Formal Proofs of Tarjan's Algorithm in Why3, Coq, and Isabelle

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Abstract

Comparing provers on a formalization of the same problem is always a valuable exercise. In this paper, we present the formal proof of correctness of a non-trivial algorithm from graph theory that was carried out in three proof assistants: Why3, Coq, and Isabelle.

1 Introduction

In this paper, we consider Tarjan's algorithm [25] for discovering the strongly connected components in a directed graph and present a formal proof of its correctness in three different systems: Why3, Coq and Isabelle/HOL. The algorithm is treated at an abstract level with a functional programming style manipulating finite sets, stacks and mappings, but it respects the linear time behaviour of the original presentation. It would not be difficult to derive and prove correct an efficient implementation with imperative programs and concrete data types such as integers, linked lists and mutable arrays from our presentation.

To our knowledge this is the first time that the formal correctness proof of a non-trivial program is carried out in three very different proof assistants: Why3 is based on a first-order logic with inductive predicates and automatic provers, Coq on an expressive theory of higher-order logic and dependent types, and Isabelle/HOL combines higher-order logic with automatic provers. We do not claim that our proof is the simplest possible one, and we will discuss the design and implementation of other proofs in the conclusion, but our proof is indeed elegant and follows Tarjan's presentation. Crucially for our comparison, the algorithm is defined at the same level of abstraction in all three systems, and the proof relies on the same arguments in the three formal systems. Note

that a similar exercise but for a much more elementary proof (the irrationality of square root of 2) and using many more proof assistants (17) was presented in [29].

Formal and informal proofs of algorithms about graphs were already performed in [21, 27, 22, 11, 15, 26, 17, 24, 23, 13, 7]. Some of them are part a larger library, others focus on the treatment of pointers or about concurrent algorithms. In particular, Lammich and Neumann [15] give a proof of Tarjan's algorithm within their framework for verifying graph algorithms in Isabelle/HOL. In our formalization, we are aiming for a simple, direct, and readable proof.

It is not possible to expose here the details of the full proofs in the three systems, but the interested reader can access and run them on the Web [6, 8, 18]. In this paper, we recall the principles of the algorithm in section 2; we describe the proofs in the three systems in sections 3, 4, and 5 by emphasizing the differences induced by the logic which are used; we conclude in sections 6 and 7 by commenting the developments and advantages of each proof system.

2 The algorithm

The algorithm [25] performs a depth-first search on the set *vertices* of all vertices in the graph. Every vertex is visited once and is assigned a serial number of its visit. The algorithm maintains an environment e containing four fields: a stack *e.stack*, a set *e.sccs* of strongly connected components, a new fresh serial number *e.sn*, and a function *e.num* which records the serial numbers assigned to vertices. The field *e.stack* contains the visited vertices which are not part of the components already stored in *e.sccs*. Vertices are pushed onto the stack in the order of their visit.

The depth-first search is organized by two mutually recursive functions *dfs1* and *dfs*. The function *dfs* takes as argument a set r of roots and an environment e. It returns a pair consisting of an integer and the modified environment. If the set of roots is empty, the returned integer is $+\infty$. Otherwise the returned integer is the minimum of the results of the calls to *dfs1* on non-visited vertices in r and of the serial numbers of the already visited ones.

The main procedure *tarjan* initializes the environment with an empty stack, an empty set of strongly connected components, the fresh number 0 and the constant function giving the number -1 to each vertex. The result is the set of components returned by the function *dfs* called on all vertices in the graph.

```
let rec dfs1 x e =
  let n0 = e.sn in
  let (n1, e1) = dfs (successors x)
        (add_stack_incr x e) in
  if n1 < n0 then (n1, e1) else
    let (s2, s3) = split x e1.stack in
    (+∞, {stack = s3;
       sccs = add (elements s2) e1.sccs;
       sn = e1.sn; num = set_infty s2 e1.num})
```

```
with dfs r e = if is_empty r then (+∞, e) else
  let x = choose r in
  let r' = remove x r in
  let (n1, e1) = if e.num[x] ≠ -1
     then (e.num[x], e) else dfs1 x e in
  let (n2, e2) = dfs r' e1 in (min n1 n2, e2)

let tarjan () =
  let e = {stack = Nil; sccs = empty;
           sn = 0; num = const (-1)} in
  let (_, e') = dfs vertices e in e'.sccs
```

The heart of the algorithm is in the body of *dfs1* which visits a new vertex *x*. The auxiliary function *add_stack_incr* updates the environment by pushing *x* on the stack, assigning it the current fresh serial number, and incrementing that number in view of future calls. The function *dfs1* performs a recursive call to *dfs* for the successor vertices of *x* as roots and the updated environment. If the returned integer value *n1* is less than the number assigned to *x*, the function simply returns *n1* and the current environment. Otherwise, the function declares that a new strongly connected component has been found, consisting of all vertices that are contained on top of *x* in the current stack. Therefore the stack is popped until *x*; the popped vertices are stored as a new set in *e.sccs*; and their numbers are all set to $+\infty$, ensuring that they do not interfere with future calculations of min values. The auxiliary functions *split* and *set_infty* are used to carry out these updates.

```
let add_stack_incr x e = let n = e.sn in
  {stack = Cons x e.stack; sccs = e.sccs;
      sn = n+1; num = e.num[x ← n]}

let rec set_infty s f = match s with Nil → f
  | Cons x s' → (set_infty s' f)[x ← +∞]  end

let rec split x s = match s with Nil → (Nil, Nil)
  | Cons y s' → if x = y then (Cons x Nil, s')
    else let (s1', s2) = split x s' in
         (Cons y s1', s2)  end
```

Figure 1 illustrates the behavior of the algorithm by an example. We presented the algorithm as a functional program, using data structures available in the Why3 standard library [3]. For lists we have the constructors *Nil* and *Cons*; the function *elements* returns the set of elements of a list. For finite sets, we have the empty set *empty*, and the functions *add* to add an element to a set, *remove* to remove an element from a set, *choose* to pick an arbitrary element in a (non-empty) set, and *is_empty* to test for emptiness. We also use maps with functions *const* denoting the constant function, `_[_]` to access the value of an element, and `_[_ ← _]` for creating a map obtained from an existing map by setting an element to a given value. We also define an abstract type *vertex* for vertices and a constant *vertices* for the finite set of all vertices in the graph. The type *env* of environments is a record with the four fields *stack*, *sccs*, *sn* and *num* as described above.

