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Abstract

We study the long time behavior of the solution to some McKean-Vlasov stochastic differential equation (SDE) driven by a Poisson process. In neuroscience, this SDE models the asymptotic dynamic of the membrane potential of a spiking neuron in a large network. We prove that for a small enough interaction parameter, any solution converges to the unique (in this case) invariant probability measure. To this aim, we first obtain global bounds on the jump rate and derive a Volterra type integral equation satisfied by this rate. We then replace temporary the interaction part of the equation by a deterministic external quantity (we call it the external current). For constant current, we obtain the convergence to the invariant probability measure. Using a perturbation method, we extend this result to more general external currents. Finally, we prove the result for the non-linear McKean-Vlasov equation.

Keywords McKean-Vlasov SDE · Long time behavior · Mean-field interaction · Volterra integral equation · Piecewise deterministic Markov process
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1 Introduction

We study a model of network of neurons. For each $N \in \mathbb{N}$, we consider a Piecewise-Deterministic Markov Process (PDMP) $\mathbf{X}^N_t = (X^{1,N}_t, \cdots, X^{N,N}_t) \in \mathbb{R}^N_+$. For $i \in \{1, \cdots, N\}$, $X^{i,N}_t$ models the membrane potential of a neuron (say neuron i) in the network. It emits spikes at random times. The spiking rate of neuron i at time t is $f(X^{i,N}_t)$: it only depends on the potential of neuron i. When the neuron i emits a spike, say at time τ, its potential is reset ($X^{i,N}_{\tau+} = 0$) and the potential of the other neurons increases by an amount $\frac{J}{N}$, where the connection strength $J \geq 0$ is fixed:

$$\forall j \neq i, \quad X^{j,N}_{\tau_+} = X^{j,N}_{\tau_-} + \frac{J}{N}.$$

Between two spikes, the potentials evolve according to the one dimensional equation

$$\frac{dX^{i,N}_t}{dt} = b(X^{i,N}_t).$$

The functions b and f are assumed to be smooth. This process is indeed a PDMP, in particular Markov (see [10]). Equivalently, the model can be described using a system of SDEs driven by Poisson measures. Let $(\mathbf{N}^i(du, dz))_{i=1,\cdots,N}$ be a family of N independent Poisson measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dudz$. Let $(X^{i,N}_0)_{i=1,\cdots,N}$ be a family of N random variables on

$\mathbb{R}_+$, *i.i.d.* of law ν and independent of the Poisson measures. Then $(X^{i,N})$ is a *càdlàg* process solution of coupled SDEs:

$$\begin{aligned}\forall i, \quad X_t^{i,N} =& X_0^{i,N} + \int_0^t b(X_u^{i,N})du + \frac{J}{N}\sum_{j\neq i}\int_0^t\int_{\mathbb{R}_+} \mathbb{1}_{\{z\leq f(X_{u-}^{j,N})\}}\mathbf{N}^j(du,dz) \\ &- \int_0^t\int_{\mathbb{R}_+} X_{u-}^{i,N}\mathbb{1}_{\{z\leq f(X_{u-}^{i,N})\}}\mathbf{N}^i(du,dz).\end{aligned} \quad (1)$$

When the number of neurons N goes to infinity, it has been proved (see [11, 18]) for specific linear functions b and under few assumptions on f that $X_t^{1,N}$ - i.e. the first coordinate of the solution to (1) - converges in law to the solution of the McKean-Vlasov SDE:

$$X_t = X_0 + \int_0^t b(X_u)du + J\int_0^t \mathbb{E}\, f(X_u)du - \int_0^t\int_{\mathbb{R}_+} X_{u-}\mathbb{1}_{\{z\leq f(X_{u-})\}}\mathbf{N}(du,dz), \quad (2)$$

where, $\mathcal{L}(X_0) := \mathcal{L}(X_0^{1,N}) = \nu$ and $\mathbf{N}$ is a Poisson measure on $\mathbb{R}_+\times\mathbb{R}_+$ with intensity measure $dudz$. The measure $\mathbf{N}$ and X_0 are independent.

Equation (2) is a mean-field equation and is the current object of interest. Note that the drift part of (2) involves the law of the solution in the term $\mathbb{E}\, f(X_u)$: the equation is non-linear in the sense of McKean-Vlasov. Here, we study existence and uniqueness of the solution of (2) and its long time behavior.

Let $\nu(t,\cdot)$ be the law of X_t at time $t\geq 0$. It is a weak solution of the following Fokker-Planck PDE:

$$\begin{cases} \dfrac{\partial}{\partial t}\nu(t,x) = -\dfrac{\partial}{\partial x}[(b(x)+Jr_t)\nu(t,x)] - f(x)\nu(t,x), \quad x>0 \\ \nu(0,\cdot)=\nu, \quad \nu(t,0) = \dfrac{r_t}{b(0)+Jr_t}, \quad r_t = \displaystyle\int_0^\infty f(x)\nu(t,x)dx. \end{cases} \quad (3)$$

This model with a noisy threshold is known in the physic literature as the "Escape Noise" model (see [19, Chap. 9] for references and biological considerations). From a mathematical point of view, it has been first studied in [11] and has been the object of further developments in [18]. The function $f:\mathbb{R}_+\to\mathbb{R}_+$ can be considered of the type $f(x) = (\frac{x}{\vartheta})^\xi$ for large $\xi>0$ and some soft threshold $\vartheta>0$. In this situation, if the potential of the neuron is equal to x then the neuron has a small probability to spike between t and $t+dt$ if $x<\vartheta$ and a large probability if $x>\vartheta$. Such a choice of f mimics the standard Integrate-And-Fire model with a fixed (deterministic) threshold around ϑ.

Results on the existence of solution to (2), in a slightly different context (in particular, with $b(x)\sim_\infty -\kappa x$ for $\kappa\geq 0$), have been obtained in [11]: the authors explored the case where the initial condition ν is compactly supported. This property is preserved at any time $t>0$. So, the behavior of the solution with a rate function f locally Lipschitz continuous is similar to the case with a function f globally Lipschitz continuous. When the initial condition is not compactly supported, the situation is more delicate. In [18], the authors proved existence and path-wise uniqueness of the solution to (2) (in a slightly different setting than ours) using an ad-hoc distance.

Note that the global existence results obtained for this model differ from those obtained for the "standard" Integrate-and-Fire model with a fixed deterministic threshold. This situation, studied for instance in [4, 6, 12, 13], corresponds (informally) to the choice $f(x) = +\infty\mathbb{1}_{\{x\geq\vartheta\}}$, $\vartheta>0$ being the fixed threshold. In these papers, a diffusion part is included in the modeling. In [4], the authors proved that a blow-up phenomenon appears when the law of the initial condition is concentrated close to the threshold ϑ: the jump rate of the solution diverges to infinity in finite time. Here, the situation is completely different: the jump rate is uniformly bounded in time (see Theorem 5). In [4], the authors have obtained results on the stability of the solution for the diffusive model with a deterministic threshold (see also [5] for a variant).

Very little is known about the long time behavior of the solutions to (2). One can study it by considering the long time behavior of the finite particles system (1) and then apply the propagation of chaos to extend the results to the McKean-Vlasov equation (2). This strategy has been developed in [29, 2] for diffusive problems. The long time behavior of the particles system (1) has been studied in [16, 22] (again in a slightly different setting but the methods could be adapted to our case): the authors proved that the particles system is Harris-ergodic and consequently converges weakly to its unique invariant probability measure. However, transferring the long time behavior of the particles system to the McKean-Vlasov equation is possible if the propagation of chaos holds *uniformly in time*. In [11, 18], the propagation of chaos is only proved on compact time interval $[0, T]$ and their estimates diverge as T goes to infinity. Because Equation (2) may have multiple invariant probability measures, there is no hope in general to prove such uniform propagation of chaos.

Coupling methods are also used to study the long time behavior of SDEs. In [1], the authors have studied the TCP (a linear PDMP) which is close to (2). The size of the jumps is $-x/2$ in the TCP and $-x$ in our setting, x being the position of the process just before the jump. The main difference is the non-linearity: we failed to adapt their methods when the interactions are non-zero ($J > 0$).

Butkovsky studied in [3] the long time behavior of some McKean-Vlasov diffusion SDE of the form:

$$\forall t \geq 0,\ X_t = X_0 + \int_0^t [b_1(X_u) + \epsilon b_2(X_u, \mu_u)]\, du + W_t,\ \mu_u = \mathcal{L}(X_u), \tag{4}$$

where $(W_t)_{t\geq 0}$ is a Brownian motion. Here the drift terms b_1 and b_2 are assumed to be globally Lipschitz and b_2 is assumed uniformly bounded with respect to its two parameters. The author proved that if the parameter ϵ is small enough, (4) has a unique invariant probability measure which is globally stable. The case $\epsilon > 0$ (and small) is treated as a perturbation of the case $\epsilon = 0$ using a Girsanov transform. It could be interesting to see how this method could be adapted to SDE driven by Poisson measures, but we did not pursue this path.

Another approach consists in studying the non-linear Fokker-Planck equation (3). Such non-linear transport equations with boundary conditions have been studied in the context of population dynamics (see for instance [21, 28, 30, 26]). In [21], the authors have characterized the stationary solutions of the PDE and found a criterion of local stability for stationary solutions. They derived a Volterra integral equation and used it to obtain the stability criteria. More recently, [24, 25, 23] have re-explored these models for neuroscience applications (see [8, 7] for a rigorous derivation of some of these PDEs using Hawkes processes).

PDE (3) differs from theirs in the sense that we have a non-linear transport term (theirs is constant and equal to one) and our boundary condition is more complex. The long time behavior of the PDE (3) has been successfully studied in [18] and in [14] in the case where $b \equiv 0$. In this situation, one can simplify the PDE (3) with a simpler boundary condition

$$\nu(t, 0) = \frac{1}{J}.$$

The authors proved that if the density of the initial condition satisfies this boundary condition and regularity assumptions, then $\nu(t, .)$ converges to the density of the invariant probability measure as t goes to infinity. The convergence holds in L^1 or in stronger norms (see [14]). For $b \neq 0$, the boundary condition is more delicate and their methods cannot be easily applied.

Actually the long time behavior of the solution to (2) may be remarkably intricate. Depending on the choice of f, b and J, equation (2) may have multiple invariant probability measures. Even if the invariant probability measure is unique, it is not necessarily a stable one and oscillations may appear (see Examples page 7). In [15], the authors have numerically illustrated this phenomenon in a setting close to ours.

Our main result describes the long time behavior of the solution to (2) in the weakly connected regime (Theorem 9). If the connection strength J is small enough, we prove that (2) has a unique invariant probability measure which is globally stable. We give the explicit expression of this non-trivial invariant distribution and starting from any initial condition X_0, we prove the convergence in law of X_t to it, exponentially fast, as t goes to infinity. We argue that this result is very general: it does not depend on the explicit shape of the functions f or b. For stronger connection strengths J, such a result cannot hold true in general as equation (2) may have multiple invariant probability measures.

Note that we prove convergence in law, which is weaker than convergence in L^1. On the other hand, we require very few on the initial condition, in particular, we do not assume the existence of a density for the initial condition in Theorem 9. We also provide a new proof for the existence and uniqueness of the solution to (2), based on a Picard iteration scheme (see Theorem 5). As in [18], we do not require the initial condition to be compactly supported. One of the main difficulty to study (2) (or its PDE version (3)) is that there is no simple autonomous equation for the jump rate $t \mapsto \mathbb{E}\, f(X_t)$. To overcome this difficulty, we introduce a "linearized" version of (2) for which we can derive a closed equation of the jump rate.

Fix a $s \geq 0$ and let $(a_u)_{u \geq s}$ be a continuous deterministic non-negative function, called the external current. It replaces the interaction $J\mathbb{E}f(X_u)$ in (2). We consider the linear non-homogeneous SDE:

$$\forall t \geq s, \quad Y_t^{s,\nu,(a.)} = Y_s^{s,\nu,(a.)} + \int_s^t b(Y_u^{s,\nu,(a.)})du + \int_s^t a_u du - \int_s^t \int_{\mathbb{R}_+} Y_{u-}^{s,\nu,(a.)} \mathbb{1}_{\{z \leq f(Y_{u-}^{s,\nu,(a.)})\}} \mathbf{N}(du, dz), \tag{5}$$

where $\mathcal{L}(Y_s^{s,\nu,(a.)}) = \nu$. Under quite general assumptions on b and f, this SDE has a path-wise unique solution (see Lemma 14). We denote the jump rate of this SDE by:

$$\forall t \geq s, \quad r_{(a.)}^{\nu}(t,s) := \mathbb{E}\, f(Y_t^{s,\nu,(a.)}). \tag{6}$$

Moreover, taking $s = 0$ and $Y_0^{0,\nu,(a.)} = X_0$, it holds that $(Y_t^{0,\nu,(a.)})_{t \geq 0}$ is a solution to (2) if it satisfies the closure condition

$$\forall t \geq 0, \quad a_t = J r_{(a.)}^{\nu}(t, 0). \tag{7}$$

Conversely, any solution to (2) is a solution to (5) with $a_t = J\,\mathbb{E}\, f(X_t)$. We prove that the function $r_{(a.)}^{\nu}$ satisfies a Volterra integral equation

$$\forall t \geq s, \quad r_{(a.)}^{\nu}(t,s) = K_{(a.)}^{\nu}(t,s) + \int_s^t K_{(a.)}(t,u) r_{(a.)}^{\nu}(u,s)du, \tag{8}$$

where the kernels $K_{(a.)}^{\nu}$ and $K_{(a.)}$ are explicit in terms of ν, a, b and f (see (18) and (19)).

Our main tool is this Volterra equation: we use it with a Picard iteration scheme to "recover" the non-linear equation (2). The McKean-Vlasov equation (2), its "linearized" non-homogeneous version (5), the Fokker-Planck PDE (3) and the Volterra equation (8) are different ways to investigate this mean-field problem, each of these interpretations having their own strength and weakness. Here, we use mainly the Volterra equation (8) and the non-homogeneous SDE (5). To prove that equation (2) has a path-wise unique solution, we rely on the Volterra equation (8) and show that the following mapping:

$$(a_t)_{t \geq 0} \mapsto J r_{(a.)}^{\nu}(., 0) := [t \mapsto J\,\mathbb{E}\, f(Y_t^{0,\nu,(a.)})], \tag{9}$$

is contracting on $\mathcal{C}([0,T], \mathbb{R}_+)$ for all $T > 0$. It then follows that the fixed point of this mapping satisfies the closure condition (7) and can be used to define a solution to (2). Conversely any solution to (2) defines a fixed point of this mapping and one proves strong uniqueness for (2).

Finally, we prove our main result concerning the long time behavior of the solution to (2). Let us detail the structure of the proof. First, we give in Proposition 30 the long time behavior of the solution to the linear equation (5) with a constant current ($a_t \equiv a$). Any solution converges in law to a unique invariant probability measure ν_a^{∞} (Proposition 26). In that case, the Volterra equation (8) is of convolution type and it is possible to study finely its solution using Laplace transform techniques. Second, we prove, for small J, the uniqueness of a constant current a^* such that

$$\forall t \geq 0, \quad a^* = J\mathbb{E}f(Y_t^{0,\nu_{a^*}^{\infty},a^*}).$$

Third, we extend the previous convergence result to non-constant currents (a_t) satisfying

$$|a_t - a^*| \leq Ce^{-\lambda t}. \tag{10}$$

Using a perturbation method, we prove that

$$Y_t^{0,\nu,(a.)} \xrightarrow[t\to\infty]{\mathcal{L}} \nu_{a^*}^{\infty}.$$

Fourth, in Theorem 9, we give the long time behavior of the solution to the non-linear equation (2) for small J. Here, we use a fixed point argument.

The layout of the paper is as follows. Our main results are given in Section 2. In Section 3, we gather technical results. In Section 4, we study the non-homogeneous linear equation (5) and derive the Volterra equation satisfied by the jump rate. In Section 5, we characterize the invariant probability measures of (2). In Section 7 we study the long time behavior of the solution to (5) with a constant current a. In Section 8, we introduce the perturbation method. Finally Section 9 is devoted to the proof of our main result (Theorem 9).

2 Notations and results

Let us introduce some notations and definitions. For $s \geq 0$ and a probability measure ν on $\mathbb{R}_+$, let $Y_s^{s,\nu,(a.)}$ be a ν-distributed random variable, independent of a Poisson measure $\mathbf{N}(du, dz)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ of intensity measure $dudz$. We consider the canonical filtration $(\mathcal{F}_t^s)_{t\geq s}$ associated to the Poisson measure $\mathbf{N}$ and to the initial condition $Y_s^{s,\nu,(a.)}$, that is the completion of

$$\sigma\{Y_s^{s,\nu,(a.)}, \mathbf{N}([s,r]\times A): \ s \leq r \leq t, \ A \in \mathcal{B}(\mathbb{R}_+)\}.$$

Definition 1. *Let $s \geq 0$ and consider $(a_t): [s,\infty) \to \mathbb{R}_+$ a measurable locally integrable function ($\forall t \geq s, \ \int_s^t a_u du < \infty$).*

- *A process $(Y_t^{s,\nu,(a.)})_{t\geq s}$ is said to be a solution of the non-homogeneous linear equation* (5) *with a current $(a_t)_{t\geq s}$ if the law of $Y_s^{s,\nu,(a.)}$ is ν, $(Y_t^{s,\nu,(a.)})_{t\geq s}$ is $(\mathcal{F}_t^s)_{t\geq s}$-adapted, càdlàg, a.s. $\forall t \geq s, \ \int_s^t f(Y_u^{s,\nu,(a.)})du < \infty$ and* (5) *holds a.s.*
- *An $(\mathcal{F}_t^0)_{t\geq 0}$-adapted càdlàg process $(X_t)_{t\geq 0}$ is said to solve the non-linear SDE* (2) *if $t \mapsto \mathbb{E} f(X_t)$ is measurable locally integrable and if $(X_t)_{t\geq 0}$ is a solution of* (5) *with $s = 0$, $Y_0^{0,\nu,(a.)} = X_0$ and $\forall t \geq 0, \ a_t = J\,\mathbb{E} f(X_t)$.*

Let $t \geq s \geq 0$. We denote by $Y_t^{s,\nu,(a.)}$ a solution to the linear non-homogeneous SDE (5) driven by $(a_t)_{t\geq s} \in \mathcal{C}([s,\infty), \mathbb{R}_+)$ starting with law ν at time s. We denote its associated jump rate by: $r_{(a.)}^{\nu}(t,s) := \mathbb{E} f(Y_t^{s,\nu,(a.)})$. For any measurable function g, we write $\nu(g) := \int_0^\infty g(x)\nu(dx) = \mathbb{E}\, g(Y_s^{s,\nu,(a.)})$ whenever this integral makes sense.

Between its random jumps, the SDE (5) is reduced to a non-homogeneous ODE. Let us introduce its flow $\varphi_{t,s}^{(a.)}(x)$, which by definition is the solution of:

$$\forall t \geq s, \ \frac{d}{dt}\varphi_{t,s}^{(a.)}(x) = b(\varphi_{t,s}^{(a.)}(x)) + a_t \tag{11}$$
$$\varphi_{s,s}^{(a.)}(x) = x.$$

If $a_t \equiv a$, we denote $\varphi_t^a(x) = \varphi_{t,0}^{(a.)}(x)$.

Assumptions 2. *We assume that $b: \mathbb{R}_+ \to \mathbb{R}$ is a locally Lipschitz function with $b(0) > 0$ and that b is bounded from above:*

$$\exists C_b \geq 0: \ \forall x \geq 0, \ b(x) \leq C_b. \tag{12}$$

We assume moreover that there is a positive constant C_φ such that for all $(a_t)_{t\geq 0}$, $(d_t)_{t\geq 0} \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ we have

$$\forall x \geq 0,\ \forall s \leq t,\ |\varphi^{(a.)}_{t,s}(x) - \varphi^{(d.)}_{t,s}(x)| \leq C_\varphi \int_s^t |a_u - d_u| du. \tag{13}$$

The assumption $b(0) > 0$ implies that for all $x, t, s \in \mathbb{R}_+$, we have $\varphi^{(a.)}_{t,s}(x) \in \mathbb{R}_+$.

Assumptions 3. *We assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{C}^1$ convex increasing function with $f(0) = 0$ and satisfies:*

3.1. there exists a constant C_f such that

$$\forall x, y \geq 0,\ f(x+y) \leq C_f(1 + f(x) + f(y)) \text{ and } f'(x+y) \leq C_f(1 + f'(x) + f'(y)).$$

3.2. for all $\theta \geq 0$, $\sup_{x\geq 0}\{\theta f'(x) - f(x)\} < \infty$.
Define $\psi(\theta) := \sup_{x\geq 0}\{\theta f'(x) - \frac{1}{2} f^2(x)\} < \infty$. We also assume that

$$\lim_{\theta\to+\infty} \frac{\psi(\theta)}{\theta^2} = 0.$$

3.3. Finally we assume that there is a constant $C_{b,f} > 0$ such that

$$\forall x \geq 0,\ |b(x)| \leq C_{b,f}(1 + f(x)).$$

Note that these assumptions ensure that $f(x) > 0$ for all $x > 0$.

