}\mathbb{E}^2 f(Y_t^{s,\nu}) \\ &\leq \frac{1}{2}[2\psi(\bar{a} + C_b) - \mathbb{E}^2 f(Y_t^{s,\nu})],\end{aligned}$$

where in the last line, we used our Assumption 4.2. Setting $M(\bar{a}) := \max(\sqrt{2\psi(\bar{a} + C_b)}, \frac{\kappa}{J})$ we have:

$$\mathbb{E}\, f(Y_t^{s,\nu}) \leq \frac{1}{2}[M(\bar{a})^2 - \mathbb{E}^2 f(Y_t^{s,\nu})].$$

It follows from the next two lemmas that:

$$\nu(f) \leq M(\bar{a}) \implies [\forall t \geq s\ \ \mathbb{E}\, f(Y_t^{s,\nu}) \leq M(\bar{a})].$$

Let now see how to choose the constant $\bar{a}$. Our Assumption 4.2 yields:

$$\lim_{\theta \to \infty} \frac{J\sqrt{2\psi(\theta)}}{\theta} = 0,$$

so we can find a constant $\bar{a}_0$ such that:

$$\forall a \geq \bar{a}_0,\ \frac{J\sqrt{2\psi(a)}}{a} \leq 1/2.$$

Without loss of generality, we can choose $\bar{a}_0 \geq C_b$ and assume $\bar{a}_0$ is an increasing function of J. We choose $\bar{a} := \max(\kappa, \bar{a}_0)$. $\bar{a}$ is an increasing function of J and of κ and it yields:

$$JM(\bar{a}) \leq \max(\kappa, J\sqrt{2\psi(\bar{a} + C_b)}) \leq \max(\kappa, \frac{\bar{a} + C_b}{2}) \leq \bar{a}.$$

We deduce that:

$$\left[\sup_{t \geq s} a_t \leq \bar{a} \text{ and } J\nu(f) \leq \bar{a}\right] \implies \left\{\begin{array}{l}\frac{d}{dt}\mathbb{E}\, f(Y_t^{s,\nu}) \leq \frac{1}{2}\left[\frac{\bar{a}^2}{J^2} - \mathbb{E}^2 f(Y_t^{s,\nu})\right] \\ \mathbb{E}\, f(Y_s^{s,\nu}) \leq \frac{\bar{a}}{J}.\end{array}\right\} \implies \sup_{t \geq s} J r^{\nu}_{(a.)}(t, s) \leq \bar{a}.$$

□

Lemma 23. *Let $M > 0$ and $x_0 \in \mathbb{R}_+$. The following ODE:*

$$\dot{x} = \frac{1}{2}(M^2 - x^2),\ x(0) = x_0$$

has a unique solution given by:

$$x_t = \left\{\begin{array}{ll} M\frac{e^{Mt} - A}{e^{Mt} + A},\ A = \frac{M - x_0}{M + x_0} & \text{if } x_0 < M \\ M & \text{if } x_0 = M \\ M\frac{e^{Mt} + A}{e^{Mt} - A},\ A = \frac{x_0 - M}{M + x_0} & \text{if } x_0 > M.\end{array}\right.$$

Proof. A direct computation shows that the given formula is indeed a solution. Uniqueness follows from the next lemma. □

Lemma 24. *Let $y \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ be such that:*

$$\dot{y} \leq \frac{1}{2}(M^2 - y^2), \; y_0 = x_0.$$

Then it holds that: $\forall t \geq 0, \; y_t \leq x_t$, where x is the solution of the ODE of the previous lemma.

Proof. Let $z_t := y_t - x_t$. We have $z_0 = 0$. Assume there is a $r > 0$ such that $z_r > 0$. Define:

$$s := \sup\{t \in [0, r] : z_t = 0\}.$$

It follows from the continuity of z that $z_s = 0$. Because $z_r > 0$ we have:

$$0 \leq s < r.$$

Furthermore, z is non-negative on $[s, r]$, so $\forall t \in [s, r]$:

$$\dot{z}_t \leq \frac{1}{2}(x_t^2 - y_t^2) = -\frac{1}{2} z_t.(x_t + y_t) \leq 0.$$

We used here the fact that both x and y are non-negative functions. To conclude, we get the following contradiction:

$$0 = z_s \geq z_r > 0.$$

□

We have proved that $t \mapsto \mathbb{E}\, f(Y_t^{s,\nu})$ is $\mathcal{C}^1$ and bounded on $\mathbb{R}_+$. The same methods can be applied to the non-linear equation (2).

Lemma 25. *Grant assumptions 2, 4 and 5. Consider $(X_t)_{t\geq 0}$ a solution of the non-linear equation (2) in the sense of Definition 1. Then $t \mapsto \mathbb{E}\, f(X_t) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$. Moreover, there a finite constant $\bar{r} > 0$ (only depending on b, f and J) such that:*

$$\sup_{t\geq 0} \mathbb{E}\, f(X_t) \leq \max(\bar{r}, \mathbb{E}\, f(X_0)), \quad \limsup_{t\to\infty} \mathbb{E}\, f(X_t) \leq \bar{r}.$$

$\bar{r}$ can be chosen to be an increasing function of J.

Proof. By applying the same argument than in the proof of Lemma 15 it is clear that:

$$t \mapsto \mathbb{E}\, f(X_t), t \mapsto \mathbb{E}\, f'(X_t), t \mapsto \mathbb{E}\, f^2(X_t) \text{ and } t \mapsto \mathbb{E}\, |b(X_t)| f'(X_t)$$

are locally bounded. Applying the Itô's formula and taking expectation yields:

$$\mathbb{E}\, f(X_t) = \mathbb{E}\, f(X_0) + \int_0^t \mathbb{E}\, f'(X_u) b(X_u) du + J \int_0^t \mathbb{E}\, f'(X_u)\, \mathbb{E}\, f(X_u) du - \int_0^t \mathbb{E}\, f^2(X_u) du. \tag{22}$$

We deduce that $t \mapsto \mathbb{E}\, f(X_t)$ is continuous. We define $\forall t \geq 0, \; a_t := \mathbb{E}\, f(X_t)$, from Lemma 14, it is clear that:

$$a.s. \; \forall t \geq 0, \; X_t = Y_t^{0,\nu},$$

where $(Y_t^{0,\nu})_{t\geq 0}$ is the solution of (5) driven by $(a_t)_{t\geq 0}$. In particular, Lemma 21 applies and the functions $t \mapsto \mathbb{E}\, f'(X_t)$, $t \mapsto \mathbb{E}\, f^2(X_t)$ and $t \mapsto \mathbb{E}\, f'(X_t) b(X_t)$ are continuous. From equation (22), we deduce that $t \mapsto \mathbb{E}\, f(X_t) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ and that:

$$\frac{d}{dt} \mathbb{E}\, f(X_t) = \mathbb{E}\, f'(X_t) b(X_t) + J\, \mathbb{E}\, f'(X_t)\, \mathbb{E}\, f(X_t) - \mathbb{E}\, f^2(X_t).$$

We have:

1. $\mathbb{E}\, f'(X_t) b(X_t) - \frac{1}{4} \mathbb{E}\, f^2(X_t) \leq \frac{1}{2}[2C_b\, \mathbb{E}\, f'(X_t) - \frac{1}{2} \mathbb{E}\, f^2(X_t)] \leq \frac{1}{2}\psi(2C_b)$, using our Assumptions 2.3 and 4.2.

2.

$$J\,\mathbb{E}\,f'(X_t)\,\mathbb{E}\,f(X_t) - \frac{1}{4}\,\mathbb{E}\,f^2(X_t) \le J\,\mathbb{E}\,f'(X_t)\,\mathbb{E}\,f(X_t) - \frac{1}{4}\,\mathbb{E}^2\,f(X_t)$$
$$\le \mathbb{E}\,f(X_t)[J\,\mathbb{E}\,f'(X_t) - \frac{1}{8}\,\mathbb{E}\,f(X_t) - \frac{1}{8}\,\mathbb{E}\,f(X_t)]$$
$$\le 2\beta,$$

where $\beta = \sup_{x\ge 0} Jf'(x) - \frac{1}{8}f(x) < \infty$ (by Assumption 4.2). Note that β is a non-decreasing function of J.

Combining 1 and 2 yields:

$$\frac{d}{dt}\,\mathbb{E}\,f(X_t) \le \frac{1}{2}[(\psi(2C_b) + 4\beta) - \mathbb{E}\,f^2(X_t)].$$

We define: $\bar{r} := \sqrt{\psi(2C_b) + 4\beta}$, Lemmas 23 and 24 yields to:

$$\sup_{t\ge 0} \mathbb{E}\,f(X_t) \le \max(\bar{r}, \mathbb{E}\,f(X_0)), \quad \limsup_{t\to\infty} \mathbb{E}\,f(X_t) \le \bar{r}.$$

□

4.4 Existence and uniqueness of the solution of the non-linear SDE: proof of Theorem 6

We now prove that equation (2) has a unique strong solution $(X_t)_{t\ge 0}$. Let $J > 0$ (the case $J = 0$ has already been treated in Lemma 14 by choosing $(a_t)_{t\ge 0} \equiv 0$). Let ν, the initial condition, satisfying Assumption 5, be fixed. We grant Assumptions 2 and 4. Let $T > 0$ be a fixed horizon time. Thanks to Theorem 22 with $\kappa := \max(J\,\mathbb{E}\,f(X_0), J\bar{r})$, we build the following application:

$$\begin{array}{cccc} \Phi & : & \mathcal{C}^T_{\bar{a}} & \to & \mathcal{C}^T_{\bar{a}} \\ & & (a_t)_t & \mapsto & Jr^{\nu}_{(a.)}(\cdot, 0) := (Jr^{\nu}_{(a.)}(t,0))_t \ , \end{array} \tag{23}$$

where $\mathcal{C}^T_{\bar{a}} := \{(a_t)_t \in \mathcal{C}([0,T], \mathbb{R}_+) : \ \sup_{t\in[0,T]} a_t \le \bar{a}\}$. $r^{\nu}_{(a.)}(t,0) := \mathbb{E}\,f(Y^{0,\nu}_t)$ is defined by equation (5) (using $s = 0$). The constant $\bar{a}$ is given by Theorem 22: in particular $\bar{a}$ does not depend on T. We equip $\mathcal{C}^T_{\bar{a}}$ with the sup norm $||(a_t)_t||_{\infty,T} := \sup_{t\in[0,T]} |a_t|$: $(\mathcal{C}^T_{\bar{a}}, ||.||_{\infty,T})$ is a complete metric space. We now prove that the application Φ defined by (23) is contracting. Let $(a_t)_t, (d_t)_t \in \mathcal{C}^T_{\bar{a}}$; we denote by $r^{\nu}_{(a.)}(t,s)$ and $r^{\nu}_{(d.)}(t,s)$ their corresponding jump rate, t belongs to $[s,T]$. $r^{\nu}_{(a.)}$ and $r^{\nu}_{(d.)}$ both satisfies the Volterra equation of the type (15). It follows that the difference $\Delta := r^{\nu}_{(a.)} - r^{\nu}_{(d.)}$ satisfies:

$$\begin{aligned} \Delta =& K^{\nu}_{(a.)} - K^{\nu}_{(d.)} + K_{(a.)} * (r^{\nu}_{(a.)} - r^{\nu}_{(d.)}) + (K_{(a.)} - K_{(d.)}) * r^{\nu}_{(d.)} \\ =& W + K_{(a.)} * \Delta \text{ with } W := K^{\nu}_{(a.)} - K^{\nu}_{(d.)} + (K_{(a.)} - K_{(d.)}) * r^{\nu}_{(d.)} \end{aligned}$$

Consequently, Δ is solution of a non-homogeneous Volterra equation with a kernel $K_{(a.)}$ and a forcing term W:

$$\Delta = W + K_{(a.)} * \Delta. \tag{24}$$

This equation can solved explicitly in term of $r_{(a.)}$, the "resolvent" of $K_{(a.)}$:

Lemma 26. *Let $r_{(a.)}(t,s)$ be the solution of the following Volterra equation:*

$$r_{(a.)} = K_{(a.)} + K_{(a.)} * r_{(a.)}.$$

Then $\Delta = W + r_{(a.)} * W$.

Proof. Using the same argument as in Lemma 20, we see that the Volterra equation (24) admits a unique solution. Furthermore, we have: $W + K_{(a.)} * (W + r_{(a.)} * W) = W + K_{(a.)} * W + (r_{(a.)} - K_{(a.)}) * W = W + r_{(a.)} * W$. □

Lemma 27. *There exists a constant Θ_T only depending on T, f, b and $\bar{a}$ such that:*

$$\forall 0 \leq s \leq t \leq T,\ \forall x \in \mathbb{R}_+,\ |K^x_{(a.)} - K^x_{(d.)}|(t,s) \leq \Theta_T(1 + f'(x) + f(x) + f'(x)f(x)) \int_s^t |a_u - d_u| du.$$

Proof.

$$\begin{aligned} |K^x_{(a.)} - K^x_{(d.)}|(t,s) &= |f(\varphi^{(a.)}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}(x))du\right) - f(\varphi^{(d.)}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{(d.)}_{u,s}(x))du\right)| \\ &\leq |f(\varphi^{(a.)}_{t,s}(x)) - f(\varphi^{(d.)}_{t,s}(x))| \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}(x))du\right) \\ &+ f(\varphi^{(d.)}_{t,s}(x))| \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}(x))du\right) - \exp\left(-\int_s^t f(\varphi^{(d.)}_{u,s}(x))du\right)| \\ &:= M + N \end{aligned}$$

Using Assumptions 2.2, 2.3 and 4.1, we deduce that:

$$\begin{aligned} M &\leq |f(\varphi^{(a.)}_{t,s}(x)) - f(\varphi^{(d.)}_{t,s}(x))| \\ &\leq f'(x + C^{\bar{a}}_b T)|\varphi^{(a.)}_{t,s}(x) - \varphi^{(d.)}_{t,s}(x)| \\ &\leq C_f(1 + f'(x) + f'(C^{\bar{a}}_b T))C_\varphi \int_s^t |a_u - d_u| du. \end{aligned}$$

Furthermore, using that $\forall A, B \geq 0: \quad |e^{-A} - e^{-B}| = e^{-\min(A,B)}(1 - e^{-|A-B|}) \leq |A - B|$, it yields:

$$\begin{aligned} N &\leq C_f[1 + f(x) + f(C^{\bar{a}}_b T)] \int_s^t |f(\varphi^{(a.)}_{u,s}(x)) - f(\varphi^{(d.)}_{u,s}(x))| du \\ &\leq C_f[1 + f(x) + f(C^{\bar{a}}_b T)] f'(x + C^{\bar{a}}_b T) C_\varphi \int_s^t \int_s^r |a_u - d_u| du dr \\ &\leq T C_\varphi C_f^2 [1 + f(x) + f(C^{\bar{a}}_b T)][1 + f'(x) + f'(C^{\bar{a}}_b T)] \int_s^t |a_u - d_u| du. \end{aligned}$$