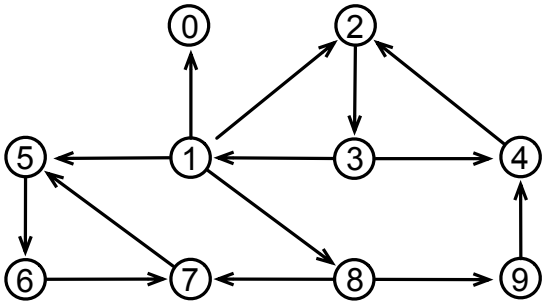


Figure 1: An example: the vertices are numbered and pushed onto the stack in the order of their visit by the recursive function *dfs1*. When the first component $\{0\}$ is discovered, vertex 0 is popped; similarly when the second component $\{5, 6, 7\}$ is found, its vertices are popped; finally all vertices are popped when the third component $\{1, 2, 3, 4, 8, 9\}$ is found. Notice that there is no cross-edge to a vertex with a number less than 5 when the second component is discovered. Similarly in the first component, there is no edge to a vertex with a number less than 0. In the third component, there is no edge to a vertex less than 1 since we have set the number of vertex 0 to $+\infty$ when 0 was popped.

```
type vertex
constant vertices: set vertex
function successors vertex : set vertex
type env = {stack: list vertex;
            sccs: set (set vertex);
            sn: int; num: map vertex int}
```

For a correspondence between our presentation and the imperative programs used in standard textbooks, the reader is referred to [7]. The present version can be directly translated into Coq or Isabelle functions, and it respects the linear running time behaviour of the algorithm, since vertices could be easily implemented by integers, $+\infty$ by the cardinal of *vertices*, finite sets by lists of integers and mappings by mutable arrays (see for instance [6]).

Like many algorithms on graphs, Tarjan's algorithm is not easy to understand and even looks a bit magical. In the original presentation, the integer value returned by the function *dfs1* is given by the following formula when called on vertex *x*.

$$\begin{aligned} \mathit{LOWLINK}(x) = \min\{\mathit{num}[y] \;\mid\; & x \stackrel{*}{\Longrightarrow} z \hookrightarrow y \\ & \wedge\ x \text{ and } y \text{ are in the same connected component}\} \end{aligned}$$

This expression is evaluated on the spanning tree (forest) corresponding to one run of *dfs*. The relation $x \Longrightarrow z$ means that z is a son of x in the spanning tree, the relation $\stackrel{*}{\Longrightarrow}$ is its transitive and reflexive closure, and $z \hookrightarrow y$ means that there is a cross-edge between z and y in the spanning tree. In figure 2, $\Longrightarrow$ is drawn in thick lines and $\hookrightarrow$ in dotted lines; a table of the values of the *LOWLINK* function is also shown. Thus the integer value returned by *dfs1* is the minimum of the numbers of vertices in the same

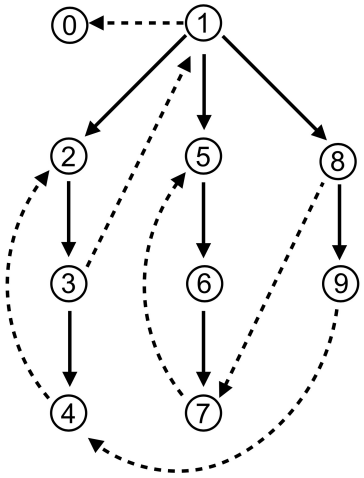
connected component accessible by just one cross-edge by all descendants of *x* visited in the recursive calls. If none, $+\infty$ is returned (here is a slight simplification w.r.t. the original algorithm). Notice that the result may be the number of a vertex which is not an ancestor of *x* in the spanning tree. Take for instance, vertices 8 or 9 in figure 2.

The algorithm relies on the existence of a base with a minimal serial number for each connected component, the members of which are among its descendants in the spanning tree. The reason is that a cross-edge reaches from x either an ancestor of x, or a descendant of a grandson in the spanning tree, or a cousin to the left of x. Intuitively, cross-edges never go right in the spanning tree. Therefore these bases are organized as a Christmas tree, and each connected component is a prefix of one sub-tree of which the root is its base.

Thus for each environment *e* in the algorithm, the working stack *e.stack* corresponds to a cut of the spanning tree where connected components to its left are pruned and stored in *e.sccs*. In this stack, any vertex can reach any vertex higher in the stack. And if a vertex is a base of a connected component, no cross-edge can reach some vertex lower than this base in the stack, otherwise that last vertex would be in the same connected component with a strictly lower serial number.

We therefore have to organize the proofs of the algorithm around these arguments. To maintain these invariants we will distinguish, as is common for depth-first search algorithms, three sets of vertices: white vertices are the non-visited ones, black vertices are those that are already fully visited, and gray vertices are those that are still being visited. Clearly, these sets are disjoint and white vertices can be considered as forming the complement in *vertices* of the union of the gray and black ones.

The previously mentioned invariant properties can now be expressed for vertices in the stack: no such vertex is white, any vertex can reach all vertices higher in the stack, any vertex can reach some gray vertex lower in the stack. Moreover, vertices in the stack respect the numbering order, i.e. a vertex *x* is lower than *y* in the stack if and only



0	0
1	1
2	1
3	1
4	2
5	5
6	5
7	5
8	4
9	4

Figure 2: Spanning forest and the *LOWLINK* function.

if the number assigned to x is strictly less than the number assigned to y.

3 The proof in Why3

The Why3 system comprises the language WhyML for writing programs and a many sorted first-order logic with inductive data types and inductive predicates to express the logical assertions. The system generates proof obligations w.r.t. the assertions, pre- and post-conditions and lemmas inserted in the WhyML program. The system is interfaced with off-the-shelf automatic provers (we mainly use Alt-Ergo, CVC, E-prover and Z3) and also interactive proof assistants such as Coq or Isabelle.

