Reduced Order Finite Time Observers for Time-Varying Nonlinear Systems

Frédéric Mazenc, Saeed Ahmed, Michael Malisoff

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Abstract— **We construct finite-time reduced order observers for a broad class of nonlinear time-varying continuous-time systems. We illustrate our results using a tracking problem for nonholonomic systems in chained form.**

Index Terms— **Observer, stability, time-varying**

I. INTRODUCTION

The problem of estimating the values of solutions of systems when some variables cannot be measured is of great relevance from the theoretical and applied points of view. Asymptotic observers, such as for instance the celebrated Luenberger observer [5], [6], are very popular and many observers for families of nonlinear systems have been constructed. However, they provide a useful estimate only after a transient period during which they cannot be used, which can be a disadvantage in some applications like fault detection where a finite-time state estimation is desirable [12].

To obtain an exact estimate of the solutions of a system in an arbitrary short amount of time, finite time observers have been proposed. Some use nonsmooth functions; see for instance [4], [11]. Their designs are based on homogeneous properties which preclude the possibility of deriving smooth observers from this technique. Another type of finite time observers has been developed. They are smooth and use past values of the output or dynamic extensions. They have been proposed a few decades ago for linear systems; see in particular [2], [13], and our other references below. More recently, finite time observers were designed for nonlinear systems, e.g., in [9], [10], and [14]. They apply to systems whose vector field is time-invariant when the output is set to zero and provide estimates of all the state variables.

Since systems are frequently time-varying and tracking problems can be recast into stabilization problems of equilibria of time-varying systems, and since the measured components of the state do not need to be estimated, this paper adapts the main results of [9] and [14] to construct finite-time reduced order observers for a family of nonlinear time-varying systems. The observer we will build only gives estimates of the unmeasured variables, as does the asymptotic observer proposed for instance in [3] and [1, Chapt. 4, Sec. 4.4.3]. This feature presents the following technical advantages. This provides a technical advantage over other observers that would require fundamental solutions of time-varying linear systems whose dimensions equal that of the original systems, because of the difficulty of finding formulas for fundamental solutions of higher dimensional time-varying linear systems. To the best of our knowledge, finite-time reduced order observers for nonlinear time-varying systems are proposed for the first time in the present paper.

Mazenc is with EPI DISCO INRIA-Saclay, Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), CNRS, CentraleSupélec, Université Paris-Sud, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France, frederic.mazenc@l2s.centralesupelec.fr.

Ahmed is with Department of Electrical and Electronics Engineering, Graduate School of Engineering and Science, Bilkent University, Ankara 06800, Turkey, ahmed@ee.bilkent.edu.tr.

Malisoff is with the Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA, malisoff@lsu.edu.

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After providing an introductory result in Section II, we state and prove our main reduced order finite-time observer design for time-varying systems in Section III. Our illustration in Section IV demonstrates our approach, in a tracking problem for a nonholonomic system in chained form, and we conclude with ideas for future research in Section V.

We use standard notation, which will be simplified whenever no confusion can arise from the context and the dimensions of our Euclidean spaces are arbitrary, unless otherwise noted. Any $k \times l$ matrix whose entries are all 0 is denoted 0. The Euclidean norm, and the induced norm of matrices, are denoted by $|\cdot|$, and I is the identity matrix. For any constant $\tau > 0$, any continuous function $\varphi : [-\tau, +\infty) \to \mathbb{R}^n$ and all $t \geq 0$, we define φ_t by $\varphi_t(m) = \varphi(t+m)$ for all $m \in [-\tau, 0]$. We denote by C_{in} the set $C([-\tau, 0])$ of all continuous functions $\varphi : [-\tau, 0] \to \mathbb{R}^n$. For any continuous function $\Omega : [-\tau, +\infty) \to \mathbb{R}^{n\times n}$, we let Φ_Ω denote the function such that

$$\frac{\partial \Phi_\Omega}{\partial t}(t, t_0) = -\Phi_\Omega(t, t_0)\Omega(t) \tag{1}$$

and $\Phi_\Omega(t_0, t_0) = I$ for all $t \in \mathbb{R}$ and $t_0 \in \mathbb{R}$. Then $\mathcal{M}(t, s) = \Phi_\Omega^{-1}(t, s)$ is the fundamental solution associated to Ω for the system $\dot{x} = \Omega(t)x$; see [15, Lemma C.4.1].