Assumptions 4. *We assume that the law of the initial condition is a probability measure ν satisfying $\nu(f^2) < \infty$.*

Let us give our main results.

Theorem 5. *Under Assumptions 2, 3 and 4, the non-linear SDE* (2) *has a path-wise unique solution $(X_t)_{t\geq 0}$ in the sense of Definition 1. Furthermore, there is a finite constant $\bar{r} > 0$ (only depending on b, f and J) such that:*

$$\sup_{t\geq 0} \mathbb{E}\, f(X_t) \leq \max(\bar{r}, \mathbb{E}\, f(X_0)), \quad \limsup_{t\to\infty} \mathbb{E}\, f(X_t) \leq \bar{r}.$$

The upper-bound $\bar{r}$ can be chosen to be an increasing function of J.

Notation 6. *Denote for all $a \geq 0$ the probability measure*

$$\nu^\infty_a(dx) := \frac{\gamma(a)}{b(x) + a} \exp\left(-\int_0^x \frac{f(y)}{b(y) + a} dy\right) \mathbb{1}_{\{x\in[0,\sigma_a]\}} dx, \tag{14}$$

where $\gamma(a)$ is the normalization

$$\gamma(a) := \left[\int_0^{\sigma_a} \frac{1}{b(x) + a} \exp\left(-\int_0^x \frac{f(y)}{b(y) + a} dy\right) dx\right]^{-1}. \tag{15}$$

The upper bound σ_a of the support of ν^∞_a is given by $\sigma_a := \lim_{t\to\infty} \varphi^a_t(0) \in \mathbb{R}^*_+ \cup \{+\infty\}$.

Remark 7. *1. For all $a \geq 0$, $\gamma(a) = \nu^\infty_a(f)$.*

2. *We prove in Proposition 26 that for any $a \geq 0$, ν_a^∞ is the unique invariant probability measure of* (5) *with $a_t \equiv a$.*

Proposition 8. *The probability measure ν_a^∞ is an invariant measure of* (2) *iff*

$$\frac{a}{\gamma(a)} = J. \tag{16}$$

Moreover, define $J_m := \sup\{J_0 \geq 0 : \ \forall J \in [0, J_0]$ equation (16) *has a unique solution$\}$, then $J_m > 0$. Consequently, for all $0 \leq J < J_m$ the non-linear process* (2) *has a unique invariant probability measure.*

We now state our main result: the convergence to the unique invariant probability measure for weak enough interactions.

Theorem 9. *Under Assumptions 2, 3, 4, there exists strictly positive constants J^* and λ (both only depending on b and f) satisfying*

$$0 < J^* < J_m, \ 0 < \lambda < f(\sigma_0),$$

(J_m and σ_0 are defined in Proposition 8) and such that for any $0 \leq J \leq J^*$, there is a constant $D > 0$:

$$\forall t \geq 0, \ |\mathbb{E} f(X_t) - \gamma(a^*)| \leq De^{-\lambda t}.$$

Here, $(X_t)_{t \geq 0}$ is the solution of the non-linear SDE (2) *starting with law ν and a^* is the unique solution of* (16)*. The constant D only depends on b, f, $\mathbb{E} f(X_0)$, J and λ.*

Moreover, it holds that X_t converges in law to $\nu_{a^*}^\infty$ at an exponential speed. If $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a bounded Lipschitz-continuous function, it holds that

$$\exists D' > 0, \forall t \geq 0, \quad |\mathbb{E}\, \phi(X_t) - \nu_{a^*}^\infty(\phi)| \leq D' e^{-\lambda t},$$

where the constant D' only depends on b, f, J, ν, λ and ϕ through its infinite norm and its Lipschitz constant.

Note that in Theorem 9, the unique invariant probability measure is globally stable: for weak enough interactions, starting from any initial condition, the system converges to its steady state.

Examples

Given the following constants $p \geq 1$, $\mu > 0$ and $\kappa \geq 0$, define, for all $x \geq 0$:

$$f(x) := x^p, \ b(x) = \mu - \kappa x.$$

Then (b, f) satisfies the Assumptions 2 and 3. In that case, the flow is given by

$$\varphi_{t,s}^{(a.)}(x) = xe^{-\kappa(t-s)} + \frac{\mu}{\kappa}[1 - e^{-\kappa(t-s)}] + \int_s^t e^{-\kappa(t-u)} a_u du.$$

We have $\forall x, y \in \mathbb{R}_+, \ f(x+y) \leq 2^{p-1}(f(x) + f(y))$. A similar estimate holds for f'. Moreover $\psi(\theta) = \frac{1}{2}\theta^{\frac{2p}{p+1}}(p-1)^{\frac{p-1}{p+1}}(1+p)$, so Assumption 3.2. holds.

Consequently, Theorem 9 applies. When $\kappa > 0$, the invariant probability measures are compactly supported and not necessarily unique. Consider for instance $b(x) = \mu - x$, $f(x) = x^2$. If μ is small enough, a numerical study shows that there exists $0 < a_1 < a_2 < \infty$ such that the function $a \mapsto \frac{a}{\gamma(a)}$ is increasing on $[0, a_1]$, decreasing on $[a_1, a_2]$ and finally increasing on $[a_2, \infty)$. Thus, if $J \in (a_1, a_2)$, the non-linear equation (2) admits exactly 3 non-trivial invariant probability measures. A numerical study shows that only two of the three are locally stable (bi-stability).

Another interesting example is the following. Assume $b(x) = 2 - 2x$ and $f(x) = x^{10}$. Then, a numerical study shows that the function $a \mapsto \frac{a}{\gamma(a)}$ is increasing on $\mathbb{R}_+$ and consequently for all $J \geq 0$, (2) admits a unique invariant probability measure. But if $J \in [0.7, 1.05]$ a further numerical analysis shows that the law of the solution of (2) asymptotically oscillates, betraying that the invariant probability measure is not locally stable. Those examples emphasis on the fact that the condition J small enough is required for Theorem 9 to hold.

Remark 10. *Assumption 2 is crucial to obtain our result on the long time behavior (Theorem 9). It restricts us to $\kappa \geq 0$. If $b(x) = \mu - \kappa x$ with $\kappa < 0$ then Assumption 2 does not hold.*

3 Technical notations and technical lemmas

The following standard results on the ODE (11) will be useful all along:

Lemma 11. *Assume b satisfies Assumption 2. Then:*

1. *For all $x \geq 0$ and $s \geq 0$, the ODE (11) has a unique solution $t \mapsto \varphi^{(a.)}_{t,s}(x)$ defined on $[s, \infty)$. This is the flow associated to the drift b and to the external current $(a_t)_{t \geq 0}$.*
2. *Given (a_t) and (d_t) in $\mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, the flow satisfies the following comparison principle:*
$$[\forall t \geq 0,\ a_t \geq d_t] \implies [\forall x \geq y \geq 0,\ \forall t \geq s \geq 0,\ \varphi^{(a.)}_{t,s}(x) \geq \varphi^{(d.)}_{t,s}(y)].$$
3. *The flow grows at most linearly with respect to the initial condition:*
$$\forall a \geq 0, \quad \forall x \geq 0,\ \forall t \geq 0, \qquad \varphi^a_t(x) \leq x + C^a_b t, \qquad \textit{where} \qquad C^a_b := C_b + a.$$
4. *The function $(t, s) \mapsto \varphi^{(a.)}_{t,s}(0)$ is continuous on $\{(t, s) : 0 \leq s \leq t < \infty\}$.*
5. *For any constant current $a \geq 0$, the flow converges to a limit as t goes to infinity (possibly equal to $+\infty$):*
$$\forall a \geq 0,\ \forall x \geq 0,\ \lim_{t \to +\infty} \varphi^a_t(x) := \sigma^x_a \in \mathbb{R}^*_+ \cup \{+\infty\}. \tag{17}$$
It holds that $\inf_{a,x \geq 0} \sigma^x_a > 0$. Moreover if we define:
$$\sigma_a := \inf\{x \geq 0 :\ b(x) + a = 0\} \in \mathbb{R}^*_+ \cup \{+\infty\},$$
we have: $\sigma^0_a = \sigma_a$.

Remark 12. 1. *Assumption 3.2. ensures that f does not grow too fast in the sense that for all $\epsilon > 0$, there is a constant $C_\epsilon > 0$, such that: $\forall x \geq 0\ f(x) \leq C_\epsilon e^{\epsilon x}$.*

2. *Using that f is increasing and continuous, we have, for all $a \geq 0$:*
$$\lim_{t \to \infty} f(\varphi^a_t) = f(\sigma_a) \geq f(\sigma_0) > 0.$$

We show that the jump rate $r^\nu_{(a.)}$ of the non-homogeneous SDE (5), satisfies the Volterra equation (8) where the kernels $K^\nu_{(a.)}$ and $K_{(a.)}$ are defined by

$$\forall t \geq s \geq 0,\ K^\nu_{(a.)}(t, s) := \int_0^\infty f(\varphi^{(a.)}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}(x)) du\right) \nu(dx), \tag{18}$$
$$K_{(a.)}(t, s) := K^{\delta_0}_{(a.)}(t, s). \tag{19}$$

Given two "kernels" α and β, it is convenient to follow the notation of [20] and define:

$$\forall t \geq s,\ (\alpha * \beta)(t,s) := \int_s^t \alpha(t,u)\beta(u,s)du. \tag{20}$$

The Volterra equation (8) becomes

$$r^\nu_{(a.)} = K^\nu_{(a.)} + K_{(a.)} * r^\nu_{(a.)}. \tag{21}$$

Similarly to (18) and (19), we define the kernels

$$\forall t \geq s,\ H^\nu_{(a.)}(t,s) := \int_0^\infty \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}(x))du\right)\nu(dx), \quad H_{(a.)} := H^{\delta_0}_{(a.)}, \quad \forall x \geq 0,\ H^x_{(a.)} := H^{\delta_x}_{(a.)}. \tag{22}$$

From the definition, one can check directly the following relation

$$1 * K^\nu_{(a.)} = 1 - H^\nu_{(a.)}. \tag{23}$$

To shorten notations, we shall also write $r_{(a.)}(t,s) := r^{\delta_0}_{(a.)}(t,s)$.
When the input current $(a_t)_{t\geq 0}$ is constant and equal to a, equation (5) is homogeneous and we write

$$\forall t \geq 0,\ Y^{\nu,a}_t := Y^{0,\nu,a}_t,\ r^\nu_a(t) := r^\nu_{(a.)}(t,0),\ K^\nu_a(t) := K^\nu_{(a.)}(t,0),\ H^\nu_a(t) := H^\nu_{(a.)}(t,0),\ \varphi^a_t(x) := \varphi^{(a.)}_{t,0}(x).$$

Note that in this homogeneous situation, the operation $*$ corresponds to the classical convolution operation. In particular this operation is commutative in the homogeneous setting and equation (21) is a *convolution Volterra equation*.

Remark 13. *For any $(a.) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ and any probability measure ν, we have*

$$\forall t \geq s \geq 0:\ H^\nu_{(a.)}(t,s) \leq H_0(t-s).$$

4 Study of the non-linear SDE (2) and of its linearized version (5)

4.1 On the non-homogeneous linear SDE (5)

Fix $s \geq 0$ and let $(a_t) : [s,\infty) \to \mathbb{R}_+$ be a continuous function. We consider the non-homogeneous linear SDE (5). We always assume that ν, the law of the initial condition $Y^{s,\nu,(a.)}_s$, satisfies Assumptions 4.

Lemma 14. *Grant Assumptions 2, 3 and 4. Then the SDE* (5) *has a path-wise unique solution on* $[s,\infty)$ *in the sense of Definition 1.*

Proof. We give a direct proof by considering the jumps of $Y^{s,\nu,(a.)}_t$ and by solving the equation between the jumps.

- **Step 1**: we grant Assumptions 2, 4 and assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is measurable and bounded. There exists a constant $0 < K < \infty$ such that:

$$\sup_{x\geq 0} f(x) \leq K.$$

  In this case, the solution of (5) can be constructed in the following way. Define by induction:

$$\tau_0 := \inf\{t \geq s :\ \int_s^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \leq f(\varphi^{(a.)}_{u,s}(Y^{s,\nu,(a.)}_s))\}} \mathbf{N}(du,dz) > 0\},$$

$$\forall n \geq 0,\ \tau_{n+1} := \inf\{t \geq \tau_n :\ \int_{\tau_n}^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \leq f(\varphi^{(a.)}_{u,\tau_n}(0))\}} \mathbf{N}(du,dz) > 0\}.$$

Using that $f \le K$, it follows that $a.s. \lim_{n\to\infty} \tau_n = +\infty$. We define:

$$Y_t^{s,\nu,(a.)} = \varphi_{t,s}^{(a.)}(Y_s^{s,\nu,(a.)})\mathbb{1}_{t\in[s,\tau_0)} + \sum_{n\ge 1} \varphi_{t,\tau_n}^{(a.)}(0)\mathbb{1}_{t\in[\tau_n,\tau_{n+1})},$$

and we can directly verify that $t \mapsto Y_t^{s,\nu,(a.)}$ is almost surely a solution of (5).

Uniqueness of equation (5) follows immediately from Lemma 11 (point 1): two solutions have to be equal almost surely before the first jump, from which we deduce that the two solutions have to jump at the same time. By induction on the number of jumps, the two trajectories are almost surely equal.

- **Step 2**: We now come back to the general case where f is not assumed to be bounded and we adapt the strategy of [18], proof of Proposition 2. We grant Assumptions 2, 3 and 4.

  We use Step 1 with $f^K(x) := f(\min(x, K))$ for some $K > 0$. Let us denote $Y_t^{s,\nu,(a.),K}$ the solution of (5) where f has been replaced by f^K. The boundedness of f^K implies the path-wise uniqueness of $Y_t^{s,\nu,(a.),K}$. We introduce $\zeta_K := \inf\{t \ge 0 : \ |Y_t^{s,\nu,(a.),K}| \ge K\}$, it holds that $Y_t^{s,\nu,(a.),K} = Y_t^{s,\nu,(a.),K+1}$ for all $t \in [0, \zeta_K]$ and all $K \in \mathbb{N}$. Moreover, $\zeta_K < \zeta_{K+1}$. We define $\zeta := \sup_K \zeta_K$ and deduce the existence and uniqueness of a solution $t \mapsto Y_t^{s,\nu,(a.)}$ of (5) on $[0, \zeta[$ such that $\limsup_{t\to\zeta} Y_t^{s,\nu,(a.)} = \infty$ on the event $\{\zeta < \infty\}$. But any solution of (5) satisfies for all $t \ge s$, $Y_t^{s,\nu,(a.)} \le \varphi_{t,s}^{(a.)}(Y_s^{s,\nu,(a.)}) < \infty$ a.s. and so it holds that $\zeta = +\infty$ a.s.

□

Lemma 15. *Grant Assumptions 2, 3 and 4. Let $(Y_t^{s,\nu,(a.)})_{t\ge s}$ be the solution of* (5)*. The functions $t \mapsto \mathbb{E} f(Y_t^{s,\nu,(a.)})$, $t \mapsto \mathbb{E} f'(Y_t^{s,\nu,(a.)})$, $t \mapsto \mathbb{E} f'(Y_t^{s,\nu,(a.)})|b(Y_t^{s,\nu,(a.)})|$ and $t \mapsto \mathbb{E} f^2(Y_t^{s,\nu,(a.)})$ are locally bounded on $[s,\infty)$. Moreover, $t \mapsto \mathbb{E} f(Y_t^{s,\nu,(a.)}) =: r_{(a.)}^{\nu}(t,s)$ is continuous on $[s,\infty)$.*

Proof. Consider the interval $[s, T]$ for some $T > 0$. Let $A := \sup_{t\in[s,T]} a_t$. It is clear that

$$\forall t \in [s,T], a.s.\ Y_t^{s,\nu,(a.)} \le Y_s^{s,\nu,(a.)} + \int_s^t [b(Y_u^{s,\nu,(a.)}) + a_u]du \le Y_s^{s,\nu,(a.)} + C_T,$$

with $C_T := (C_b + A)(T - s)$. We used here that b is bounded from above (Assumption 2). Using that f^2 is non-decreasing and Assumption 3.1., we have:

$$a.s.\ f^2(Y_t^{s,\nu,(a.)}) \le f^2(Y_s^{s,\nu,(a.)} + C_T) \le C_f^2(1 + f(C_T) + f(Y_s^{s,\nu,(a.)}))^2.$$

Using Assumption 4, we deduce that $t \mapsto \mathbb{E} f^2(Y_t^{s,\nu,(a.)})$ is bounded on $[s,T]$. By the Cauchy–Schwarz inequality, this implies that $t \mapsto \mathbb{E} f(Y_t^{s,\nu,(a.)})$ is also bounded on $[s,T]$. Finally, using the Assumption 3.2. (with $\theta = 1$), there is a constant C such that for all $x \ge 0$ $f'(x) \le C + f(x)$. Assumption 3.3. thus yields

$$\forall x \ge 0\ f'(x)|b(x)| \le C_{b,f}(1 + f(x))(C + f(x)),$$

and so this proves that $t \mapsto \mathbb{E} f'(Y_t^{s,\nu,(a.)})|b(Y_t^{s,\nu,(a.)})|$ is also bounded on $[s,T]$. We now apply the Itô formula (see for instance Theorem 32 of [27, Chap. II]) to $Y_t^{s,\nu,(a.)}$. It gives for any $\epsilon > 0$

$$f(Y_{t+\epsilon}^{s,\nu,(a.)}) = f(Y_t^{s,\nu,(a.)}) + \int_t^{t+\epsilon} f'(Y_u^{s,\nu,(a.)})[b(Y_u^{s,\nu,(a.)}) + a_u]du - \int_t^{t+\epsilon}\int_0^\infty f(Y_{u-}^{s,\nu,(a.)})\mathbb{1}_{\{z\le f(Y_{u-}^{s,\nu,(a.)})\}}\mathbf{N}(du,dz).$$

Taking the expectation, it follows that

$$\mathbb{E} f(Y_{t+\epsilon}^{s,\nu,(a.)}) - \mathbb{E} f(Y_t^{s,\nu,(a.)}) = \int_t^{t+\epsilon} \mathbb{E} f'(Y_u^{s,\nu,(a.)})[b(Y_u^{s,\nu,(a.)}) + a_u]du - \int_t^{t+\epsilon} \mathbb{E} f^2(Y_u^{s,\nu,(a.)})du,$$

from which we deduce that $t \mapsto \mathbb{E} f(Y_t^{s,\nu,(a.)})$ is locally Lipschitz and consequently continuous.

□

4.2 The Volterra equation

Along this section, we grant Assumptions 2, 3 and 4. Let $s \geq 0$ and $(a_t)_{t\geq s} \in \mathcal{C}([s,\infty),\mathbb{R}_+)$ be fixed. We consider $(Y_t^{s,\nu,(a.)})_{t\geq s}$ the path-wise unique solution of equation (5) driven by the current $(a_t)_{t\geq s}$. Following [18], we define:

$$\tau_{s,t} := \sup\{u \in [s,t] : Y_u^{s,\nu,(a.)} \neq Y_{u-}^{s,\nu,(a.)}\},$$

the time of the last jump before t, with the convention that $\tau_{s,t} = s$ if there is no jump during $[s,t]$. It follows directly from (5) that:

$$\forall t \geq s,\ a.s.\ Y_t^{s,\nu,(a.)} = \varphi_{t,s}^{(a.)}(Y_s^{s,\nu,(a.)})\mathbb{1}_{\{\tau_{s,t}=s\}} + \varphi_{t,\tau_{s,t}}^{(a.)}\mathbb{1}_{\{\tau_{s,t}>s\}}.$$

We also define:

$$\forall t \geq s,\ J_t := \int_s^t \int_0^\infty \mathbb{1}_{\{z \leq f(Y_{u-}^{s,\nu,(a.)})\}}\mathbf{N}(du,dz),$$

the number of jumps between s and t.

Lemma 16. *For all $t \geq u \geq s \geq 0$, we have*

$$\mathbb{P}(J_t = J_u|\mathcal{F}_u) = H_{(a.)}^{Y_u^{s,\nu,(a.)}}(t,u)\ a.s.$$

where $H_{(a.)}^x$ is given by (22).