Combining the two estimates, we get the result. □

Proof of Theorem 6. We now write Θ_T for any constant that depends only on T, on the initial condition ν, on b, f, J and on $\bar{a}$ and that can change from line to line. Using our Assumption 5, it follows that:

$$\forall (a_t)_t, (d_t)_t \in \mathcal{C}^T_{\bar{a}},\ \forall t \in [0,T]:\ |K^\nu_{(a.)} - K^\nu_{(d.)}|(t,0) \leq \Theta_T \int_0^t |a_u - d_u| du.$$

Moreover,

$$|(K_{(a.)} - K_{(d.)}) * r_{(d.)}|(t,0) = |\int_0^t (K_{(a.)} - K_{(d.)})(t,u) r_{(d.)}(u) du| \leq \frac{\bar{a}}{J} \Theta_T (1 + f'(0)).T \int_0^t |a_u - d_u| du.$$

Consequently, there is a constant Θ_T such that:

$$\forall (a_t)_t, (d_t)_t \in \mathcal{C}^T_{\bar{a}},\ \forall t \in [0,T]:\ |W|(t,0) \leq \Theta_T \int_0^t |a_u - d_u| du.$$

Using the formula in Lemma 26, we deduce that:

$$\begin{aligned} |\Delta(t,0)| \leq& |W|(t,0) + \int_0^t r_{(a.)}(t,u)|W|(u,0) du \\ \leq& \Theta_T.(1 + T\frac{\bar{a}}{J}) \int_0^t |a_u - d_u| du. \end{aligned}$$

We have proved that there is a constant Θ_T such that:

$$\forall (a_t)_t, (d_t)_t \in \mathcal{C}^T_{\bar{a}}, \forall t \in [0,T]: \ ||Jr^{\nu}_{(a.)}(\cdot,0) - Jr^{\nu}_{(d.)}(\cdot,0)||_{\infty,t} \leq \Theta_T \int_0^t ||a-d||_{\infty,u} du.$$

This estimate is sufficient to prove the Theorem 6 by a classical Picard argument: the application Φ^n is contracting for n large enough and we deduce that Φ has a unique fix point $(a^*_t)_t$. It is then easy to check that $(Y^{0,\nu}_t)_{t\in[0,T]}$, driven by the current (a^*) and with initial condition $Y^{0,\nu}_0 = X_0$, defines a solution of (2) up to time T. This proves existence of a strong solution to (2). Now, if $(X_t)_{t\geq 0}$ is a strong solution of (2) in the sense of Definition 1, let $\forall t \geq 0,\ a_t := J\,\mathbb{E}\, f(X_t)$. We have $\sup_{t\geq 0} a_t \leq \max(J\bar{r}, J\,\mathbb{E}\, f(X_0)) \leq \bar{a}$ and consequently $(a_t)_{t\in[0,T]} \in \mathcal{C}^T_{\bar{a}}$. Moreover, it is clear that $(X_t)_{t\geq 0}$ solves (5) with $a_t := J\,\mathbb{E}\, f(X_t)$ and $Y^{0,\nu}_0 := X_0$. We deduce that (a_t) is the unique fixed point of Φ: $\forall t \in [0,T] : a_t = a^*_t$. Consequently, by Lemma 14, we have: $a.s.\ \forall t \in [0,T]\ X_t = Y^{0,\nu}_t$. This proves path-wise uniqueness and ends the proof of Theorem 6. □

5 The invariant measures: proof of Theorem 8

We now study the invariant measures of the non-linear process (2). We follow the strategy of [18]: we first study the linear process driven by a constant current a and show that it has a unique invariant measure. We then use this result to study the invariant measures of the non-linear equation (2). Let $a \geq 0$ and $(Y^\nu_t)_t$ the solution of the following SDE:

$$Y^\nu_t = Y^\nu_0 + \int_0^t b(Y^\nu_u)du + at - \int_0^t \int_{\mathbb{R}_+} Y^\nu_{u-} \mathbb{1}_{\{z \leq f(Y^\nu_{u-})\}} \mathbf{N}(du,dz) \tag{25}$$

Equation (25) is equation (5) with $\forall t \geq 0,\ a_t = a$ and $s = 0$.

Theorem 28. *Grant Assumptions 2, 4, and 7. Then the SDE* (25) *has a unique invariant measure ν^∞_a given by equation* (19)*:*

$$\nu^\infty_a(dx) := \frac{\gamma(a)}{b(x)+a} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right) \mathbb{1}_{\{x\in[0,\sigma_a]\}}dx,$$

where $\gamma(a)$ is the normalizing factor given by (20)*. Moreover we have $\nu^\infty_a(f) = \gamma(a)$.*

A proof of this theorem can be found in [18] (proposition 21) with $b(x) := -\lambda x$ and with slightly different assumptions on f. We give here a proof based on different arguments. Note that the general method introduced by [9] to find the stationary measures of a PDMP can be applied here; we use a method introduced in this paper to prove the uniqueness part.

Proof. Let us first check that the probability measure ν^∞_a is indeed an invariant measure of (25).
Claim 1: with our assumptions on f and b, ν^∞_a satisfies Assumption 5.
First, using $b(0) > 0$, we have: $\forall a \geq 0,\ \sigma_a \geq \sigma_0 > 0$. The changes of variable $x = \varphi^a_t$ and $y = \varphi^a_u$ yield

$$\begin{aligned}\int_0^{\sigma_a} \frac{f^2(x)}{b(x)+a} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right)dx &= \int_0^\infty f^2(\varphi^a_t) \exp\left(-\int_0^{\varphi^a_t} \frac{f(y)}{b(y)+a}dy\right)dt \\ &= \int_0^\infty f^2(\varphi^a_t) \exp\left(-\int_0^t f(\varphi^a_u)du\right)dt.\end{aligned}$$

This last integral is finite by Assumptions 4 and 7.

Claim 2: We have: $K^{\nu^\infty_a}_a(t) = \gamma(a)H_a$.

$$\begin{aligned}K^{\nu^\infty_a}_a(t) &= \int_0^\infty K^x_a(t)\nu^\infty_a(dx) \\ &= \int_0^{\sigma_a} f(\varphi^a_t(x)) \exp\left(-\int_0^t f(\varphi^a_u(x))du\right) \frac{\gamma(a)}{b(x)+a} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right)dx.\end{aligned}$$

By the change of variable $y = \varphi_u^a$ it yields:

$$K_a^{\nu_a^\infty}(t) = \int_0^{\sigma_a} f(\varphi_t^a(x)) \exp\left(-\int_0^t f(\varphi_u^a(x))du\right) \frac{\gamma(a)}{b(x)+a} \exp\left(-\int_0^{t(x)} f(\varphi_u^a)du\right)dx,$$

where $t(x)$ is the unique $t \geq 0$ such that $\varphi_t^a = x$. We now make the change of variable $x = \varphi_s^a$ we obtain (using the semi-group property satisfied by φ_t^a):

$$\begin{aligned}
K_a^{\nu_a^\infty}(t) &= \gamma(a) \int_0^\infty f(\varphi_t^a(\varphi_s^a)) \exp\left(-\int_0^t f(\varphi_u^a(\varphi_s^a))du\right) \exp\left(-\int_0^s f(\varphi_u^a)du\right)ds \\
&= \gamma(a) \int_0^\infty f(\varphi_{t+s}^a) \exp\left(-\int_0^{t+s} f(\varphi_u^a)du\right)ds \\
&= \gamma(a) \int_t^\infty f(\varphi_\theta^a) \exp\left(-\int_0^\theta f(\varphi_u^a)du\right)d\theta \\
&= \gamma(a) \left[-\exp\left(-\int_0^\theta f(\varphi_u^a)du\right)\right]_t^\infty \\
&= \gamma(a)[H_a(t) - \lim_{\theta\to\infty} \exp\left(-\int_0^\theta f(\varphi_u^a)du\right)]
\end{aligned}$$

Using Assumption 7, we have: $\lim_{\theta\to\infty} \exp\left(-\int_0^\theta f(\varphi_u^a)du\right) = 0$ and the claim is proved.

We consider now $(Y_t^{\nu_a^\infty})_{t\geq 0}$ the solution of equation (25) starting from $\mathcal{L}(Y_0^{\nu_a^\infty}) = \nu_a^\infty$. Theorems 19 applies, so $r_a(t) = \mathbb{E}\, f(Y_t^{\nu_a^\infty})$ is the unique solution of the Volterra equation:

$$r_a = K_a^{\nu_a^\infty} + K_a * r_a$$

But using Claim 2 and the relation (17), it yields:

$$\gamma(a) = K_a^{\nu_a^\infty} + K_a * \gamma(a). \tag{26}$$

By uniqueness (Lemma 20) we deduce that: $\forall t \geq 0,\ r_a(t) = \gamma(a)$.

Finally, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. Using theorem 19 we have:

$$\begin{aligned}
\mathbb{E}\,\phi(Y_t^{\nu_a^\infty}) &= \gamma(a) \int_0^t \phi(\varphi_{t-u}^a) H_a(t-u)du + \int_0^\infty \phi(\varphi_t^a(x)) H_a^x(t) \nu_a^\infty(dx) \\
&= \gamma(a) \int_0^t \phi(\varphi_u^a) H_a(u)du + \int_0^{\sigma_a} \phi(\varphi_t^a(x)) \exp\left(-\int_0^t f(\varphi_u^a(x))du\right) \frac{\gamma(a)}{b(x)+a} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right)dx
\end{aligned}$$

The change of variables $y = \varphi_u^a$ and $x = \varphi_\theta^a$ yields to:

$$\begin{aligned}
\mathbb{E}\,\phi(Y_t^{\nu_a^\infty}) &= \gamma(a) \int_0^t \phi(\varphi_u^a) H_a(u)du + \gamma(a) \int_0^\infty \phi(\varphi_t^a(\varphi_\theta^a)) \exp\left(-\int_0^t f(\varphi_u^a(\varphi_\theta^a))du\right) \exp\left(-\int_0^\theta f(\varphi_u^a)du\right)d\theta \\
&= \gamma(a) \int_0^t \phi(\varphi_u^a) H_a(u)du + \gamma(a) \int_t^\infty \phi(\varphi_u^a) \exp\left(-\int_0^u f(\varphi_\theta^a)d\theta\right)du \\
&= \gamma(a) \int_0^\infty \phi(\varphi_u^a) H_a(u)du \\
&= \mathbb{E}\,\phi(Y_t^{\nu_a^\infty}).
\end{aligned}$$

This proves that $\forall t \geq 0,\ \mathcal{L}(Y_t^{\nu_a^\infty}) = \nu_a^\infty$ and consequently ν_a^∞ is an invariant measure of (25).
It remains to prove that the invariant measure is unique. Let ν be an invariant measure of (25) and let g be

a $\mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R})$ compactly supported test function. We follow the method of [9] (proof of theorem 3(a)) and define:

$$\lambda_g(x) := \int_0^\infty g(\varphi_t^a(x)) \exp\left(-\int_0^t f(\varphi_r^a(x)) dr\right) dt.$$

Claim 3: $\mathcal{L}\lambda_g(x) := \lambda_g(0) f(x) - g(x)$, where $\mathcal{L}$ is the generator of the Markov process (25) defined in equation (11):

$$\forall \phi \in \mathcal{C}_b^1(\mathbb{R}_+)\ \mathcal{L}\phi(x) := \phi'(x)[b(x) + a] + (\phi(0) - \phi(x)) f(x).$$

Let us see how Claim 3 implies uniqueness of the invariant measure. Using that ν is an invariant measure, the Itô's formula yields:

$$0 = \frac{d}{dt} \mathbb{E}\, \lambda_g(Y_t^\nu) = \mathbb{E}\, \mathcal{L}\lambda_g(Y_t^\nu) = \lambda_g(0) p_\nu - \mathbb{E}\, g(Y_t^\nu),$$

with $p_\nu := \mathbb{E}\, f(Y_t^\nu) = \nu(f)$. Note that this equality ensures that $p_\nu < +\infty$. The same computations can be done with $g \equiv 1$, giving:

$$\lambda_1(0) p_\nu = 1,$$

and it follows that:

$$\int_0^\infty g(x) \nu(dx) = \frac{\lambda_g(0)}{\lambda_1(0)} = \int_0^\infty g(x) \nu_a^\infty(dx).$$

The class of compactly supported $\mathcal{C}^\infty(\mathbb{R}_+, \mathbb{R})$ is determining the probability measures on $\mathbb{R}_+$ so necessarily $\nu = \nu_a^\infty$.

Proof of Claim 3.

Using the semi-group property of $\varphi_t^a(x)$ it yields:

$$\lambda_g(\varphi_t^a(x)) = \exp\left(\int_0^t f(\varphi_u^a(x)) du\right)[\lambda_g(x) - \int_0^t g(\varphi_u^a(x)) \exp\left(-\int_0^u f(\varphi_\theta^a(x)) d\theta\right) du].$$

It follows that:

$$\frac{d}{dt}\lambda_g(\varphi_t^a(x))|_{t=0} = \lambda_g'(x)(b(x) + a) = f(x)\lambda_g(x) - g(x),$$

which gives the result. $\square$

The next lemma characterizes the invariant measures of equation (2).

Lemma 29. *The invariant measures of the non-linear equation* (2) *are* $\{\nu_a^\infty :\ a \geq 0,\ \frac{a}{\gamma(a)} = J\}$.

Proof. Let ν be an invariant measure of the non-linear equation (2). It follows that:

$$\forall t \geq 0,\ \ \mathbb{E}\, f(X_t) = \nu(f) =: p.$$

Let $a := Jp$. $(X_t)_{t\geq 0}$ solves (25) and ν is an invariant measure of equation (25). It implies that $\nu = \nu_\infty^a$. Moreover $p = \gamma(a)$ and so necessarily $\frac{a}{\gamma(a)} = J$.

Conversely, let $a \geq 0$ such that $\frac{a}{\gamma(a)} = J$ and let $Y_0^\nu \sim \nu_a^\infty$. Let (Y_t^ν) be the solution of SDE (25). We have seen that $\mathbb{E}\, f(Y_t^\nu) = \gamma(a)$, it follows that $a = J\, \mathbb{E}\, f(X_t)$ and consequently $(Y_t^\nu)_{t\geq 0}$ is also a solution of the non-linear SDE (2) and ν_a^∞ is one of its invariant measure. $\square$

The problem of finding the invariant measures of the mean-field equation has been reduced to finding the solutions of the scalar equation (21). When J is small enough, we can prove that this scalar equation has a unique solution, which concludes the proof of Theorem 8.