There are numerous libraries that can be used in the Why3 library, for integer arithmetic, polymorphic lists, finite sets and mappings, etc. There is also a small theory for paths in graphs. Here we define graphs, paths and strongly connected components as follows.

```
axiom successors_vertices: ∀x. mem x vertices →
  subset (successors x) vertices
predicate edge (x y: vertex) =
  mem x vertices ∧ mem y (successors x)
inductive path vertex (list vertex) vertex =
  | Path_empty: ∀x: vertex. path x Nil x
  | Path_cons: ∀x y z: vertex, l: list vertex.
      edge x y → path y l z → path x (Cons x l) z

predicate reachable (x y: vertex) = ∃l. path x l y
predicate in_same_scc (x y: vertex)  =
  reachable x y ∧ reachable y x
predicate is_subscc (s: set vertex) =
  ∀x y. mem x s → mem y s → in_same_scc x y
predicate is_scc (s: set vertex) = not is_empty s
  ∧ is_subscc s
  ∧ (∀s'. subset s s' → is_subscc s' → s == s')
```

where *mem* and *subset* denote membership and the subset relation for finite sets.

We add two ghost fields in environments for the black and gray sets of vertices. These fields are used in the proofs but not used in the calculation of the connected components, which is checked by the type-checker of the language.

```
type env = {ghost black: set vertex;
    ghost gray: set vertex;
    stack: list vertex; sccs: set (set vertex);
    sn: int; num: map vertex int}
```

The functions now become:

```
let rec dfs1 x e =
  let n0 = e.sn in
  let (n1, e1) = dfs (successors x)
        (add_stack_incr x e) in
  if n1 < n0 then (n1, add_black x e1) else
```

```
    let (s2, s3) = split x e1.stack in
    (+∞, {stack = s3;
       black = add x e1.black; gray = e.gray;
       sccs = add (elements s2) e1.sccs;
       sn = e1.sn; num = set_infty s2 e1.num})
with dfs r e = ... (* unmodified *)
let tarjan () =
  let e = {black = empty; gray = empty;
    stack = Nil; sccs = empty; sn = 0;
    num = const (-1)} in
  let (_, e') = dfs vertices e in e'.sccs
```

with a new function *add_black* turning a vertex from gray to black and the modified *add_stack_incr* adding a new gray vertex with a fresh serial number to the current stack.

```
let add_stack_incr x e =
  let n = e.sn in
  {black = e.black; gray = add x e.gray;
   stack = Cons x e.stack; sccs = e.sccs;
   sn = n+1; num = e.num[x ←n]}
let add_black x e =
 {black = add x e.black; gray = remove x e.gray;
  stack = e.stack; sccs = e.sccs;
  sn = e.sn; num = e.num}
```

The main invariant ($\mathcal{I}$) of our program states that the environment is well-formed:

```
predicate wf_env (e: env) =
  let {stack = s; black = b; gray = g} = e in
  wf_color e ∧ wf_num e ∧
  simplelist s ∧ no_black_to_white b g ∧
  (∀x y. lmem x s → lmem y s →
     e.num[x] ≤ e.num[y] → reachable x y) ∧
  (∀y. lmem y s → ∃x. mem x g ∧
     e.num[x] ≤ e.num[y] ∧ reachable y x) ∧
  (∀cc. mem cc e.sccs ↔
              subset cc e.black ∧ is_scc cc)
```

where *lmem* stands for membership in a list. The well-formedness property is the conjunction of seven clauses. The two first clauses express quite elementary conditions about the colored sets of vertices and the numbering function. We do not express them formally here (see [7, 6] for a detailed description). The third clause states that there are no repetitions in the stack, and the fourth that there is no edge from a black vertex to a white vertex. The next two clauses formally express the property already stated above: any vertex in the stack reaches all higher vertices and any vertex in the stack can reach a lower gray vertex. The last clause states that the *sccs* field is the set of all connected components all of whose vertices are black. Since at the end of the *tarjan* function, all vertices are black, the *sccs* field will contain exactly the set of all strongly connected components.

Our functions *dfs1* and *dfs* modify the environment in a monotonic way. Namely they augment the set of the fully visited vertices (the black ones); they keep invariant

the set of the ones currently under visit (the gray set); they increase the stack with new black vertices; they also discover new connected components and they keep invariant the serial numbers of vertices in the stack,

```
predicate subenv (e e': env) =
  subset e.black e'.black ∧ e.gray == e'.gray
  ∧ (∃s. e'.stack = s ++ e.stack ∧
         subset (elements s) e'.black)
  ∧ subset e.sccs e'.sccs
  ∧ (∀x. lmem x e.stack → e.num[x] = e'.num[x])
```

Once these invariants are expressed, it remains to locate them in the program text and to add assertions which help to prove them. The pre-conditions of *dfs1* are quite natural: the vertex x must be a white vertex of the graph, and it must be reachable from all gray vertices. Moreover invariant $(\mathcal{I})$ must hold. The post-conditions of *dfs1* are of three kinds. Firstly $(\mathcal{I})$ and the monotony property *subenv* hold in the resulting environment. Vertex x is black at the end of *dfs1*. Finally we express properties of the integer value n returned by this function which should be $LOWLINK(x)$ as announced previously. Notice that we do not know yet the connected component of x, but the definition of *LOWLINK* still works thanks to the numbering with $+\infty$ of the visited vertices not in its component. In this proof, we give three implicit properties for characterizing the resulting n value. First, the returned value is never higher than the number of x in the final environment. Secondly, the returned value is either $+\infty$ or the number of a vertex in the stack reachable from x. Finally, if there is an edge from a vertex y' in the new part of the stack to a vertex y in its old part, the resulting value n must be lower than the number of y.

```
let rec dfs1 x e  =
(* pre-condition *)
requires {mem x vertices}
requires {∀y. mem y e.gray → reachable y x}
requires {not mem x (union e.black e.gray)}
requires {wf_env e} (* I *)
(* post-condition *)
returns {(_, e') → wf_env e' ∧ subenv e e'}
returns {(_, e') → mem x e'.black}
returns {(n, e') → n ≤ e'.num[x]}
returns {(n, e') → n = +∞ ∨ num_of_reachable n x e'}
returns {(n, e') → ∀y. xedge_to e'.stack e.stack y
                     → n ≤ e'.num[y]}
```

where the auxiliary predicates used in these post-conditions are formally defined in the following way.

```
predicate num_of_reachable (n: int) (x: vertex)
  (e: env) = ∃y. lmem y e.stack ∧ n = e.num[y] ∧
     reachable x y
predicate xedge_to (s1 s3: list vertex)
  (y: vertex) = (∃s2. s1 = s2 ++ s3 ∧
    ∃y'. lmem y' s2 ∧ edge y' y) ∧ lmem y s3
```

Notice that when the integer result n of *dfs1* is infinite, the number of x must also be infinite, meaning that its connected component has been found. Also notice that the

definition of *xedge_to* fits the definition of *LOWLINK* when the cross edge ends at a vertex residing in the stack before the call of *dfs1*. The pre- and post-conditions for the function *dfs* are quite similar up to a generalization to sets of vertices which are the roots of the algorithm (see [6]).