II. INTRODUCTORY RESULT

Before proving our general result for time-varying systems in Section III, we provide an introductory result that shows how to construct reduced order finite-time observers for a simple family of systems. We consider the system

$$\dot{x}(t) = Ax(t) + \delta(t) \tag{2}$$

with x valued in $\mathbb{R}^n$ and where $\delta : [0, +\infty) \to \mathbb{R}^n$ is a piecewise-continuous locally bounded function. We assume that the output

$$y(t) = Cx(t) \tag{3}$$

is valued in $\mathbb{R}^p$ for any dimension $p \leq n$ and C is of full rank. Also, we assume that the pair (A, C) is observable. Since C is of full rank, there exist matrices A_1 and A_2, a linear change of coordinates

$$x_T = C_T x = \begin{bmatrix} y \\ x_r \end{bmatrix} \tag{4}$$

and functions δ_i for $i = 1, 2$ that are piecewise-continuous with respect to their first argument and linear with respect to y such that the x_T system can be written as follows:

$$\begin{cases} \dot{y}(t) &= A_1 x_r(t) + \delta_1(t, y(t)) \\ \dot{x}_r(t) &= A_2 x_r(t) + \delta_2(t, y(t)) \ . \end{cases} \tag{5}$$

Since the pair (A, C) is observable, it follows that the pair (A_2, A_1) is observable; see [6, pp. 304-306]. Since (A_2, A_1) is observable, one can use [9, Lemma 1] to prove that there are a matrix $L \in \mathbb{R}^{(n-p)\times p}$ and a constant $\tau > 0$ such that

$$M_\tau = e^{-A_2\tau} - e^{-H\tau} \tag{6}$$

with the choice $H = A_2 + LA_1$ is invertible.

In terms of the new variable

$$x_s = x_r + Ly \tag{7}$$

we can then use simple calculations to obtain

$$\dot{x}_s(t) = (A_2 + LA_1)x_r(t) + \delta_2(t, y(t)) + L\delta_1(t, y(t)). \tag{8}$$

From the definition of H, it follows that

$$\dot{x}_s(t) = Hx_s(t) + Ky(t) + \delta_3(t, y(t)) \tag{9}$$

where $K = -(A_2 + LA_1)L$ and $\delta_3 = \delta_2 + L\delta_1$. By integrating the second equation in (5) and (9), we obtain

$$\begin{aligned} &x_r(t-\tau) = e^{-A_2\tau} x_r(t) \\ &- \int_{t-\tau}^{t} e^{A_2(t-m-\tau)} \delta_2(m, y(m)) \mathrm{d}m \end{aligned} \tag{10}$$

and

$$\begin{aligned} &x_s(t-\tau) = e^{-H\tau} x_s(t) \\ &- \int_{t-\tau}^{t} e^{H(t-m-\tau)} [Ky(m) + \delta_3(m, y(m))] \mathrm{d}m \end{aligned} \tag{11}$$

for all $t \geq \tau$. From the definition (7), we deduce that

$$\begin{aligned} &x_r(t-\tau) = e^{-H\tau} x_r(t) + e^{-H\tau} Ly(t) - Ly(t-\tau) \\ &- \int_{t-\tau}^{t} e^{H(t-m-\tau)} [Ky(m) + \delta_3(m, y(m))] \mathrm{d}m \end{aligned} \tag{12}$$

for all $t \geq \tau$. Subtracting (12) from (10) and then using the definition of M_τ in (6), we deduce that

$$\begin{aligned} &M_\tau x_r(t) = e^{-H\tau} Ly(t) - Ly(t-\tau) \\ &- \int_{t-\tau}^{t} e^{H(t-m-\tau)} [Ky(m) + \delta_3(m, y(m))] \mathrm{d}m \\ &+ \int_{t-\tau}^{t} e^{A_2(t-m-\tau)} \delta_2(m, y(m)) \mathrm{d}m \end{aligned} \tag{13}$$

for all $t \geq \tau$. Since M_τ is invertible, it follows that

$$x_r(t) = \hat{x}_r(t) \tag{14}$$

holds for all $t \geq \tau$, where

$$\begin{aligned} &\hat{x}_r(t) \\ &= -M_\tau^{-1} \int_{t-\tau}^{t} e^{H(t-m-\tau)} [Ky(m) + \delta_2(m, y(m)) \\ &\quad + L\delta_1(m, y(m))] \mathrm{d}m \\ &+ M_\tau^{-1} \int_{t-\tau}^{t} e^{A_2(t-m-\tau)} \delta_2(m, y(m)) \mathrm{d}m \\ &+ M_\tau^{-1} \left[e^{-H\tau} Ly(t) - Ly(t-\tau) \right] \ . \end{aligned} \tag{15}$$

Thus, when δ_1 and δ_2 are known, the formula (15) provides the exact value of $x_r(t)$ for all $t \geq \tau$.