Lemma 30. *Equation* (21) *has at least one solution* $a^* > 0$*. Moreover, there is a constant* $J_0 > 0$ *such that for all* $J \in [0, J_0]$ (21) *has a unique solution.*

Proof. Let $\Gamma(a) := \gamma(a)^{-1} = \int_0^{\sigma_a} \frac{1}{b(x)+a} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right)dx$.
Claim 1: $\lim_{a\to+\infty} \sigma_a = +\infty$.
By Assumption 2.3 one has: $\sup_{x\geq 0} b(x) \leq C_b$. Let $a > C_b$:

$$\begin{aligned}
\sigma_a &= \inf\{x \geq 0 : \ b(x) + a \geq 0\} \\
&= \inf\{x \geq 0 : \ |b(x)| \geq a\} \\
&\geq \inf\{x \geq 0 : \ C_{b,f}(1 + f(x)) \geq a\} \text{ using Assumption 4.3} \\
&\geq \inf\{x \geq 0 : \ Ce^x \geq a\} \text{ using } \forall x \geq 0,\ f(x) \leq Ce^x \text{ for some } C > 0 \\
&= \log(a) - \ln(C) \to_{a\to\infty} +\infty.
\end{aligned}$$

Claim 2: $\lim_{a\to+\infty} a\Gamma(a) = +\infty$.

$$\begin{aligned}
a\Gamma(a) &= \int_0^{\sigma_a} \frac{1}{1+\frac{b(x)}{a}} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a^*}dy\right)dx \\
&\geq \int_0^{\sigma_a} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right)dx.
\end{aligned}$$

Let $M > 0$, by the Claim 1, for a large enough one has $\sigma_a \geq M$. For such a, it follows that:

$$a\Gamma(a) \geq \int_0^{M} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right)dx \to_{a\to\infty} = M,$$

and so $\forall M > 0, \liminf_{a\to+\infty} a\Gamma(a) \geq M$ which proves the claim.
Let $U(a) := a\Gamma(a)$. One has $U(0) = 0$, $\lim_{a\to+\infty} U(a) = +\infty$ and U is continuous on $\mathbb{R}_+$: it follows that equation $U(a) = J$ has at least one solution a^*. Moreover, one can show easily that the function $a \mapsto U(a) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ and $U'(0) = \Gamma(0) > 0$: there $a_0 > 0$ such that $a \mapsto U(a)$ is strictly increasing on $[0, a_0]$. Using $\lim_{a\to+\infty} U(a) = +\infty$, we can find a_1 such that: $\forall a \geq a_1, U(a) \geq 1$. Finally let $J_0 := \min_{a\in[a_0,a_1]} U(a) > 0$. Let $J < J_0$, it is clear that the equation $U(a) = J$ has exactly one solution $a^* \in [0, a_0]$. □

6 Convergence in law of the time marginals: proof of Theorem 9

We now assume that Assumptions 2, 4, 5 and 7 hold and consider $(X_t)_{t\geq 0}$ the solution of (2) starting from $\mathcal{L}(X_0) = \nu$. Let $a_t = J\,\mathbb{E}\,f(X_t)$, we assume that:

$$\forall t \geq 0,\ |a_t - a^*| \leq Ce^{-\lambda t},$$

for some constant $C, \lambda > 0$ and some $a^* \geq 0$ satisfying (21): $\frac{a^*}{\gamma(a^*)} = J$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a bounded Lipschitz-continuous function, with Lipschitz constant l_ϕ. We consider $\lambda' \in]0, \min(\lambda, f(\sigma_0))[$. We now prove that there is a constant D (only depending on C, b, f, J λ', $||\phi||_\infty$ and l_ϕ) such that:

$$\forall t \geq 0,\ \ |\mathbb{E}\,\phi(X_t) - \nu^\infty_{a^*}(\phi)| \leq De^{-\lambda' t}.$$

We have $a.s.$ $X_t = Y_t^{0,\nu}$ and by formula (19), it yields:

$$\begin{aligned}
\mathbb{E}\,\phi(X_t) &= \int_0^t \phi(\varphi_{t,u}^{(a.)})H_{(a.)}(t,u)r^\nu_{(a.)}(u,0)du + \int_0^\infty \phi(\varphi_{t,0}^{(a.)}(x))H^x_{(a.)}(t,0)\nu(dx) \\
&= M_t + N_t
\end{aligned}$$

The second term goes to zero exponentially fast in t. Using Assumption 2.2 and the monotonicity of f yields:

$$N_t \leq ||\phi||_\infty H_0(t) = ||\phi||_\infty \exp\left(-\int_0^t f(\varphi_u)du\right).$$

By Assumption 7, $\lim_{u\to\infty} f(\varphi_u) = f(\sigma_0)$. and using the fact that $\lambda' < f(\sigma_0)$, one can find a constant D such that:

$$N_t \leq De^{-\lambda' t}.$$

Moreover, one has:

$$\begin{aligned}\int_0^\infty \phi(x)\nu_{a^*}^\infty(dx) &= \int_0^\infty \phi(\varphi_t^{a^*})\gamma(a^*)H_{a^*}(t)dt \\ &= \int_0^t \phi(\varphi_{t,u}^{a^*})H_{a^*}(t,u)\gamma(a^*)du + \int_t^\infty \phi(\varphi_u^{a^*})\gamma(a^*)H_{a^*}(u)du.\end{aligned}$$

Again the second term goes exponentially fast to zero. This follows from:

$$\begin{aligned}\int_t^\infty \phi(\varphi_u^{a^*})\gamma(a^*)H_{a^*}(u)du &\leq ||\phi||_\infty\gamma(a^*)\int_t^\infty \frac{f(\varphi_u^{a^*})}{\inf_{v\geq t} f(\varphi_v^{a^*})}\exp\left(-\int_0^u f(\varphi_\theta^{a^*})d\theta\right)du \\ &= \frac{||\phi||_\infty\gamma(a^*)}{\inf_{v\geq t} f(\varphi_v^{a^*})}\exp\left(-\int_0^t f(\varphi_\theta^{a^*})d\theta\right) \\ &\leq De^{-\lambda' t},\end{aligned}$$

for some constant D. It remains to show that

$$\Delta := |\int_0^t \phi(\varphi_{t,u}^{(a.)})H_{(a.)}(t,u)r_{(a.)}^\nu(u,0)du - \int_0^t \phi(\varphi_{t,u}^{a^*})H_{a^*}(t,u)\gamma(a^*)du|$$

goes to zero exponentially fast. One has:

$$\begin{aligned}\Delta \leq &\frac{\bar{a}}{J}\int_0^t |\phi(\varphi_{t,u}^{(a.)}) - \phi(\varphi_{t,u}^{a^*})|H_0(t,u)du + \frac{||\phi||_\infty\bar{a}}{J}\int_0^t |H_{(a.)}(t,u) - H_{a^*}(t,u)|du \\ &+ ||\phi||_\infty\int_0^t H_0(t,u)|r_{(a.)}^\nu(u,0) - \gamma(a^*)|du \\ :=&\alpha_t + \beta_t + \theta_t.\end{aligned}$$

Using the fact that $|a_t - a^*| \leq Ce^{-\lambda' t}$ $(\lambda' < \lambda)$, it follows that:

$$\begin{aligned}\theta_t &\leq \frac{||\phi||_\infty C}{J}\int_0^t H_0(t,u)e^{-\lambda' u}du \\ &= \frac{||\phi||_\infty C}{J}e^{-\lambda' t}\int_0^t H_0(t,u)e^{\lambda'(t-u)}du \\ &\leq \left[\frac{||\phi||_\infty C}{J}\int_0^\infty H_0(u)e^{\lambda' u}du\right]e^{-\lambda' t} =: De^{-\lambda' t}.\end{aligned}$$

The fact that $u \mapsto H_0(u)e^{\lambda' u}$ belongs to $L^1(\mathbb{R}_+)$ follows from $\lambda' < f(\sigma_0)$. Moreover, Assumption 2.4 yields

$$\begin{aligned}\alpha_t &\leq \frac{\bar{a}l_\phi}{J}\int_0^t |\varphi_{t,u}^{(a.)} - \varphi_{t,u}^{a^*}|H_0(t,u)du \\ &\leq \frac{\bar{a}l_\phi C_\varphi}{J}\int_0^t\int_u^t |a_\theta - a^*|d\theta H_0(t,u)du.\end{aligned}$$

Using that $\int_u^t |a_\theta - a^*|d\theta \leq C\int_u^t e^{-\lambda'\theta}d\theta = \frac{Ce^{-\lambda' t}}{\lambda'}[e^{\lambda'(t-u)} - 1] \leq \frac{Ce^{-\lambda' t}}{\lambda'}e^{\lambda'(t-u)}$, one has:

$$\begin{aligned}\alpha_t &\leq \frac{\bar{a}l_\phi C_\varphi C}{J\lambda'}e^{-\lambda' t}\int_0^t e^{\lambda'(t-u)}H_0(t,u)du \\ &\leq \left[\frac{\bar{a}l_\phi C_\varphi C}{J\lambda'}\int_0^\infty H_0(u)e^{\lambda' u}du\right]e^{-\lambda' t} =: De^{-\lambda' t}.\end{aligned}$$

Finally, using the inequality $|e^{-A} - e^{-B}| \leq e^{-\min(A,B)}|A - B|$, one has:

$$\beta_t \leq \frac{||\phi||_\infty\bar{a}}{J}\int_0^t H_0(t-u)\int_u^t |f(\varphi_{\theta,u}^{(a.)}) - f(\varphi_{\theta,u}^{a^*})|d\theta du,$$

and moreover:

$$\int_u^t |f(\varphi^{(a.)}_{\theta,u}) - f(\varphi^{a^*}_{\theta,u})| d\theta \le C_\varphi f'(\varphi^{\bar a}_{t,u}) \int_u^t \int_u^\theta |a_s - a^*| ds d\theta$$
$$\le C_\varphi C f'(\varphi^{\bar a}_{t,u}) \int_u^t \int_u^\theta e^{-\lambda' s} ds d\theta$$
$$\le C_\varphi \frac{C}{\lambda'} f'(\varphi^{\bar a}_{t,u}) e^{-\lambda' t}(t-u) e^{\lambda'(t-u)}.$$

Note that Assumption 2.3 implies that $f'(\varphi^{\bar a}_{t,u}) \le f'(C^{\bar a}_b (t-u))$ and from Assumption 4.2, it yields that:

$$\forall \epsilon > 0, \exists A_\epsilon : \ \forall x \ge 0, f'(x) \le A_\epsilon e^{\epsilon x}.$$

Choosing $\epsilon := (f(\sigma_0) - \lambda')/2$, one has:

$$\int_u^t |f(\varphi^{(a.)}_{\theta,u}) - f(\varphi^{a^*}_{\theta,u})| d\theta \le A_\epsilon C_\varphi \frac{C}{\lambda'} (t-u) e^{(\lambda'+\epsilon)(t-u)} e^{-\lambda' t},$$

and we deduce that:

$$\beta_t \le \left[\frac{A_\epsilon C C_\varphi ||\phi||_\infty \bar a}{\lambda' J} \int_0^{+\infty} H_0(u) u e^{(\lambda'+\epsilon)u} du \right] e^{-\lambda' t} =: D e^{-\lambda' t}.$$

Combining the three estimates, we have proved the result.

7 Long time behavior with constant drift

7.1 Study of the Volterra equation

In the case where (a_t) is constant and equal to a, the Volterra equation (15) is a linear homogeneous convolution Volterra equation. For such equations, it is very natural to use Laplace transform techniques as convolutions become scalar products with this transformation. Furthermore, our "kernel" K_a and the "forcing term" K_a^ν are non-negative. Volterra equation with positive kernels have been studied in the context of Renewal theory. The main reference on this question is a paper of Feller [17]. The author uses essentially Tauberian theorems for Laplace transform to study the asymptotic properties of the solution. The non-negativity of both the kernel and the forcing term are key to apply these Tauberian theorems. These simple methods are very efficient and can be applied to our case to show quite easily that:

$$\mathbb{E}\, f(Y_t^\nu) \to_{t\to\infty} \gamma(a). \tag{27}$$

However, these methods do not give a good estimate of the rate of convergence. To be more precise, following [17, Theorem 3] we can prove that:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}\, f(Y_u^\nu) du = \gamma(a) \ ,$$

without rate of convergence. Note that such weak result can be obtained directly using ergodic theory: indeed this time average convergence holds almost surely without the expectation according to Birkhoff's ergodic theorem. This follows from the uniqueness of the invariant measure and to the fact that this invariant measure is absolutely continuous with respect to the Lebesgue measure on $[0, \sigma_a]$.

Using more advanced Tauberian theorems such as the one of Haar, it would be possible to prove (27) with a polynomial rate of convergence, namely:

$$\forall n \ge 0, \lim_{t\to\infty} t^n |\, \mathbb{E}\, f(Y_t^\nu) - \gamma(a)| = 0.$$

We refer to [17, Theorem 4] for this method. However, as we will see, in our case the rate of convergence is of exponential type. We can relate it to the parameters of the problem. In order to achieve the optimal rate of convergence, we use general methods from the Volterra integral equation theory, and especially the so called "Whole-line Palay-Wiener" Theorem. To apply these methods, we have first to study finely the properties of the Laplace transform of the kernel K_a.

Along this section, we grant Assumptions 2, 4, 5 and 7. Let $a \geq 0$ be a constant non-negative current. We first consider the case where the initial condition ν is δ_0, that is: $Y_0 = 0$ *a.s.* Let Y_t be the solution of the SDE (5) with $(a.) \equiv a$. Applying formula (15) we find that $r_a(t) := \mathbb{E} f(Y_t)$ satisfies the following Volterra equation:

$$r_a = K_a + K_a * r_a, \tag{28}$$

with $K_a(t) = f(\varphi_t^a) \exp\left(-\int_0^t f(\varphi_u^a) du\right)$. We remind the reader that H_a is defined by $H_a(t) := \exp\left(-\int_0^t f(\varphi_u^a) du\right)$ and that H_a and K_a satisfies the following relation, that we exploit later:

$$\forall t \geq 0, \quad \int_0^t K_a(s) ds = 1 - H_a(t) \tag{29}$$

Definition 31 (Laplace transform)**.** *Let $g : \mathbb{R}_+ \mapsto \mathbb{R}$ be a measurable function. The Laplace transform of g is the following function:*

$$\widehat{g}(z) := \int_0^\infty e^{-zt} g(t) dt,$$

defined for all $z \in \mathbb{C}$ for which the integral exists.