Proof. We have $\{J_t = J_u\} = \{\int_u^t\int_0^\infty \mathbb{1}_{\{z\leq f(Y_{\theta-}^{s,\nu,(a.)})\}}\mathbf{N}(d\theta,dz) = 0\}$. Moreover, $\mathcal{F}_u$ and $\sigma\{\mathbf{N}([u,\theta]\times A) : \theta \in [u,t], A \in \mathcal{B}(\mathbb{R}_+)\}$ are independent. It follows from the Markov property satisfied by $(Y_t^{s,\nu,(a.)})_{t\geq s}$ that:

$$a.s.\ \mathbb{P}(J_t = J_u|\mathcal{F}_u) = \Phi(Y_u^{s,\nu,(a.)})$$

where: $\Phi(x) := \mathbb{P}(\int_u^t\int_0^\infty \mathbb{1}_{\{z\leq f(\varphi_{\theta,u}^{(a.)}(x))\}}\mathbf{N}(d\theta,dz) = 0) = H_{(a.)}^x(t,u)$. ☐

Lemma 17 (See also [18], Proposition 25)**.** *For all $t > s$, the law of $\tau_{s,t}$ is given by:*

$$\mathcal{L}(\tau_{s,t})(du) = H_{(a.)}^\nu(t,s)\delta_s(du) + r_{(a.)}^\nu(u,s)H_{(a.)}(t,u)\mathbb{1}_{\{s<u<t\}}du.$$

Proof. First, from Lemma 16, it follows that:

$$\mathbb{P}(\tau_{s,t} = s) = \mathbb{P}(J_t = J_s) = \mathbb{E}(H_{(a.)}^{Y_s^{s,\nu,(a.)}}(t,s)) = H_{(a.)}^\nu(t,s).$$

Let now $u \in (s,t]$ and $h > 0$ such that: $s < u-h < u \leq t$. We have:

$$\mathbb{P}(\tau_{s,t} \in (u-h,u]) = \mathbb{P}(J_u > J_{u-h}, J_t = J_u) = \mathbb{E}(\mathbb{1}_{\{J_u>J_{u-h}\}}\mathbb{P}(J_t = J_u|\mathcal{F}_u)) = \mathbb{E}(\mathbb{1}_{\{J_u>J_{u-h}\}}H_{(a.)}^{Y_u^{s,\nu,(a.)}}(t,u)).$$

Let $A := \sup_{u\in[s,t]} a_u$. On the event $\{J_u > J_{u-h}\}$, the process jumps at least once during $(u-h,u]$ and so, by Lemma 11 (point 2), we have $Y_u^{s,\nu,(a.)} \in [0,\varphi_{u,u-h}^{(a.)}] \subset [0,\varphi_h^A]$. It follows that

$$|\mathbb{P}(\tau_{s,t} \in (u-h,u]) - \mathbb{E}(\mathbb{1}_{\{J_u>J_{u-h}\}}H_{(a.)}(t,u))| \leq \sup_{x\in[0,\varphi_h^A]} |H_{(a.)}^x(t,u) - H_{(a.)}(t,u)|\mathbb{P}(J_u > J_{u-h}).$$

From the following Lemma 18, we have:

$$\lim_{h\downarrow 0}\frac{1}{h}\mathbb{P}(J_u > J_{u-h}) = r_{(a.)}^\nu(u,s).$$

Using Lemma 11 (point 4), $x \mapsto H^x_{(a.)}(t,u)$ is continuous at $x = 0$. From the continuity of $h \mapsto \varphi^A_h$ at $h = 0$, it yields:

$$\lim_{h\downarrow 0} \frac{1}{h} |\mathbb{P}(\tau_{s,t} \in (u-h,u]) - \mathbb{E}(\mathbb{1}_{\{J_u > J_{u-h}\}} H_{(a.)}(t,u))| = 0.$$

Combining the two results, we obtain the stated formula:

$$\lim_{h\downarrow 0} \frac{1}{h} \mathbb{P}(\tau_{s,t} \in (u-h,u]) = r^\nu_{(a.)}(u,s) H_{(a.)}(t,u).$$

This proves the result. ☐

Lemma 18 (See also [18], Lemma 23)**.** *For all $u \in (s,t]$ we have:*

$$\lim_{h\downarrow 0} \frac{1}{h} \mathbb{P}(J_u > J_{u-h}) = r^\nu_{(a.)}(u,s).$$

Proof. Again let $A := \sup_{u\in[s,t]} a_u < \infty$. We have:

$$\begin{aligned}
&|hr^\nu_{(a.)}(u,s) - \mathbb{P}(J_u > J_{u-h})| \\
&\quad \leq \left| hr^\nu_{(a.)}(u,s) - hr^\nu_{(a.)}(u-h,s) \right| + \left| hr^\nu_{(a.)}(u-h,s) - \mathbb{E} \int_{u-h}^u f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu,(a.)}_{u-h}))d\theta \right| \\
&\quad + \left| \mathbb{E} \int_{u-h}^u f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu,(a.)}_{u-h}))d\theta - \left[1 - \mathbb{E} \exp\left(- \int_{u-h}^u f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu,(a.)}_{u-h}))d\theta \right) \right] \right| \\
&\quad =: \Delta^1_h + \Delta^2_h + \Delta^3_h.
\end{aligned}$$

From the continuity of $u \mapsto r^\nu_{(a.)}(u,s)$ (Lemma 15) it follows that $\lim_{h\downarrow 0} \frac{\Delta^1_h}{h} = 0$. Moreover,

$$\Delta^2_h = \left| \int_{u-h}^u \mathbb{E} f(Y^{s,\nu,(a.)}_{u-h})d\theta - \mathbb{E} \int_{u-h}^u f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu,(a.)}_{u-h}))d\theta \right|.$$

Assumption 2 gives

$$\forall y \geq 0,\ \forall \theta \in [u-h,u],\ 0 \leq \varphi^{(a.)}_{\theta,u-h}(y) - y \leq \varphi^A_h(y) - y \leq C^A_b h.$$

We deduce that

$$\Delta^2_h \leq h \int_{u-h}^u \mathbb{E}\, g_h(Y^{s,\nu,(a.)}_{u-h}) C^A_b d\theta,$$

with $g_h(x) := \sup_{y\in[0,C^A_b h]} f'(x+y) = f'(x + C^A_b h)$. Using Assumption 3.1., we have $f'(x + C^A_b h) \leq C_f(1 + f'(C^A_b h) + f'(x))$. *It* follows that $\mathbb{E}\, g_h(Y^{s,\nu,(a.)}_{u-h}) \leq C_f(1 + f(C^A_b h) + \mathbb{E} f'(Y^{s,\nu,(a.)}_{u-h}))$. The function $t \mapsto \mathbb{E} f'(Y^{s,\nu,(a.)}_t)$ being locally bounded, we deduce that $\limsup_{h\downarrow 0} \frac{\Delta^2_h}{h} = 0$. Finally, using that $\forall x \geq 0,\ |x - (1 - e^{-x})| \leq x^2$ we have

$$\Delta^3_h \leq \mathbb{E} \left(\int_{u-h}^u f(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu,(a.)}_{u-h}))d\theta \right)^2.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\Delta^3_h \leq h\, \mathbb{E} \int_{u-h}^u f^2(\varphi^{(a.)}_{\theta,u-h}(Y^{s,\nu,(a.)}_{u-h}))d\theta \leq h^2\, \mathbb{E} f^2(Y^{s,\nu,(a.)}_{u-h} + C^A_b h).$$

Using $\forall x \geq 0, \forall y \in [0, C^A_b t],\ f^2(x+y) \leq C^2_f(1 + f(C^A_b t) + f(x))^2$ (Assumption 3.1.) and the fact that $t \mapsto \mathbb{E} f^2(Y^{s,\nu,(a.)}_t)$ and $t \mapsto \mathbb{E} f(Y^{s,\nu,(a.)}_t)$ are locally bounded (as seen in the Lemma 15), one can find a constant C_t such that

$$\Delta^3_h \leq C_t h^2.$$

This shows that $\lim_{h\downarrow 0} \frac{\Delta^3_h}{h} = 0$. Combining the three results ends the proof. ☐

Proposition 19 (See also [18], Theorem 12)**.** *Grant Assumptions 2, 3 and 4 . Let* $s \geq 0$ *and* $(a_t)_{t \geq s} \in \mathcal{C}([s,\infty), \mathbb{R}_+)$*. Let* $Y_t^{s,\nu,(a.)}$ *be the solution of equation* (5)*, starting from* $\mathcal{L}(Y_s^{s,\nu,(a.)}) = \nu$*. Let* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous non-negative function. It holds that*

$$\mathbb{E}\,\phi(Y_t^{s,\nu,(a.)}) = \int_s^t \phi(\varphi_{t,u}^{(a.)}(0)) H_{(a.)}(t,u) r_{(a.)}^{\nu}(u,s) du + \int_0^\infty \phi(\varphi_{t,s}^{(a.)}(x)) H_{(a.)}^x(t,s) \nu(dx).$$

In particular, $r_{(a.)}^{\nu}(t,s) = \mathbb{E}\, f(Y_t^{s,\nu,(a.)})$ *solves the Volterra equation* (21)

$$r_{(a.)}^{\nu} = K_{(a.)}^{\nu} + K_{(a.)} * r_{(a.)}^{\nu}.$$

Proof. We have, for all $t \geq s$

$$\begin{aligned}
\mathbb{E}\,\phi(Y_t^{s,\nu,(a.)}) &= \mathbb{E}\,\phi(Y_t^{s,\nu,(a.)}) \mathbb{1}_{\{\tau_{s,t}=s\}} + \mathbb{E}\,\phi(Y_t^{s,\nu,(a.)}) \mathbb{1}_{\{\tau_{s,t}>s\}} \\
&= \mathbb{E}\,\phi(\varphi_{t,s}^{(a.)}(Y_s^{s,\nu,(a.)})) \mathbb{1}_{\{\tau_{s,t}=s\}} + \mathbb{E}\,\phi(\varphi_{t,\tau_{s,t}}^{(a.)}(0)) \mathbb{1}_{\{\tau_{s,t}>s\}} \\
&:= \alpha_t + \beta_t.
\end{aligned}$$

Using Lemma 16, it follows that

$$\alpha_t = \mathbb{E}[\phi(\varphi_{t,s}^{(a.)}(Y_s^{s,\nu,(a.)}))\, \mathbb{P}(J_t = J_s | \mathcal{F}_s)] = \mathbb{E}[\phi(\varphi_{t,s}^{(a.)}(Y_s^{s,\nu,(a.)})) H_{(a.)}^{Y_s^{s,\nu,(a.)}}(t,s)] = \int_0^\infty \phi(\varphi_{t,s}^{(a.)}(x)) H_{(a.)}^x(t,s) \nu(dx).$$

Moreover, using Lemma 17, we have $\beta_t = \int_s^t \phi(\varphi_{t,u}^{(a.)}(0)) r_{(a.)}^{\nu}(u,s) H_{(a.)}(t,u) du$. Taking $\phi = f$ we obtain the Volterra equation (21). ☐

Note that using Lemma 17, $\int_s^t \mathcal{L}(\tau_{s,t})(du) = 1$ gives:

$$H_{(a.)}^{\nu} + H_{(a.)} * r_{(a.)}^{\nu} = 1.$$

This last formula is interesting by itself but does not characterize the jump rate $r_{(a.)}^{\nu}$. We prefer to work with (21) because, as shown in the next lemma, this Volterra equation admits a unique solution.

Lemma 20. *Let* $s \geq 0$ *be fixed,* $(a_t)_{t \geq s} \in \mathcal{C}([s,\infty), \mathbb{R}_+)$*. Then equation* (21) *has a unique continuous solution* $t \mapsto r_{(a.)}^{\nu}(t,s)$ *on* $[s,\infty)$*.*

Proof. Fix $T > s$. It is sufficient to prove the existence and uniqueness result on $[s,T]$. We consider the Banach space $(\mathcal{C}([s,T],\mathbb{R}), ||\cdot||_{\infty,T})$ and define on this space the following operator: $\Gamma : r \mapsto K_{(a.)}^{\nu} + K_{(a.)} * r$. Let $A := \sup_{t\in[s,T]} a_t$, we have: $M_s^T = \sup_{s \leq u \leq t \leq T} K_{(a.)}(t,u) < \infty$. This follows from:

$$\forall s \leq u \leq t \leq T,\ K_{(a.)}(t,u) \leq f(\varphi_{t,u}^{(a.)}(0)) \leq f(C_b^A(T-s)) < \infty.$$

It is clear (using Assumptions 3.1. and 4) that the operator $\Gamma : \mathcal{C}([s,T],\mathbb{R}) \to \mathcal{C}([s,T],\mathbb{R})$ is well defined. Given $n \in \mathbb{N}$, the iteration Γ^n is an affine operator with linear part $\Gamma_0^n : r \mapsto (K_{(a.)})^{*(n)} * r$. We prove that Γ^n is contracting for n large enough, which is equivalent to proving that Γ_0^n is contracting for n large enough. By induction, it is easily shown that

$$\forall r \in \mathcal{C}([s,T],\mathbb{R}), \forall n \in \mathbb{N}\ \ ||\Gamma_0^n(r)||_{\infty,t} := \sup_{u\in[s,t]} |(\Gamma_0^n(r))(u,s)| \leq \frac{||r||_{\infty,T}(M_s^T(t-s))^n}{n!}.$$

Consequently $\forall r \in \mathcal{C}([s,T],\mathbb{R}), \forall n \in \mathbb{N}, ||\Gamma_0^n(r)||_{\infty,T} \leq \frac{(M_s^T(T-s))^n}{n!}||r||_{\infty,T}$ and Γ_0^n is contracting for n large enough. We deduce that the operator Γ^n is also contracting and has a unique fixed point in $\mathcal{C}([s,T],\mathbb{R})$. It is also a fixed point of Γ. This proves that (21) has a unique solution in $\mathcal{C}([s,T],\mathbb{R})$. ☐

We shall need the following well-known result on Volterra equation:

Lemma 21. *Consider $k, w : \mathbb{R}_+^2 \to \mathbb{R}$ two continuous kernels. The Volterra equation $x = w + k * x$ has a unique solution given by $x = w + r * w$, where $r : \mathbb{R}_+^2 \to \mathbb{R}$ is the "resolvent" of k, i.e. the unique solution of*

$$r = k + k * r.$$

Proof. It is clear from the proof of the preceding lemma that both Volterra equations have a unique solution. Moreover, we have: $w + k * (w + r * w) = w + k * w + (r - k) * w = w + r * w$. By uniqueness, we deduce that $x = w + r * w$. □

4.3 The jump rate is uniformly bounded

Lemma 22. *Grant Assumptions 2, 3 and 4 . Let $s \geq 0$ and $(a_t)_{t \geq s} \in \mathcal{C}([s, \infty), \mathbb{R}_+)$. Let $Y_t^{s,\nu,(a.)}$ be the solution of equation (5), starting from $\mathcal{L}(Y_s^{s,\nu,(a.)}) = \nu$. Then the functions $t \mapsto \mathbb{E} f'(Y_t^{s,\nu,(a.)})$, $t \mapsto \mathbb{E} f'(Y_t^{s,\nu,(a.)}) b(Y_t^{s,\nu,(a.)})$ and $t \mapsto \mathbb{E} f^2(Y_t^{s,\nu,(a.)})$ are continuous on $[s, \infty)$.*

Proof. The proof relies on Proposition 19. Consider the interval $[s, T]$ for some fixed $T > s \geq 0$ and let $A := \sup_{t \in [s,T]} a_t$. Let $\phi \in \{f', f'b, f^2\}$. By Lemma 11 (point 4), the function $(t, u) \mapsto \phi(\varphi_{t,u}^{(a.)}(0)) H_{(a.)}(t, u) r_{(a.)}^{\nu}(u, s)$ is uniformly continuous on $\{(t, u) : s \leq u \leq t \leq T\}$. Consequently

$$t \mapsto \int_s^t \phi(\varphi_{t,u}^{(a.)}(0)) H_{(a.)}(t, u) r_{(a.)}^{\nu}(u, s) du \text{ is continuous on } [s, T].$$

The continuity of $t \mapsto \int_0^\infty \phi(\varphi_{t,s}^{(a.)}(x)) H_{(a.)}^x(t, s) \nu(dx)$ follows from the Dominated Convergence Theorem. For instance, for $\phi \equiv f'$, one has

$$\forall t \in [s, T], \forall x \geq 0,\ f'(\varphi_{t,s}^{(a.)}(x)) \leq f'(\varphi_{t-s}^{A}(x)) \leq f'(x + C_b^A(t-s)) \leq C_f(f'(x) + 1 + f'(C_b^A(T - s))),$$

from which the result follows easily using Assumption 4 and Assumption 3.2.. The same method can be applied for $\phi(x) := f'(x)b(x)$ (using Assumption 3.3.) and for $\phi(x) := f^2(x)$. □

Proposition 23. *Grant Assumptions 2, 3 and 4. Let $s, J \geq 0$ be fixed. Given any $\kappa \geq 0$, there is a constant $\bar{a} \geq \kappa$ only depending on b, f, J and κ such that:*

$$\forall (a_t)_{t \geq s} \in \mathcal{C}([s, \infty), \mathbb{R}_+),\ \left\{ \sup_{t \geq s} a_t \leq \bar{a} \text{ and } J\nu(f) \leq \bar{a} \right\} \implies \sup_{t \geq s} J r_{(a.)}^{\nu}(t, s) \leq \bar{a}.$$

Moreover, $\bar{a}$ can be chosen to be an increasing function of J and κ.

Proof. Assume $\sup_{t \geq s} a_t \leq \bar{a}$ for some $\bar{a} > 0$ that we specify later. Applying the Itô formula and taking expectations yields

$$\forall t \geq s,\ \mathbb{E} f(Y_t^{s,\nu,(a.)}) = \mathbb{E} f(Y_s^{s,\nu,(a.)}) + \int_s^t \mathbb{E} f'(Y_u^{s,\nu,(a.)})[b(Y_u^{s,\nu,(a.)}) + a_u] du - \int_s^t \mathbb{E} f^2(Y_u^{s,\nu,(a.)}) du.$$

Lemma 22 implies that $t \mapsto \mathbb{E} f(Y_t^{s,\nu,(a.)})$ is $\mathcal{C}^1$ and

$$\forall t \geq s,\ \frac{d}{dt} \mathbb{E} f(Y_t^{s,\nu,(a.)}) = \mathbb{E} f'(Y_t^{s,\nu,(a.)})(b(Y_t^{s,\nu,(a.)}) + a_t) - \mathbb{E} f^2(Y_t^{s,\nu,(a.)}).$$

Using (12), the Cauchy-Schwarz inequality gives

$$\begin{aligned} \frac{d}{dt} \mathbb{E} f(Y_t^{s,\nu,(a.)}) &\leq \left\{ [\bar{a} + C_b] \mathbb{E} f'(Y_t^{s,\nu,(a.)}) - \frac{1}{2} \mathbb{E} f^2(Y_t^{s,\nu,(a.)}) \right\} - \frac{1}{2} \mathbb{E}^2 f(Y_t^{s,\nu,(a.)}) \\ &\leq \frac{1}{2} [2\psi(\bar{a} + C_b) - \mathbb{E}^2 f(Y_t^{s,\nu,(a.)})], \end{aligned}$$

where in the last line, we used Assumption 3.2.. Setting $M(\bar{a}) := \sqrt{2\psi(\bar{a}+C_b)}$ and using the sign of the right hand side, we conclude that

$$\nu(f) \leq M(\bar{a}) \implies [\forall t \geq s \ \mathbb{E}\, f(Y_t^{s,\nu,(a.)}) \leq M(\bar{a})].$$

To complete the proof, we need to check that for any $\kappa \geq 0$, any $J \geq 0$, there is a constant $\bar{a} \geq \kappa$ such that $JM(\bar{a}) \leq \bar{a}$. This follows easily from Assumption 3.2., which gives

$$\lim_{\theta\to\infty} \frac{J\sqrt{2\psi(\theta)}}{\theta} = 0.$$

It is clear that $\bar{a}(J)$ can be chosen to be a non-decreasing function of J and κ. We deduce that:

$$\left[\sup_{t\geq s} a_t \leq \bar{a} \text{ and } J\nu(f) \leq \bar{a}\right] \implies \left\{\begin{aligned} &\frac{d}{dt}\,\mathbb{E}\, f(Y_t^{s,\nu,(a.)}) \leq \frac{1}{2}\left[\frac{\bar{a}^2}{J^2} - \mathbb{E}^2\, f(Y_t^{s,\nu,(a.)})\right] \\ &\mathbb{E}\, f(Y_s^{s,\nu,(a.)}) \leq \frac{\bar{a}}{J}. \end{aligned}\right\} \implies \sup_{t\geq s} Jr^{\nu}_{(a.)}(t,s) \leq \bar{a}.$$

□

We have proved that $t \mapsto \mathbb{E}\, f(Y_t^{s,\nu,(a.)})$ is $\mathcal{C}^1$ and bounded on $\mathbb{R}_+$. The same methods can be applied to the non-linear equation (2).