We now add seven assertions in the body of the *dfs1* function to help the automatic provers. In contrast, the function *dfs* needs no extra assertions in its body. In *dfs1*, when the number *n0* of *x* is strictly greater than the number resulting from the call to its successors, the first assertion states that vertex *x* can reach a strictly lower vertex in the current stack and the second assertion states that a lower gray vertex is reachable and that thus the connected component of *x* is not fully black at end of *dfs1*. That key assertion is proved from the first one by transitivity of reachability. (We understand here why the algorithm only takes care of a single cross-edge: for any visited vertex *x*, the spanning tree must contain at least one back-edge to a strict ancestor of *x* when $n1 < n0$.) The next four assertions show that the connected component *(elements s2)* of *x* is on top of *x* in the stack when $n1 \geq n0$, and that then *x* is the base of that connected component. The seventh assertion helps proving that the coloring constraint is preserved at the end of *dfs1*.

```
let n0 = e.sn in
let (n1, e1) =
  dfs (successors x) (add_stack_incr x e) in
if n1 < n0 then begin
  assert{∃y. y ≠ x ∧ precedes x y e1.stack ∧
     reachable x y};
  assert{∃y. y ≠ x ∧ mem y e1.gray ∧
     e1.num[y] < e1.num[x] ∧ in_same_scc x y};
  (n1, add_black x e1) end
else
  let (s2, s3) = split x e1.stack in
  assert{is_last x s2 ∧ s3 = e.stack ∧
     subset (elements s2) (add x e1.black)};
  assert{is_subscc (elements s2)};
  assert{∀y. in_same_scc y x → lmem y s2};
  assert{is_scc (elements s2)};
  assert{inter e.gray (elements s2) == empty};
  (+∞, {black = add x e1.black; gray = e.gray;
     stack = s3; sccs = add (elements s2) e1.sccs;
     sn = e1.sn; num = set_infty s2 e1.num})
```

where *inter* is set intersection, and *precedes* and *is_last* are two auxiliary predicates defined below.

```
predicate is_last (x: α) (s: list α) =
  ∃s'. s = s' ++ Cons x Nil
predicate precedes (x y: α) (s: list α) =
  ∃s1 s2. s = s1 ++ (Cons x s2) ∧ lmem y (Cons x s2)
```

All proofs are discovered by the automatic provers except for two proofs carried out interactively in Coq. One is the proof of the black extension of the stack in case $n1 < n0$. The provers could not work with the existential quantifier, although the Coq proof is quite short. The second Coq proof is the fifth assertion in the body of

provers	Alt-Ergo	CVC4	E-prover	Z3	#VC	#PO
49 lemmas	1.91	26.11	3.33		70	49
split	0.09	0.16			6	6
add_stack_incr	0.01				1	1
add_black	0.02				1	1
set_infty	0.03				1	1
dfs1	77.89	150.2	19.99	13.67	79	20
dfs	4.71	3.52		0.26	58	25
tarjan	0.85				15	5
total	85.51	179.99	23.32	13.93	231	108

Table 1: Performance results of the provers (in seconds, on a 3.3 GHz Intel Core i5 processor). Total time is 341.15 seconds. The two last columns contain the numbers of verification conditions and proof obligations. Notice that there may be several VCs per proof obligation.

dfs1, which asserts that any y in the connected component of x belongs to *s2*. It is a maximality assertion which states that the set *(elements s2)* is a complete connected component. The proof of that assertion is by contradiction. If y is not in *s2*, there must be an edge from x' in *s2* to some y' not in *s2* such that x reaches x' and y' reaches y. There are three cases, depending on the position of y'. Case 1 is when y' is in *sccs*: this is not possible since x would then be in *sccs* which contradicts x being gray. Case 2 is when y' is an element of *s3*: the serial number of y' is strictly less than the one of x which is *n0*. If $x' \neq x$, the cross-edge from x' to y' contradicts $n1 \geq n0$ (post-condition 5); if $x' = x$, then y' is a successor of x and again it contradicts $n1 \geq n0$ (post-condition 3). Case 3 is when y' is white, which is impossible since x' is black in *s2*.

The figures of the Why3 proof are listed in table 1. Alt-Ergo 2.2 and CVC4 1.5 proved the bulk of the proof obligations.[1] The proof uses 49 lemmas that were all proved automatically, but with an interactive interface providing hints to apply inlining, splitting, or induction strategies. This includes 13 lemmas on sets, 16 on lists, 5 on lists without repetitions, 3 on paths, 5 on connected components and 6 very specialized lemmas directly involved in the proof obligations of the algorithm. Among the lemmas, a critical one is the lemma *xpath_xedge* on paths which reduces a predicate on paths to a predicate on edges. In fact, most of the Why3 proof works on edges which are handled more robustly by the automatic provers than paths. The two Coq proofs are 16 and 141 lines long (the Coq files of 677 and 721 lines include preambles that are automatically generated during the translation from Why3 to Coq). The interested reader is refered to [6] where the full proof is available.

The proof explained so far does not show that the functions terminate; we have only shown the partial correctness of the algorithm. But after adding two lemmas about union and difference for finite sets, termination is automatically proved by the following lexicographic ordering on the number of white vertices and roots.

```
let rec dfs1 x e  =
variant {cardinal (diff vertices
```

[1] In addition to the results reported in the table, Spass was used to discharge one proof obligation.

```
            (union e.black e.gray)), 0}
with dfs r e =
variant {cardinal (diff vertices
            (union e.black e.gray)), 1, cardinal r}
```

4 The proof in Coq

Coq is a proof assistant based on type theory. It uses the calculus of constructions, a higher order lambda-calculus, to express formulae and proofs. Some basic notions of graph theory are provided by the Mathematical Component Library [16]. The formalization in Coq follows closely what has been done in Why3, so we mostly highlight differences. It is parameterized by a finite type *V* for the vertices and a function *successors* that represents the graph, i.e. *(successors v)* gives all the successors of the vertex *v* in the graph.