Note that the Laplace transform of H_a and K_a is well defined for all $z \in \mathbb{C}$, $\Re(z) > -f(\sigma_a)$. This follows from the fact that $\forall \lambda < f(\sigma_a)$, $\exists C_\lambda :\ \forall t \geq 0$, $H_a(t) \leq C_\lambda e^{-\lambda t}$. Consequently, $\forall \lambda > -f(\sigma_a)$, $\int_0^\infty e^{\lambda t} H_a(t) dt < \infty$. The same holds for K_a.

Integrating by parts the Laplace transform of K_a shows that:

$$\forall z \in \mathbb{C},\ \Re(z) > -f(\sigma_a) \implies \widehat{K_a}(z) = 1 - z\widehat{H_a}(z). \tag{30}$$

It is also useful to introduce the following Banach space:

Definition 32. *Let $L_\lambda = \{f \in \mathcal{B}(\mathbb{R}_+, \mathbb{R}) : ||f||_{\lambda,1} < \infty\}$ the space of Borel-measurable function from $\mathbb{R}_+$ to $\mathbb{R}$, equipped with:*

$$||f||_{\lambda,1} = \int_{\mathbb{R}_+} |f(s)| e^{\lambda s} ds.$$

$(L_\lambda, ||f||_{\lambda,1})$ is a Banach space: this follows from the general theory of L^p space, see for instance [29, Th. 3.11] with $p = 1$ and $\mu(dx) := e^{\lambda x} dx$; where dx denotes the Lebesgue measure on $\mathbb{R}_+$.

The long time behavior of $r_a(t)$ is closely related to the location of the zeros of $\widehat{K_a} - 1$, and thus to the location of the zeros of $\widehat{H_a}$. It is the purpose of the following section to study the location of these zeros.

7.2 On the zeros of $\widehat{H_a}$

As we have seen, $\widehat{K_a}(z) = 1 - z\widehat{H_a}(z)$ and consequently for any non-null complex number z with $\Re(z) > -f(\sigma_a)$ it holds that: $\widehat{K_a}(z) = 1 \iff \widehat{H_a}(z) = 0$. It is readily seen that $\widehat{K_a}(0) = 1$ and $\widehat{H_a}(0) = \int_0^\infty H_a(t) dt \neq 0$. Note also that $\widehat{H_a}(z) = 0 \iff \widehat{H_a}(\bar{z}) = 0$, so it is sufficient to locate the zeros of $\widehat{H_a}$ in the region $\Im(z) \geq 0$.

Lemma 33. $\forall z \in \mathbb{C},\ \Re(z) \geq 0 \implies \widehat{H_a}(z) \neq 0$.

Proof. First, it follows from for the non-negativity of K_a that:

$$|\widehat{K_a}(z)| \leq \int_0^\infty |e^{-tz}| K_a(t) dt < \int_0^\infty K_a(t) dt = 1 \text{ if } \Re(z) > 0.$$

It yields: $\Re(z) > 0 \implies \widehat{H_a}(z) \neq 0$. Moreover, (following [17] proof of Theorem 4, (b)) if $z = iy$, $y > 0$ then:

$$\begin{aligned} 1 - \widehat{K_a}(iy) &= iy\widehat{H_a}(iy) \\ &= \int_0^\infty (1 - e^{-iyt})K_a(t)dt \\ &= \int_0^\infty (1 - \cos(yt))K_a(t)dt + i\int_0^\infty \sin(yt)K_a(t)dt. \end{aligned}$$

But then $\widehat{K_a}(iy) = 1$ for some $y > 0$ would implies that for Lebesgue almost every $t \geq 0$ $(1 - \cos(yt))K_a(t) = 0$, that is Lebesgue almost everywhere, $K_a(t) = 0$, which obviously contradict our assumptions. It follows that $\forall y > 0,\ \widehat{H_a}(iy) \neq 0$. □

Lemma 34. *The zeros of $\widehat{H_a}$ are isolated, that is:*

$$\widehat{H_a}(z_0) = 0 \implies \exists r(z_0) > 0 : \ \forall\ |z - z_0| \leq r(z_0),\ z \neq z_0 : \ \widehat{H_a}(z) \neq 0.$$

Proof. This directly follows from the fact that $z \mapsto \widehat{H_a}(z)$ is an holomorphic function on $\Re(z) > -f(\sigma_a)$, consequently its zeros are isolated. □

Lemma 35. *The zeros of $\widehat{H_a}$ are within a "cone":*

$$\forall z \in \mathbb{C},\ \Re(z) > -f(\sigma_a),\ z = x + iy,\ \widehat{H_a}(z) = 0 \implies |y| \leq \phi_a(x),$$

where $\phi_a(x) := ||K'_{a,x}||_1$ and $K_{a,x}(t) := e^{-xt}K_a(t)$, $K'_{a,x}(t) := \frac{d}{dt}K_{a,x}(t)$.

Proof. Let $z = x + iy$, $y > 0, x > -f(\sigma_a)$. We have:

$$\widehat{K_a}(z) = \int_0^\infty e^{-zt}K_a(t)dt = \int_0^\infty e^{-iyt}K_{a,x}(t)dt = \int_0^\infty \frac{e^{-iyt}}{iy}K'_{a,x}(t)dt.$$

The last equality follows by an integration by part. It yields:

$$|\widehat{K_a}(z)| \leq \frac{||K'_{a,x}||_1}{|y|}.$$

We deduce that $|y| > ||K'_x||_1 \implies \widehat{K_a}(z) \neq 1 \implies \widehat{H_a}(z) \neq 0$. □

Consequently, from Lemmas 33, 34 and Lemma 35, we can define the abscissa corresponding to the "first" zero of $\widehat{H_a}$:

$$\lambda_a^* := -\sup\{\Re(z)|\ \Re(z) > -f(\sigma_a),\ \widehat{H_a}(z) = 0\}, \tag{31}$$

with the convention that $\lambda_a^* = f(\sigma_a)$ if the set of zeros is empty. We have proved that:

$$0 < \lambda_a^* \leq f(\sigma_a) \leq \infty.$$

λ_a^* is key in our problem as it gives the speed of convergence to the invariant measure. λ_a^* depends only on a, b and f.

7.3 Convergence with optimal rate

Our goal in this section is to prove the following theorem:

Theorem 36. *The equation (28) has a unique solution r_a of the form: $r_a = \gamma(a) + \xi_a$. Moreover,*

$$\forall \lambda \in [0, \lambda_a^*[: \quad \xi_a \in L_\lambda,$$

where λ_a^* is defined by (31).

This result can be deduced from general theorems of the Volterra equations theory. For instance, one can apply [20, Th. 2.4, Chap. 7]. However, this last result is written for general measure kernels in weighted spaces and its proof is somehow difficult to follow. In our setting the proof given by [20] simplifies a lot and we give it here for completeness. We use the following so-called "Whole Line Palay-Wiener" Theorem which is one of the most important ingredients of the convolution Volterra integral equations theory.

Theorem 37 (Whole-line Palay-Wiener)**.** *Let $k \in L^1(\mathbb{R}, \mathbb{R})$. There exists a function $r \in L^1(\mathbb{R}, \mathbb{R})$ satisfying the equation*

$$r = k + k * r$$

if and only if:

$$\forall y \in \mathbb{R},\ \widehat{k}(iy) := \int_{\mathbb{R}} e^{-iyt} k(t)dt \neq 1.$$

Note that in this theorem '*' denotes the standard convolution on $\mathbb{R}$ and that $\widehat{k}(iy)$ is actually the Fourier transform of k evaluated at $y \in \mathbb{R}$.

Proof. See [20, Th. 4.3, Chap. 2]. We prove later, in details, an extension of this theorem (see Theorem 49). □

Proof of Theorem 36. Let σ_- and σ_+ be any real numbers such that:

$$-\lambda_a^* < \sigma_- < 0 < \sigma_+ < \infty.$$

We first extend r_a, K_a and H_a to the whole line by defining: $\forall t \in \mathbb{R} :\ r_a(t) := r_a(t)\mathbb{1}_{\{t \geq 0\}},\ K_a(t) := K_a(t)\mathbb{1}_{\{t \geq 0\}}$ and $H_a(t) := H_a(t)\mathbb{1}_{\{t \geq 0\}}$. We have from (28):

$$\forall t \in \mathbb{R},\ r_a(t) = K_a(t) + K_a * r_a(t) = K_a(t) + \int_{\mathbb{R}} K_a(t-u) r_a(u) du.$$

We also define $\forall t \in \mathbb{R},\ r_{a,\Delta}(t) := e^{-\Delta t} r_a(t),\quad K_{a,\Delta}(t) := e^{-\Delta t} K_a(t)$ for any $\Delta \in \mathbb{R}$. Note that $K_{a,\sigma_-} \in L^1(\mathbb{R})$ and that $\forall y \in \mathbb{R},\ \widehat{K}_{a,\sigma_-}(iy) = \widehat{K_a}(\sigma_- + iy) \neq 1$ (by definition of λ_a^*). We can apply Theorem 37: there exists $\xi_{a,\sigma_-} \in L^1(\mathbb{R})$ such that:

$$\xi_{a,\sigma_-} = K_{a,\sigma_-} + K_{a,\sigma_-} * \xi_{a,\sigma_-}$$

We define $\xi_a := e^{\sigma_- t} \xi_{a,\sigma_-}(t)$. It holds that: $\int_{\mathbb{R}} |\xi_a(u)| e^{-\sigma_- u} du < \infty$ and moreover:

$$\xi_a = K_a + K_a * \xi_a.$$

From the continuity of K_a, it automatically holds that ξ_a is continuous. Following the proof of [20, Th. 2.4, Chap. 7], let ϕ be any non-negative $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$ function vanishing outside of $[-1, 1]$ such that $\int \phi = 1$ and define $\forall n \geq 1,\ \phi^n(t) := n\phi(nt),\ \xi^n_{a,\sigma_-}(t) := (\phi^n * \xi_a)(t) e^{-\sigma_- t},\ r^n_{a,\sigma_+}(t) := (\phi^n * r_a)(t) e^{-\sigma_+ t}$. It holds:

$$\xi^n_{a,\sigma_-} \in L^1(\mathbb{R}),\quad \widehat{\xi^n_{a,\sigma_-}}(iy) = \left[\widehat{\phi^n} \frac{\widehat{K_a}}{1 - \widehat{K_a}}\right](iy + \sigma_-),$$

$$r^n_{a,\sigma_+} \in L^1(\mathbb{R}),\quad \widehat{r^n_{a,\sigma_+}}(iy) = \left[\widehat{\phi^n} \frac{\widehat{K_a}}{1 - \widehat{K_a}}\right](iy + \sigma_+).$$

We can now use the Fourier inverse formula for $L^1(\mathbb{R})$ functions to get:

$$\forall t \in \mathbb{R},\ \xi^n_{a,\sigma_-}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iyt} \left[\widehat{\phi^n} \frac{\widehat{K_a}}{1 - \widehat{K_a}}\right](iy + \sigma_-) dy$$

$$\forall t \in \mathbb{R},\ r^n_{a,\sigma_+}(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iyt} \left[\widehat{\phi^n} \frac{\widehat{K_a}}{1 - \widehat{K_a}}\right](iy + \sigma_+) dy,$$

or after the changes of variable $z = iy + \sigma_-$ and $z = iy + \sigma_+$:

$$\forall t \in \mathbb{R},\ \xi^n_{a,\sigma_-}(t)e^{\sigma_- t} := \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma_- - iT}^{\sigma_- + iT} e^{zt}\widehat{\phi^n}(z)\frac{\widehat{K_a}(z)}{z\widehat{H_a}(z)}dz$$

$$\forall t \in \mathbb{R},\ r^n_{a,\sigma_+}(t)e^{\sigma_+ t} := \lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma_+ - iT}^{\sigma_+ + iT} e^{zt}\widehat{\phi^n}(z)\frac{\widehat{K_a}(z)}{z\widehat{H_a}(z)}dz.$$

Let Γ_T be the closed curve in the complex plane composed of four straight lines that join the points $\sigma_- - iT$, $\sigma_- + iT$, $\sigma_+ + iT$, and $\sigma_+ - iT$ in the anti-clockwise direction, then it follows from the residue theorem that:

$$\int_{\Gamma_T} e^{zt}\widehat{\phi^n}(z)\frac{\widehat{K_a}(z)}{z\widehat{H_a}(z)}dz = 2\pi i\frac{\widehat{K_a}(0)\widehat{\phi^n}(0)}{\widehat{H_a}(0)} = 2\pi i\gamma(a). \tag{32}$$

Indeed, $\widehat{H_a}(0) = \int_0^\infty H_a(t)dt = \int_0^\infty \exp\left(-\int_0^t f(\varphi^a_u du)\right)dt = \frac{1}{\gamma(a)}$. Moreover, one can find $T_0 > 0$ such that for all z in the strip $\Re(z) \in [\sigma_-, \sigma_+]$ with $|\mathcal{I}(z)| \geq T_0$, we have:

$$|\widehat{K_a}(z)| \leq \frac{1}{2}.$$

For such z, it follows that:

$$|\frac{\widehat{K_a}(z)}{z\widehat{H_a}(z)}| = |\frac{\widehat{K_a}(z)}{1-\widehat{K_a}(z)}| \leq 1.$$

We can also find a constant M_n such that z in the strip $\Re(z) \in [\sigma_-, \sigma_+]$, $z \neq 0$:

$$|\widehat{\phi^n}(z)| \leq \frac{M_n}{|z|}.$$

Hence the integrand of (32) goes to zero as $|\mathcal{I}(z)| \to \infty$ uniformly in the strip $\Re(z) \in [\sigma_-, \sigma_+]$. Therefore we can take the limit $T \to \infty$ in (32) and we get:

$$\forall t \in \mathbb{R},\ \forall n \geq 1:\ r^n_{a,\sigma_+}(t)e^{\sigma_+ t} = \gamma(a) + \xi^n_{a,\sigma_-}(t)e^{\sigma_- t}$$

Finally, letting n goes to infinity, it follows from the continuity of ξ_a and of r_a that:

$$\forall t \in \mathbb{R},\ r_a(t) = \gamma(a) + \xi_a(t).$$

The theorem is proved by choosing $\sigma_- = -\lambda$. □

The speed of convergence obtained in this theorem is optimal. To see this, assume that $\lambda^*_a < \sigma_a$ and choose $\sigma_- < -\lambda^*_a$. The given proof can be mimicked except that the integral (32) now involves terms of the order $e^{-\lambda^*_a t}$ corresponding to the roots of $\widehat{H_a}$ with real part equal to $-\lambda^*_a$. We deduce that: $\limsup_{t\to\infty} |r(t) - \gamma(a)|e^{\lambda^*_a t} \neq 0$.