III. Main Result for Time-Varying Systems

A. Statement of Main Result and Remarks

In this section, we show how the finite-time observer design of the previous section adapts to time-varying nonlinear systems of the type

$$\begin{cases} \dot{z}(t) &= A_1(t) x_r(t) + \delta_1(t, z(t)) \\ \dot{x}_r(t) &= A_2(t) x_r(t) + \delta_2(t, z(t)) \end{cases} \tag{16}$$

where z is valued in $\mathbb{R}^p$, x_r is valued in $\mathbb{R}^{n-p}$, the output is

$$y(t) = z(t) + \epsilon(t) \tag{17}$$

where $\epsilon(t)$ is a piecewise continuous function that is bounded by a constant $\overline{\epsilon} \geq 0$, the functions A_i for $i = 1$ and 2 are piecewise continuous and bounded, and δ_1 and δ_2 are functions that are piecewise continuous with respect to t and locally Lipschitz with respect to z and such that there is a nonnegative valued continuous function $\overline{\delta}$ such that

$$|\delta_1(t, z)| + |\delta_2(t, z)| \leq \overline{\delta}(|z|) \tag{18}$$

for all $t \geq 0$ and $z \in \mathbb{R}^p$.

Remark 1: The special structure of the system (16) does not limit the family of linear systems to which our approach applies because, as explained in the previous section, any system of the type

$$\dot{X} = A(t)X + \mathcal{F}(t, Y) \tag{19}$$

with an output $Y = CX$ with C of full rank can be transformed through a linear time-invariant change of coordinates into a system of the form (16). The term $\epsilon(t)$ in (17) represents a disturbance in the measurements, which is of interest because in practice, measurements are frequently affected by disturbances. □

We introduce the following assumption:

Assumption 1: There are a constant $\tau > 0$ and a bounded function L of class C^1 with a bounded first derivative such that for all $t \in \mathbb{R}$, the matrix

$$\Lambda(t) = \Phi_{A_2}(t, t-\tau) - \Phi_H(t, t-\tau) \tag{20}$$

with $H(t) = A_2(t) + L(t)A_1(t)$ is invertible. □

See the remarks below on ways to check the preceding assumption. Let us define

$$\begin{aligned} &\delta_3(t, z) = L(t)\delta_1(t, z) + \delta_2(t, z) \text{ and} \\ &\delta_4(t, z) = -[D(t)z + \delta_3(t, z)], \end{aligned} \tag{21}$$

where

$$D(t) = \dot{L}(t) - H(t)L(t). \tag{22}$$

Let

$$\begin{aligned} &\hat{x}_r(t) = \\ &\Lambda(t)^{-1} \int_{t-\tau}^{t} [\Phi_{A_2}(m, t-\tau)\delta_2(m, y(m) - \epsilon(m)) \\ &+ \Phi_H(m, t-\tau)\delta_4(m, y(m) - \epsilon(m))] \, \mathrm{d}m \\ &+ \Lambda(t)^{-1} [\Phi_H(t, t-\tau)L(t)(y(t) - \epsilon(t)) \\ &- L(t-\tau)(y(t-\tau) - \epsilon(t-\tau))] \end{aligned} \tag{23}$$

for all $t \geq 0$. We will prove:

Theorem 1: Let (16) satisfy Assumption 1. Then

$$x_r(t) = \hat{x}_r(t) \tag{24}$$

holds for all $t \geq \tau$.

Remark 2: In general, one can check easily that Assumption 1 is satisfied when $n - p = 1$ because then Φ_{A_2} and Φ_H take the simple one dimensional forms

$$\Phi_{A_2}(t, t_0) = e^{-\int_{t_0}^{t} A_2(m)\mathrm{d}m} \text{ and } \Phi_H(t, t_0) = e^{-\int_{t_0}^{t} H(m)\mathrm{d}m}.$$

When $n - p > 1$, checking this assumption may be more difficult because then formulas for Φ_{A_2} and Φ_H would be more difficult to compute. However, it is easier to determine explicit expressions for Φ_{A_2} and Φ_H than for Φ_A where A is the function in (19) because the dimension of A is larger than those of A_2 and H. This can be an advantage of the reduced-order approach over the full order one. □

Remark 3: If there are a function L, two constants $\tau > 0$ and $\varpi \in (0, 1)$ such that $|\Phi_{A_2}(t, t-\tau)^{-1}\Phi_H(t, t-\tau)| \leq \varpi$ for all $t \geq 0$, then Assumption 1 is satisfied and Λ^{-1} is bounded. Indeed, in this case the matrix $I - \Phi_{A_2}(t, t-\tau)^{-1}\Phi_H(t, t-\tau)$ is invertible for all $t \geq 0$ and

$$\begin{array}{l} \left[I - \Phi_{A_2}(t, t-\tau)^{-1}\Phi_H(t, t-\tau)\right]^{-1} = \\ \sum\limits_{k=0}^{\infty} \left[\Phi_{A_2}(t, t-\tau)^{-1}\Phi_H(t, t-\tau)\right]^k \end{array} \tag{25}$$

for all $t \geq 0$, which implies that

$$\left|\left[I - \Phi_{A_2}(t, t-\tau)^{-1}\Phi_H(t, t-\tau)\right]^{-1}\right| \leq \frac{1}{1-\varpi} \tag{26}$$