The environment that is passed around in the algorithm is defined as a record with five fields:

```
Record env := Env {
 black : {set V};
 stack : seq V;
 esccs : {set {set V}};
    sn : nat;
   num : {ffun V -> nat}}.
```

Note that with respect to Why3, we have no ghost mechanism available for the *black* field and we do not hold gray vertices. They are globally defined as the elements of the stack that are not black. Also, we restrict ourselves to natural numbers, representing the integer n in Why3 by the natural number $n+1$ in Coq.

Our definition of the algorithm is very similar to the one of Why3. The only difference is the way recursion is handled. We untangle the mutually recursive function *tarjan* into two separate functions The first one *dfs1* treats a vertex x and the second one *dfs* a set of vertices *roots* in an environment e.

```
Definition dfs1 dfs x e :=
 let: (m1, e1) :=
  dfs [set y in successors x] (add_stack x e) in
 if m1 < sn e then (m1, add_black x e1)
 else (∞, add_sccs x e1).

Definition dfs dfs1 dfs roots e :=
 if [pick x in roots] isn't Some x then (∞, e)
 else let roots' := roots :\ x in
      let: (m1, e1) :=
        if num e x ≠ 0 then (num e x, e) else dfs1 x e in
      let: (m2, e2) := dfs roots' e1 in (minn m1 m2, e2).
```

Then, the two functions are glued together in a recursive function *tarjan_rec* where the parameter n controls the maximal recursive height.

```
Fixpoint tarjan_rec n :=
 if n is n1.+1 then
  dfs (dfs1 (tarjan_rec n1)) (tarjan_rec n1)
 else fun r e ⇒ (∞, e).

Let N := #|V| * #|V|.+1 + #|V|.
Definition tarjan := sccs (tarjan_rec N setT e0).2.
```

If n is not zero (i.e. it is a successor of some *n1*), *tarjan_rec* calls *dfs* taking care that its parameters can only use recursive call to *tarjan_rec* with a smaller recursive height, here *n1*. This ensures termination. A dummy value is returned in the case where n is zero. As both *dfs* and *dfs1* cannot be applied more than the number of vertices, the value N encodes the lexicographic product of the two maximal heights. It gives *tarjan_rec* enough fuel to never encounter the dummy value so *tarjan* correctly terminates the computation. This last statement is of course proved formally later.

The invariants are essentially the same as in the Why3 proof. There are just packaged in a different way so we can express more easily intermediate lemmas between the different packages. We first group together the properties about connectivity

```
Record wf_graph e := WfGraph {
 wf_stack_to_stack :
  {in stack e &, ∀ x y,
   (num e x ≤ num e y) -> gconnect x y};
 wf_stack_to_grays :
  {in stack e, ∀ y,
   ∃ x, [∧ x ∈ grays e, (num e x ≤ num e y) & gconnect y x]
}.
```

The main invariant then collects all the properties

```
Record invariants (e : env) := Invariants {
       inv_wf_color : wf_color e;
        inv_wf_num : wf_num e;
       inv_wf_graph : wf_graph e;
 wf_noblack_towhite : noblack_to_white e;
          inv_sccs : sccs e = black_gsccs e;
}.
```

Pre-conditions are stored in a record and are similar to the ones defined in Why3: all the gray vertices of e are connected to all the elements of *roots*. and all the invariants hold.

```
Definition access_to e (roots : {set V}) :=
 {in gray e & roots, ∀ x y, gconnect x y}.

Record pre_dfs (roots : {set V}) (e : env) := PreDfs {
 pre_access_to  : access_to e roots;
 pre_invariants : invariants e
}.
```

The post-conditions are expressed slightly differently mostly because we take advantage of the expressivity of big operators [1]. The *bigcup* operator (typeset as $\backslash bigcup\_$)

is defined in the Mathematical Component Library and represents indexed union of sets. The *bigmin* operator (typeset as $\backslash min\_$) represents the minimum of a set of natural numbers (and should be included in future version of the Library). Defining the minimum of the empty set is a bit problematic since one would like to preserve the property that the minimum of a subset is never smaller than the minimum of the full set. This is why the *bigmin* does not work directly on sets of natural numbers but on sets of elements of an ordinal type I_n (the type of all the natural numbers smaller than n). This type has the key property of having a maximal element n. This is the value given to the minimum of the empty set. In our use case, as ∞ is defined as the number of vertices plus one, we simple take $n = \infty$.

The post-conditions are then expressed by a record that states that the invariants hold, the next environment is an extension of the old one, the new white vertices have been decremented by the vertices that are reachable from the roots by white vertices and finally the returned value *m* is exactly the smallest number from all the vertices that have lost their white color.

```
Record post_dfs (roots : {set V}) (e e' : env) (m : nat) := PostDfs {
 post_invariants : invariants e';
 post_subenv     : subenv e e';
 post_whites     :
  whites e' = white e :\: \bigcup_(x in roots) wreach e x;
 post_num        :
  m = \min_(y in \bigcup_(x in roots) wreach e x)
        @inord ∞ (num e' y);
}.
```

Note that we have defined the predicate *wreach* to express the reachability through white vertices and we are using the explicit cast *inord* to turn a number associated to a vertex into an element of I_∞.

Now we can state the correctness of *dfs* and *dfs1*

```
Definition dfs_correct
   (dfs : {set V} -> env -> nat * env) roots e :=
 pre_dfs roots e ->
 let (m, e') := dfs roots e in post_dfs roots e e' m.
Definition dfs1_correct
   (dfs1 : V -> env -> nat * env) x e :=
 (x ∈ white e) -> pre_dfs [set x] e ->
 let (m, e') := dfs1 x e in post_dfs [set x] e e' m.
```

where *[set x]* represents the set whose only element is *x*. The two central theorems to prove are then

```
Lemma dfs_is_correct dfs1 dfsrec (roots : {set V}) e :
 (∀ x, x ∈ roots -> dfs1_correct dfs1 x e) ->
 (∀ x, x ∈ roots -> ∀ e1, white e1 \subset white e ->
    dfs_correct dfsrec (roots :\ x) e1) ->
 dfs_correct (dfs dfs1 dfsrec) roots e.
Lemma dfs1_is_correct dfs (x : V) e :
 (dfs_correct dfs [set y | edge x y] (add_stack x e)) ->
 dfs1_correct (dfs1 dfs) x e.
```

$l = 1$	$l \leq 10$	$l \leq 20$	$l \leq 30$	$l = 35$	$l = 70$	$l = 328$
37	25	5	3	1	1	1

Table 2: Sizes (numbers l of lines) of the 73 proofs in the file *tarjan_num*.