7.4 Long time behavior starting from initial condition ν: proof of Theorem 10

We now turn back to the general case where the initial condition can be any probability measure satisfying Assumption 5, and give the proof of Theorem 10.

Proof of Theorem 10. Let $r^\nu_a(t) = \mathbb{E}\, f(Y^\nu_t)$ with $\mathcal{L}(Y_0) = \nu$. r^ν_a is the unique solution of the Volterra equation:

$$r^\nu_a = K^\nu_a + K_a * r^\nu_a.$$

If we choose ν to be the invariant measure ν^∞_a, we get: $\gamma(a) = K^{\nu^\infty_a}_a + K_a * \gamma(a)$. Consequently, it follows that:

$$r^\nu_a - \gamma(a) = K^\nu_a - K^{\nu^\infty_a}_a + K_a * (r^\nu_a - \gamma(a)).$$

We can solve this equation in term of r_a, the "resolvent" of K_a (as we did Lemma 26) and obtain:

$$\begin{aligned} r_a^\nu - \gamma(a) &= K_a^\nu - K_a^{\nu_a^\infty} + r_a * (K_a^\nu - K_a^{\nu_a^\infty}) \\ &= K_a^\nu - K_a^{\nu_a^\infty} + \xi_a * (K_a^\nu - K_a^{\nu_a^\infty}) + \gamma(a) * (K_a^\nu - K_a^{\nu_a^\infty}), \end{aligned}$$

where $r_a = \xi_a + \gamma(a)$ is the solution of the Volterra equation $r_a = K_a + K_a * r_a$. Using that $\gamma(a) * K_a^\nu = \gamma(a)(1 - H_a^\nu)$ (see equation 17), it yields:

$$r_a^\nu - \gamma(a) = K_a^\nu - K_a^{\nu_a^\infty} + \gamma(a)(H_a^{\nu_a^\infty} - H_a^\nu) + \xi_a * (K_a^\nu - K_a^{\nu_a^\infty}).$$

We now write Θ any constant only depending on λ, f, b and a and which may change from line to line. It is clear that for any $0 < \lambda < f(\sigma_a)$:

$$|H_a^{\nu_a^\infty} - H_a^\nu|(t) \le \int_0^\infty H_a^x(t)|\nu - \nu_a^\infty|(dx) \le \int_0^\infty H_a(t)|\nu - \nu_a^\infty|(dx) \le \Theta e^{-\lambda t} \int_0^\infty |\nu - \nu_a^\infty|(dx).$$

Similarly, for any $0 < \lambda < f(\sigma_a)$:

$$\begin{aligned} |K_a^\nu - K_a^{\nu_a^\infty}|(t) &\le \int_0^\infty f(\varphi_t^a(x)) H_a^x(t)|\nu - \nu_a^\infty|(dx) \le \int_0^\infty f(x + C_b^a t) H_a(t)|\nu - \nu_a^\infty|(dx) \\ &\le C_f \int_0^\infty [1 + f(x) + f(C_b^a t)] H_a(t)|\nu - \nu_a^\infty|(dx) \le \Theta e^{-\lambda t} \int_0^\infty (1 + f(x))|\nu - \nu_a^\infty|(dx). \end{aligned}$$

We used here our Assumption 4.1. Let now $0 < \lambda < \lambda_a^*$, it holds that:

$$|\xi_a * (K_a^\nu - K_a^{\nu_a^\infty})|(t) \le \int_0^t |\xi_a(t-u)||K_a^\nu - K_a^{\nu_a^\infty}|(u) du \le \Theta e^{-\lambda t} \int_0^\infty (1 + f(x))|\nu - \nu_a^\infty|(dx).$$

Combining the three estimates, one deduces that:

$$|r_a^\nu(t) - \gamma(a)| \le \Theta e^{-\lambda t} \int_0^\infty (1 + f(x))|\nu - \nu_a^\infty|(dx).$$

It remains to prove that

$$Y_t^\nu \xrightarrow[t\to\infty]{\mathcal{L}} \nu_a^\infty. \tag{33}$$

Note that $(Y_t^\nu)_{t\ge0}$ is the solution of (2) with $J = 0$ and a drift $b(x) + a$. Applying Theorem 9, (33) follows. □

8 Long time behavior with a general drift

In this section, we generalize the results obtained in Section 7 to non constant current. We consider the process (5) driven by a current (a_t) and assume that this current (a_t) is converging at an exponential rate to a limit value a. We seek to prove that the jump rate of this process is converging to $\gamma(a)$ and estimate the speed of convergence. This "perturbation" analysis will be useful to study the long time behavior of the solution of the non-linear McKean-Vlasov equation (2) with small interactions.

Along this section, we grant Assumptions 2, 4, 5 and 7. We fix a constant $s \ge 0$ and consider $(a_t)_{t\ge s}$ such that:

Assumptions 38. *$(a_t)_{t\ge s}$ is a continuous function satisfying:*

1. *$\sup_{t\ge s} a_t \le \bar{a}$ for some constant $\bar{a} > 0$.*
2. *We assume that $(a_t)_{t\ge s}$ converges to a at an exponential rate:*

$$\exists C \ge 0, \lambda > 0 : \ \forall t \ge s, \ |a_t - a| \le C e^{-\lambda t}. \tag{34}$$

These constants C and λ are important in our analysis. Any mention of C and λ in this section always refer to these two constants.

3. *Furthermore, we assume that*
$$0 < \lambda < \min(\lambda_a^*, f(\sigma_0)),$$
where $\lambda_a^* > 0$ *is given in* (31) *(we recall that* λ_a^* *only depends on* f*,* b *and* a*) and* σ_0 *is defined in Assumption 7.*

Let $r^{\nu}_{(a.)}(t,s) := \mathbb{E}\, f(Y_t^{s,\nu})$, where $Y_t^{s,\nu}$ is the solution of the SDE (5) driven by the current (a_t) and starting at time s with law ν. The goal of this section is to prove the following theorem:

Theorem 39. *If* C *is small enough, then:*
$$\exists D \geq 0,\ \forall t \geq s,\ |r^{\nu}_{(a.)}(t,s) - \gamma(a)| \leq De^{-\lambda(t-s)}\ ,$$
where $\gamma(a)$ *is given by* (20).

Note that the exponential decay rate λ is conserved (this is the same λ as in (34)). We make efforts to keep track of the constant D and to relate it to C. As in Section 7 it is useful to split the study in two parts: the case where the initial condition is a Dirac mass at 0 and the general case. We thus consider $r_{(a.)}$ the unique solution of the following Volterra equation:
$$r_{(a.)} = K_{(a.)} + K_{(a.)} * r_{(a.)}, \tag{35}$$
where $K_{(a.)}$ is define by (14).

It is also useful to introduce a Banach space adapted to this non-homogeneous setting.

8.1 An adapted Banach algebra

Definition 40. *A function* $K : (\mathbb{R}_+)^2 \to \mathbb{R}$ *is said to be a Volterra Kernel with weight* λ *if:* K *is Borel measurable,* $\forall s > t : K(t,s) = 0$ *a.e. and* $||K||_{\lambda,1} < \infty$ *with :*
$$||K||_{\lambda,1} := \operatorname*{ess\,sup}_{t\geq 0} \int_{\mathbb{R}_+} |K(t,s)|e^{\lambda(t-s)}ds.$$
We define $\mathcal{V}_\lambda$ *the set of Volterra kernels with weight* λ*. We also define for* $K \in \mathcal{V}_\lambda$*:*
$$||K||_{\lambda,\infty} = \operatorname*{ess\,sup}_{t,s\geq 0} |K(t,s)e^{\lambda(t-s)}| \in \mathbb{R}_+ \cup \{+\infty\}.$$

Lemma 41. *Let* $a, b \in \mathcal{V}_\lambda$*. We define:*
$$\forall t \geq s \geq 0,\ (a*b)(t,s) := \int_s^t a(t,u)b(u,s)du. \tag{36}$$
Then $a * b \in \mathcal{V}_\lambda$ *and* $||a*b||_{\lambda,1} \leq ||a||_{\lambda,1}||b||_{\lambda,1}$*. Moreover, we have* $\forall a,b,c \in \mathcal{V}_\lambda$*,* $\forall \alpha \in \mathbb{R}$*:*
$$(a*b)*c = a*(b*c),\ a*(b+c) = a*b + a*c,\ (a+b)*c = a*c + b*c,$$
$$a*(\alpha b) = \alpha(a*b) = (\alpha a)*b.$$

Theorem 42. *The space* $(\mathcal{V}_\lambda, ||\cdot||_{\lambda,1})$ *is a Banach space.*

For the proof of the previous lemma and of this theorem, we refer to [20], Theorem 2.4 and Proposition 2.7 (i) of Chapter 9 with $p = +\infty$ and $J = \mathbb{R}_+$. To be more precise, the results are only written and proved for $\lambda = 0$ (no exponential weight) but the proof given can be readily extended in our case with an exponential weight of the form $e^{\lambda(t-s)}$.

Lemma 43 (Connection with the case $\forall t \geq 0,\ a_t = a$)**.** *Let* $g \in L_\lambda$*. We define:*
$$\forall t,s \in \mathbb{R}_+ :\ \tilde{g}(t,s) := g(t-s)\mathbb{1}_{t\geq s}.$$
Then $\tilde{g} \in \mathcal{V}_\lambda$ *and* $||g||_{\lambda,1} = ||\tilde{g}||_{\lambda,1}$.

This result allows us to always see an element of L_λ as an element of $\mathcal{V}_\lambda$. We always make this identification in an implicit way. **Note that the algebra L_λ is commutative whereas $\mathcal{V}_\lambda$ is not**.

8.2 The perturbation method

We want to solve equation (35) and show that its solution can be written in the form:

$$r_{(a.)} = \gamma(a) + \xi_{(a.)}, \quad \xi_{(a.)} \in \mathcal{V}_\lambda.$$

We achieved this when (a_t) is constant (see Theorem 36). Let's see how to extend this result for small perturbations, i.e. for small C. The algebra $\mathcal{V}_\lambda$ does not have any neutral element (in fact the neutral element would be a dirac distribution) but assume for the sake of this heuristic approach that I is a neutral element of the algebra (e.g. $k * I = I * k = k \ \forall k \in \mathcal{V}_\lambda$). Equation (35) can be rewritten as:

$$(I - K_{(a.)}) * (I + r_{(a.)}) = I. \tag{37}$$

In particular (taking $(a.) \equiv a$), we have $(I - K_a) * (I + r_a) = (I + r_a) * (I - K_a) = I$. Furthermore,

$$I - K_{(a.)} = (I - K_a) * (I - (I + r_a)\bar{K}_{(a.)}),$$

with $\bar{K}_{(a.)} := K_{(a.)} - K_a \in \mathcal{V}_\lambda$. Equation (37) becomes:

$$(I - K_a) * (I - (I + r_a) * \bar{K}_{(a.)}) * (I + r_{(a.)}) = I.$$

We multiply by $I + r_a$ on the left of each side, and we get:

$$(I - (I + r_a) * \bar{K}_{(a.)}) * (I + r_{(a.)}) = I + r_a.$$

We now expand this equation - the neutral element I disappears and we get:

$$r_{(a.)} - (\bar{K}_{(a.)} + r_a * \bar{K}_{(a.)}) * r_{(a.)} = r_a + \bar{K}_{(a.)} + r_a * \bar{K}_{(a.)},$$

or after defining $\Delta_K := \bar{K}_{(a.)} + r_a * \bar{K}_{(a.)}$:

$$r_{(a.)} = r_a + \Delta_K * r_{(a.)} + \Delta_K. \tag{38}$$

It is natural to consider the resolvent equation associated to (38), that is the solution of the following equation:

$$\Delta_r = \Delta_K + \Delta_K * \Delta_r. \tag{39}$$

Expanding Δ_K, we see that:

$$\begin{aligned}\Delta_K &= \bar{K}_{(a.)} + (\gamma(a) + \xi_a) * \bar{K}_{(a.)} \\ &= \bar{K}_{(a.)} + \xi_a * \bar{K}_{(a.)} + \gamma(a) * (K_{(a.)} - K_a).\end{aligned}$$

The key point to note is that $\gamma(a) * K_{(a.)}$ can be computed explicitly in term of $H_{(a.)}$ using the relation (17):

$$\gamma(a) * K_{(a.)} = \gamma(a)(1 - H_{(a.)}).$$

It follows that:

$$\gamma(a) * (K_{(a.)} - K_a) = -\gamma(a)\bar{H}_{(a.)}, \quad \text{with } \bar{H}_{(a.)} := H_{(a.)} - H_a.$$

We deduce that:

$$\Delta_K = \bar{K}_{(a.)} + \xi_a * \bar{K}_{(a.)} - \gamma(a)\bar{H}_{(a.)} \in \mathcal{V}_\lambda.$$

Moreover, as we will see later, the norms of $\bar{K}_{(a.)}$ and of $\bar{H}_{(a.)}$ are controlled by C, e.g. the "size" of the perturbation. Let C small enough such that $||\Delta_K||_{\lambda,1} < 1$ so that we can solve (39) in $\mathcal{V}_\lambda$. The solution $\Delta_r \in \mathcal{V}_\lambda$ satisfies:

$$||\Delta_r||_{\lambda,1} \leq \frac{||\Delta_K||_{\lambda,1}}{1 - ||\Delta_K||_{\lambda,1}}.$$

Going back to (38), the solution is now given by:

$$r_{(a.)} = r_a + \Delta_K + \Delta_r * (r_a + \Delta_K).$$

We obtain the following formula:

$$r_{(a.)} = r_a + \Delta_r + \Delta_r * r_a. \tag{40}$$

The end of this section is devoted to make the above argument rigorous and to see how the constants propagate. We recall that our Banach algebra does not have any neutral element so the proof have to be adapted. We begin by estimating the norms of $\bar{K}_{(a.)}$ and $\bar{H}_{(a.)}$.

8.2.1 Norm of $\bar{K}_{(a.)}$ and of $\bar{H}_{(a.)}$

We recall that $\forall t \geq s \geq 0$:

$$\bar{K}_{(a.)}(t,s) := K_{(a.)}(t,s) - K_a(t,s) := f(\varphi^{(a.)}_{t,s}) \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}) du\right) - f(\varphi^a_{t-s}) \exp\left(-\int_s^t f(\varphi^a_{u-s}) du\right),$$
$$\bar{H}_{(a.)}(t,s) := H_{(a.)}(t,s) - H_a(t,s) := \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}) du\right) - \exp\left(-\int_s^t f(\varphi^a_{u-s}) du\right).$$

Lemma 44. *There is a function $\eta \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ such that:*

1. $\eta \in L^1(\mathbb{R}_+)$.
2. $\forall t \geq s \geq 0$:

$$|\bar{K}_{(a.)}(t,s)| \leq Ce^{-\lambda t}\eta(t-s),$$
$$|\bar{H}_{(a.)}(t,s)| \leq Ce^{-\lambda t}\eta(t-s).$$

The constants C and λ are the same as the one appearing in (34). The function η only depends on $b, \bar{a}, f$ and λ (in particular it does not depend on C). Furthermore, we can choose η such that $||\eta||_1$ is a non-decreasing functions of $\bar{a}$.