for all $t \geq 0$. We deduce that $\Lambda(t)^{-1}$ is well-defined for all $t \geq 0$ and

$$\begin{array}{l} \left|\Lambda(t)^{-1}\right| = \\ \left|\left(I - \Phi_{A_2}(t, t-\tau)^{-1}\Phi_H(t, t-\tau)\right)^{-1} \Phi_{A_2}(t, t-\tau)^{-1}\right| \\ \leq \frac{\left|\Phi_{A_2}(t, t-\tau)^{-1}\right|}{1-\varpi}, \end{array}$$

which is bounded by a constant because $\Phi_{A_2}(t, t-\tau)^{-1}$ is bounded. Also, the function Λ^{-1} is bounded if the system is periodic because then Λ^{-1} is continuous and periodic. □

Remark 4: When ϵ is unknown, the exact estimate (24) cannot be used because ϵ is present in (23). Fortunately, we can use equation (24) to obtain the approximate observer

$$\begin{array}{l} x_r^*(t) = \\ \Lambda(t)^{-1} \int_{t-\tau}^{t} \left[\Phi_{A_2}(m, t-\tau)\delta_2(m, y(m))\right. \\ \left.+\Phi_H(m, t-\tau)\delta_4(m, y(m))\right] \mathrm{d}m \\ +\Lambda(t)^{-1} \left[\Phi_H(t, t-\tau)L(t)y(t) - L(t-\tau)y(t-\tau)\right]. \end{array} \tag{27}$$

Since the functions A_2 and H are bounded, we can easily use Gronwall's inequality to check that $\sup_{m \in [t-\tau, t]} |\Phi_H(m, t-\tau)|$ and $\sup_{m \in [t-\tau, t]} |\Phi_{A_2}(m, t-\tau)|$ are bounded functions of t. We deduce that if δ_1 and δ_2 are globally Lipschitz with respect to their second argument and $\Lambda(t)^{-1}$ is bounded, then there is a constant $l_a > 0$ such that

$$|x_r^*(t) - x_r(t)| \leq l_a |\epsilon|_\infty \tag{28}$$

for all $t \geq \tau$. □

B. Proof of Theorem 1

The variable

$$x_v(t) = \Phi_{A_2}(t, 0)x_r(t) \tag{29}$$

is such that

$$\begin{array}{rcl} \dot{x}_v(t) & = & -\Phi_{A_2}(t, 0)A_2 x_r(t) \\ & & +\Phi_{A_2}(t, 0)[A_2(t)x_r(t) + \delta_2(t, z(t))] \\ & = & \Phi_{A_2}(t, 0)\delta_2(t, z(t)). \end{array} \tag{30}$$

By integrating (30) between $t - \tau$ and $t \geq \tau$, we obtain

$$x_v(t) = x_v(t-\tau) + \int_{t-\tau}^{t} \Phi_{A_2}(m, 0)\delta_2(m, z(m))\mathrm{d}m. \tag{31}$$

From the definition of x_v, it straightforwardly follows that

$$\begin{array}{l} \Phi_{A_2}(t-\tau, 0)^{-1}\Phi_{A_2}(t, 0)x_r(t) = x_r(t-\tau) \\ + \int_{t-\tau}^{t} \Phi_{A_2}(t-\tau, 0)^{-1}\Phi_{A_2}(m, 0)\delta_2(m, z(m))\mathrm{d}m. \end{array} \tag{32}$$

For any continuous function $\Omega : \mathbb{R} \to \mathbb{R}^{n \times n}$, the function Ψ_Ω defined as $\Psi_\Omega(t, t_0) = \Phi_\Omega(t, t_0)^\top$ satisfies

$$\frac{\partial \Psi_\Omega}{\partial t}(t, t_0) = -\Omega(t)^\top \Psi_\Omega(t, t_0). \tag{33}$$

It follows from the semigroup property of flow maps that $\Psi_\Omega(t, 0) = \Psi_\Omega(t, t-\tau)\Psi_\Omega(t-\tau, 0)$. Consequently $\Psi_\Omega(t, 0)^\top = \Psi_\Omega(t-\tau, 0)^\top \Psi_\Omega(t, t-\tau)^\top$. Also, we have $\Psi_\Omega(m, 0) = \Psi_\Omega(m, t-\tau)\Psi_\Omega(t-\tau, 0)$, which implies that $\Psi_\Omega(m, 0)^\top = \Psi_\Omega(t-\tau, 0)^\top \Psi_\Omega(m, t-\tau)^\top$ for all $m \geq t-\tau$. It follows that $\Phi_\Omega(t, 0) = \Phi_\Omega(t-\tau, 0)\Phi_\Omega(t, t-\tau)$ and $\Phi_\Omega(m, 0) = \Phi_\Omega(t-\tau, 0)\Phi_\Omega(m, t-\tau)$. From the preceding identities and (32), we deduce that

$$\begin{array}{l} \Phi_{A_2}(t, t-\tau)x_r(t) = x_r(t-\tau) \\ + \int_{t-\tau}^{t} \Phi_{A_2}(m, t-\tau)\delta_2(m, z(m))\mathrm{d}m. \end{array} \tag{34}$$