They simply state that the results of *dfs* and *dfs1* are correct if their respective recursive calls are correct. The proof of the first lemma is straightforward since *dfs* simply iterates on a list. It is mostly some book-keeping between what is known and what needs to be proved. This is done in about 70 lines. The second one is more intricate and requires 328 lines. Gluing these two theorems together and proving termination gives us an extra 20 lines to prove the theorem

```
Theorem tarjan_rec_terminates n roots e :
 n ≥ #|white e| * #|V|.+1 + #|roots| ->
 dfs_correct (tarjan_rec n) roots e.
```

From this last theorem the correctness of *tarjan* follows directly.

Some quantitative information should be added. The COQ contribution is composed of three files. The *bigmin* file defines the *bigmin* operator and is 160 lines long. The *extra* file defines some notions of graph theory that were not available in the current Mathematical Component Library. For example, it is where conditional reachability is defined. This file is 350 lines long. The main file is *tarjan_num* and is 1185 lines long. It is compiled in 10 seconds with a memory footprint of 900 Mb (half of which is resident) on a Intel® i7 2.60GHz quad-core laptop running Linux. The proofs are performed in the SSREFLECT proof language [12] with very little automation. The proof script is mostly procedural, alternating book-keeping tactics (*move*) with transformational ones (mostly *rewrite* and *apply*), but often intermediate steps are explicitly declared with the *have* tactic. There are more than a hundred of such intermediate steps in the 700 lines of proof of the file *tarjan_num*. Table 2 gives the distribution of the numbers of lines of these proofs. Most of them are one-liners and the only complicated proof is the one corresponding to the lemma *dfs1_is_correct*.

5 The proof in Isabelle/HOL

Isabelle/HOL [19] is the encoding of simply typed higher-order logic in the logical framework Isabelle [20]. Unlike Why3, it is not primarily intended as an environment for program verification and does not contain specific syntax for stating pre- and post-conditions or intermediate assertions in function definitions. Logics and formalisms for program verification have been developed within Isabelle/HOL (e.g., [14]), but they target imperative rather than functional programming, so we simply formalize the algorithm as an Isabelle function. Isabelle/HOL provides an extensive library of data structures and proofs; in this development we mainly rely on the set and list libraries. We start by introducing a *locale*, fixing parameters and assumptions for the remainder of the proof. We explicitly assume that the set of vertices is finite: by default, sets may be infinite in Isabelle/HOL.

```
locale graph =
  fixes vertices :: ν set
    and successors :: ν ⇒ ν set
  assumes finite vertices
      and ∀v ∈ vertices. successors v ⊆ vertices
```

We introduce reachability in graphs using an inductive predicate definition, rather than via an explicit reference to paths as in the Why3 definition. Isabelle then generates appropriate induction theorems for use in proofs.

```
inductive reachable where
  reachable x x
| ⟦y ∈ successors x; reachable y z⟧ ⟹ reachable x z
```

The definition of strongly connected components mirrors that used in Why3. The following lemma states that SCCs are disjoint; its one-line proof is found automatically using *sledgehammer* [2], which heuristically selects suitable lemmas from the set of available facts (including Isabelle's library), invokes several automatic provers, and in case of success reconstructs a proof that is checked by the Isabelle kernel.

```
lemma scc-partition:
  assumes is-scc S and is-scc S' and x ∈ S ∩ S'
  shows   S = S'
```

Environments are represented by records, similar to the formalization in Why3, except that there is no distinction between regular and "ghost" fields. Also, the definition of the well-formedness predicate closely mirrors that used in Why3.[2]

```
record ν env =
  black :: ν set              gray :: ν set
  stack :: ν list             sccs :: ν set set
  sn :: nat                   num :: ν ⇒ int

definition wf_env where wf_env e ≡
  wf_color e ∧ wf_num e
∧ distinct (stack e) ∧ no_black_to_white e
∧ (∀x y. y ≼ x in (stack e) ⟶ reachable x y)
∧ (∀y ∈ set (stack e). ∃g ∈ gray e.
      y ≼ g in (stack e) ∧ reachable y g)
∧ sccs e = { C . C ⊆ black e ∧ is_scc C }
```

The definition of the two mutually recursive functions *dfs1* and *dfs* again closely follows their representation in Why3.

```
function (domintros) dfs1 and dfs where
  dfs1 x e =
   (let (n1,e1) = dfs (successors x)
                      (add_stack_incr x e) in
     if n1 < int (sn e) then (n1, add_black x e1)
     else (let (l,r) = split_list x (stack e1) in
```

[2]We use infix syntax and the symbol ≼ to denote precedence. The correspondence between numbers of vertices in the stack and precedence is asserted by the invariant *wf_num*.

```
      (+∞,
       (| black = insert x (black e1),
         gray = gray e, stack = r, sn = sn e1,
         sccs = insert (set l) (sccs e1),
         num = set_infty l (num e1) |) ))) and
dfs roots e =
 (if roots = {} then (+∞, e)
  else (let x = SOME x. x ∈ roots;
            res1 = (if num e x ≠ -1
                    then (num e x, e)
                    else dfs1 x e);
            res2 = dfs (roots - {x}) (snd res1)
        in  (min (fst res1) (fst res2),
                 snd res2) ))
```