Lemma 24. *Grant Assumptions 2, 3 and 4. Consider $(X_t)_{t\geq 0}$ a solution of the non-linear equation (2) in the sense of Definition 1. Then $t \mapsto \mathbb{E}\, f(X_t) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R})$ and there is a finite constant $\bar{r} > 0$ (only depending on b, f and J) such that:*

$$\sup_{t\geq 0} \mathbb{E}\, f(X_t) \leq \max(\bar{r}, \mathbb{E}\, f(X_0)), \quad \limsup_{t\to\infty} \mathbb{E}\, f(X_t) \leq \bar{r}.$$

Moreover, $\bar{r}$ can be chosen to be an increasing function of J.

Proof. By applying the same argument as in the proof of Lemma 15 it is clear that the functions

$$t \mapsto \mathbb{E}\, f(X_t), t \mapsto \mathbb{E}\, f'(X_t), t \mapsto \mathbb{E}\, f^2(X_t) \text{ and } t \mapsto \mathbb{E}\, |b(X_t)| f'(X_t)$$

are locally bounded. Applying the Itô formula and taking expectations yields

$$\mathbb{E}\, f(X_t) = \mathbb{E}\, f(X_0) + \int_0^t \mathbb{E}\, f'(X_u) b(X_u) du + J\int_0^t \mathbb{E}\, f'(X_u)\,\mathbb{E}\, f(X_u) du - \int_0^t \mathbb{E}\, f^2(X_u) du. \quad (24)$$

We deduce that $t \mapsto \mathbb{E}\, f(X_t)$ is continuous. Define for all $t \geq 0$, $a_t := \mathbb{E}\, f(X_t)$. From Lemma 14, it is clear that:

$$a.s.\ \forall t \geq 0,\ X_t = Y_t^{0,\nu,(a.)},$$

where $(Y_t^{0,\nu,(a.)})_{t\geq 0}$ is the solution of (5) driven by $(a_t)_{t\geq 0}$. In particular, Lemma 22 applies and the functions $t \mapsto \mathbb{E}\, f'(X_t)$, $t \mapsto \mathbb{E}\, f^2(X_t)$ and $t \mapsto \mathbb{E}\, f'(X_t) b(X_t)$ are continuous. From equation (24), we deduce that $t \mapsto \mathbb{E}\, f(X_t) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ and

$$\frac{d}{dt}\,\mathbb{E}\, f(X_t) = \mathbb{E}\, f'(X_t) b(X_t) + J\,\mathbb{E}\, f'(X_t)\,\mathbb{E}\, f(X_t) - \mathbb{E}\, f^2(X_t).$$

We have:

1. $\mathbb{E}\, f'(X_t) b(X_t) - \frac{1}{4}\mathbb{E}\, f^2(X_t) \leq \frac{1}{2}[2C_b\,\mathbb{E}\, f'(X_t) - \frac{1}{2}\mathbb{E}\, f^2(X_t)] \leq \frac{1}{2}\psi(2C_b)$, using Assumptions 12 and 3.2..
2. $$\begin{aligned} J\,\mathbb{E}\, f'(X_t)\,\mathbb{E}\, f(X_t) - \frac{1}{4}\mathbb{E}\, f^2(X_t) &\leq J\,\mathbb{E}\, f'(X_t)\,\mathbb{E}\, f(X_t) - \frac{1}{4}\mathbb{E}^2\, f(X_t) \\ &\leq \mathbb{E}\, f(X_t)[J\,\mathbb{E}\, f'(X_t) - \frac{1}{8}\mathbb{E}\, f(X_t) - \frac{1}{8}\mathbb{E}\, f(X_t)] \\ &\leq 2\beta^2, \end{aligned}$$
   where $\beta := \sup_{x\geq 0} Jf'(x) - \frac{1}{8}f(x) < \infty$ (by Assumption 3.2.). We used $\sup_{y\geq 0} y(\beta - \frac{1}{8}y) \leq 2\beta^2$ to obtain the last inequality. Note that β is a non-decreasing function of J.

Combining the points 1 and 2 gives

$$\frac{d}{dt}\,\mathbb{E}\, f(X_t) \leq \frac{1}{2}[(\psi(2C_b) + 4\beta^2) - \mathbb{E}\, f^2(X_t)]. \tag{25}$$

We define: $\bar{r} := \sqrt{\psi(2C_b) + 4\beta^2}$ and deduce that

$$\sup_{t\geq 0} \mathbb{E}\, f(X_t) \leq \max(\bar{r}, \mathbb{E}\, f(X_0)), \quad \limsup_{t\to\infty} \mathbb{E}\, f(X_t) \leq \bar{r}.$$

□

4.4 Existence and uniqueness of the solution of the non-linear SDE: proof of Theorem 5

We now prove that equation (2) has a unique strong solution $(X_t)_{t\geq 0}$. Let $J > 0$ (the case $J = 0$ has already been treated in Lemma 14 by choosing $(a_t)_{t\geq 0} \equiv 0$). Let ν, the initial condition, satisfying Assumption 4, be fixed. We grant Assumptions 2 and 3. Let $T > 0$ be a fixed horizon time. Thanks to Proposition 23 with $\kappa := \max(J\,\mathbb{E}\, f(X_0), J\bar{r})$, we build the following application:

$$\begin{array}{cccc} \Phi & : & \mathcal{C}^T_{\bar{a}} & \to & \mathcal{C}^T_{\bar{a}} \\ & & (a_t)_t & \mapsto & J r^\nu_{(a.)}(\cdot, 0), \end{array} \tag{26}$$

where $\mathcal{C}^T_{\bar{a}} := \{(a_t)_t \in \mathcal{C}([0,T], \mathbb{R}_+) : \ \sup_{t\in[0,T]} a_t \leq \bar{a}\}$. The function $r^\nu_{(a.)}(t,0) := \mathbb{E}\, f(Y^{0,\nu,(a.)}_t)$ is defined by equation (5) (using $s = 0$). The constant $\bar{a}$ is given by Proposition 23: in particular $\bar{a}$ does not depend on T. We equip $\mathcal{C}^T_{\bar{a}}$ with the sup norm $||(a_t)_t||_{\infty,T} := \sup_{t\in[0,T]} |a_t|$. The metric space $(\mathcal{C}^T_{\bar{a}}, ||\cdot||_{\infty,T})$ is complete. We now prove that the application Φ defined by (26) is contracting. Let $(a_t)_t, (d_t)_t \in \mathcal{C}^T_{\bar{a}}$; we denote by $r^\nu_{(a.)}(t,s)$ and $r^\nu_{(d.)}(t,s)$ their corresponding jump rate, where t belongs to $[s,T]$. Both $r^\nu_{(a.)}$ and $r^\nu_{(d.)}$ satisfy the Volterra equation (21). It follows that the difference $\Delta := r^\nu_{(a.)} - r^\nu_{(d.)}$ satisfies:

$$\begin{aligned} \Delta &= K^\nu_{(a.)} - K^\nu_{(d.)} + K_{(a.)} * (r^\nu_{(a.)} - r^\nu_{(d.)}) + (K_{(a.)} - K_{(d.)}) * r^\nu_{(d.)} \\ &= W + K_{(a.)} * \Delta \text{ with } W := K^\nu_{(a.)} - K^\nu_{(d.)} + (K_{(a.)} - K_{(d.)}) * r^\nu_{(d.)} \end{aligned}$$

Consequently, Δ solves the following non-homogeneous Volterra equation with kernel $K_{(a.)}$

$$\Delta = W + K_{(a.)} * \Delta. \tag{27}$$

Using Lemma 21, this equation can be solved explicitly in terms of $r_{(a.)}$, the "resolvent" of $K_{(a.)}$

$$\Delta = W + r_{(a.)} * W. \tag{28}$$

Lemma 25. *There exists a constant Θ_T only depending on T, f, b and $\bar{a}$ such that, for all $a, d \in \mathcal{C}^T_{\bar{a}}$:*

$$\forall\, 0 \leq s \leq t \leq T,\ \forall x \in \mathbb{R}_+,\ |K^{\delta_x}_{(a.)} - K^{\delta_x}_{(d.)}|(t,s) \leq \Theta_T(1 + f'(x) + f(x) + f'(x)f(x)) \int_s^t |a_u - d_u| du.$$

Proof. Fix (a_t) and (d_t) in $\mathcal{C}^T_{\bar{a}}$. We have

$$\begin{aligned} |K^{\delta_x}_{(a.)} - K^{\delta_x}_{(d.)}|(t,s) &= \left| f(\varphi^{(a.)}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}(x))du\right) - f(\varphi^{(d.)}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{(d.)}_{u,s}(x))du\right)\right| \\ &\leq \left| f(\varphi^{(a.)}_{t,s}(x)) - f(\varphi^{(d.)}_{t,s}(x))\right| \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}(x))du\right) \\ &\quad + f(\varphi^{(d.)}_{t,s}(x)) \left| \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}(x))du\right) - \exp\left(-\int_s^t f(\varphi^{(d.)}_{u,s}(x))du\right)\right| \\ &=: M + N. \end{aligned}$$

Assumptions 2 and 3.1. together with Lemma 11 (2) give

$$\begin{aligned} M &\leq |f(\varphi_{t,s}^{(a.)}(x)) - f(\varphi_{t,s}^{(d.)}(x))| \\ &\leq f'(x + C_b^{\bar{a}} T)|\varphi_{t,s}^{(a.)}(x) - \varphi_{t,s}^{(d.)}(x)| \\ &\leq C_f(1 + f'(x) + f'(C_b^{\bar{a}} T))C_\varphi \int_s^t |a_u - d_u| du. \end{aligned}$$

Furthermore, using that $\forall A, B \geq 0: \quad |e^{-A} - e^{-B}| \leq |A - B|$, we have

$$\begin{aligned} N &\leq C_f[1 + f(x) + f(C_b^{\bar{a}} T)] \int_s^t |f(\varphi_{u,s}^{(a.)}(x)) - f(\varphi_{u,s}^{(d.)}(x))| du \\ &\leq C_f[1 + f(x) + f(C_b^{\bar{a}} T)] f'(x + C_b^{\bar{a}} T) C_\varphi \int_s^t \int_s^r |a_u - d_u| du dr \\ &\leq T C_\varphi C_f^2 [1 + f(x) + f(C_b^{\bar{a}} T)][1 + f'(x) + f'(C_b^{\bar{a}} T)] \int_s^t |a_u - d_u| du. \end{aligned}$$

Combining the two estimates, we get the result. □

Proof of Theorem 5. We now write Θ_T for any constant that depends only on T, on the initial condition ν, on b, f, J and on $\bar{a}$ and that can change from line to line. By Assumptions 3 and 4, it follows that:

$$\forall (a_t), (d_t) \in \mathcal{C}_{\bar{a}}^T, \ \forall t \in [0, T]: \ |K_{(a.)}^\nu - K_{(d.)}^\nu|(t, 0) \leq \Theta_T \int_0^t |a_u - d_u| du.$$

Moreover, since $\sup_{t \in [0,T]} r_{(d.)}(t, 0) \leq \frac{\bar{a}}{J}$ by Proposition 23, we have

$$|(K_{(a.)} - K_{(d.)}) * r_{(d.)}|(t, 0) = |\int_0^t (K_{(a.)} - K_{(d.)})(t, u) r_{(d.)}(u) du| \leq \frac{\bar{a}}{J} \Theta_T (1 + f'(0)) T \int_0^t |a_u - d_u| du.$$

Consequently, there is a constant Θ_T such that

$$\forall (a_t), (d_t) \in \mathcal{C}_{\bar{a}}^T, \ \forall t \in [0, T]: \ |W|(t, 0) \leq \Theta_T \int_0^t |a_u - d_u| du.$$

Using the formula (28), we deduce that

$$\begin{aligned} |\Delta(t, 0)| \leq& |W|(t, 0) + \int_0^t r_{(a.)}(t, u)|W|(u, 0) du \\ \leq& \Theta_T (1 + T \frac{\bar{a}}{J}) \int_0^t |a_u - d_u| du. \end{aligned}$$

We have proved that there is a constant Θ_T such that:

$$\forall (a_t), (d_t) \in \mathcal{C}_{\bar{a}}^T, \forall t \in [0, T], \quad ||J r_{(a.)}^\nu(\cdot, 0) - J r_{(d.)}^\nu(\cdot, 0)||_{\infty, t} \leq \Theta_T \int_0^t ||a - d||_{\infty, u} du.$$

This estimate is sufficient to prove Theorem 5 by a classical Picard/Gronwall argument. We deduce that Φ has a unique fixed point $(a_t^*)_t$. It is then easy to check that $(Y_t^{0,\nu,(a^*.)})_{t \in [0,T]}$, driven by the current (a^*) and with initial condition $Y_0^{0,\nu,(a^*.)} = X_0$, defines a solution of (2) up to time T. This proves existence of a strong solution to (2). Now, if $(X_t)_{t \geq 0}$ is a strong solution of (2) in the sense of Definition 1, let $\forall t \geq 0, \ a_t := J \,\mathbb{E} f(X_t)$. We have $\sup_{t \geq 0} a_t \leq \max(J \bar{r}, J \,\mathbb{E} f(X_0)) \leq \bar{a}$ and consequently $(a_t)_{t \in [0,T]} \in \mathcal{C}_{\bar{a}}^T$. Moreover, it is clear that $(X_t)_{t \geq 0}$ solves (5) with $a_t := J \,\mathbb{E} f(X_t)$ and $Y_0^{0,\nu,(a.)} := X_0$. We deduce that (a_t) is the unique fixed point of Φ: $\forall t \in [0, T]: a_t = a_t^*$. Consequently, by Lemma 14, we have: $a.s.\ \forall t \in [0, T] \ X_t = Y_t^{0,\nu,(a.)}$. This proves path-wise uniqueness and ends the proof of Theorem 5. □

5 The invariant probability measures: proof of Proposition 8

We now study the invariant probability measures of the non-linear process (2). We follow the strategy of [18]: we first study the linear process driven by a constant current a and show that it has a unique invariant probability measure. We then use this result to study the invariant probability measures of the non-linear equation (2). Let $a \geq 0$ and $(Y_t^{\nu,a})_t$ the solution of the following SDE:

$$Y_t^{\nu,a} = Y_0^{\nu,a} + \int_0^t b(Y_u^{\nu,a})du + at - \int_0^t \int_{\mathbb{R}_+} Y_{u-}^{\nu,a} \mathbb{1}_{\{z \leq f(Y_{u-}^{\nu,a})\}} \mathbf{N}(du, dz) \tag{29}$$

Equation (29) is equation (5) with $\forall t \geq 0,\ a_t = a$ and $s = 0$.

Proposition 26. *Grant Assumptions 2 and 3. Then the SDE* (29) *has a unique invariant probability measure* ν_a^∞ *given by equation* (14)*:*

$$\nu_a^\infty(dx) := \frac{\gamma(a)}{b(x)+a} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right) \mathbb{1}_{\{x \in [0,\sigma_a]\}} dx,$$

where $\gamma(a)$ *is the normalizing factor given by* (15)*. Moreover we have* $\nu_a^\infty(f) = \gamma(a)$*.*

A proof of this result can be found in [18, Prop. 21] with $b(x) := -\kappa x$ and with slightly different assumptions on f. We give here a proof based on different arguments. Note that the general method introduced by [9] to find the stationary measures of a PDMP can be applied here; we use a method introduced in this paper to prove the uniqueness part.

Proof. Let us first check that the probability measure ν_a^∞ is indeed an invariant measure of (29).
Claim 1 The probability measure ν_a^∞ satisfies Assumption 4.
First $b(0) > 0$ yields to $\forall a \geq 0,\ \sigma_a \geq \sigma_0 > 0$. The function $t \mapsto \varphi_t^a(0)$ is a bijection from $\mathbb{R}_+$ to $[0, \sigma_a)$. Consequently, the changes of variable $x = \varphi_t^a(0)$ and $y = \varphi_u^a(0)$ give

$$\int_0^{\sigma_a} \frac{f^2(x)}{b(x)+a} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right)dx = \int_0^\infty f^2(\varphi_t^a(0)) \exp\left(-\int_0^t f(\varphi_u^a(0))du\right)dt.$$

This last integral is finite by Remark 12.
Claim 2 We have: $K_a^{\nu_a^\infty}(t) = \gamma(a) H_a(t)$.
We recall that $H_a(t) = H_a^{\delta_0}(t, 0)$. We have, for all $t \geq 0$:

$$K_a^{\nu_a^\infty}(t) = \int_0^{\sigma_a} f(\varphi_t^a(x)) \exp\left(-\int_0^t f(\varphi_u^a(x))du\right) \frac{\gamma(a)}{b(x)+a} \exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right)dx. \tag{30}$$

The change of variable $y = \varphi_u^a(0)$ yields:

$$K_a^{\nu_a^\infty}(t) = \int_0^{\sigma_a} f(\varphi_t^a(x)) \exp\left(-\int_0^t f(\varphi_u^a(x))du\right) \frac{\gamma(a)}{b(x)+a} \exp\left(-\int_0^{t(x)} f(\varphi_u^a(0))du\right)dx,$$

where $t(x)$ is the unique $t \geq 0$ such that $\varphi_t^a(0) = x$. We now make the change of variable $x = \varphi_s^a(0)$ we obtain (using the semi-group property satisfied by $\varphi_t^a(0)$):

$$\begin{aligned}
K_a^{\nu_a^\infty}(t) &= \gamma(a) \int_0^\infty f(\varphi_t^a(\varphi_s^a(0))) \exp\left(-\int_0^t f(\varphi_u^a(\varphi_s^a(0)))du\right) \exp\left(-\int_0^s f(\varphi_u^a(0))du\right)ds \\
&= \gamma(a) \int_t^\infty f(\varphi_\theta^a(0)) \exp\left(-\int_0^\theta f(\varphi_u^a(0))du\right)d\theta \\
&= \gamma(a) \left[H_a(t) - \lim_{\theta \to \infty} \exp\left(-\int_0^\theta f(\varphi_u^a(0))du\right)\right].
\end{aligned}$$

Using Remark 2, we have: $\lim_{\theta\to\infty} \exp\left(-\int_0^\theta f(\varphi_u^a(0))du\right) = 0$ and the claim is proved.

We now consider $(Y_t^{\nu_a^\infty,a})_{t\geq 0}$ the solution of equation (29) starting from $\mathcal{L}(Y_0^{\nu_a^\infty,a}) = \nu_a^\infty$. Proposition 19 applies, so $r_a^{\nu_a^\infty}(t) = \mathbb{E}\, f(Y_t^{\nu_a^\infty,a})$ is the unique solution of the Volterra equation

$$r_a^{\nu_a^\infty} = K_a^{\nu_a^\infty} + K_a * r_a^{\nu_a^\infty}.$$

Using Claim 2 and the relation (23), we verify that the constant function $\gamma(a)$ is a solution of

$$K_a^{\nu_a^\infty} + K_a * \gamma(a) = \gamma(a)H_a + \gamma(a)(1-H_a) = \gamma(a).$$

By uniqueness (Lemma 20), we deduce that $\forall t\geq 0,\ r_a^{\nu_a^\infty}(t) = \gamma(a)$.