Also, the choice $x_s(t) = x_r(t) + L(t)z(t)$ gives

$$\begin{array}{rcl} \dot{x}_s(t) & = & A_2(t)x_r(t) + \delta_2(t, z(t)) + \dot{L}(t)z(t) \\ & & +L(t)[A_1(t)x_r(t) + \delta_1(t, z(t))] \\ & = & H(t)x_r(t) + \dot{L}(t)z(t) + \delta_3(t, z(t)) \\ & = & H(t)x_s(t) + [\dot{L}(t) - H(t)L(t)]z(t) \\ & & +\delta_3(t, z(t)). \end{array} \tag{35}$$

Arguing as we did to prove (34), we obtain

$$\begin{array}{l} \Phi_H(t, t-\tau)x_s(t) = x_s(t-\tau)+ \\ \int_{t-\tau}^{t} \Phi_H(m, t-\tau)[D(m)z(m) + \delta_3(m, z(m))]\mathrm{d}m \end{array} \tag{36}$$

where D is the function defined in (22), for all $t \geq \tau$. From the definition of x_s, we deduce that

$$\begin{array}{l} \Phi_H(t, t-\tau)[x_r(t) + L(t)z(t)] \\ = x_r(t-\tau) + L(t-\tau)z(t-\tau) \\ + \int_{t-\tau}^{t} \Phi_H(m, t-\tau)[D(m)z(m) + \delta_3(m, z(m))]\mathrm{d}m. \end{array} \tag{37}$$

By reorganizing the terms, we obtain

$$\begin{array}{l} \Phi_H(t, t-\tau)x_r(t) = x_r(t-\tau) \\ -\Phi_H(t, t-\tau)L(t)z(t) + L(t-\tau)z(t-\tau) \\ + \int_{t-\tau}^{t} \Phi_H(m, t-\tau)[D(m)z(m) + \delta_3(m, z(m))]\mathrm{d}m. \end{array} \tag{38}$$

By subtracting (38) from (34), we obtain

$$\begin{array}{l}\Lambda(t)x_r(t) = \int_{t-\tau}^{t} \Phi_{A_2}(m, t-\tau)\delta_2(m, z(m))\mathrm{d}m \\ -\int_{t-\tau}^{t} \Phi_H(m, t-\tau)[D(m)z(m)+\delta_3(m, z(m))]\mathrm{d}m \\ +\Phi_H(t, t-\tau)L(t)z(t) - L(t-\tau)z(t-\tau)\end{array} \quad (39)$$

for all $t \geq \tau$. Assumption 2 ensures that for all $t \in \mathbb{R}$, $\Lambda(t)$ is invertible, which implies that (24) is satisfied.

IV. Illustration

A. The studied problem

To illustrate Theorem 1, let us consider the following system from [7, p. 137]:

$$\left\{\begin{array}{lcl}\dot{\xi}_4 &=& \xi_3 v_1 \\ \dot{\xi}_3 &=& \xi_2 v_1 \\ \dot{\xi}_2 &=& v_2 \\ \dot{\xi}_1 &=& v_1\end{array}\right. \quad (40)$$

with $(\xi_1, \xi_2, \xi_3, \xi_4)$ valued in $\mathbb{R}^4$ and the input (v_1, v_2) valued in $\mathbb{R}^2$. Then (40) is a nonholonomic system in chained form.

We assume that the variables ξ_4, ξ_3 and ξ_1 are measured, but that ξ_2 is not measured. Also, we assume that there is no disturbance in the measurement of ξ_2. Let us consider the problem of making the system (40) track

$$(\xi_{1r}(t), \xi_{2r}(t), \xi_{3r}(t), \xi_{4r}(t)) = (2\sin(t), 0, 0, 0) \ . \quad (41)$$

We can recast this as a problem of stabilizing the origin of a time-varying system using the classical time-varying change of variables

$$x_1 = \xi_1 - \xi_{1r}(t) \quad (42)$$

and by selecting the feedback

$$v_1(t, x_1) = -2\mathrm{sign}(x_1)\sqrt{|x_1|} + 2\cos(t) \quad (43)$$

where sign is the function defined by $\mathrm{sign}(m) = \frac{m}{|m|}$ when $m \neq 0$ and $\mathrm{sign}(0) = 0$. They result in

$$\left\{\begin{array}{lcl}\dot{\xi}_4 &=& \xi_3\left[-2\mathrm{sign}(x_1)\sqrt{|x_1|} + 2\cos(t)\right] \\ \dot{\xi}_3 &=& \xi_2\left[-2\mathrm{sign}(x_1)\sqrt{|x_1|} + 2\cos(t)\right] \\ \dot{\xi}_2 &=& v_2(t) \\ \dot{x}_1 &=& -2\mathrm{sign}(x_1)\sqrt{|x_1|} \ .\end{array}\right. \quad (44)$$