The **function** keyword introduces the definition of a recursive function. Isabelle checks that the definition is well-formed and generates appropriate simplification and induction theorems. Because HOL is a logic of total functions, it introduces two proof obligations: the first one requires the user to prove that the cases in the function definitions cover all type-correct arguments; this holds trivially for the above definitions. The second obligation requires exhibiting a well-founded ordering on the function parameters that ensures the termination of recursive function invocations, and Isabelle provides a number of heuristics that work in many cases. However, the functions defined above will in fact not terminate for arbitrary calls, in particular for environments that assign sequence number -1 to non-white vertices. The *domintros* attribute instructs Isabelle to consider these functions as "partial". More precisely, it introduces an explicit predicate representing the domains for which the functions are defined. This "domain condition" appears as a hypothesis in the simplification rules that mirror the function definitions so that the user can assert the equality of the left- and right-hand sides of the definitions only if the domain predicate holds. Isabelle also proves (mutually inductive) rules for proving when the domain condition is guaranteed to hold. Our first objective is therefore to establish sufficient conditions that ensure the termination of the two functions. Assuming the domain condition, we prove that the functions never decrease the set of colored vertices and that vertices are never explicitly assigned the number -1 by our functions. Denoting the union of gray and black vertices as *colored*, we define the predicate

```
definition colored_num where colored_num e ≡
  ∀v ∈ colored e. v ∈ vertices ∧ num e v ≠ -1
```

and show that this predicate is an invariant of the functions. We can then show that the triple defined as

```
  (vertices - colored e, {x}, 1)
  (vertices - colored e, roots, 2)
```

for the arguments of *dfs1* and *dfs*, respectively, decreases w.r.t. lexicographical ordering on finite subset inclusion and $<$ on natural numbers across recursive function calls, provided that *colored_num* holds when the function is called and x is a white vertex

(resp., *roots* is a set of vertices). These conditions are therefore sufficient to ensure that the domain condition holds:[3]

```
theorem dfs1_dfs_termination:
  ⟦x ∈ vertices - colored e; colored_num e⟧ ⟹
      dfs1_dfs_dom (Inl(x,e))
  ⟦roots ⊆ vertices; colored_num e⟧ ⟹
      dfs1_dfs_dom (Inr(roots,e))
```

The proof of partial correctness follows the same ideas as the proof presented for Why3. We define the pre- and post-conditions of the two functions as predicates in Isabelle. For example, the predicates for *dfs1* are defined as follows:

```
definition dfs1_pre where dfs1_pre e ≡
    wf_env e ∧ x ∈ vertices ∧ x ∉ colored e
  ∧ (∀g ∈ gray e. reachable g x)

definition dfs1_post where dfs1_post x e res ≡
    let n = fst res; e' = snd res
    in  wf_env e' ∧ subenv e e' ∧ roots ⊆ colored e'
      ∧ (∀x ∈ roots. n ≤ num e' x)
      ∧ (n = +∞ ∨ (∃x ∈ roots. ∃y in set (stack e').
                      num e' y = n ∧ reachable x y))
```

We now show the following theorems:

- The pre-condition of each function establishes the pre-condition of every recursive call appearing in the body of that function. For the second recursive call in the body of *dfs* we also assume the post-condition of the first recursive call.
- The pre-condition of each function, plus the post-conditions of each recursive call in the body of that function, establishes the post-condition of the function.

Combining these results, we establish partial correctness:

```
theorem dfs_partial_correct:
  ⟦dfs1_dfs_dom (Inl(x,e)); dfs1_pre x e⟧ ⟹
      dfs1_post x e (dfs1 x e)
  ⟦dfs1_dfs_dom (Inr(roots,e)); dfs_pre roots e⟧ ⟹
      dfs_post roots e (dfs roots e)
```

We define the initial environment and the overall function.

```
definition init_env where init_env ≡
  (| black = {},     gray = {},
    stack = [],      sccs = {},
    sn = 0,          num = λ_. -1 |)
definition tarjan where tarjan ≡
  sccs (snd (dfs vertices init_env))
```

[3]Observe that Isabelle introduces a single operator corresponding to the two mutually recursive functions whose domain is the disjoint sum of the domains of both functions.

$i = 1$	$i \leq 5$	$i \leq 10$	$i \leq 20$	$i \leq 30$	$i = 35$	$i = 43$	$i = 48$
28	8	4	1	2	1	1	1

Table 3: Distribution of interactions in the Isabelle proofs.

It is trivial to show that the arguments to the call of *dfs* in the definition of *tarjan* satisfy the pre-condition of *dfs*. Putting together the theorems establishing termination and partial correctness, we obtain the desired total correctness results.

```
theorem dfs_correct:
  dfs1_pre x e ⟹ dfs1_post x e (dfs1 x e)
  dfs_pre roots e ⟹ dfs_post roots e (dfs roots e)
theorem tarjan_correct:
  tarjan = { C . is_scc C ∧ C ⊆ vertices }
```

The intermediate assertions appearing in the Why3 code guided the overall proof: they are established either as separate lemmas or as intermediate steps within the proofs of the above theorems. Similar as in the Coq proof, the overall induction proof was explicitly decomposed into individual lemmas as laid out above. In particular, whereas Why3 identifies the predicates that can be used from the function code and its annotation with pre- and post-conditions, these assertions appear explicitly in the intermediate lemmas used in the proof of theorem *dfs_partial_correct*. The induction rules that Isabelle generated from the function definitions was helpful for finding the appropriate decomposition of the overall correctness proof.

Despite the extensive use of *sledgehammer* for invoking automatic back-end provers, including the SMT solvers CVC4 and Z3, from Isabelle, we found that in comparison to Why3, significantly more user interactions were necessary in order to guide the proof. Although many of those were straightforward, a few required thinking about how a given assertion could be derived from the facts available in the context. Table 3 indicates the distribution of the number of interactions used for the proofs of the 46 lemmas the theory contains. These numbers cannot be compared directly to those shown in Table 2 for the Coq proof because an Isabelle interaction is typically much coarser-grained than a line in a Coq proof. As in the case of Why3 and Coq, the proofs of partial correctness of *dfs1* (split into two lemmas following the case distinction) and of *dfs* required the most effort. It took about one person-month to carry out the case study, starting from an initial version of the Why3 proof. Processing the entire Isabelle theory on a laptop with a 2.7 GHz Intel® Core i5 (dual-core) processor and 8 GB of RAM takes 35 seconds of CPU time.

6 General comments about the proof

Our proof refers to colors, finite sets, and the stack, although the general argument seems to be about properties of strongly connected components in spanning trees. The algorithmician would explain the algorithm with spanning trees as in Tarjan's article. It would be nice to extract a program from such a proof, but beside the fact that this

is not so easy, programmers like to understand the proof in terms of variables and data that their program is using.