Finally, let $\phi:\mathbb{R}_+\to\mathbb{R}_+$ be a measurable function. Using Proposition 19, we have:

$$\begin{aligned}\mathbb{E}\,\phi(Y_t^{\nu_a^\infty,a}) &= \gamma(a)\int_0^t \phi(\varphi_{t-u}^a(0))H_a(t-u)du + \int_0^\infty \phi(\varphi_t^a(x))H_a^x(t)\nu_a^\infty(dx)\\ &= \gamma(a)\int_0^t \phi(\varphi_u^a(0))H_a(u)du\\ &\quad + \int_0^{\sigma_a} \phi(\varphi_t^a(x))\exp\left(-\int_0^t f(\varphi_u^a(x))du\right)\frac{\gamma(a)}{b(x)+a}\exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right)dx.\end{aligned}$$

The change of variables $y=\varphi_u^a(0)$ and $x=\varphi_\theta^a(0)$ yields

$$\begin{aligned}\mathbb{E}\,\phi(Y_t^{\nu_a^\infty,a}) &= \gamma(a)\int_0^t \phi(\varphi_u^a(0))H_a(u)du\\ &\quad + \gamma(a)\int_0^\infty \phi(\varphi_t^a(\varphi_\theta^a(0)))\exp\left(-\int_0^t f(\varphi_u^a(\varphi_\theta^a(0)))du\right)\exp\left(-\int_0^\theta f(\varphi_u^a(0))du\right)d\theta\\ &= \gamma(a)\int_0^t \phi(\varphi_u^a(0))H_a(u)du + \gamma(a)\int_t^\infty \phi(\varphi_u^a(0))\exp\left(-\int_0^u f(\varphi_\theta^a(0))d\theta\right)du\\ &= \gamma(a)\int_0^\infty \phi(\varphi_u^a(0))H_a(u)du\\ &= \nu_a^\infty(\phi).\end{aligned}$$

This proves that $\forall t\geq 0,\ \mathcal{L}(Y_t^{\nu_a^\infty,a}) = \nu_a^\infty$ and consequently ν_a^∞ is an invariant probability measure of (29). Moreover, we have

$$\nu_a^\infty(f) = \gamma(a)\int_0^{\sigma_a}\frac{f(x)}{b(x)+a}\exp\left(-\int_0^x \frac{f(y)}{b(y)+a}dy\right)dx = \gamma(a).$$

It remains to prove that the invariant probability measure is unique. Following [10] and [9], we define $\mathcal{B}^{ac}(\mathbb{R}_+)$ the set of bounded function $h:\mathbb{R}_+\to\mathbb{R}$ such that for all $x\in\mathbb{R}_+$, the function $t\mapsto h(\varphi_t^a(x))$ is absolutely continuous on $\mathbb{R}_+$. For $h\in\mathcal{B}^{ac}(\mathbb{R}_+)$, we define $\mathcal{H}h(x) := \frac{d}{dt}h(\varphi_t^a(x))\big|_{t=0}$.

Claim 3 Let $h\in\mathcal{B}^{ac}(\mathbb{R}_+)$, then for all $x\geq 0$ we have

$$\frac{d}{dt}\mathbb{E}\,h(Y_t^{\delta_x,a})\Big|_{t=0} = \mathcal{L}h(x) \quad\text{with}\quad \mathcal{L}h(x) := \mathcal{H}h(x) + (h(0)-h(x))f(x).$$

Let $\tau_1^x = \inf\{t\geq 0:\ Y_t^{\delta_x,a}\neq Y_{t-}^{\delta_x,a}\}$ and $\tau_2^x = \inf\{t\geq \tau_1^x:\ Y_t^{\delta_x,a}\neq Y_{t-}^{\delta_x,a}\}$ be the times of the first and second jumps of $(Y_t^{\delta_x,a})$. We have

$$\mathbb{E}\,h(Y_t^{\delta_x,a}) = \mathbb{E}\,h(Y_t^{\delta_x,a})\mathbb{1}_{\{t<\tau_1^x\}} + \mathbb{E}\,h(Y_t^{\delta_x,a})\mathbb{1}_{\{\tau_1^x\leq t<\tau_2^x\}} + \mathbb{E}\,h(Y_t^{\delta_x,a})\mathbb{1}_{\{t\geq\tau_2^x\}} =: \alpha_t+\beta_t+\theta_t.$$

By Lemma 16, we have $\alpha_t = h(\varphi_t^a(x))\mathbb{P}(t<\tau_1^x) = h(\varphi_t^a(x))H_a^x(t)$. It follows that $\frac{d}{dt}\alpha_t\big|_{t=0} = \mathcal{H}h(x) - h(x)f(x)$. Moreover using that the density of τ_1^x is $s\mapsto K_a^x(s)$ it holds that $\beta_t =$

$\int_0^t h(\varphi^a_{t-s}(0))K^x_a(s)H^0_a(t-s)ds$. We deduce that $\frac{d}{dt}\beta_t\big|_{t=0} = h(0)f(x)$. Then, using that h is bounded, we have $\theta_t \leq ||h||_\infty \int_0^t \int_0^t K^x_a(u)K^0_a(s-u)duds \in \mathcal{O}(t^2)$. This proves Claim 3.

Let g be a bounded measurable function. We follow the method of [9] (proof of Theorem 3(a)) and define

$$\forall x \geq 0, \quad \lambda_g(x) := \int_0^\infty g(\varphi^a_t(x)) \exp\left(-\int_0^t f(\varphi^a_r(x))dr\right)dt.$$

Claim 4 The function λ_g belongs to $\mathcal{B}^{ac}(\mathbb{R}_+)$ and satisfies $\mathcal{H}\lambda_g(x) = f(x)\lambda_g(x) - g(x)$. Using the semi-group property of $\varphi^a_t(x)$ we have

$$\lambda_g(\varphi^a_t(x)) = \exp\left(\int_0^t f(\varphi^a_u(x))du\right)\left[\lambda_g(x) - \int_0^t g(\varphi^a_u(x)) \exp\left(-\int_0^u f(\varphi^a_\theta(x))d\theta\right)du\right].$$

This proves that λ_g is in $\mathcal{B}^{ac}(\mathbb{R}_+)$ with $\frac{d}{dt}\lambda_g(\varphi^a_t(x)) = f(\varphi^a_t(x))\lambda_g(\varphi^a_t(x)) - g(\varphi^a_t(x))$ and gives the stated formula.

Consider now ν an invariant probability measure with $\nu(f) < \infty$. The Markov property at time $t = 0$ together with Claim 3 shows that $\frac{d}{dt}\mathbb{E}\,\lambda_g(Y^{\nu,a}_t)\big|_{t=0} = \frac{d}{dt}\int_0^\infty \mathbb{E}\,\lambda_g(Y^{\delta_x,a}_t)\nu(dx)\Big|_{t=0} = \nu(\mathcal{L}\lambda_g)$. The exchange of the derivative at time $t = 0$ and the integral on $\mathbb{R}_+$ is legitimate thanks to the Dominated Convergence Theorem. Claim 4 and the fact that ν is an invariant measure then show that

$$0 = \frac{d}{dt}\mathbb{E}\,\lambda_g(Y^{\nu,a}_t)\Big|_{t=0} = \lambda_g(0)\nu(f) - \nu(g).$$

The same computations can be done with $g \equiv 1$, giving $\lambda_1(0)\nu(f) = 1$. It follows that

$$\nu(g) = \frac{\lambda_g(0)}{\lambda_1(0)} = \int_0^\infty g(x)\nu^\infty_a(dx).$$

We deduce that necessarily $\nu = \nu^\infty_a$. □

The next lemma characterizes the invariant probability measures of (2).

Lemma 27. *The invariant probability measures of the non-linear equation* (2) *are* $\{\nu^\infty_a \mid a = J\gamma(a), a \in \mathbb{R}_+\}$.

Proof. Let ν be an invariant probability measure of (2) and $\mathcal{L}(X_0) = \nu$. We have

$$\forall t \geq 0, \;\; \mathbb{E}\,f(X_t) = \nu(f) =: p.$$

Let $a := Jp$. The process $(X_t)_{t\geq 0}$ solves (29) and ν is an invariant probability measure of equation (29). It implies that $\nu = \nu^\infty_a$. Moreover $p = \gamma(a)$ and so necessarily $\frac{a}{\gamma(a)} = J$.

Conversely, let $a \geq 0$ such that $\frac{a}{\gamma(a)} = J$. Let $(Y^{\nu^\infty_a,a}_t)$ be the solution of (29) with $\mathcal{L}(Y^{\nu^\infty_a,a}_0) = \nu^\infty_a$. We have seen that $\mathbb{E}\,f(Y^{\nu^\infty_a,a}_t) = \gamma(a)$, it follows that $a = J\,\mathbb{E}\,f(Y^{\nu^\infty_a,a}_t)$. Consequently $(Y^{\nu^\infty_a,a}_t)_{t\geq 0}$ solves (2) and ν^∞_a is one of its invariant probability measure. □

The problem of finding the invariant probability measures of the mean-field equation (2) has been reduced to finding the solutions of the scalar equation (16). When J is small enough, we can prove that it has a unique solution, which concludes the proof of Proposition 8.

Lemma 28. *Equation* (16) *has at least one solution* $a^* > 0$. *Moreover, there is a constant* $J_0 > 0$ *such that for all* $J \in [0, J_0]$ (16) *has a unique solution.*

Proof. Recall (15). By the changes of variable $y = \varphi^a_u(0)$ and $x = \varphi^a_t(0)$, it holds that

$$\gamma(a)^{-1} = \int_0^\infty \exp\left(-\int_0^t f(\varphi^a_u(0))du\right)dt. \tag{31}$$

In particular, the function $a \mapsto \gamma(a)$ is non-decreasing. Furthermore, using that $b(x) \leq C_b$, we have

$$\frac{a}{\gamma(a)} \geq a \int_0^\infty \exp\left(-\int_0^t f((a+C_b)u)du\right)dt$$
$$\geq \frac{a}{a+C_b}\int_0^\infty \exp\left(-\frac{1}{a+C_b}\int_0^\theta f(z)dz\right)d\theta.$$

We deduce that $\lim_{a\to+\infty} a\gamma(a)^{-1} = +\infty$. Let $U(a) := a\gamma(a)^{-1}$. One has $U(0) = 0$, $\lim_{a\to+\infty} U(a) = +\infty$ and U is continuous on $\mathbb{R}_+$. It follows that the equation $U(a) = J$ has at least one solution a^*. Moreover, one can show that the function U has a derivative at $a = 0$ and $U'(0) = 1/\gamma(0) > 0$. Consequently, there is $a_0 > 0$ such that U is strictly increasing on $[0, a_0]$. Using $\lim_{a\to+\infty} U(a) = +\infty$, we can find a_1 such that: $\forall a \geq a_1, U(a) \geq 1$. Finally let $J_0 := \min_{a\in[a_0,a_1]} U(a) > 0$. Let $J < J_0$, it is clear that the equation $U(a) = J$ has exactly one solution $a^* \in [0, a_0]$. □

6 The convergence of the jump rate implies the convergence in law of the time marginals

The goal of this section is to prove that controlling the behavior of the jump rate $t \mapsto \mathbb{E} f(X_t)$ can be sufficient to deduce the asymptotic law of (X_t), solution of (2).

Proposition 29. *Grant Assumptions 2, 3, 4. Let $(X_t)_{t\geq 0}$ be the solution of the non-linear equation* (2). *Assume that there exist constants $\lambda, C > 0$ and $a^* \geq 0$ (that may depend on b, f, ν, and J) such that:*

$$\forall t \geq 0,\ |\mathbb{E} f(X_t) - \gamma(a^*)| \leq Ce^{-\lambda t},$$

and that a^* satisfies equation (16)*: $\frac{a^*}{\gamma(a^*)} = J$. Then*

$$X_t \xrightarrow[t\to\infty]{\mathcal{L}} \nu^\infty_{a^*}.$$

Moreover, if $\phi : \mathbb{R}_+ \to \mathbb{R}$ is any bounded Lipschitz-continuous function, it holds that

$$\forall 0 < \lambda' < \min(\lambda, f(\sigma_0)),\ \exists D > 0 \quad s.t. \quad \forall t \geq 0, \quad |\mathbb{E}\,\phi(X_t) - \nu^\infty_{a^*}(\phi)| \leq De^{-\lambda' t},$$

where the constant D only depends on $b, f, J, C, \nu, \lambda'$ and ϕ through its infinite norm and its Lipschitz constant.

Proof. Let $(X_t)_{t\geq 0}$ be the solution of (2) and $\phi : \mathbb{R}_+ \to \mathbb{R}$ a bounded Lipschitz-continuous function, with Lipschitz constant l_ϕ. Consider $\lambda' \in (0, \min(\lambda, f(\sigma_0)))$. We denote by D any constant only depending on $b, f, J, C, \nu, \lambda'$, $||\phi||_\infty$ and l_ϕ which shall change from line to line. Define for all $t \geq 0$, $a_t := J\,\mathbb{E} f(X_t)$. It holds that $(X_t)_{t\geq 0}$ is a solution of (5) with driving current $(a_.)$. Denote $r^\nu_{(a.)}(t, 0) = \mathbb{E} f(X_t)$. By Proposition 19, we have

$$\mathbb{E}\,\phi(X_t) = \int_0^t \phi(\varphi^{(a.)}_{t,u}(0))H_{(a.)}(t,u)r^\nu_{(a.)}(u,0)du + \int_0^\infty \phi(\varphi^{(a.)}_{t,0}(x))H^x_{(a.)}(t,0)\nu(dx)$$

Using Remarks 13 and 12 (2), together with the fact that $\lambda' < f(\sigma_0)$, we deduce that

$$\forall t \geq 0, \quad \int_0^\infty \phi(\varphi^{(a.)}_{t,0}(x))H^x_{(a.)}(t,0)\nu(dx) \leq De^{-\lambda' t}$$

for some constant D. Moreover, one has, using the change of variable $x = \varphi^{a^*}_v(0)$

$$\nu^\infty_{a^*}(\phi) = \int_0^{\sigma_{a^*}} \phi(x)\nu^\infty_{a^*}(dx) = \int_0^\infty \phi(\varphi^{a^*}_v(0))\gamma(a^*)H_{a^*}(v)dv$$
$$= \int_0^t \phi(\varphi^{a^*}_{t,u}(0))H_{a^*}(t,u)\gamma(a^*)du + \int_t^\infty \phi(\varphi^{a^*}_v(0))\gamma(a^*)H_{a^*}(v)dv.$$

The last equality is obtained with the change of variable $v = t-u$. The second term is controlled by

$$\int_t^\infty \phi(\varphi_u^{a^*}(0))\gamma(a^*)H_{a^*}(u)du \leq ||\phi||_\infty \gamma(a^*)\int_t^\infty \frac{f(\varphi_u^{a^*}(0))}{\inf_{v\geq t} f(\varphi_v^{a^*}(0))}\exp\left(-\int_0^u f(\varphi_\theta^{a^*}(0))d\theta\right)du$$
$$= \frac{||\phi||_\infty \gamma(a^*)}{\inf_{v\geq t} f(\varphi_v^{a^*}(0))}\exp\left(-\int_0^t f(\varphi_\theta^{a^*}(0))d\theta\right)$$
$$\leq De^{-\lambda' t},$$

for some constant D. We used again Remark 12. It remains to show that

$$\Delta := \left|\int_0^t \phi(\varphi_{t,u}^{(a.)}(0))H_{(a.)}(t,u)r_{(a.)}^\nu(u,0)du - \int_0^t \phi(\varphi_{t,u}^{a^*}(0))H_{a^*}(t,u)\gamma(a^*)du\right|$$

goes to zero exponentially fast. One has

$$\Delta \leq \int_0^t \left|\phi(\varphi_{t,u}^{(a.)}(0)) - \phi(\varphi_{t,u}^{a^*}(0))\right| H_{(a.)}(t,u)r_{(a.)}^\nu(u,0)du + \int_0^t \left|H_{(a.)}(t,u) - H_{a^*}(t,u)\right| \phi(\varphi_{t,u}^{a^*}(0))r_{(a.)}^\nu(u,0)du$$
$$+ \int_0^t H_{a^*}(t,u)\phi(\varphi_{t,u}^{a^*}(0))\left|r_{(a.)}^\nu(u,0) - \gamma(a^*)\right| du$$
$$=: \alpha_t + \beta_t + \theta_t.$$

Using that for all $t \geq 0$, $|r_{(a.)}^\nu(t,0) - \gamma(a^*)| \leq Ce^{-\lambda' t}$ $(\lambda' < \lambda)$ and Remark 13, we obtain:

$$\theta_t \leq C||\phi||_\infty \int_0^t H_0(t,u)e^{-\lambda' u}du$$
$$= C||\phi||_\infty e^{-\lambda' t}\int_0^t H_0(t-u)e^{\lambda'(t-u)}du$$
$$\leq \left[C||\phi||_\infty \int_0^\infty H_0(u)e^{\lambda' u}du\right]e^{-\lambda' t} =: De^{-\lambda' t}.$$

The fact that $u \mapsto H_0(u)e^{\lambda' u}$ belongs to $L^1(\mathbb{R}_+)$ follows from $\lambda' < f(\sigma_0)$. By Theorem 5, one can find a constant $\bar{p}$ (with $\gamma(a^*) \leq \bar{p}$) such that:

$$\forall t \geq 0, \ \ \mathbb{E}\, f(X_t) = r_{(a.)}^\nu(t,0) \leq \bar{p}.$$

Moreover, Assumption 2 and Remark 13 give

$$\alpha_t \leq \bar{p}l_\phi \int_0^t |\varphi_{t,u}^{(a.)}(0) - \varphi_{t,u}^{a^*}(0)|H_0(t,u)du$$
$$\leq \bar{p}l_\phi C_\varphi \int_0^t\int_u^t |a_\theta - a^*|d\theta H_0(t,u)du.$$

Using that $\int_u^t |a_\theta - a^*|d\theta \leq JC\int_u^t e^{-\lambda'\theta}d\theta \leq \frac{JCe^{-\lambda' u}}{\lambda'}$, one has

$$\alpha_t \leq \frac{\bar{p}l_\phi C_\varphi JC}{\lambda'}e^{-\lambda' t}\int_0^t e^{\lambda'(t-u)}H_0(t-u)du$$
$$\leq \left[\frac{\bar{p}l_\phi C_\varphi JC}{\lambda'}\int_0^\infty H_0(u)e^{\lambda' u}du\right]e^{-\lambda' t} =: De^{-\lambda' t}.$$

Finally, using the inequality $|e^{-A} - e^{-B}| \leq e^{-\min(A,B)}|A-B|$ together with Remark 13, we obtain

$$\beta_t \leq ||\phi||_\infty \bar{p}\int_0^t H_0(t-u)\int_u^t \left|f(\varphi_{\theta,u}^{(a.)}(0)) - f(\varphi_{\theta,u}^{a^*}(0))\right|d\theta du.$$

Setting $\bar{a} := J\bar{p}$, we have moreover, using that f' is non-decreasing and Lemma 2

$$\int_u^t \left| f(\varphi_{\theta,u}^{(a.)}(0)) - f(\varphi_{\theta,u}^{a^*}(0)) \right| d\theta \leq f'(\varphi_{t,u}^{\bar{a}}) \int_u^t \left| \varphi_{\theta,u}^{(a.)}(0) - \varphi_{\theta,u}^{a^*}(0) \right| d\theta.$$

Assumption 2 yields

$$\begin{aligned}
\int_u^t \left| f(\varphi_{\theta,u}^{(a.)}(0)) - f(\varphi_{\theta,u}^{a^*}(0)) \right| d\theta &\leq C_\varphi f'(\varphi_{t,u}^{\bar{a}}(0)) \int_u^t \int_u^\theta |a_s - a^*| ds d\theta \\
&\leq C_\varphi J C f'(\varphi_{t,u}^{\bar{a}}(0)) \int_u^t \int_u^\theta e^{-\lambda' s} ds d\theta \\
&\leq C_\varphi \frac{JC}{\lambda'} f'(\varphi_{t,u}^{\bar{a}}(0))(t-u) e^{\lambda'(t-u)} e^{-\lambda' t}.
\end{aligned}$$

We used the fact that

$$\int_u^t \int_u^\theta e^{-\lambda' s} ds d\theta = \int_u^t \frac{e^{-\lambda' u} - e^{-\lambda' \theta}}{\lambda'} d\theta \leq \frac{(t-u)e^{-\lambda' u}}{\lambda'}.$$

Note that Lemma 3 implies that $f'(\varphi_{t,u}^{\bar{a}}(0)) \leq f'(C_b^{\bar{a}}(t-u))$ and using Remark 12(1) we have

$$\forall \epsilon > 0, \exists A_\epsilon : \ \forall x \geq 0, f'(x) \leq A_\epsilon e^{\epsilon x}.$$

Choosing $\epsilon := (f(\sigma_0) - \lambda')/2$, we obtain

$$\int_u^t \left| f(\varphi_{\theta,u}^{(a.)}(0)) - f(\varphi_{\theta,u}^{a^*}(0)) \right| d\theta \leq A_\epsilon C_\varphi \frac{JC}{\lambda'} (t-u) e^{(\lambda'+\epsilon)(t-u)} e^{-\lambda' t},$$

and we deduce that

$$\beta_t \leq \left[\frac{A_\epsilon J C_\varphi C ||\phi||_\infty \bar{a}}{\lambda'} \int_0^{+\infty} H_0(u) u e^{(\lambda'+\epsilon)u} du \right] e^{-\lambda' t} =: D e^{-\lambda' t}.$$

Combining the three estimates, we have proved the result. □

7 Long time behavior with constant drift

The goal of this section is to study the rate of convergence to the invariant probability measure when $J = 0$ (no interaction). We use Laplace transform techniques to characterize the convergence. We state here the main result of the section.