We require v_2 to be bounded, i.e., that there is a constant $\overline{v}_2 > 0$ such that $|v_2(t)| \leq \overline{v}_2$ for all $t \geq 0$. As an immediate consequence, it follows that the finite escape time phenomenon does not occur and $\xi_2(t)$ is bounded by an affine function of t. By integrating the last equation of (44), we deduce that $\sqrt{|x_1(t)|} = \sqrt{|x_1(0)|} - t$ when $t \in [0, \sqrt{|x_1(0)|}]$ and $x_1(t) = 0$ for all $t \geq \sqrt{|x_1(0)|}$. We deduce that for all $t \geq \sqrt{|x_1(0)|}$, we have $\xi_2(t)x_1(t) = 0$.

B. Observer Design

Since $\xi_2(t)x_1(t) = 0$ holds for all $t \geq \sqrt{|x_1(0)|}$, and since only ξ_2 is not measured, our next objective is now to construct an observer for the two dimensional system

$$\left\{\begin{array}{lcl}\dot{\xi}_3 &=& 2\cos(t)\xi_2 \\ \dot{\xi}_2 &=& v_2(t)\end{array}\right. \quad (45)$$

for all $t \geq \sqrt{|x_1(0)|}$. With the notation of the previous section, this system can be rewritten as

$$\left\{\begin{array}{lcl}\dot{z}(t) &=& A_1(t)x_r(t) + \delta_1(t, z(t)) \\ \dot{x}_r(t) &=& \delta_2(t, z(t))\end{array}\right. \quad (46)$$

with $x_r(t) = \xi_2(t)$, the output $z(t) = \xi_3(t)$,

$$\begin{array}{l}A_1(t) = 2\cos(t) \ , \ A_2(t) = 0 \ , \ \epsilon(t) = 0, \\ \delta_1(t, z) = 0 \ , \text{ and } \delta_2(t, z) = v_2(t).\end{array} \quad (47)$$

We next check that Theorem 1 applies, with $\tau = 2\pi$ and $y = \xi_3$. Choosing $L(t) = -2\cos(t)$, we obtain $H(t) = A_2(t) + L(t)A_1(t) = -4\cos(t)^2$. Hence,

$$\Phi_H(t, s) = e^{4\int_s^t \cos^2(\ell)\mathrm{d}\ell} \quad (48)$$

for all $t \in \mathbb{R}$ and $s \in \mathbb{R}$. Since $\Phi_{A_2}(t, s) = 1$, the function Λ from (20) in Assumption 1 is $\Lambda(t) = 1 - e^{4\pi}$ for all $t \in \mathbb{R}$. We conclude that Assumption 1 is satisfied. Thus Theorem 1 applies, and provides the estimate

$$\begin{array}{l}\hat{x}_r(t) = \frac{2\cos(t)}{1-e^{4\pi}}\left[-e^{4\pi}\xi_3(t) + \xi_3(t-2\pi)\right] \\ +\frac{1}{1-e^{4\pi}}\int_{t-2\pi}^{t}\left(1 - e^{4\int_{t-2\pi}^{m}\cos^2(\ell)\mathrm{d}\ell}\right)v_2(m)\mathrm{d}m \\ +\frac{2}{1-e^{4\pi}}\int_{t-2\pi}^{t} e^{4\int_{t-2\pi}^{m}\cos^2(\ell)\mathrm{d}\ell}(4\cos^3(m) \\ \quad -\sin(m))\xi_3(m)\mathrm{d}m.\end{array} \quad (49)$$

We conclude that

$$\begin{array}{l}\xi_2(t) = \frac{2\cos(t)}{1-e^{4\pi}}\left[-e^{4\pi}\xi_3(t) + \xi_3(t-2\pi)\right] \\ +\frac{1}{1-e^{4\pi}}\int_{t-2\pi}^{t}\left(1 - e^{4\int_{t-2\pi}^{m}\cos^2(\ell)\mathrm{d}\ell}\right)v_2(m)\mathrm{d}m \\ +\frac{2}{1-e^{4\pi}}\int_{t-2\pi}^{t} e^{4\int_{t-2\pi}^{m}\cos^2(\ell)\mathrm{d}\ell}(4\cos^3(m) \\ \quad -\sin(m))\xi_3(m)\mathrm{d}m\end{array} \quad (50)$$

for all $t \geq \max\left\{2\pi, \sqrt{|x_1(0)|}\right\}$.