A first version of the formal proof used *ranks* in the working stack and a flat representation of environments by adding extra arguments to functions for the black, gray, sccs sets and the stack. That was perfect for the automatic provers of Why3. But after remodelling the proof in Coq and Isabelle/HOL, it was simpler to gather these extra arguments in records and have a single extra argument for environments. Also *ranks* disappeared leaving space to the *num* function and the precedence relation, which are easier to understand. The automatic provers have more difficulties with the inlining of environments, but with a few hints they could still succeed.

Finally, coloring of vertices is usual for graph algorithms, but we are aware that a proof without colors is feasible and has indeed been done without colors in Coq (see [8]). The stack used in our algorithm is also not necessary since it is just used to efficiently output new strongly connected components. The proof can be implemented with just finite sets, and the components will be obtained by computing differences between visited sets of vertices. However, the stack-based formulation ensures that the algorithm works in linear time, and then it must be present in the proof, and its content must be related to the visited sets of vertices.

There is thus a balance between the concision of the proof and its relation to the real algorithm. In our presentation, we therefore have allowed for a few redundancies.

7 Conclusion

The formal proof expressed in this article was initially designed and implemented in Why3 [7] after a long process, nearly a 2-year half-time work with many attempts of proofs about various graph algorithms (depth first search, Kosaraju strong connectivity, bi-connectivity, articulation points, minimum spanning tree). The big advantage of Why3 is the clear separation between programs and the logic with Hoare-logic style assertions and pre-/post-conditions about functions. It makes the correctness proof quite readable for a programmer. Also first-order logic is easy to understand. Moreover, one can prove partial correctness without caring about termination.

Another important feature of Why3 is its interface with off-the-shelf theorem provers (mainly SMT provers). Thus the system benefits of the current technology in theorem provers and clerical sub-goals can be delegated to these provers, which makes the overall proof shorter and easier to understand. Although the proof must be split in more elementary pieces, this has the benefit of improving its readability. Several hints about inlining or induction reasoning are still needed. Also, despite a useful XML file that records sessions and facilitates incremental proofs, sometimes seemingly minor modifications to the formulation of the algorithm or the predicates may result in the provers no longer being able to handle a proof obligation automatically.

The COQ and Isabelle proofs were inspired from the Why3 proof. Their development therefore required much less time although their text is longer. The COQ proof uses the SSREFLECT macros and the Mathematical Components library, which helps reduce the size of the proof compared to classical COQ. The proof also uses the bigops library and several other higher-order features which makes the proof more abstract

and closer to Tarjan's original proof.

In Coq, recursion cannot be used without proving termination. This requires an agile treatment of mutually recursive definitions of functions. Partial correctness can be proved by considering the functionals of which the recursive definitions are the fixed point, and passing as arguments the pre/post-conditions of these functions. Moreover the recursive definitions take as extra argument the number of recursive calls, in order to postpone the termination argument.

Our Coq proof does not use significant automation[4]. All details are explicitly expressed, but many of them were already present in the Mathematical Components library. Moreover, a proof certificate is produced and a functional program could in principle be extracted. The absence of automation makes the system very stable to use since the proof script is explicit, but it requires a higher degree of expertise from the user. Still, this lack of automation gives the user a direct feedback of how well the definitions work together. This led us to develop an alternative and more concise (50% shorter) formalization without colors [8].

The Isabelle/HOL proof was the last one to be implemented. It closely follows the Why3 proof, and can be seen as a mid-point between the Why3 and Coq proofs. It uses higher order logic and the level of abstraction is close to the one of the Coq proof, although more readable in this case study. The proof makes use of Isabelle's extensive support for automation. In particular, *sledgehammer* [2] was very useful for finding individual proof steps. It heuristically selects lemmas and facts available in the context and then calls automatic provers (SMT solvers and superposition-based provers for first-order logic). When one of these provers finds a proof, sledgehammer attempts to find a proof that can be certified by the Isabelle kernel, using various proof methods such as combinations of rewriting and first-order reasoning (blast, fastforce etc.), calls to the *metis* prover or reconstruction of SMT proofs through the *smt* proof method. Unlike in Why3, the automatic provers used to find the initial proof are not part of the trusted code base because ultimately the proof is checked by the kernel. The price to pay is that the degree of automation in Isabelle is still significantly lower compared to Why3. Adapting the proof to modified definitions was fast: the Isabelle/jEdit GUI eagerly processes the proof script and quickly indicates those steps that require attention.

The Isabelle proof also faces the termination problem to achieve general consistency. Since termination cannot be ensured for arbitrary arguments, the treatment of termination is delayed with the use of the *domintros* predicate. The proofs of termination and of partial correctness are independent; in particular, we obtain a weaker predicate ensuring termination than the one used for partial correctness. Although the basic principle of the termination proof is very similar to the Coq proof and relies on considering functionals of which the recursive functions are fixpoints, the technical formulation is more flexible because we rely on proving well-foundedness of an appropriate relation rather than computing an explicit upper bound on the number of recursive calls.

One strong point of Isabelle/HOL is its nice LaTeX output and the flexibility of its

[4]Hammers exist for Coq [9, 10] but unfortunately they currently perform badly when used in conjunction with the Mathematical Components library.

parser, supporting mathematical symbols. Combined with the hierarchical Isar proof language [28], the proof is in principle understandable without actually running the system, although some familiarity with the system is still required.

In the end, the three systems Why3, COQ, and Isabelle/HOL are mature, and each one has its own advantages w.r.t. readability, expressivity, stability or mechanization. Coming up with invariants that are both strong enough and understandable was by far the hardest part in this work. This effort requires creativity and understanding, although proof assistants provide some help: missing predicates can be discovered by understanding which parts of the proof fail. We think that formalizing the proof in all three systems was very rewarding and helped us better understand the state of the art in computer-aided deductive program verification. It could be also interesting to experiment this proof in other formal systems and establish comparisons based on this quite challenging example[5].

Another interesting work would be to verify an implementation of this algorithm with imperative programs and concrete data structures. This will complexify the proof, since mutable variables and mutable data structures have to be considered. Several attempts were already exposed [4, 5, 14] and it would be interesting to also develop them simultaneously in various formal systems and to understand how these proofs can be derived from ours.

A final and totally different remark is about teaching of algorithms. Do we want students to formally prove algorithms, or to present algorithms with assertions, pre- and post-conditions, and make them prove these assertions informally as exercises? In both cases, we believe that our work could make a useful contribution.

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