Proof.

$$\begin{aligned}|\bar{K}_{(a.)}(t,s)| \leq & |f(\varphi^{(a.)}_{t,s}) - f(\varphi^a_{t,s})| \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}) du\right) \\ & + f(\varphi^a_{t,s})| \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}) du\right) - \exp\left(-\int_s^t f(\varphi^a_{u,s}) du\right)| \\ := & M_1 + M_2.\end{aligned}$$

From our assumption (34) it yields:

$$\begin{aligned}|f(\varphi^{(a.)}_{t,s}) - f(\varphi^a_{t,s})| &\leq f'(\varphi^{\bar{a}}_{t,s})|\varphi^{(a.)}_{t,s} - \varphi^a_{t,s}| \leq f'(C^{\bar{a}}_b(t-s))C_\varphi \int_s^t |a_u - a| du \\ &\leq f'(C^{\bar{a}}_b(t-s))C_\varphi C \int_s^t e^{-\lambda u} du = Ce^{-\lambda t} \cdot f'(C^{\bar{a}}_b(t-s))C_\varphi \frac{e^{\lambda(t-s)} - 1}{\lambda}.\end{aligned}$$

Moreover setting $\lambda' := \frac{\lambda + f(\sigma_0)}{2}$ and using the fact that $f(\varphi_u) \to f(\sigma_0)$ as $u \to \infty$ it yields that:

$$\begin{aligned}\exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}) du\right) &\leq \exp\left(-\int_s^t f(\varphi_{u,s}) du\right) = \exp\left(-\int_0^{t-s} f(\varphi_u) du\right) \\ &\leq D(b,f,\lambda')e^{-\lambda'(t-s)},\end{aligned}$$

for some finite constant $D(b,f,\lambda')$. Let $\alpha(u) := D(b,f,\lambda')e^{-\lambda' u} f'(C^{\bar{a}}_b u)C_\varphi \frac{e^{\lambda u} - 1}{\lambda}$, we have:

$$M_1 \leq Ce^{-\lambda t}\alpha(t-s),$$

and $\alpha \in L^1(\mathbb{R}_+)$. Moreover, the inequality $|e^{-A} - e^{-B}| \leq e^{-\min(A,B)}|A - B|$ yields:

$$\begin{aligned}M_2 &\leq f(\varphi^{\bar{a}}_{t,s}) \exp\left(-\int_0^{t-s} f(\varphi_u) du\right) \left|\int_s^t f(\varphi^{(a.)}_{u,s}) - f(\varphi^a_{u,s}) du\right| \\ &\leq D(b,f,\lambda')e^{-\lambda'(t-s)} f(C^{\bar{a}}_b(t-s)) f'(C^{\bar{a}}_b(t-s)) \int_s^t |\varphi^{(a.)}_{u,s} - \varphi^a_{u,s}| du.\end{aligned}$$

One has:

$$\int_s^t |\varphi^{(a.)}_{u,s} - \varphi^a_{u,s}|du \leq C_\varphi \int_s^t \int_s^u |a_\theta - a|d\theta du \leq CC_\varphi \int_s^t \int_s^u e^{-\lambda\theta} d\theta du = Ce^{-\lambda t} \cdot \frac{C_\varphi}{\lambda}\left[(t-s)e^{\lambda(t-s)} - \frac{e^{\lambda(t-s)}-1}{\lambda}\right].$$

Consequently it yields that:

$$M_2 \leq Ce^{-\lambda t}\beta(t-s),$$

with:

$$\beta(u) := D(b,f,\lambda')e^{-\lambda' u} f(C_b^{\bar{a}}u)f'(C_b^{\bar{a}}u)\frac{C_\varphi}{\lambda}\left[ue^{\lambda u} - \frac{e^{\lambda u}-1}{\lambda}\right].$$

$\beta \in L^1(\mathbb{R}_+)$ and setting $\eta := \alpha + \beta$ concludes the proof for $\bar{K}_{(a.)}$. The same computations give a similar result for $\bar{H}_{(a.)}$. □

These estimates are sharp enough to give the following result:

Lemma 45. *Denote by* 1 *the constant kernel equal to* 1.

1. $\bar{K}_{(a.)} \in \mathcal{V}_\lambda$ *and* $||\bar{K}_{(a.)}||_{\lambda,1} \leq C||\eta||_1$.
2. $\bar{K}_{(a.)} * 1 \in \mathcal{V}_\lambda$ *and* $||\bar{K}_{(a.)} * 1||_{\lambda,1} \leq C||\eta||_1$.

The exact same estimates holds for $\bar{H}_{(a.)}$ *and* $\bar{H}_{(a.)} * 1$.

Proof. Using the preceding lemma, we have:

$$||\bar{K}_{(a.)}||_{\lambda,1} := \sup_{t\geq 0}\int_0^t |\bar{K}_{(a.)}|(t,s)e^{\lambda(t-s)}ds \leq \sup_{t\geq 0}\int_0^t Ce^{-\lambda s}\eta(t-s)ds \leq C||\eta||_1,$$

proving point 1. For point 2, we have $\forall t \geq s \geq 0$, $(\bar{K}_{(a.)} * 1)(t,s) := \int_s^t \bar{K}_{(a.)}(t,u)du$. It follows from the previous lemma that:

$$|\bar{K}_{(a.)} * 1|(t,s) \leq C\int_s^t e^{-\lambda t}\eta(t-u)du \leq Ce^{-\lambda t}||\eta||_1,$$

from which we deduce that:

$$||\bar{K}_{(a.)} * 1||_{\lambda,1} := \sup_{t\geq 0}\int_0^t |\bar{K}_{(a.)} * 1|(t,s)e^{\lambda(t-s)}ds \leq \sup_{t\geq 0}\int_0^t Ce^{-\lambda t}||\eta||_1 e^{\lambda(t-s)}ds = C||\eta||_1.$$

□

8.2.2 Proof of the perturbation method

We are now ready to prove the following theorem:

Theorem 46. *Assume* C *satisfies:*

$$0 < C < \frac{1}{||\eta||_1(1 + ||\xi_a||_{\lambda,1} + \gamma(a))}, \tag{41}$$

and let $\alpha \in]0,1[$ *be such that:* $C = \frac{\alpha}{||\eta||_1(1+||\xi_a||_{\lambda,1}+\gamma(a))}$. *Define* $\Delta_K := \bar{K}_{(a.)} + \xi_a * \bar{K}_{(a.)} - \gamma(a)\bar{H}_{(a.)}$. *It holds that* $||\Delta_K||_{\lambda,1} \leq \alpha$. *Let* Δ_r *be the solution of the Volterra equation* (39)*:*

$$\Delta_r = \Delta_K + \Delta_K * \Delta_r.$$

Then, $\Delta_r \in \mathcal{V}_\lambda$ *and* $||\Delta_r||_{\lambda,1} \leq \frac{\alpha}{1-\alpha}$. *Moreover,* $\Delta_r * 1 \in \mathcal{V}_\lambda$ *and* $||\Delta_r * 1||_{\lambda,1} \leq \frac{\alpha}{1-\alpha}$.

Let $r_{(a.)}(t,s)$ *be the jump rate associated to the current* $(a_t)_{t\geq 0}$. *Then formula* (40) *holds, that is:*

$$r_{(a.)} = r_a + \Delta_r + \Delta_r * r_a.$$

Consequently, $r_{(a.)} = \gamma(a) + \xi_{(a.)}$ *with:*

$$\xi_{(a.)} = \xi_a + \Delta_r + \Delta_r * \xi_a + \gamma(a)(\Delta_r * 1) \in \mathcal{V}_\lambda.$$

Furthermore,

$$||\xi_{(a.)}||_{\lambda,1} \leq ||\xi_a||_{\lambda,1} + \frac{\alpha}{1-\alpha}[1 + ||\xi_a||_{\lambda,1} + \gamma(a)].$$

Proof. Assume C satisfies (41) for some $\alpha \in]0,1[$. Let $\Delta_K := \bar{K}_{(a.)} + \xi_a * \bar{K}_{(a.)} - \gamma(a)\bar{H}_{(a.)}$. It is readily seen that $||\Delta_K||_{\lambda,1} \leq \alpha < 1$. Consequently equation (39) admits a solution Δ_r given by:

$$\Delta_r = \sum_{k\geq 1} \Delta_K^{*k} \in \mathcal{V}_\lambda.$$

We used here the fact that the space $\mathcal{V}_\lambda$ is a Banach algebra. Moreover:

$$||\Delta_r||_{\lambda,1} \leq ||\Delta_K||_{\lambda,1} + ||\Delta_r||_{\lambda,1}||\Delta_K||_{\lambda,1} \leq \alpha + \alpha||\Delta_r||_{\lambda,1} \text{ implying that: } ||\Delta_r||_{\lambda,1} \leq \frac{\alpha}{1-\alpha}.$$

$\Delta_r * 1$ satisfies the following Volterra equation:

$$\Delta_r * 1 = (\Delta_K * 1) + \Delta_K * (\Delta_r * 1) \tag{42}$$

with $\Delta_K * 1 = (\bar{K}_{(a.)} * 1) + \xi_a * (\bar{K}_{(a.)} * 1) + \gamma(a)(\bar{H}_{(a.)} * 1)$. It follows from Lemma 45 that $\Delta_K * 1 \in \mathcal{V}_\lambda$ and $||\Delta_K * 1||_{\lambda,1} \leq \alpha$. From $||\Delta_K||_{\lambda,1} < 1$, it follows that equation (42) has its solution in $\mathcal{V}_\lambda$ and:

$$\Delta_r * 1 \in \mathcal{V}_\lambda, \quad ||\Delta_r * 1||_{\lambda,1} \leq \frac{\alpha}{1-\alpha}.$$

It remains to check that the proposed formula (40) is indeed the solution of (35). Let $r := r_a + \Delta_r + \Delta_r * r_a$. We first check that r satisfies equation (38):

$$\begin{aligned}\Delta_K * r &= \Delta_K * r_a + (\Delta_r - \Delta_K) + (\Delta_r - \Delta_K) * r_a \\ &= \Delta_r * r_a + \Delta_r - \Delta_K \\ &= r - r_a - \Delta_K.\end{aligned}$$

Using that $\Delta_K = \bar{K}_{(a.)} + r_a * \bar{K}_{(a.)}$, the preceding equality gives:

$$r - (\bar{K}_{(a.)} + r_a * \bar{K}_{(a.)}) * r = r_a + \bar{K}_{(a.)} + r_a * \bar{K}_{(a.)}.$$

We multiply this equation by K_a on the left and obtain:

$$K_a * r - r_a * \bar{K}_{(a.)} * r = r_a * K_{(a.)}$$

We substitute this equality in (38) and finally get

$$r = K_{(a.)} + K_{(a.)} * r$$

By uniqueness, it follows that $r = r_{(a.)}$. The end of the proof follows easily. □

8.3 Proof of Theorem 39

We now prove Theorem 39, the main result of this section. Let us first reformulate it:

Theorem 47. *Let* $(Y_t^{s,\nu})_{t\geq s}$ *be the solution to the non-homogeneous equation* (5) *with current* (a_t)*, starting from initial condition* ν *at time* s*. Let* $r_{(a.)}^\nu(t,s) := \mathbb{E} f(Y_t^{s,\nu})$*. Assume Assumption 38 holds and that the constant* C *satisfies the inequality* (41) *with some* $\alpha \in]0,1[$.

Then it holds that:

$$\forall t \geq s, \ |r_{(a.)}^\nu(t,s) - \gamma(a)| \leq De^{-\lambda(t-s)}$$

with:

$$D := \frac{1 + \gamma(a) + ||\xi_a||_{\lambda,1}}{1-\alpha}||K_{(a.)}^\nu||_{\lambda,\infty} + \gamma(a)||H_{(a.)}^\nu||_{\lambda,\infty}.$$

Proof. $r^\nu_{(a.)}$ solves the Volterra equation $r^\nu_{(a.)} = K^\nu_{(a.)} + K_{(a.)} * r^\nu_{(a.)}$. Its solution is given by:

$$r^\nu_{(a.)} = K^\nu_{(a.)} + r_{(a.)} * K^\nu_{(a.)}$$

By Theorem 46, we know that $r_{(a.)} = \gamma(a) + \xi_{(a.)}$, with $\xi_{(a.)} \in \mathcal{V}_\lambda$. Furthermore using that $\gamma(a) * K^\nu_{(a.)} = \gamma(a)[1 - H^\nu_{(a.)}]$, we deduce that:

$$r^\nu_{(a.)} = \gamma(a) + K^\nu_{(a.)} + \xi_{(a.)} * K^\nu_{(a.)} - \gamma(a) H^\nu_{(a.)}.$$

Using that $\lambda < f(\sigma_0)$ (Assumption 38) we find:

$$||H^\nu_{(a.)}||_{\lambda,\infty} := \sup_{t,s} H^\nu_{(a.)}(t,s) e^{\lambda(t-s)} < \infty,\ ||K^\nu_{(a.)}||_{\lambda,\infty} := \sup_{t,s} K^\nu_{(a.)}(t,s) e^{\lambda(t-s)} < \infty.$$

We obtain:

$$\begin{aligned}\forall t \geq s,\ |r^\nu_{(a.)} - \gamma(a)| e^{\lambda(t-s)} &\leq ||K^\nu_{(a.)}||_{\lambda,\infty} + \gamma(a)||H^\nu_{(a.)}||_{\lambda,\infty} + e^{\lambda(t-s)} \int_s^t |\xi_{(a.)}|(t,u) K^\nu_{(a.)}(u,s) du \\ &\leq ||K^\nu_{(a.)}||_{\lambda,\infty} + \gamma(a)||H^\nu_{(a.)}||_{\lambda,\infty} + ||K^\nu_{(a.)}||_{\lambda,\infty} \int_s^t |\xi_{(a.)}|(t,u) e^{\lambda(t-u)} du \\ &\leq ||K^\nu_{(a.)}||_{\lambda,\infty} + \gamma(a)||H^\nu_{(a.)}||_{\lambda,\infty} + ||K^\nu_{(a.)}||_{\lambda,\infty} ||\xi_{(a.)}||_{\lambda,1}.\end{aligned}$$

Using Theorem 46 to get an estimate of $||\xi_{(a.)}||_{\lambda,1}$, we deduce the result. □

9 Long time behavior for small interactions: proof of the Theorem 12

9.1 Some uniform estimates

We now turn to the proof of Theorem 12. It is convenient to first extend the results obtained in Section 7: we need estimates that work uniformly in the input current a. In this section, we grant Assumptions 2, 4, 5 and 7.