Proposition 30. *Grant Assumptions 2, 3 and 4. Let $(Y_t^{\nu,a})_{t\geq 0}$ be the solution of (5), driven by a constant current $(a_t) \equiv a$, $a \geq 0$; starting at time $s = 0$ with law ν. One can find a constant $\lambda_a^* \in (0, f(\sigma_a)]$ (only depending on b, f and a) such that for any $0 < \lambda < \lambda_a^*$ it holds*

$$\forall t \geq 0, \quad |\mathbb{E} f(Y_t^{\nu,a}) - \gamma(a)| \leq D e^{-\lambda t} \int_0^\infty [1 + f(x)] |\nu - \nu_a^\infty|(dx), \tag{32}$$

where D is a constant only depending on f, b, a and λ. Moreover, one has

$$Y_t^{\nu,a} \xrightarrow[t\to\infty]{\mathcal{L}} \nu_a^\infty.$$

Remark 31. *In the above result, λ_a^* is explicitly known in terms of f, b and a (see its expression* (35)*) and is optimal (see Remark 40). Note also that* (32) *states explicitly the dependence on the initial distribution ν through its distance to the invariant measure ν_a^∞.*

7.1 Study of the Volterra equation

In the case where (a_t) is constant and equal to a, the Volterra equation (21) is a linear homogeneous convolution Volterra equation. If moreover the initial condition ν is δ_0, the kernel $r_a(t) := \mathbb{E}\, f(Y_t^{\delta_0, a})$ satisfies

$$r_a = K_a + K_a * r_a, \tag{33}$$

For such equations, it is very natural to use Laplace transform techniques as convolutions become scalar products with this transformation. Furthermore, the "kernel" K_a and the "forcing term" K_a^ν are non-negative. Volterra equation with positive kernels have been studied in the context of Renewal theory. The main reference on this question is a paper of Feller [17]. We refer to [17, Th. 4] for this method. However, in our case the rate of convergence is exponential. In order to achieve the optimal rate of convergence, we use general methods from the Volterra integral equation theory, and especially the so called "Whole-line Palay-Wiener" Theorem.

Along this section, we grant Assumptions 2, 3 and 4.

Definition 32 (Laplace transform)**.** *Let $g : \mathbb{R}_+ \mapsto \mathbb{R}$ be a measurable function. The Laplace transform of g is the following function*

$$\widehat{g}(z) := \int_0^\infty e^{-zt} g(t) dt,$$

defined for all $z \in \mathbb{C}$ for which the integral exists.

Note that the Laplace transforms of H_a and K_a are well defined for all $z \in \mathbb{C}$ with $\Re(z) > -f(\sigma_a)$. This follows from the fact that $\forall \lambda < f(\sigma_a),\ \sup_{t\geq 0} H_a(t) e^{\lambda t} < \infty$. The same holds for K_a. Integrating by parts the Laplace transform of K_a shows that

$$\forall z \in \mathbb{C},\ \Re(z) > -f(\sigma_a) \implies \widehat{K_a}(z) = 1 - z\widehat{H_a}(z). \tag{34}$$

It is also useful to introduce the following Banach space

Definition 33. *For any $\lambda \in \mathbb{R}$, let $L_\lambda = \{f \in \mathcal{B}(\mathbb{R}_+, \mathbb{R}) : ||f||_{\lambda,1} < \infty\}$ the space of Borel-measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$, equipped with the norm*

$$||f||_{\lambda,1} = \int_{\mathbb{R}_+} |f(s)| e^{\lambda s} ds.$$

The long time behavior of r_a is related to the location of the poles of $\widehat{r}_a$. Equation (33) gives

$$\forall \Re(z) > 0 \quad \widehat{r}_a(z) = \frac{\widehat{K}_a(z)}{1 - \widehat{K}_a(z)}.$$

This suggests to study the location of the zeros of $1 - \widehat{K}_a(z) = z\widehat{H}_a(z)$.

7.2 On the zeros of $\widehat{H_a}$

Lemma 34. $\forall z \in \mathbb{C},\ \Re(z) \geq 0 \implies \widehat{H}_a(z) \neq 0.$

Proof. Remark first that H_a being a real-valued function, $\widehat{H}_a(z) = 0$ iff $\widehat{H}_a(\bar{z}) = 0$, so it is sufficient to locate the zeros of $\widehat{H}_a$ in the region $\Im(z) \geq 0$. Next, it follows from for the non-negativity of K_a that

$$|\widehat{K}_a(z)| \leq \int_0^\infty |e^{-tz}| K_a(t) dt < \int_0^\infty K_a(t) dt = 1 \text{ if } \Re(z) > 0.$$

It yields $\Re(z) > 0 \implies \widehat{H}_a(z) \neq 0$. Moreover, following [17] proof of Theorem 4, (b), if $z = iy$, $y > 0$ then

$$iy\widehat{H}_a(iy) = 1 - \widehat{K}_a(iy) = \int_0^\infty (1 - \cos(yt))K_a(t)dt + i\int_0^\infty \sin(yt)K_a(t)dt.$$

Consequently, $\widehat{K}_a(iy) = 1$ for some $y > 0$ would imply that for Lebesgue almost every $t \geq 0$, $(1-\cos(yt))K_a(t) = 0$, that is, a.e. $K_a(t) = 0$. It obviously contradicts the assumption $f(x) > 0$ for $x > 0$. It follows that $\forall y > 0,\ \widehat{H}_a(iy) \neq 0$. Finally for $z = 0$, we have $\widehat{H}_a(0) = \int_0^\infty H_a(t)dt \neq 0$. □

Lemma 35. *The zeros of $\widehat{H}_a$ are isolated.*

Proof. This directly follows from the fact that $\widehat{H}_a$ is an holomorphic function on $\Re(z) > -f(\sigma_a)$ and thus its zeros are isolated. □

Lemma 36. *For all $z \in \mathbb{C}$, it holds that*

$$|\widehat{K}_a(z)| \leq \frac{\phi_a(\Re(z))}{|\Im(z)|},$$

where for all $x \in \mathbb{R}$, $\phi_a(x) := ||K'_{a,x}||_1$ and $K_{a,x}(t) := e^{-xt}K_a(t)$, $K'_{a,x}(t) := \frac{d}{dt}K_{a,x}(t)$.

Consequently, the zeros of $\widehat{H}_a$ are within a "cone":

$$\forall z \in \mathbb{C},\ \Re(z) > -f(\sigma_a),\ z = x + iy,\ \widehat{H}_a(z) = 0 \implies |y| \leq \phi_a(x).$$

Proof. Let $z = x + iy$, $y > 0, x > -f(\sigma_a)$. We have

$$\widehat{K}_a(z) = \int_0^\infty e^{-zt}K_a(t)dt = \int_0^\infty e^{-iyt}K_{a,x}(t)dt = \int_0^\infty \frac{e^{-iyt}}{iy}K'_{a,x}(t)dt.$$

The last equality follows by an integration by part. It yields

$$|\widehat{K}_a(z)| \leq \frac{||K'_{a,x}||_1}{|y|}.$$

We deduce that for $|y| > ||K'_{a,x}||_1$, we have $\widehat{K}_a(z) \neq 1$ and also $\widehat{H}_a(z) \neq 0$. □

Consequently, from Lemmas 34, 35 and 36, we can define the abscissa of the "first" zero of $\widehat{H}_a$:

$$\lambda_a^* := -\sup\{\Re(z)|\ \Re(z) > -f(\sigma_a),\ \widehat{H}_a(z) = 0\}, \tag{35}$$

with the convention that $\lambda_a^* = f(\sigma_a)$ if the set of zeros is empty. We have proved that

$$0 < \lambda_a^* \leq f(\sigma_a) \leq \infty.$$

The parameter λ_a^* is key here as it gives the speed of convergence to the invariant probability measure. It only depends on a, b and f.

7.3 Convergence with optimal rate

Our goal in this section is to prove the following proposition

Proposition 37. *Eq.* (33) *has a unique solution r_a of the form:*

$$r_a = \gamma(a) + \xi_a \quad \textit{with} \quad \forall\lambda \in [0, \lambda_a^*), \quad \xi_a \in L_\lambda.$$

The constant $\lambda_a^* > 0$ is defined by (35)*.*

This result can be deduced from general theorems of the Volterra equations theory. For instance, one can apply [20, Th. 2.4, Chap. 7]. However, this last result is written for general measure kernels in weighted spaces and its proof is somehow difficult to follow. In our setting, the proof given by [20] simplifies a lot and we give it here for completeness. We use the following so-called "Whole Line Palay-Wiener" Theorem which is one of the most important ingredients of the convolution Volterra integral equations theory.

Theorem 38 (Whole-line Palay-Wiener)**.** *Let $k \in L^1(\mathbb{R},\mathbb{R})$. There exists a function $x \in L^1(\mathbb{R},\mathbb{R})$ satisfying the equation*

$$\forall t \geq 0,\ x(t) = k(t) + \int_{\mathbb{R}} k(t-u)x(u)du$$

if and only if

$$\forall y \in \mathbb{R},\ \widehat{k}(iy) := \int_{\mathbb{R}} e^{-iyt}k(t)dt \neq 1.$$

Note that here $\widehat{k}(iy)$ is actually the Fourier transform of k evaluated at $y \in \mathbb{R}$.

Proof. See [20, Th. 4.3, Chap. 2]. We prove later, in details, an extension of this theorem (see Proposition 51). ☐

Proof of Proposition 37. Let σ_- and σ_+ be any real numbers such that:

$$-\lambda_a^* < \sigma_- < 0 < \sigma_+ < \infty.$$

We first extend r_a, K_a and H_a to the whole line by defining: $\forall t \in \mathbb{R},\ r_a(t) := r_a(t)\mathbb{1}_{\{t\geq 0\}},\ K_a(t) := K_a(t)\mathbb{1}_{\{t\geq 0\}}$ and $H_a(t) := H_a(t)\mathbb{1}_{\{t\geq 0\}}$. We have from (33)

$$\forall t \in \mathbb{R},\ r_a(t) = K_a(t) + \int_{\mathbb{R}} K_a(t-u)r_a(u)du. \tag{36}$$

For any $\triangle \in \mathbb{R}$, we also define $r_{a,\triangle}(t) := e^{-\triangle t}r_a(t),\quad K_{a,\triangle}(t) := e^{-\triangle t}K_a(t)$. Note that $K_{a,\sigma_-} \in L^1(\mathbb{R})$ and that $\forall y \in \mathbb{R},\ \widehat{K}_{a,\sigma_-}(iy) = \widehat{K}_a(\sigma_- + iy) \neq 1$ (by definition of λ_a^*). We can apply Theorem 38: there exists $\xi_{a,\sigma_-} \in L^1(\mathbb{R})$ such that

$$\forall t \in \mathbb{R},\ \xi_{a,\sigma_-}(t) = K_{a,\sigma_-}(t) + \int_{\mathbb{R}} K_{a,\sigma_-}(t-u)\xi_{a,\sigma_-}(u)du. \tag{37}$$

We define $\xi_a(t) := e^{\sigma_- t}\xi_{a,\sigma_-}(t)$. We have $\int_{\mathbb{R}} |\xi_a(u)|e^{-\sigma_- u}du < \infty$ and (37) reads

$$\forall t \in \mathbb{R},\ \xi_a(t) = K_a(t) + \int_{\mathbb{R}} K_a(t-u)\xi_a(u)du.$$

Remark 39. *The function ξ_a is not null on $\mathbb{R}_-$ (see formula* (39) *just below).*

We have, using equalities (36) and (37)

$$\xi_{a,\sigma_-} \in L^1(\mathbb{R}),\quad \widehat{\xi_{a,\sigma_-}}(iy) = \left[\frac{\widehat{K_a}}{1-\widehat{K_a}}\right](iy+\sigma_-),$$

$$r_{a,\sigma_+} \in L^1(\mathbb{R}),\quad \widehat{r_{a,\sigma_+}}(iy) = \left[\frac{\widehat{K_a}}{1-\widehat{K_a}}\right](iy+\sigma_+).$$

We can now use the Fourier inverse formula for $L^1(\mathbb{R})$ functions to get

$$\xi_{a,\sigma_-}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iyt}\left[\frac{\widehat{K_a}}{1-\widehat{K_a}}\right](iy+\sigma_-)dy \quad \text{and} \quad r_{a,\sigma_+}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iyt}\left[\frac{\widehat{K_a}}{1-\widehat{K_a}}\right](iy+\sigma_+)dy,$$

or after the changes of variable $z = iy + \sigma_-$ and $z = iy + \sigma_+$:

$$\xi_a(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_- - iT}^{\sigma_- + iT} e^{zt} \frac{\widehat{K}_a(z)}{1 - \widehat{K}_a(z)} dz \quad \text{and} \quad r_a(t) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_+ - iT}^{\sigma_+ + iT} e^{zt} \frac{\widehat{K}_a(z)}{1 - \widehat{K}_a(z)} dz.$$

Let Γ_T be the closed curve in the complex plane composed of four straight lines that join the points $\sigma_- - iT$, $\sigma_- + iT$, $\sigma_+ + iT$, and $\sigma_+ - iT$ in the anti-clockwise direction. It follows from the residue theorem that

$$\int_{\Gamma_T} e^{zt} \frac{\widehat{K}_a(z)}{1 - \widehat{K}_a(z)} dz = \int_{\Gamma_T} e^{zt} \frac{\widehat{K}_a(z)}{z\widehat{H}_a(z)} dz = 2\pi i \frac{\widehat{K}_a(0)}{\widehat{H}_a(0)} = 2\pi i \gamma(a). \tag{38}$$

The last equality follows from

$$\widehat{H}_a(0) = \int_0^\infty H_a(t)dt = \int_0^\infty \exp\left(-\int_0^t f(\varphi_u^a)du\right)dt \overset{(31)}{=} \frac{1}{\gamma(a)}.$$

By Lemma 36, for all z in the strip $\Re(z) \in [\sigma_-, \sigma_+]$, $z \neq 0$, we have

$$|\widehat{K}_a(z)| \leq \frac{\phi_a(\sigma_-)}{|\Im(z)|}.$$

We deduce that

$$\lim_{T\to\pm\infty} \int_{\sigma_- + iT}^{\sigma_+ + iT} e^{zt} \frac{\widehat{K}_a(z)}{1 - \widehat{K}_a(z)} dz = 0.$$

Therefore we can take the limit $T \to \infty$ in (38) and obtain

$$\forall t \in \mathbb{R},\ r_a(t) = \gamma(a) + \xi_a(t). \tag{39}$$

The proposition is proven by choosing $\sigma_- = -\lambda$. □

Remark 40. *The speed of convergence obtained in this result is optimal if $\lambda_a^* < f(\sigma_a)$ (i.e. $\widehat{H}_a$ has at least one complex zero with $\Re(z) > -f(\sigma_a)$) in the sense that*

$$\forall \lambda > \lambda_a^*,\ r_a - \gamma(a) \notin L_\lambda.$$

To see this, assume that $\lambda_a^* < f(\sigma_a)$ and choose σ_- such that $-f(\sigma_a) < \sigma_- < -\lambda_a^*$. The previous proof can be mimicked except that the residues of equation (38) now involves terms of the order $e^{-\lambda_a^* t}$ - corresponding to the roots of $\widehat{H}_a$ with real part equal to $-\lambda_a^*$.

7.4 Long time behavior starting from initial condition ν: proof of Proposition 30

We now come back to the general case where the initial condition can be any probability measure satisfying Assumption 4, and we give the proof of Proposition 30.

Proof of Proposition 30. Note that, we only consider here the convolutions on $[0, t]$ denoted by $*$ (and no more the convolution on $\mathbb{R}$). Let $r_a^\nu(t) = \mathbb{E} f(Y_t^{\nu,a})$ with $\mathcal{L}(Y_0) = \nu$. The function r_a^ν is the unique solution of the Volterra equation

$$r_a^\nu = K_a^\nu + K_a * r_a^\nu.$$

If we choose ν to be the invariant probability measure ν_a^∞, we get $\gamma(a) = K_a^{\nu_a^\infty} + K_a * \gamma(a)$ and

$$r_a^\nu - \gamma(a) = K_a^\nu - K_a^{\nu_a^\infty} + K_a * (r_a^\nu - \gamma(a)).$$

We can solve this equation in terms of r_a, the "resolvent" of K_a (using Lemma 21) and obtain

$$\begin{aligned} r_a^\nu - \gamma(a) &= K_a^\nu - K_a^{\nu_a^\infty} + r_a * (K_a^\nu - K_a^{\nu_a^\infty}) \\ &= K_a^\nu - K_a^{\nu_a^\infty} + \xi_a * (K_a^\nu - K_a^{\nu_a^\infty}) + \gamma(a) * (K_a^\nu - K_a^{\nu_a^\infty}), \end{aligned}$$

where $r_a = \xi_a + \gamma(a)$, see (39), is the solution of the Volterra equation $r_a = K_a + K_a * r_a$. Using (23), we have $\gamma(a) * K_a^\nu = \gamma(a)(1 - H_a^\nu)$ and thus

$$r_a^\nu - \gamma(a) = K_a^\nu - K_a^{\nu_a^\infty} + \gamma(a)(H_a^{\nu_a^\infty} - H_a^\nu) + \xi_a * (K_a^\nu - K_a^{\nu_a^\infty}).$$

We now write Θ any constant only depending on λ, f, b and a and which may change from line to line. It is clear that for any $0 < \lambda < f(\sigma_a)$

$$|H_a^{\nu_a^\infty} - H_a^\nu|(t) \leq \int_0^\infty H_a^x(t)|\nu - \nu_a^\infty|(dx) \leq \int_0^\infty H_a(t)|\nu - \nu_a^\infty|(dx) \leq \Theta e^{-\lambda t} \int_0^\infty |\nu - \nu_a^\infty|(dx).$$

Similarly, for any $0 < \lambda < f(\sigma_a)$,

$$\begin{aligned} |K_a^\nu - K_a^{\nu_a^\infty}|(t) &\leq \int_0^\infty f(\varphi_t^a(x))H_a^x(t)|\nu - \nu_a^\infty|(dx) \leq \int_0^\infty f(x + C_b^a t)H_a(t)|\nu - \nu_a^\infty|(dx) \\ &\leq C_f \int_0^\infty [1 + f(x) + f(C_b^a t)]H_a(t)|\nu - \nu_a^\infty|(dx) \leq \Theta e^{-\lambda t} \int_0^\infty (1 + f(x))|\nu - \nu_a^\infty|(dx). \end{aligned}$$

We used here Assumption 3.1.. Let now $0 < \lambda < \lambda_a^*$. Using $\xi_a \in L_\lambda$, it holds that

$$|\xi_a * (K_a^\nu - K_a^{\nu_a^\infty})|(t) \leq \int_0^t |\xi_a(t-u)||K_a^\nu - K_a^{\nu_a^\infty}|(u)du \leq \Theta e^{-\lambda t} \int_0^\infty (1 + f(x))|\nu - \nu_a^\infty|(dx).$$

Combining the three estimates, one deduces that

$$|r_a^\nu(t) - \gamma(a)| \leq \Theta e^{-\lambda t} \int_0^\infty (1 + f(x))|\nu - \nu_a^\infty|(dx).$$

It remains to prove that $\lim_{t\to\infty} \mathcal{L}(Y_t^{\nu,a}) = \nu_a^\infty$. The process $(Y_t^{\nu,a})_{t\geq 0}$ is the solution of (2) with $\tilde{b}(x) = b(x) + a$ and $J = 0$. Obviously, 0 solves (16). Applying Proposition 29 ends the proof. □

8 Long time behavior with a general drift

In this section, we generalize the results obtained in Section 7 to non constant currents. We consider the process (5) driven by a current (a_t) assuming to converge exponentially fast to a. We seek to prove that the jump rate of this process is converging to $\gamma(a)$ and estimate the speed of convergence. This "perturbation" analysis will be useful to study the long time behavior of the solution of the non-linear McKean-Vlasov equation (2) with small interactions. We consider a non-negative continuous function $(a_t)_{t\geq 0}$ such that

Assumptions 41. *1. $\sup_{t\geq 0} a_t \leq \bar{a}$ for some constant $\bar{a} > 0$.*

2. There exist $a \geq 0$, $C \geq 0$ and $\lambda \in (0, \min(\lambda_a^*, f(\sigma_0)))$, where σ_0 and λ_a^* are defined by (17) *and* (35)*, such that*

$$\forall t \geq 0, \quad |a_t - a| \leq C e^{-\lambda t}. \tag{40}$$

Note that the values of C and λ are important in this analysis. Any mention of C and λ in this section refer to these two constants.