C. Output feedback tracking

In this section, we illustrate how our estimate (50) can be used to solve a tracking problem that we described in Section IV-A. We design a state feedback for

$$\left\{\begin{array}{lcl}\dot{\xi}_4 &=& 2\cos(t)\xi_3 \\ \dot{\xi}_3 &=& 2\cos(t)\xi_2 \\ \dot{\xi}_2 &=& v_2(t) \ .\end{array}\right. \quad (51)$$

Let us define

$$\begin{array}{lcl}\zeta_1 &=& \xi_4 - 2\sin(t)\xi_3 - \cos(2t)\xi_2 \\ \zeta_2 &=& \xi_3 - 2\sin(t)\xi_2 \\ \zeta_3 &=& \xi_2 \ .\end{array} \quad (52)$$

Simple calculations (based on the double angle formula for the sine function) give

$$\left\{\begin{array}{lcl}\dot{\zeta}_1 &=& -\cos(2t)v_2(t) \\ \dot{\zeta}_2 &=& -2\sin(t)v_2(t) \\ \dot{\zeta}_3 &=& v_2(t) \ .\end{array}\right. \quad (53)$$

Then the derivative of the positive definite quadratic function $\nu(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{2}\left[\zeta_1^2 + \zeta_2^2 + \zeta_3^2\right]$ along all trajectories of (53) is $\dot{\nu}(t) = \left[-\cos(2t)\zeta_1 - 2\sin(t)\zeta_2 + \zeta_3\right]v_2(t)$. Thus with

$$v_2(t) = -\sigma\left(\left[-\cos(2t)\zeta_1 - 2\sin(t)\zeta_2 + \zeta_3\right]\right) \quad (54)$$

where $\sigma(s) = \frac{s}{\sqrt{1+s^2}}$, we obtain

$$\begin{array}{l}\dot{\nu}(t) = -\left[-\cos(2t)\zeta_1 - 2\sin(t)\zeta_2 + \zeta_3\right] \\ \times\sigma\left([-\cos(2t)\zeta_1 - 2\sin(t)\zeta_2 + \zeta_3]\right).\end{array} \tag{55}$$

We now use the LaSalle Invariance Principle to check that (53) in closed loop with (54) is uniformly globally asymptotically stable to 0, as follows. Consider any solution $(\zeta_1(t), \zeta_2(t), \zeta_3(t))$ of (53)-(54) such that

$$-\cos(2t)\zeta_1(t) - 2\sin(t)\zeta_2(t) + \zeta_3(t) = 0$$

for all $t \geq 0$, Then $v_2(t) = 0$ and so also $\dot{\zeta}_i(t) = 0$ for all $t \geq 0$ and $i = 1$ to 3. Consequently,

$$-\cos(2t)\zeta_1(0) - 2\sin(t)\zeta_2(0) + \xi_3(0) = 0 \tag{56}$$

for all $t \geq 0$. We deduce easily that necessarily, $\zeta_i(0) = 0$ for $i = 1$ to 3 (by differentiating through (56) with respect to t and then setting $t = 0$ in the result).

Consequently $(\zeta_1(t), \zeta_2(t), \zeta_3(t)) = (0, 0, 0)$ for all $t \geq 0$. Then, we deduce from the LaSalle Invariance Principle that (54) renders the origin of (53) uniformly globally asymptotically stable. It follows that the bounded feedback

$$\begin{array}{l}v_2(t, \xi_2, \xi_3, \xi_4) = \\ \sigma\left(\cos(2t)(\xi_4 - 2\sin(t)\xi_3 - \cos(2t)\xi_2)\right. \\ \left.+2\sin(t)(\xi_3 - 2\sin(t)\xi_2) - \xi_2\right)\end{array} \tag{57}$$

renders the origin of (51) uniformly globally exponentially stable. By grouping terms, we obtain

$$\begin{array}{l}v_2(t, \xi_2, \xi_3, \xi_4) = \sigma\left(\cos(2t)\xi_4 + 2[1 - \cos(2t)]\sin(t)\xi_3\right. \\ \left.-[\cos^2(2t) + 4\sin^2(t) + 1]\xi_2\right).\end{array}$$

Then (57) and (50) lead us to the stabilizing output feedback

$$\begin{array}{l}v_2(t) = \\ \sigma\Big(\cos(2t)\xi_4 + 2[1 - \cos(2t)]\sin(t)\xi_3 \\ -\mathcal{L}(t)\int_{t-2\pi}^{t}\left(1 - e^{\mathcal{M}(m,t)}\right)v_2(m)\mathrm{d}m \\ -2\mathcal{L}(t)\int_{t-2\pi}^{t} e^{\mathcal{M}(m,t)}(4\cos^3(m) - \sin(m))\xi_3(m)\mathrm{d}m \\ -2\cos(t)\mathcal{L}(t)\left[-e^{4\pi}\xi_3(t) + \xi_3(t - 2\pi)\right]\Big)\end{array} \tag{58}$$

where $\mathcal{M}(m, t) = 2(m - t + 2\pi) + \sin(2m) - \sin(2t)$ and

$$\mathcal{L}(t) = \frac{\cos^2(2t) + 4\sin^2(t) + 1}{1 - e^{4\pi}}. \tag{59}$$

Notice that v_2 is a solution of an implicit equation.