Lemma 48. *Let $\bar{a} > 0$. It holds that:*

$$\inf_{a \in [0,\bar{a}]} \lambda^*_a > 0.$$

Proof. We define the following function g:

$$\begin{aligned} g \ : \ [0,\bar{a}] &\to [0, f(\sigma_0)] \\ a &\mapsto -\sup\{\Re(z) |\ \widehat{H}_a(z) = 0,\ \Re(z) > -f(\sigma_0)\}. \end{aligned}$$

It is clear by definition of λ^*_a and by the results of Section 7 that:

$$\forall a \in [0,\bar{a}],\ 0 < g(a) \leq \lambda^*_a.$$

Consequently to prove the lemma it is sufficient to show that:
Claim: g is lower semi-continuous, that is:

$$\forall a_0 \in [0,\bar{a}],\ \liminf_{a \to a_0} g(a) \geq g(a_0).$$

Proof of the claim: we fix $a_0 \in [0,\bar{a}]$. We have $g(a_0) > 0$. Fix λ such that: $0 < \lambda < g(a_0)$. One can find $R > 0$ large enough such that:

$$\forall a \in [0,\bar{a}], \forall z : -\lambda \leq \Re(z) < 0, \widehat{H}_a(z) = 0 \implies |\Im(z)| < R. \tag{43}$$

Indeed let $R := \sup_{a\in[0,\bar{a}]} \phi_a(-\lambda)$ where ϕ_a is defined by Lemma 35. The function $a \mapsto \phi_a(-\lambda)$ being continuous on $[0,\bar{a}]$, it holds that $R < \infty$ and by Lemma 35, (43) holds. We denote by U the closed (plain) rectangle delimited by the points $-\lambda - iR$, $-\lambda + iR$, iR and $-iR$. By definition of $g(a_0)$, we have $\widehat{H}_{a_0} \neq 0$ on U and by the continuity of $z \mapsto \widehat{H}_{a_0}(z)$, it yields:

$$m_{a^0} := \inf_{z\in U} |\widehat{H}_{a_0}(z)| > 0.$$

Moreover, we can find a constant C_U such that:

$$\forall a \in [0,\bar{a}], \forall z \in U, |\widehat{H}_a(z) - \widehat{H}_{a_0}(z)| \leq C_U|a - a_0|.$$

This follows from:

$$|\widehat{H}_a(z) - \widehat{H}_{a_0}(z)| = |\int_0^\infty e^{-zt}(H_a(t) - H_{a_0}(t))dt| \leq \int_0^\infty e^{\lambda t}|H_a(t) - H_{a_0}(t)|dt,$$

and:

$$\begin{aligned}\forall t \geq 0,\ |H_a(t) - H_{a_0}(t)| = |\exp\left(-\int_0^t f(\varphi_u^a)du\right) - \exp\left(-\int_0^t f(\varphi_u^{a_0})du\right)| \leq H_0(t)\int_0^t |f(\varphi_u^a) - f(\varphi_u^{a_0})|du \\ \leq H_0(t)f'(\varphi_t^{\bar{a}})\int_0^t |\varphi_u^a - \varphi_u^{a_0}|du \leq C_\varphi f'(C_b^{\bar{a}}t)H_0(t)\frac{t^2}{2}|a - a_0|.\end{aligned}$$

Using that $\lambda < f(\sigma_0)$, it holds that the function $\theta : t \mapsto e^{\lambda t}C_\varphi f'(C_b^{\bar{a}}t)H_0(t)\frac{t^2}{2}$ belongs to $L^1(\mathbb{R}_+)$ and we deduce that $|\widehat{H}_a(z) - \widehat{H}_{a_0}(z)| \leq ||\theta||_{L^1}|a - a_0|$.

Let now $a \in [0,\bar{a}]$ be such that $|a - a_0| \leq \frac{m_{a_0}}{2C_U}$. It yields that:

$$\forall z : -\lambda \leq \Re(z) < 0, |\widehat{H}_a(z)| \geq ||\widehat{H}_a(z) - \widehat{H}_{a_0}(z)| - |\widehat{H}_{a_0}(z)|| \geq \frac{m_{a_0}}{2} > 0.$$

We deduce that $g(a) \geq -\lambda$. We have proved that:

$$\liminf_{a\to a_0} g(a) \geq \lambda,\ \forall 0 < \lambda < g(a_0),$$

from which the claim follows. □

Note that in the case where $f(x) := x^p$ for some $p \geq 1$ and $b(x) = \mu > 0$ we can show, by a scaling argument, that $\lambda_a^* = (1 + \frac{a}{\mu})^{\frac{p}{p+1}}\lambda_\mu^*$: the Lemma 48 is trivial for this example as the minimum is reached for $a = 0$.

Theorem 49 (Whole-line Palay-Wiener, an extension)**.** *Let $\bar{a} > 0$ and for all $a \in [0,\bar{a}]$, let $k_a \in L^1(\mathbb{R},\mathbb{R})$. Assume that:*

1. $\exists \eta \in L^1(\mathbb{R},\mathbb{R}_+)$*:* $\forall a \in [0,\bar{a}]$,
$$\forall 0 < \epsilon < 1, \forall x \in \mathbb{R} : |k_a(x) - k_a(x - \epsilon)| \leq \epsilon\eta(x).$$
2. $\exists \theta \in L^1(\mathbb{R},\mathbb{R}_+)$*:* $\forall a \in [0,\bar{a}]$,
$$\forall x \in \mathbb{R} : |k_a(x)| \leq \theta(x).$$
3. $\forall a \in [0,\bar{a}]$, $\forall y \in \mathbb{R}$ *let:* $\widehat{k}_a(iy) := \int_\mathbb{R} e^{-iyt}k_a(t)dt$. *We assume that:*
$$\inf_{a\in[0,\bar{a}],y\in\mathbb{R}} |1 - \widehat{k}_a(iy)| > 0.$$

Then for all $a \in [0,\bar{a}]$, there exists a function $r_a \in L^1(\mathbb{R},\mathbb{R})$ satisfying the equation

$$r_a = k_a + k_a * r_a.$$

Moreover, $\sup_{a\in[0,\bar{a}]} ||r_a||_{L^1} < \infty$.

Proof. We follow the proof of the theorem 4.3 in [20, Chap. 2] and emphasis on the differences. For any $p \geq 1$, define:

$$\forall a \in [0, \bar{a}], k_a^{\infty}(t) := k_a - \zeta_p * k_a,$$

where $\zeta_p(t) := p\zeta(pt)$ and ζ is the Fejer Kernel defined by: $\forall t \in \mathbb{R}, \zeta(t) := \frac{1}{\pi t^2}(1 - \cos(t))$. The Fourier transform of ζ is $\widehat{\zeta}(iy) = (1 - |y|)\mathbb{1}_{\{|y| \leq 1\}}$.

Claim 1: There is an integer $p > 0$ such that $\forall a \in [0, \bar{a}]$,

$$||k_a^{\infty}||_{L^1} \leq 1/2 \text{ and } \forall |y| \geq p : \ \widehat{k}_a^{\infty}(iy) = \widehat{k}_a(iy).$$

Proof of the claim: it is clear that with this choice of ζ, $\forall |y| \geq p$, $\widehat{k}_a^{\infty}(iy) = \widehat{k}_a(iy)$. Moreover, using $\int_{\mathbb{R}} \zeta_p(s)ds = 1$, we have:

$$\begin{aligned} ||k_a^{\infty}||_{L^1} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} k_a(t)\zeta_p(s) - k_a(t-s)\zeta_p(s)ds \right| dt \\ &\leq \int_{\mathbb{R}} \zeta(u) \int_{\mathbb{R}} |k_a(t) - k_a(t - \frac{u}{p})| dtdu. \end{aligned}$$

We used the triangular inequality, the Tonelli-Fubini Theorem (everything is non-negative) to exchange the two integrals and we made the change of variable $u = ps$. Let $R > 0$ such that $\int_{\mathbb{R}\setminus[-R,R]} \zeta(u)du \leq \frac{1}{8||\theta||_{L^1}}$. It follows that:

$$\begin{aligned} ||k_a^{\infty}||_{L^1} &\leq 1/4 + \int_{-R}^{R} \zeta(u) \int_{\mathbb{R}} |k_a(t) - k_a(t - \frac{u}{p})| dtdu \\ &\leq 1/4 + \int_{-R}^{R} \left(\int_{\mathbb{R}} |\frac{u}{p}| \eta(t)dt \right) du \\ &\leq 1/4 + \frac{R^2}{p} ||\eta||_{L^1}. \end{aligned}$$

The claim is proved by choosing an integer $p \geq 4R^2||\eta||_{L^1}$.

Along the same idea, we define $\beta(t) := 4\zeta(2t) - \zeta(t) = \frac{1}{\pi t^2}(\cos t - \cos 2t)$, $\forall \delta > 0 : \ \beta_\delta(t) = \frac{1}{\delta}\beta(\delta t)$. We also define $\forall y_0 \in \mathbb{R}$:

$$\forall t \geq 0, \ k_a^{y_0,\delta}(t) = \int_{\mathbb{R}} (\beta_\delta(t-s) - \beta_\delta(s)) e^{iy_0(t-s)} k_a(s)ds.$$

Claim 2: Given $\epsilon > 0$, one can find a constant $\delta > 0$ such that: $\forall y_0 \in \mathbb{R}, \forall a \in [0, \bar{a}]$,

$$||k_a^{y_0,\delta}||_{L^1} \leq \frac{\epsilon}{2} \text{ and } \forall |y - y_0| \leq \delta, \ \widehat{k}_a(iy) = \widehat{k}_a(iy_0) + \widehat{k_a^{y_0,\delta}}(iy).$$

Proof of the claim: by definition of $k_a^{y_0,\delta}$ it holds that:

$$\forall y \in \mathbb{R}, \widehat{k_a^{y_0,\delta}}(iy) = \widehat{\beta}_\delta(i(y - y_0))(\widehat{k}_a(iy) - \widehat{k}_a(iy_0)).$$

Moreover, $\widehat{\beta}_\delta(iy) = 1$ if $|y| \leq \delta$ and consequently the second point of the claim is satisfied. Furthermore,

$$\begin{aligned} \int_{\mathbb{R}} |k_a^{y_0,\delta}(t)| dt &\leq \int_{\mathbb{R}} |k_a(s)| \int_{\mathbb{R}} |\beta(t - \delta s) - \beta(t)| dtds \\ &\leq \int_{\mathbb{R}} \theta(s) \int_{\mathbb{R}} |\beta(t - \delta s) - \beta(t)| dtds. \end{aligned}$$

The term on the left is independent of both y_0 and a and goes to zero as δ goes to zero. This follows from the Lebesgue's dominated convergence Theorem: $\forall s \in \mathbb{R}$, $\delta \mapsto \theta(s) \int_{\mathbb{R}} |\beta(t - \delta s) - \beta(t)| dt$ goes to zero as δ goes to zero and $\theta(s) \int_{\mathbb{R}} |\beta(t - \delta s) - \beta(t)| dt \leq 2\theta(s)||\beta||_{L^1} \in L^1(\mathbb{R})$. We thus choose δ small enough such that $\int_{\mathbb{R}} \theta(s) \int_{\mathbb{R}} |\beta(t - \delta s) - \beta(t)| dtds \leq \epsilon/2$ and this proves the first point of the claim.

It follows from Claim 1 that $\forall a \in [0, \bar{a}]$, the equation $r_a^\infty = k_a + k_a^\infty * r_a^\infty$ has a unique solution $r_a^\infty \in L^1(\mathbb{R})$, with $||r_a^\infty||_{L^1} \leq 2||\theta||_{L^1}$. This solution is given by:

$$r_a^\infty = k_a + [\sum_{n\geq 1} (k_a^\infty)^{*n}] * k_a,$$

where the series is absolutely converging ($||k_a^\infty||_{L^1} \leq 1/2$). Moreover, we have:

$$\forall a \in [0, \bar{a}], \forall |y| \geq p,\ \widehat{r}_a^\infty(iy) = \frac{\widehat{k}_a(iy)}{1 - \widehat{k}_a(iy)}.$$

Similarly, we define $\epsilon := \inf_{a\in[0,\bar{a}], y\in\mathbb{R}} |1 - \widehat{k}_a(iy)| > 0$ and apply the second claim. Given $y_0 \in \mathbb{R}$ and $a \in [0, \bar{a}]$, let $A_a^{y_0} = \frac{1}{1-k_a(y_0)}$. We have $||A_a^{y_0} k_a^{y_0,\delta}||_{L^1} \leq 1/2$ and:

$$\forall y : |y - y_0| \leq \delta :\ \frac{\widehat{k}_a(iy)}{1 - \widehat{k}_a(iy)} = \frac{A_a^{y_0} \widehat{k}_a(iy)}{1 - \widehat{k_a^{y_0,\delta}}(iy) A_a^{y_0}}.$$

This follows from $1 - \widehat{k}_a(iy) = 1 - \widehat{k}_a(iy_0) - \widehat{k_a^{y_0,\delta}}(iy) = \frac{1}{A_a^{y_0}}(1 - A_a^{y_0} \widehat{k_a^{y_0,\delta}}(iy))$. We define:

$$r_a^{y_0} := A_a^{y_0} k_a + [\sum_{n\geq 1} (A_a^{y_0} k_a^{y_0,\delta})^{*k}] * A_a^{y_0} k_a,$$

which is the solution of $r_a^{y_0} = A_a^{y_0} k_a + A_a^{y_0} k_a^{y_0,\delta} * r_a^{y_0}$ (the series is absolutely converging) and:

$$||r_a^{y_0}||_{L^1} \leq \frac{2}{\epsilon} ||\theta||_{L^1}.$$

It yields:

$$\forall |y - y_0| \leq \delta : \widehat{r}_a^{y_0}(iy) = \frac{\widehat{k}_a(iy)}{1 - \widehat{k}_a(iy)}.$$

Still following [20], one can find an integer $m > 0$ such that: $\forall a \in [0, \bar{a}]$, $\forall j \in 0, \pm 1, ..., \pm m \cdot p$, there exists a function $r_a^j \in L^1(\mathbb{R})$ with $||r_a^j||_{L^1} \leq \frac{2}{\epsilon}||\theta||_{L^1}$ s.t.