Let $r_{(a.)}^\nu(t,s) = \mathbb{E}\, f(Y_t^{s,\nu,(a.)})$, where $Y_t^{s,\nu,(a.)}$ is the solution of (5) driven by the current (a_t) and starting at time s with law ν. The goal of this section is to prove that if C is small enough, then there exists an explicit constant D such that

$$\forall t \geq s \geq 0,\ |r_{(a.)}^\nu(t,s) - \gamma(a)| \leq D e^{-\lambda(t-s)}\ ,$$

where $\gamma(a)$ is given by (15). Note that the exponential decay rate λ is preserved. We make efforts to keep track of the constant D and to relate it to C. As in Section 7 it is useful to split the study in two parts: the case where the initial condition is a Dirac mass at 0 and the general case. We thus consider the unique solution $r_{(a.)}$ of the following Volterra equation:

$$r_{(a.)} = K_{(a.)} + K_{(a.)} * r_{(a.)}. \tag{41}$$

It is also useful to introduce a Banach space adapted to this non-homogeneous setting.

8.1 An adapted Banach algebra

Definition 42. *A function $K : (\mathbb{R}_+)^2 \to \mathbb{R}$ is said to be a Volterra Kernel with weight $\lambda \in \mathbb{R}$ if: K is Borel measurable, $\forall s > t : K(t,s) = 0$ a.e. and $||K||_{\lambda,1} < \infty$ with*

$$||K||_{\lambda,1} := \operatorname*{ess\,sup}_{t \geq 0} \int_{\mathbb{R}_+} |K(t,s)| e^{\lambda(t-s)} ds.$$

We define $\mathcal{V}_\lambda$ the set of Volterra kernels with weight λ. We also define for $K \in \mathcal{V}_\lambda$:

$$||K||_{\lambda,\infty} = \operatorname*{ess\,sup}_{t,s \geq 0} |K(t,s) e^{\lambda(t-s)}| \in \mathbb{R}_+ \cup \{+\infty\}.$$

Proposition 43. *The space $(\mathcal{V}_\lambda, || \cdot ||_{\lambda,1})$ is a Banach algebra. Furthermore, for all $a, b \in \mathcal{V}_\lambda$, $||a * b||_{\lambda,1} \leq ||a||_{\lambda,1} ||b||_{\lambda,1}$.*

Proposition 43 is proved in [20], Theorem 2.4 and Proposition 2.7 (i) of Chapter 9.

Lemma 44 (Connection with the time homogeneous setting)**.** *Let $g \in L_\lambda$. We define*

$$\forall t, s \in \mathbb{R}_+, \quad \tilde{g}(t,s) := g(t-s) \mathbb{1}_{t \geq s}.$$

Then $\tilde{g} \in \mathcal{V}_\lambda$ and $||g||_{\lambda,1} = ||\tilde{g}||_{\lambda,1}$.

This result allows us to consider elements of L_λ as elements of $\mathcal{V}_\lambda$. **Note that the algebra L_λ is commutative whereas $\mathcal{V}_\lambda$ is not.**

8.2 The perturbation method

Define $\bar{K}_{(a.)} := K_{(a.)} - K_a$ and $\bar{H}_{(a.)} := H_{(a.)} - H_a$.

Lemma 45. *Grant Assumptions 2, 3, 4 and 41. Then, there exists a continuous non-negative and integrable function η such that for all $t \geq s \geq 0$, one has*

$$|\bar{K}_{(a.)}(t,s)| \leq C e^{-\lambda t} \eta(t-s),$$
$$|\bar{H}_{(a.)}(t,s)| \leq C e^{-\lambda t} \eta(t-s).$$

The function η only depends on $b, \bar{a}, f$ and λ (in particular it does not depend on C). Furthermore, we can choose η such that $||\eta||_1$ is a non-decreasing function of $\bar{a}$.

Proof. Here, to simplify the notation, we write $\varphi^{(a.)}_{t,s}$ for $\varphi^{(a.)}_{t,s}(0)$. We have

$$\bar{K}_{(a.)}(t,s) = f(\varphi^{(a.)}_{t,s}) \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}) du\right) - f(\varphi^a_{t,s}) \exp\left(-\int_s^t f(\varphi^a_{u,s}) du\right)$$

$$\begin{aligned} |\bar{K}_{(a.)}(t,s)| &\leq |f(\varphi^{(a.)}_{t,s}) - f(\varphi^a_{t,s})| \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}) du\right) \\ &\quad + f(\varphi^a_{t,s}) \left| \exp\left(-\int_s^t f(\varphi^{(a.)}_{u,s}) du\right) - \exp\left(-\int_s^t f(\varphi^a_{u,s}) du\right) \right| \\ &=: M_1 + M_2. \end{aligned}$$

Assumptions 2, 3.1. and (40) give

$$|f(\varphi_{t,s}^{(a.)}) - f(\varphi_{t,s}^{a})| \le f'(\varphi_{t,s}^{\bar{a}})|\varphi_{t,s}^{(a.)} - \varphi_{t,s}^{a}| \le f'(C_b^{\bar{a}}(t-s))C_\varphi \int_s^t |a_u - a|du$$
$$\le f'(C_b^{\bar{a}}(t-s))C_\varphi C \int_s^t e^{-\lambda u}du \le Ce^{-\lambda t} f'(C_b^{\bar{a}}(t-s))C_\varphi \frac{e^{\lambda(t-s)}}{\lambda}.$$

Moreover choosing $\lambda' \in (\lambda, f(\sigma_0))$ and using the fact that $f(\varphi_u^0) \to f(\sigma_0)$ as $u \to \infty$, one obtains

$$\exp\left(-\int_s^t f(\varphi_{u,s}^{(a.)})du\right) \le \exp\left(-\int_s^t f(\varphi_{u,s}^{0})du\right) = \exp\left(-\int_0^{t-s} f(\varphi_{u}^{0})du\right)$$
$$\le D(b,f,\lambda')e^{-\lambda'(t-s)},$$

for some finite constant $D(b,f,\lambda')$. Let $\alpha(u) := \frac{D(b,f,\lambda')}{\lambda} e^{-(\lambda'-\lambda)u} f'(C_b^{\bar{a}}u)C_\varphi$, we have

$$M_1 \le Ce^{-\lambda t}\alpha(t-s),$$

and $\alpha \in L^1(\mathbb{R}_+)$. Moreover, for $A, B \ge 0$, we have $|e^{-A} - e^{-B}| \le e^{-\min(A,B)}|A-B|$. So,

$$M_2 \le f(\varphi_{t,s}^{\bar{a}}) \exp\left(-\int_0^{t-s} f(\varphi_u^0)du\right) \left|\int_s^t f(\varphi_{u,s}^{(a.)}) - f(\varphi_{u,s}^{a})du\right|$$
$$\le f(C_b^{\bar{a}}(t-s))D(b,f,\lambda')e^{-\lambda'(t-s)} f'(C_b^{\bar{a}}(t-s)) \int_s^t |\varphi_{u,s}^{(a.)} - \varphi_{u,s}^{a}|du.$$

One has

$$\int_s^t |\varphi_{u,s}^{(a.)} - \varphi_{u,s}^{a}|du \le C_\varphi \int_s^t \int_s^u |a_\theta - a|d\theta du \le CC_\varphi \int_s^t \int_s^u e^{-\lambda\theta}d\theta du \le Ce^{-\lambda t} \cdot \frac{C_\varphi}{\lambda}(t-s)e^{\lambda(t-s)}.$$

Consequently $M_2 \le Ce^{-\lambda t}\beta(t-s)$ with

$$\beta(u) := D(b,f,\lambda')e^{-(\lambda'-\lambda)u} f(C_b^{\bar{a}}u) f'(C_b^{\bar{a}}u) \frac{C_\varphi}{\lambda} u e^{\lambda u}.$$

It holds that $\beta \in L^1(\mathbb{R}_+)$ and setting $\eta := \alpha + \beta$ completes the proof for $\bar{K}_{(a.)}$. The same computations give a similar result for $\bar{H}_{(a.)}$. □

These estimates are sharp enough to give the following result:

Lemma 46. *Grant Assumptions 2, 3, 4 and 41. Let η be the function given by Lemma 45. Denote by 1 the kernel $\mathbb{1}_{t \ge s}$. Then*

1. $\bar{K}_{(a.)} \in \mathcal{V}_\lambda$ *and* $||\bar{K}_{(a.)}||_{\lambda,1} \le C||\eta||_1$.
2. $\bar{K}_{(a.)} * 1 \in \mathcal{V}_\lambda$ *and* $||\bar{K}_{(a.)} * 1||_{\lambda,1} \le C||\eta||_1$.

The exact same estimates holds for $\bar{H}_{(a.)}$ and $\bar{H}_{(a.)} * 1$.

Proof. Using Lemma 45, we have

$$||\bar{K}_{(a.)}||_{\lambda,1} := \sup_{t \ge 0} \int_0^t |\bar{K}_{(a.)}|(t,s)e^{\lambda(t-s)}ds \le \sup_{t \ge 0} \int_0^t Ce^{-\lambda s}\eta(t-s)ds \le C||\eta||_1,$$

proving point 1. For point 2, we have $\forall t \ge s \ge 0$, $(\bar{K}_{(a.)} * 1)(t,s) := \int_s^t \bar{K}_{(a.)}(t,u)du$. And Lemma 45 gives

$$||\bar{K}_{(a.)} * 1||_{\lambda,1} = \sup_{t \ge 0} \int_0^t |\bar{K}_{(a.)} * 1|(t,s)e^{\lambda(t-s)}ds \le \sup_{t \ge 0} \int_0^t Ce^{-\lambda t}||\eta||_1 e^{\lambda(t-s)}ds = C||\eta||_1.$$

□

Proposition 47. *Grant Assumptions 2, 3, 4. Assume* $(a_t)_{t\geq 0}$ *satisfies Assumption 41 and that the constant* C *is small enough:*

$$\alpha := C||\eta||_1(1 + ||\xi_a||_{\lambda,1} + \gamma(a)) < 1. \tag{42}$$

Define $\Delta_K := \bar{K}_{(a.)} + \xi_a * \bar{K}_{(a.)} - \gamma(a)\bar{H}_{(a.)}$ *and let* Δ_r *be the solution of the Volterra equation*

$$\Delta_r = \Delta_K + \Delta_K * \Delta_r. \tag{43}$$

Then

1. $\Delta_K \in \mathcal{V}_\lambda$ *with* $||\Delta_K||_{\lambda,1} \leq \alpha$ *and* $\Delta_K * 1 \in \mathcal{V}_\lambda$ *with* $||\Delta_K * 1||_{\lambda,1} \leq \alpha$.
2. $\Delta_r \in \mathcal{V}_\lambda$ *with* $||\Delta_r||_{\lambda,1} \leq \frac{\alpha}{1-\alpha}$ *and* $\Delta_r * 1 \in \mathcal{V}_\lambda$ *with* $||\Delta_r * 1||_{\lambda,1} \leq \frac{\alpha}{1-\alpha}$.
3. *Consider* $r_{(a.)}(t,s)$ *the jump rate associated to the current* $(a_t)_{t\geq 0}$. *Then*

$$r_{(a.)} = r_a + \Delta_r + \Delta_r * r_a. \tag{44}$$

Consequently, we have $r_{(a.)} = \gamma(a) + \xi_{(a.)}$ *with*

$$\xi_{(a.)} = \xi_a + \Delta_r + \Delta_r * \xi_a + \gamma(a)(\Delta_r * 1) \in \mathcal{V}_\lambda.$$

Furthermore,

$$||\xi_{(a.)}||_{\lambda,1} \leq ||\xi_a||_{\lambda,1} + \frac{\alpha}{1-\alpha}[1 + ||\xi_a||_{\lambda,1} + \gamma(a)].$$

Proof. By Lemma 46, we have $||\Delta_K||_{\lambda,1} \leq \alpha < 1$. Consequently equation (43) admits a unique solution $\Delta_r \in \mathcal{V}_\lambda$ satisfying $||\Delta_r||_{\lambda,1} \leq \frac{\alpha}{1-\alpha}$. The kernel $\Delta_r * 1$ satisfies the following Volterra equation

$$\Delta_r * 1 = (\Delta_K * 1) + \Delta_K * (\Delta_r * 1) \tag{45}$$

with $\Delta_K * 1 = (\bar{K}_{(a.)} * 1) + \xi_a * (\bar{K}_{(a.)} * 1) + \gamma(a)(\bar{H}_{(a.)} * 1)$. It follows from Lemma 46 that $\Delta_K * 1 \in \mathcal{V}_\lambda$ and $||\Delta_K * 1||_{\lambda,1} \leq \alpha$. From $||\Delta_K||_{\lambda,1} < 1$, one gets that equation (45) has its solution in $\mathcal{V}_\lambda$ and

$$\Delta_r * 1 \in \mathcal{V}_\lambda, \quad ||\Delta_r * 1||_{\lambda,1} \leq \frac{\alpha}{1-\alpha}.$$

It remains to check that $r_{(a.)}$ given by (44) is indeed the solution of (41). Let $r := r_a + \Delta_r + \Delta_r * r_a$. One has

$$\begin{aligned}\Delta_K * r &= \Delta_K * r_a + (\Delta_r - \Delta_K) + (\Delta_r - \Delta_K) * r_a \\ &= \Delta_r * r_a + \Delta_r - \Delta_K \\ &= r - r_a - \Delta_K,\end{aligned}$$

i.e. r satisfies

$$r = r_a + \Delta_K + \Delta_K * r. \tag{46}$$

Using Proposition 37 and (23), we have $\Delta_K = \bar{K}_{(a.)} + r_a * \bar{K}_{(a.)}$. Equation (46) gives

$$r - (\bar{K}_{(a.)} + r_a * \bar{K}_{(a.)}) * r = r_a + \bar{K}_{(a.)} + r_a * \bar{K}_{(a.)}.$$

We multiply this equation by K_a on the left and obtain, using that $K_a * r_a = r_a * K_a = r_a - K_a$:

$$K_a * r - r_a * \bar{K}_{(a.)} * r = r_a - K_a + r_a * \bar{K}_{(a.)}.$$

The relation $\bar{K}_{(a.)} = K_{(a.)} - K_a$ yields

$$K_a * r - r_a * \bar{K}_{(a.)} * r = r_a * K_{(a.)},$$

or equivalently

$$\Delta_K * r = K_{(a.)} * r - r_a * K_{(a.)}.$$

We substitute this equality in (46) and finally obtain

$$r = K_{(a.)} + K_{(a.)} * r.$$

By uniqueness (Lemma 20 with $\nu = \delta_0$), it follows that $r = r_{(a.)}$. The end of the proof follows easily. ☐

Remark 48. *Let us explain how the formula* (44) *was derived. The algebra* $\mathcal{V}_\lambda$ *does not have any neutral element (in fact the neutral element would be a Dirac distribution) but assume for the sake of this heuristic that* I *is a neutral element of the algebra (i.e.* $k*I = I*k = k\ \forall k \in \mathcal{V}_\lambda$*). Equation* (41) *can be rewritten as*

$$(I - K_{(a.)}) * (I + r_{(a.)}) = I. \tag{47}$$

In particular (taking $(a_t) \equiv a$*), we have* $(I-K_a)*(I+r_a) = (I+r_a)*(I-K_a) = I$*. Furthermore,*

$$I - K_{(a.)} = (I - K_a) * (I - (I + r_a) * \bar{K}_{(a.)}),$$

with $\bar{K}_{(a.)} = K_{(a.)} - K_a \in \mathcal{V}_\lambda$*. Equation* (47) *becomes* $(I-K_a)*(I-(I+r_a)*\bar{K}_{(a.)})*(I+r_{(a.)}) = I$*. We multiply by* $I+r_a$ *on the left of each side, and we get* $(I-(I+r_a)*\bar{K}_{(a.)})*(I+r_{(a.)}) = I+r_a$*.*

We now expand this equation - the neutral element I *disappears and obtain:*

$$r_{(a.)} - (\bar{K}_{(a.)} + r_a * \bar{K}_{(a.)}) * r_{(a.)} = r_a + \bar{K}_{(a.)} + r_a * \bar{K}_{(a.)}.$$

Using the definition of Δ_K *we obtain* $r_{(a.)} = r_a + \Delta_K * r_{(a.)} + \Delta_K$*. Solving this equation in terms of* Δ_r *the resolvent of* Δ_K *we have* $r_{(a.)} = r_a + \Delta_K + \Delta_r * (r_a + \Delta_K)$*. It gives the desired formula.*

We now come back to an arbitrary initial condition ν and prove the main result of this section.

Proposition 49. *Grant Assumptions 2, 3 and 4. Let* $(Y_t^{s,\nu,(a.)})_{t\geq s}$ *be the solution to the non-homogeneous equation* (5) *driven by current* $(a_t)_{t\geq 0}$ *and with distribution* ν *at time* s*. Let* $r^\nu_{(a.)}(t,s) = \mathbb{E}\, f(Y_t^{s,\nu,(a.)})$*. Assume* (a_t) *satisfies Assumption 41 and that the constant* C *satisfies the inequality* (42) *for some* $\alpha \in (0,1)$*. Then it holds that*

$$\forall t \geq s \geq 0,\ |r^\nu_{(a.)}(t,s) - \gamma(a)| \leq De^{-\lambda(t-s)},$$

with

$$D := \frac{1 + \alpha\gamma(a) + ||\xi_a||_{\lambda,1}}{1-\alpha} ||K^\nu_{(a.)}||_{\lambda,\infty} + \gamma(a)||H^\nu_{(a.)}||_{\lambda,\infty}.$$

Proof. The kernel $r^\nu_{(a.)}$ solves the Volterra equation $r^\nu_{(a.)} = K^\nu_{(a.)} + K_{(a.)} * r^\nu_{(a.)}$. By Lemma 21, its solution is

$$r^\nu_{(a.)} = K^\nu_{(a.)} + r_{(a.)} * K^\nu_{(a.)}.$$

Using Proposition 47, we know that $r_{(a.)} = \gamma(a) + \xi_{(a.)}$, with $\xi_{(a.)} \in \mathcal{V}_\lambda$. Furthermore using that $\gamma(a) * K^\nu_{(a.)} = \gamma(a)[1 - H^\nu_{(a.)}]$, we deduce that:

$$r^\nu_{(a.)} = \gamma(a) + K^\nu_{(a.)} + \xi_{(a.)} * K^\nu_{(a.)} - \gamma(a) H^\nu_{(a.)}.$$

Using that $\lambda < f(\sigma_0)$ (Assumption 41) we find

$$||H^\nu_{(a.)}||_{\lambda,\infty} = \sup_{t,s} H^\nu_{(a.)}(t,s)e^{\lambda(t-s)} < \infty,\ ||K^\nu_{(a.)}||_{\lambda,\infty} = \sup_{t,s} K^\nu_{(a.)}(t,s)e^{\lambda(t-s)} < \infty.$$

We obtain

$$\forall t \geq s,\ |r^{\nu}_{(a.)}(t,s) - \gamma(a)|e^{\lambda(t-s)} \leq ||K^{\nu}_{(a.)}||_{\lambda,\infty} + \gamma(a)||H^{\nu}_{(a.)}||_{\lambda,\infty} + e^{\lambda(t-s)} \int_s^t |\xi_{(a.)}|(t,u) K^{\nu}_{(a.)}(u,s)du$$
$$\leq ||K^{\nu}_{(a.)}||_{\lambda,\infty} + \gamma(a)||H^{\nu}_{(a.)}||_{\lambda,\infty} + ||K^{\nu}_{(a.)}||_{\lambda,\infty} \int_s^t |\xi_{(a.)}|(t,u) e^{\lambda(t-u)}du$$
$$\leq ||K^{\nu}_{(a.)}||_{\lambda,\infty} + \gamma(a)||H^{\nu}_{(a.)}||_{\lambda,\infty} + ||K^{\nu}_{(a.)}||_{\lambda,\infty}||\xi_{(a.)}||_{\lambda,1}.$$

Using the estimate of $||\xi_{(a.)}||_{\lambda,1}$ given by Proposition 47, we deduce the result. □

9 Long time behavior for small interactions: proof of Theorem 9

9.1 Some uniform estimates

We now turn to the proof of Theorem 9. It is convenient to first extend the results obtained in Section 7: we need uniform estimates in the input current a. In this section, we grant Assumptions 2, 3 and 4.