D. Simulation

We performed simulations, which show the efficiency of the approach. Fig. 1 shows the simulation of closed loop nonlinear time varying system (44) with v_2 defined in (58). Since our simulation shows good stabilization, it helps to illustrate our general theory in the special case of the system (40). We choose $x_1(0) = 0.5$ which implies that $\sqrt{|x_1(t)|} = 1/\sqrt{2} - t$ when $t \in [0, 1/\sqrt{2}]$ and $x_1(t) = 0$ for all $t \geq 1/\sqrt{2}$. This is evident from the simulation as well.

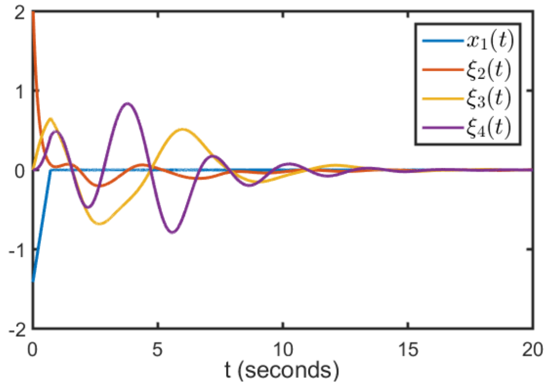

Fig. 1. Simulation results.

V. Conclusions

We proposed a new type of reduced order finite time observers. The observer applies to time-varying systems. We conjecture that it can be used to solve a problem of constructing interval observers that is similar to those in [9]. We plan to apply our observer to solve a dynamic output feedback stabilization problem for a MIMO nonlinear system. We will study other extensions. In particular, we will combine the main result of the present paper with the result of [8] and to the case where there are a delay and a disturbance in the input and where the outputs are only available on some finite time intervals.

References

[1] F. Bonnans and P. Rouchon. *Commande et Optimisation de Systèmes Dynamiques*. Les Éditions de l'Ecole Polytechnique, Palaiseau, France, 2005.

[2] R. Engel and G. Kreisselmeier. A continuous time observer which converges in finite time. *IEEE Transactions on Automatic Control*, 47(7):1202–1204, 2002.

[3] F. Friedland. Reduced-order state obervers. In H. Unbehauen, editor, *Control Systems, Robotics And Automation - Vol. VIII*, pages 26–36. Eoless Publishers Co. Ltd., Oxford, United Kingdom, 2009.

[4] F. Lopez-Ramirez, A. Polyakov, D. Efimov, and W. Perruquetti. Finite-time and fixed-time observer design: Implicit Lyapunov function approach. *Automatica*, 87:52–60, 2018.

[5] D. Luenberger. Observers for multivariable systems. *IEEE Transactions on Automatic Control*, 11(2):190–197, 1966.

[6] D. Luenberger. *Introduction to Dynamic Systems*. John Wiley and Sons, New York, 1979.

[7] M. Malisoff and F. Mazenc. *Constructions of Strict Lyapunov Functions*. Springer, New York, 2009.

[8] F. Mazenc, S. Ahmed, and M. Malisoff. Finite time estimation through a continuous-discrete observer. *International Journal of Robust and Nonlinear Control*, submitted in 2017, in review, http://www.math.lsu.edu/∼malisoff/.

[9] F. Mazenc, E. Fridman, and W. Djema. Estimation of solutions of observable nonlinear systems with disturbances. *Systems and Control Letters*, 79:47–58, 2015.

[10] P. Menold, R. Findeisen, and F. Allgower. Finite time convergent observers for nonlinear systems. In *Proceedings of the IEEE Conference on Decision and Control*, pages 5673–5678, Maui, HI, 2003.

[11] W. Perruquetti, T. Floquet, and E. Moulay. Finite time observers: application to secure communication. *IEEE Transactions on Automatic Control*, 53(1):356–360, 2008.

[12] T. Raff and F. Allgower. An impulsive observer that estimates the exact state of a linear continuous time system in predetermined fnite time. In *Proceedings of the Mediterranean Conference on Control and Automation*, pages 1–3, Athens, Greece, 2007.

[13] T. Raff and F. Allgower. An observer that converges in finite time due to measurement-based state updates. *IFAC Proceedings Volumes*, 41(2):2693–2695, 2008.

[14] F. Sauvage, M. Guay, and D. Dochain. Design of a nonlinear finite time converging observer for a class of nonlinear systems. *Journal of Control Science and Engineering*, 2007(36954), 2007.

[15] E. Sontag. *Mathematical Control Theory, Second Edition*. Springer, New York, 1998.