$$\widehat{r_a^j}(iy) = \frac{\widehat{k}_a(iy)}{1 - \widehat{k}_a(iy)}\ ,\ |y - j/m| \leq 1/m.$$

We conclude by defining:

$$r_a = \sum_{j=-m.p}^{m.p} \psi_j * (r_a^j - r_a^\infty) + r_a^\infty,$$

where $\psi_j(t) = \frac{1}{m} e^{-ijt/m} \zeta(t/m)$. It is clear that $r_a \in L^1(\mathbb{R})$ and that:

$$\sup_{a\in[0,\bar{a}]} ||r_a||_{L^1} \leq 2||\theta||_{L^1} + mp \left(\frac{2}{\epsilon} ||\theta||_{L^1} + 2||\theta||_{L^1} \right) < \infty.$$

With this choice of ψ_j, $\forall y \in \mathbb{R}, \widehat{r}_a(iy) = \frac{\widehat{k}_a(iy)}{1 - \widehat{k}_a(iy)}$ and by uniqueness of the Fourier transform, we conclude that r_a is the solution of $r_a = k_a + k_a * r_a$. We refer to [20], proof of the Theorem 4.3 for more details. □

As a consequence of the previous theorem, we have:

Lemma 50. *Let $\bar{a} > 0$, define $\lambda^* = \inf_{a\in[0,\bar{a}]} \lambda_a^*$ ($\lambda^* > 0$ by Lemma 48). Let $0 < \lambda < \lambda^*$ and consider r_a the solution of the Volterra equation $r_a = K_a + K_a * r_a$. By Theorem 36, it holds that $r_a = \gamma(a) + \xi_a$ for some $\xi_a \in L_\lambda$. Then we have $\sup_{a\in[0,\bar{a}]} ||\xi_a||_{\lambda,1} < \infty$.*

Proof. In the proof of Theorem 36, we have seen that $\forall t \geq 0,\ \xi_a(t) = e^{-\lambda t}\xi_{a,-\lambda}(t)$, where $\xi_{a,-\lambda} \in L^1(\mathbb{R})$ is the unique solution of:

$$\xi_{a,-\lambda} = K_{a,-\lambda} + K_{a,-\lambda} * \xi_{a,-\lambda},$$

with $\forall t \in \mathbb{R}, K_{a,-\lambda}(t) := e^{\lambda t}K_a(t)\mathbb{1}_{\{t\geq 0\}}$. As in the proof of Lemma 48, define $R := \sup_{a\in[0,\bar{a}]} \phi_a(-\lambda) < \infty$ such that: $\inf_{a\in[0,\bar{a}],y\in\mathbb{R}} |1 - \widehat{K_{a,-\lambda}}(iy)| = \inf_{a\in[0,\bar{a}],y\in[-R,R]} |1 - \widehat{K_{a,-\lambda}}(iy)| > 0$. We have $\forall t \geq 0,\ K_{a,-\lambda}(t) = e^{\lambda t} f(\varphi_t^a)H_a(t) \leq e^{\lambda t} f(\varphi_t^{\bar{a}})H_0(t)$ (using hypothesis 2.2 and the fact that f is non-decreasing). It follows from $\lambda < f(\sigma_0)$ that:

$$\forall t \in \mathbb{R},\ K_{a,-\lambda}(t) \leq \theta(t) := e^{\lambda t} f(C_t^{\bar{a}})H_0(t)\mathbb{1}_{\mathbb{R}_+}(t) \in L^1(\mathbb{R}).$$

Finally given any $\epsilon \in]0,1[$, we have:

$$\begin{aligned}\forall t \geq \epsilon,\ |K_{a,-\lambda}(t) - K_{a,-\lambda}(t-\epsilon)| &= |e^{\lambda t} f(\varphi_t^a)H_a(t) - e^{\lambda(t-\epsilon)} f(\varphi_{t-\epsilon}^a)H_a(t-\epsilon)| \\ &\leq |e^{\lambda t} - e^{\lambda(t-\epsilon)}|K_a(t) + |f(\varphi_t^a) - f(\varphi_{t-\epsilon}^a)|e^{\lambda t}H_a(t) + |H_a(t) - H_a(t-\epsilon)|e^{\lambda t} f(\varphi_t^a) \\ &:= A_1 + A_2 + A_3 \text{ with :}\end{aligned}$$

1. $A_1 = e^{\lambda t}K_a(t)(1 - e^{-\lambda\epsilon}) \leq \epsilon e^{\lambda t} f(\varphi_t^{\bar{a}})H_0(t) := \epsilon\eta_1(t),\ \eta_1 \in L^1(\mathbb{R}_+)$.
2. $A_2 \leq f'(\varphi_t^a)(\varphi_t^a - \varphi_{t-\epsilon}^a)e^{\lambda t}H_0(t) \leq f'(\varphi_t^{\bar{a}})e^{\lambda t}H_0(t)\int_{t-\epsilon}^t (b(\varphi_u^a) + a)du \leq \epsilon C_b^{\bar{a}} f'(\varphi_t^{\bar{a}})e^{\lambda t}H_0(t) := \epsilon\eta_2(t)$, $\eta_2 \in L^1(\mathbb{R}_+)$.
3. $A_3 \leq |1 - \exp\left(\int_{t-\epsilon}^t f(\varphi_u^a du)\right)|H_a(t)e^{\lambda t} f(\varphi_t^a) \leq H_a(t-1)\epsilon f^2(\varphi_t^a)e^{\lambda t} \leq \epsilon H_0(t-1)f^2(\varphi_t^{\bar{a}})e^{\lambda t} := \epsilon\eta_3(t)$, $\eta_3 \in L^1(\mathbb{R}_+)$.

Moreover, if $t \in [0,\epsilon]$, it is clear that:

$$|K_{a,-\lambda}(t) - K_{a,-\lambda}(t-\epsilon)| = K_{a,-\lambda}(t) \leq f(\varphi_\epsilon^{\bar{a}})e^{\lambda t}H_0(t) \leq \epsilon f'(\varphi_1^{\bar{a}})C_b^{\bar{a}}e^{\lambda t}H_0(t) := \epsilon\eta_4(t),\ \eta_4 \in L^1(\mathbb{R}_+).$$

Setting $\forall t \in \mathbb{R}\ \eta(t) := (\eta_1 + \eta_2 + \eta_3 + \eta_4)(t)\mathbb{1}_{\mathbb{R}_+}(t)$, we deduce that:

$$\forall t \in \mathbb{R}, \forall \epsilon \in [0,1], |K_{a,-\lambda}(t) - K_{a,-\lambda}(t-\epsilon)| \leq \epsilon\eta(t).$$

Consequently, Theorem 49 applies and we obtain the result. □

9.2 Proof of Theorem 12

We are now ready to give the proof of the main theorem.

- **Step 1**: In Section 4.3, we defined a non-decreasing function $J \mapsto \bar{r}(J)$ such that:

  $$\frac{d}{dt}\mathbb{E} f(X_t) \leq \frac{1}{2}[\bar{r}(J)^2 - \mathbb{E}^2 f(X_t)],$$

  where $(X_t)_{t\geq 0}$ is the solution of the non-linear equation (2) (in particular if $\mathbb{E} f(X_0) \leq \bar{r}(J)$ then $\forall t \geq 0,\ \mathbb{E} f(X_t) \leq \bar{r}(J)$). Using Theorem 22 with $\kappa := J\bar{r}(J) + 1$, there is a non-decreasing function $J \mapsto \bar{a}(J)$ such that:

  $$\forall J, s \geq 0,\ \forall (a_t)_{t\geq s} \in \mathcal{C}([s,\infty[,\mathbb{R}_+),\ [\sup_{t\geq s} a_t \leq \bar{a}(J) \text{ and } J\nu(f) \leq \bar{a}(J)] \implies \sup_{t\geq s} Jr_{(a.)}^{\nu}(t,s) \leq \bar{a}(J).$$

  Moreover, it holds that $\forall J \geq 0,\ J\bar{r}(J) < \bar{a}(J)$.

- **Step 2**: we define:

  $$\lambda^* := \inf_{a\in[0,\bar{a}(J_m)]} \lambda_a^*,$$

  where $J_m > 0$ is defined in Theorem 8. $\lambda^* > 0$ thanks to Lemma 48. We now fix λ such that $0 < \lambda < \lambda^*$.

- **Step 3**:

– Using Lemma 50, we know that the solution of the Volterra equation $r_a = K_a + K_a * r_a$ can be written in the form $r_a = \gamma(a) + \xi_a$ with $\xi_a \in L_\lambda$ and that:

$$\xi^\infty(J) := \sup_{a\in[0,\bar{a}(J)]} ||\xi_a||_{\lambda,1} < \infty.$$

It is clear that $J \mapsto \xi^\infty(J)$ is non-decreasing (as $J \mapsto \bar{a}(J)$ is non-decreasing).

– if $(a_t) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ with $\sup_{t\geq 0} a_t \leq \bar{a}$ then:

$$||K^\nu_{(a.)}||_{\lambda,\infty} \leq k^\infty(\nu(f), \bar{a}) < \infty,$$

for some function k^∞ which is increasing with respect to $\nu(f)$ and $\bar{a}$. Moreover, $||H^\nu_{(a.)}||_{\lambda,\infty} \leq h^\infty < \infty$.

– The function $\eta_{\bar{a}}$ of Lemma 44 satisfies:

$$||\eta_{\bar{a}}||_1 < \infty,\ \bar{a} \mapsto ||\eta_{\bar{a}}||_1 \text{ is non decreasing,}$$

and consequently the function $J \mapsto \eta^\infty(J) := ||\eta_{\bar{a}(J)}||_1$ is non-decreasing.

– Finally the function $a \mapsto \gamma(a)$ is non-decreasing and it follows that $\forall a \in [0, \bar{a}(J)], \gamma(a) \leq \gamma(\bar{a}(J))$.

- **Step 4**: We now assume that $\mathbb{E} f(X_s) \leq \bar{r}(J_m) + 1$ for some $s > 0$ (we shall come back to the general case in Step 6). We recall that $J_m > 0$ is defined in Theorem 8 and, as a consequence of its definition, $\forall 0 \leq J < J_m$ the equation $a\gamma^{-1}(a) = J$ has a unique solution $a^*(J) \in [0, \bar{a}(J_m)]$. We now apply Theorem 47 with $\alpha = 1/2$. Define:

$$C(J) := \frac{1}{2\eta^\infty(J)(1 + \xi^\infty(J) + \gamma(\bar{a}(J)))},\ D(J) := 2(1+\gamma(\bar{a}(J))+\xi^\infty(J))k^\infty(\bar{r}(J_m)+1, \bar{a}(J))+\gamma(\bar{a}(J))h^\infty.$$

From Step 3, it is clear that the functions $J \mapsto \frac{1}{C(J)}$ and $J \mapsto D(J)$ are non-decreasing. Consequently, we can find a constant $J^* \in]0, J_m[$ such that:

$$\forall J \in [0, J^*] : \frac{JD(J)}{C(J)} \leq 1.$$

Theorem 47 tells us that for every $0 \leq J \leq J^*$, given any $s \geq 0$ and $(a_t) \in \mathcal{C}([s, \infty[, \mathbb{R}_+)$ with $\sup_{t\geq s} a_t \leq \bar{a}(J)$ and such that:

$$\forall t \geq s,\ |a_t - a^*(J)| \leq C(J)e^{-\lambda t},$$

it holds that:

$$\forall t \geq s,\ |Jr^\nu_{(a.)}(t, s) - a^*(J)| \leq C(J)e^{-\lambda(t-s)}.$$

- **Step 5**: Let now $J \in]0, J^*]$ be fixed (the case $J = 0$ is already treated by Theorem 10). We assume that $J\,\mathbb{E} f(X_s) \leq \bar{a}(J)$ and that $\mathbb{E} f(X_s) \leq \bar{r}(J_m) + 1$ for some $s \geq 0$. Let $\nu := \mathcal{L}(X_s)$. We define recursively $a^n \in \mathcal{C}([s, \infty[, \mathbb{R}_+)$ by:

$$\forall t \geq s,\ a^0(t) := a^*(J),\ a^{n+1}(t) := Jr^\nu_{(a^n.)}(t, s).$$

By Step 4, it holds that:

$$\forall n \geq 0,\ \forall t \geq s,\ |a^n(t) - a^*(J)| \leq C(J)e^{-\lambda(t-s)}.$$

We deduce that:

$$\begin{aligned}\forall t \geq s,\ |\,\mathbb{E} f(X_t) - \gamma(a^*(J))| &\leq |\,\mathbb{E} f(X_t) - r^\nu_{(a^n.)}(t, s)| + \frac{1}{J}|a^{n+1}(t) - a^*(J)| \\ &\leq \frac{1}{J}|J\,\mathbb{E} f(X_t) - a^{n+1}(t)| + \frac{C(J)}{J}e^{-\lambda(t-s)}.\end{aligned}$$

The Picard iterations studied in part 4.4 shows that:

$$\forall t \geq s, \lim_{n\to\infty} |J\,\mathbb{E}\, f(X_t) - a^n(t)| = 0.$$

We have proved that:

$$\forall t \geq s, \ |\mathbb{E}\, f(X_t) - \gamma(a^*(J))| \leq \frac{e^{\lambda s}C(J)}{J} e^{-\lambda t}.$$

This proves the theorem under our additional assumptions on $\mathbb{E}\, f(X_s)$.

- **Step 6**: we now prove that $\mathbb{E}\, f(X_s) \leq \min(\frac{\bar{a}(J)}{J}, \bar{r}(J_m) + 1)$ for some $s \geq 0$. By Step 1 we have $\bar{r}(J) < \frac{\bar{a}(J)}{J}$ and by lemmas 23 and 24 it follows that:

$$\mathbb{E}\, f(X_t) \leq \bar{r}(J) \frac{e^{\bar{r}(J)t} + A}{e^{\bar{r}(J)t} - A},$$

for some constant $A \geq 0$. Consequently, $\limsup_{t\to\infty} \mathbb{E}\, f(X_t) \leq \bar{r}(J)$ and it follows that there is a $0 \leq s < \infty$ such that $\mathbb{E}\, f(X_s) \min(\frac{\bar{a}(J)}{J}, \bar{r}(J_m) + 1)$ (this time s can be computed explicitly). This ends the proof of Theorem 12.

Remark 51. *There is some freedom in the above construction of the constants λ and J^*. λ is chosen arbitrarily on $[0, \lambda^*[$ and the value of J^* depends both on λ and on a parameter $\alpha \in]0, 1[$, here chosen to be equals to $1/2$ (see Step 4). Depending on the type of result we want, we may optimize this construction to get either J^* or λ as large as possible.*

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