Lemma 50. *Let $\bar{a} > 0$. It holds that*

$$\inf_{a \in [0,\bar{a}]} \lambda^*_a > 0.$$

Proof. We define the function g related to the first zero of $\widehat{H}_a$ by

$$\forall a \in [0,\bar{a}], \quad g(a) := -\sup\{\Re(z) |\ \widehat{H}_a(z) = 0,\ \Re(z) > -f(\sigma_0)\}.$$

By convention, $g(a) = f(\sigma_0)$ if $\widehat{H}_a$ is not null on $\Re(z) > -f(\sigma_0)$. By definition of λ^*_a and by the results of Section 7 we know that $g(a) \in (0, \lambda^*_a]$. So, to prove the lemma, it suffices to show the following result
Claim g is lower semi-continuous, that is

$$\forall a_0 \in [0,\bar{a}],\ \liminf_{a \to a_0} g(a) \geq g(a_0).$$

Proof of the claim. Choose $a_0 \in [0,\bar{a}]$. We have $g(a_0) > 0$. Fix $\lambda \in (0, g(a_0))$. Thanks to Lemma 36, one can find $R > 0$, such that for all $a \in [0,\bar{a}]$, for all z with $\Re(z) \in [-\lambda, 0]$ and $\Im(z) \notin [-R,R]$, we have $\widehat{H}_a(z) \neq 0$. Denote $U = \{z \in \mathbb{C}, \Re(z) \in [-\lambda,0], |\Im(z)| \leq R\}$. By definition of $g(a_0)$, we have $\widehat{H}_{a_0} \neq 0$ on U and the continuity of $z \mapsto \widehat{H}_{a_0}(z)$ yields $\inf_{z\in U} |\widehat{H}_{a_0}(z)| > 0$. Moreover, $(a,z) \mapsto \widehat{H}_a(z)$ is continuous on $[0,\bar{a}] \times U$, so one can find $\delta > 0$ such that for all $|a - a_0| \leq \delta$, $z \in U$, we have $|\widehat{H}_a(z)| \neq 0$. and so $g(a) \geq \lambda$. We have proved that $\forall \lambda \in (0, g(a_0))$, $\liminf_{a\to a_0} g(a) \geq \lambda$. It ends the proof. □

Proposition 51 (Whole-line Palay-Wiener, an extension)**.** *Let $\bar{a} > 0$ and for all $a \in [0,\bar{a}]$, let $k_a \in L^1(\mathbb{R},\mathbb{R})$. Assume that*

1. *$\exists \eta \in L^1(\mathbb{R},\mathbb{R}_+)$ s.t. $\forall a \in [0,\bar{a}]$, $\forall 0 < \epsilon < 1, \forall t \in \mathbb{R}, |k_a(t) - k_a(t-\epsilon)| \leq \epsilon\eta(t)$.*
2. *$\exists \theta \in L^1(\mathbb{R},\mathbb{R}_+)$ s.t. $\forall a \in [0,\bar{a}]$, $\forall t \in \mathbb{R} : |k_a(t)| \leq \theta(t)$.*
3. *$\forall a \in [0,\bar{a}]$, $\forall y \in \mathbb{R}$ let $\widehat{k}_a(iy) = \int_{\mathbb{R}} e^{-iyt} k_a(t)dt$. We assume that*

$$\inf_{a\in[0,\bar{a}], y\in\mathbb{R}} |1 - \widehat{k}_a(iy)| > 0.$$

Then for all $a \in [0,\bar{a}]$, there exists a function $x_a \in L^1(\mathbb{R},\mathbb{R})$ satisfying the equation $x_a = k_a + k_a * x_a$ and

$$\sup_{a\in[0,\bar{a}]} ||x_a||_{L^1} < \infty.$$

Proof. We follow the proof of Theorem 4.3 in [20, Chap. 2] and emphasis on the differences. Let $\zeta(t) := \frac{1}{\pi t^2}(1-\cos(t))$ be the Fejer kernel; its Fourier transform is $\widehat{\zeta}(iy) = (1-|y|)\mathbb{1}_{\{|y|\leq 1\}}$. For any $p \geq 1$, set $\zeta_p(t) := p\zeta(pt)$ and $\forall a \in [0,\bar{a}]$,

$$k_a^\infty(t) := k_a - \zeta_p * k_a.$$

Claim 1 There is an integer $p > 0$ such that $\forall a \in [0,\bar{a}]$, $\forall |y| \geq p$, we have

$$||k_a^\infty||_{L^1} \leq 1/2 \quad \text{and} \quad \widehat{k}_a^\infty(iy) = \widehat{k}_a(iy).$$

Proof of the claim. It is clear that with this choice of ζ, $\forall |y| \geq p$, $\widehat{k}_a^\infty(iy) = \widehat{k}_a(iy)$. Moreover, using $\int_\mathbb{R} \zeta_p(s)ds = 1$, we have

$$\begin{aligned} ||k_a^\infty||_{L^1} &= \int_\mathbb{R} \left| \int_\mathbb{R} k_a(t)\zeta_p(s) - k_a(t-s)\zeta_p(s)ds \right| dt \\ &\leq \int_\mathbb{R} \zeta(u) \int_\mathbb{R} |k_a(t) - k_a(t-\frac{u}{p})|dtdu. \end{aligned}$$

We used the Tonelli-Fubini Theorem (everything is non-negative). Let $R > 0$ such that $\int_{\mathbb{R}\setminus[-R,R]} \zeta(u)du \leq \frac{1}{8||\theta||_{L^1}}$. It follows that

$$\begin{aligned} ||k_a^\infty||_{L^1} &\leq 1/4 + \int_{-R}^R \zeta(u) \int_\mathbb{R} |k_a(t) - k_a(t-\frac{u}{p})|dtdu \\ &\leq 1/4 + \int_{-R}^R \left(\int_\mathbb{R} |\frac{u}{p}|\eta(t)dt \right) du \\ &\leq 1/4 + \frac{R^2}{p}||\eta||_{L^1}. \end{aligned}$$

The claim is proved by choosing an integer $p \geq 4R^2||\eta||_{L^1}$.

Along the same idea, we define $\beta(t) := 4\zeta(2t) - \zeta(t) = \frac{1}{\pi t^2}(\cos t - \cos 2t)$. Note that $\forall |y| \leq 1$, we have $\widehat{\beta}(iy) = 1$. Then for all $\delta > 0$, we set $\beta_\delta(t) = \delta\beta(\delta t)$ and

$$\forall y_0 \in \mathbb{R}, \forall t \geq 0, \quad k_a^{y_0,\delta}(t) = \int_\mathbb{R} (\beta_\delta(t-s) - \beta_\delta(t))e^{iy_0(t-s)}k_a(s)ds.$$

Claim 2 Given $\epsilon > 0$, one can find a constant $\delta > 0$ such that: $\forall y_0 \in \mathbb{R}, \forall a \in [0,\bar{a}]$,

$$\forall |y-y_0| \leq \delta,\ \widehat{k}_a(iy) = \widehat{k}_a(iy_0) + \widehat{k_a^{y_0,\delta}}(iy) \quad \text{and} \quad ||k_a^{y_0,\delta}||_{L^1} \leq \frac{\epsilon}{2}.$$

Proof of the claim. By definition of $k_a^{y_0,\delta}$ it holds that

$$\forall y \in \mathbb{R}, \widehat{k_a^{y_0,\delta}}(iy) = \widehat{\beta}_\delta(i(y-y_0))(\widehat{k}_a(iy) - \widehat{k}_a(iy_0)).$$

Moreover, $\widehat{\beta}_\delta(iy) = 1$ if $|y| \leq \delta$ and consequently the first point of the claim is satisfied. Furthermore,

$$\begin{aligned} \int_\mathbb{R} |k_a^{y_0,\delta}(t)|dt &\leq \int_\mathbb{R} |k_a(s)| \int_\mathbb{R} |\beta(t-\delta s) - \beta(t)|dtds \\ &\leq \int_\mathbb{R} \theta(s) \int_\mathbb{R} |\beta(t-\delta s) - \beta(t)|dtds. \end{aligned}$$

The right hand side does not depend on y_0 nor a and goes to zero as δ goes to zero. This proves the second point of the claim.

It follows from Claim 1 that $\forall a \in [0, \bar{a}]$, the equation $x_a^\infty = k_a + k_a^\infty * x_a^\infty$ has a unique solution $x_a^\infty \in L^1(\mathbb{R})$ with $||x_a^\infty||_{L^1} \leq 2||\theta||_{L^1}$. Moreover, we have

$$\forall a \in [0, \bar{a}], \forall |y| \geq p, \quad \widehat{x_a^\infty}(iy) = \frac{\widehat{k}_a(iy)}{1 - \widehat{k}_a(iy)}.$$

Similarly, we define $\epsilon := \inf_{a\in[0,\bar{a}], y\in\mathbb{R}} |1 - \widehat{k}_a(iy)| > 0$ and apply the second claim. Given $y_0 \in \mathbb{R}$ and $a \in [0, \bar{a}]$, let $A_a^{y_0} = \frac{1}{1-\widehat{k}_a(iy_0)}$. We have $1 - \widehat{k}_a(iy) = 1 - \widehat{k}_a(iy_0) - \widehat{k_a^{y_0,\delta}}(iy) = \frac{1}{A_a^{y_0}}(1 - A_a^{y_0}\widehat{k_a^{y_0,\delta}}(iy))$. So,

$$\forall |y - y_0| \leq \delta, \quad \frac{\widehat{k}_a(iy)}{1 - \widehat{k}_a(iy)} = \frac{A_a^{y_0}\widehat{k}_a(iy)}{1 - A_a^{y_0}\widehat{k_a^{y_0,\delta}}(iy)}.$$

Using $||A_a^{y_0} k_a^{y_0,\delta}||_{L^1} \leq 1/2$, we can define the solution of $x_a^{y_0} = A_a^{y_0} k_a + A_a^{y_0} k_a^{y_0,\delta} * x_a^{y_0}$ and we have

$$||x_a^{y_0}||_{L^1} \leq \frac{2}{\epsilon}||\theta||_{L^1}.$$

Consequently, for all y with $|y - y_0| \leq \delta$ we have

$$\widehat{x_a^{y_0}}(iy) = \frac{\widehat{k}_a(iy)}{1 - \widehat{k}_a(iy)}.$$

Furthermore, still following [20], one can find an integer $m > 0$ such that: $\forall a \in [0, \bar{a}]$, $\forall j \in \mathbb{Z}, |j| \leq mp$, there exists a function $x_a^{j/m} \in L^1(\mathbb{R})$ with $||x_a^{j/m}||_{L^1} \leq \frac{2}{\epsilon}||\theta||_{L^1}$ such that

$$\forall |y - j/m| \leq 1/m, \quad \widehat{x_a^{j/m}}(iy) = \frac{\widehat{k}_a(iy)}{1 - \widehat{k}_a(iy)}.$$

We define $\psi_j(t) = \frac{1}{m}e^{-ijt/m}\zeta(t/m)$. We have $||\psi_j||_{L^1} = 1$. Its Fourier transform is given by

$$\widehat{\psi}_j(iy) = \begin{cases} 0 & \text{if } |y - j/m| > 1/m \\ 1 - m|y - j/m| & \text{otherwise.} \end{cases}$$

We set

$$x_a = \sum_{|j|\leq mp} \psi_j * (x_a^{j/m} - x_a^\infty) + x_a^\infty.$$

It is clear that $x_a \in L^1(\mathbb{R})$ and that

$$\sup_{a\in[0,\bar{a}]} ||x_a||_{L^1} \leq mp\left(\frac{2}{\epsilon}||\theta||_{L^1} + 2||\theta||_{L^1}\right) + 2||\theta||_{L^1} < \infty.$$

With this choice of ψ_j, $\forall y \in \mathbb{R}, \widehat{x}_a(iy) = \frac{\widehat{k}_a(iy)}{1-\widehat{k}_a(iy)}$ and by uniqueness of the Fourier transform, we conclude that x_a is the solution of $x_a = k_a + k_a * x_a$. ☐

As a consequence of the previous result, we have

Corollary 52. *Let $\bar{a} > 0$, define $\lambda^* = \inf_{a\in[0,\bar{a}]} \lambda_a^*$ ($\lambda^* > 0$ by Lemma 50). Let $0 < \lambda < \lambda^*$ and consider r_a the solution of the Volterra equation $r_a = K_a + K_a * r_a$. By Proposition 37, it holds that $r_a = \gamma(a) + \xi_a$ for some $\xi_a \in L_\lambda$. Then we have $\sup_{a\in[0,\bar{a}]} ||\xi_a||_{\lambda,1} < \infty$.*

Proof. Recall (see proof of Proposition 37) that $\xi_a(t) = e^{-\lambda t}\xi_{a,-\lambda}(t)$ and so $||\xi_a||_{\lambda,1} = ||\xi_{a,-\lambda}||_{L^1}$. We now prove that Proposition 51 applies to $\xi_{a,-\lambda}$. Indeed, it solves

$$\xi_{a,-\lambda} = K_{a,-\lambda} + K_{a,-\lambda} * \xi_{a,-\lambda},$$

with $K_{a,-\lambda}(t) := e^{\lambda t} K_a(t) \mathbb{1}_{\{t \geq 0\}}$. It remains to show that $K_{a,-\lambda}$ fulfills the assumptions of Proposition 51.

1. We use $\sup_{a \in [0,\bar{a}]} K_a(t) \leq f(\varphi_t^{\bar{a}}(0)) H_0(t)$ and $\sup_{a \in [0,\bar{a}]} |\varphi_t^a(0) - \varphi_{t-\epsilon}^a(0)| \leq \epsilon C_b^{\bar{a}}$.
2. For all $t \geq 0$ and $a \in [0, \bar{a}]$, we have

$$K_{a,-\lambda}(t) \leq \theta(t) := e^{\lambda t} f(C_t^{\bar{a}}) H_0(t) \mathbb{1}_{\mathbb{R}_+}(t) \in L^1(\mathbb{R}).$$

3. We have $\widehat{K_{a,-\lambda}}(iy) = \widehat{K}_a(-\lambda + iy)$. We conclude by Lemmas 50 and 36. □

9.2 Proof of Theorem 9

We are now ready to give the proof of the main theorem.

- **Step 1** Recall that equation (25) gives

$$\frac{d}{dt} \mathbb{E} f(X_t) \leq \frac{1}{2}[\bar{r}(J)^2 - \mathbb{E}^2 f(X_t)],$$

  where $(X_t)_{t \geq 0}$ is the solution of the non-linear equation (2) and the function $J \mapsto \bar{r}(J)$ is non-decreasing. Using Proposition 23 with $\kappa := J\bar{r}(J) + 1$, there is a non-decreasing function $J \mapsto \bar{a}(J)$ such that:

$$\forall J, s \geq 0,\ \forall (a_t)_{t \geq s} \in \mathcal{C}([s, \infty), \mathbb{R}_+),\ [\sup_{t \geq s} a_t \leq \bar{a}(J) \text{ and } J\nu(f) \leq \bar{a}(J)] \implies \sup_{t \geq s} J r_{(a.)}^{\nu}(t,s) \leq \bar{a}(J).$$

  Moreover, it holds that $\forall J \geq 0,\ J\bar{r}(J) < \bar{a}(J)$.

- **Step 2** We define

$$\lambda^* := \inf_{a \in [0, \bar{a}(J_m)]} \lambda_a^*,$$

  where $J_m > 0$ is defined in Proposition 8. Lemma 50 gives $\lambda^* > 0$. We now fix λ such that $0 < \lambda < \lambda^*$.

- **Step 3**
  - Using Corollary 52, we know that the solution of the Volterra equation $r_a = K_a + K_a * r_a$ is $r_a = \gamma(a) + \xi_a$ with $\xi_a \in L_\lambda$ and that:

$$\xi^\infty(J) := \sup_{a \in [0, \bar{a}(J)]} ||\xi_a||_{\lambda,1} < \infty.$$

    It is clear that $J \mapsto \xi^\infty(J)$ is non-decreasing (as $J \mapsto \bar{a}(J)$ is).
  - One can find a function $k^\infty : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, non-decreasing with respect to its two parameters, such that for all $(a_t) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ we have:

$$\sup_{t \geq 0} a_t \leq \bar{a} \implies ||K_{(a.)}^{\nu}||_{\lambda,\infty} \leq k^\infty(\nu(f), \bar{a}) < \infty.$$

    Moreover, one can find a constant h^∞ (only depending on λ, b and f) such that for all $(a_t) \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, we have

$$||H_{(a.)}^{\nu}||_{\lambda,\infty} \leq h^\infty.$$

    These two points follow from $\lambda < f(\sigma_0)$, Assumption 2, Remarks 12(2) and 13.

– The function $\eta_{\bar{a}}$ of Lemma 45 satisfies

$$||\eta_{\bar{a}}||_1 < \infty,\ \bar{a} \mapsto ||\eta_{\bar{a}}||_1 \text{ is non-decreasing,}$$

and consequently the function $J \mapsto ||\eta_{\bar{a}(J)}||_1$ is non-decreasing.

– Finally the normalization γ is a non-decreasing function of a (see (31)) and it follows that

$$\forall a \in [0, \bar{a}(J)], \gamma(a) \leq \gamma(\bar{a}(J)).$$

- **Step 4** Let ν be a probability measure such that $\nu(f) \leq \bar{r}(J_m) + 1$. Remind that for all $J \in (0, J_m)$ the equation $a\gamma^{-1}(a) = J$ has a unique solution $a^*(J) \in [0, \bar{a}(J_m)]$. We now apply Proposition 49 with $\alpha = 1/2$. Define:

$$C(J) := \frac{1}{2||\eta_{\bar{a}(J)}||_1(1 + \xi^\infty(J) + \gamma(\bar{a}(J)))}$$
$$D(J) := 2(1 + \gamma(\bar{a}(J)) + \xi^\infty(J))k^\infty(\bar{r}(J_m) + 1, \bar{a}(J)) + \gamma(\bar{a}(J))h^\infty.$$

From Step 3, it is clear that the functions $J \mapsto \frac{1}{C(J)}$ and $J \mapsto D(J)$ are non-decreasing. Consequently, we can find a constant $J^* \in (0, J_m)$ such that

$$\forall J \in [0, J^*], \quad \frac{JD(J)}{C(J)} \leq 1.$$

Proposition 49 tells us that for every $0 \leq J \leq J^*$, given any $(a_t)_{t\geq 0} \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ with $\sup_{t\geq 0} a_t \leq \bar{a}(J)$ and such that

$$\forall t \geq 0,\ |a_t - a^*(J)| \leq C(J)e^{-\lambda t},$$

it holds

$$\forall t \geq 0,\ |Jr^\nu_{(a.)}(t, 0) - a^*(J)| \leq C(J)e^{-\lambda t}.$$

- **Step 5** Let now $J \in (0, J^*]$ be fixed (the case $J = 0$ is already treated by Proposition 30). We assume the initial condition ν of (2) satisfies $J\nu(f) \leq \bar{a}(J)$ and that $\nu(f) \leq \bar{r}(J_m) + 1$ (we shall come back to the general case in Step 6). We define recursively $a^n \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ by

$$\forall t \geq 0,\ a^0(t) := a^*(J) \quad \text{and} \quad \forall n \geq 0, \quad a^{n+1}(t) := Jr^\nu_{(a^n.)}(t, 0).$$

By Step 4 and by induction, it holds that:

$$\forall n \geq 0,\ \forall t \geq 0,\ |a^n(t) - a^*(J)| \leq C(J)e^{-\lambda t}.$$

We deduce that:

$$\begin{aligned}\forall t \geq 0,\ |\mathbb{E} f(X_t) - \gamma(a^*(J))| &\leq |\mathbb{E} f(X_t) - r^\nu_{(a^n.)}(t, 0)| + \frac{1}{J}|a^{n+1}(t) - a^*(J)| \\ &\leq \frac{1}{J}|J\,\mathbb{E} f(X_t) - a^{n+1}(t)| + \frac{C(J)}{J}e^{-\lambda t}.\end{aligned}$$

The Picard iteration studied in Part 4.4 shows that

$$\forall t \geq 0,\ \lim_{n\to\infty} |J\,\mathbb{E} f(X_t) - a^n(t)| = 0.$$

We have proved that

$$\forall t \geq 0,\ |\mathbb{E} f(X_t) - \gamma(a^*(J))| \leq \frac{C(J)}{J}e^{-\lambda t}.$$

- **Step 6** We now prove that there exists $s \geq 0$ such that $\mathbb{E}\, f(X_s) \leq \min(\frac{\bar{a}(J)}{J}, \bar{r}(J_m)+1)$. By Step 1, we have $\limsup \mathbb{E}\, f(X_t) \leq \overline{r}(J)$. Since $\overline{r}(J) < \overline{a}(J)/J$ and since $\overline{r}(J) \leq \overline{r}\,(J_m)$, the conclusion follows. Consequently, Step 5 can be applied to the process $(X_t)_{t\geq s}$ starting with $\nu = \mathcal{L}(X_s)$. This proves the convergence of the jump rate.

The convergence of the law of X_t to the invariant measure then follows from Proposition 29. This ends the proof of Theorem 9.

Remark 53. *There is some freedom in the above construction of the constants λ and J^*. We can choose any λ in $[0, \lambda^*)$ and the value of J^* depends both on λ and on a parameter $\alpha \in (0,1)$, here chosen to be equal to $1/2$ (see Step 4). We may optimize this construction to get either J^* or λ as large as possible.*

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