Sequential Predictors for Linear Time-Varying Systems with Multiple Delays

Michael Malisoff, Frédéric Mazenc

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Abstract— **We provide uniformly globally exponentially stabilizing feedback controls for a large class of linear time-varying systems that contain an arbitrary number of different delays, using a new sequential predictors approach. This allows different delays in different components of the input. We illustrate our work in an example from identification theory, and in an Euler-Lagrange system arising from two-link manipulator systems.**

Key Words: Delays, stability, time-varying systems

I. Introduction

Sequential predictors provide a useful framework for delay compensation under input delays. They can help solve feedback stabilization problems using controls that are free of distributed terms, without imposing bounds on the allowable input delays; see the pioneering work [4], and [12], [13] and [23]. Prior sequential predictor results covered nonlinear systems with constant delays [13], linear time-varying systems with time-varying delays [12], and time-varying systems with sampling and measurement delays [23]. However, the existing sequential predictor works for time-varying systems left open a complementary problem of feedback stabilization when there several different delays, except for results in [25] that produce distributed terms in the controls. This note provides a method to overcome this challenge, using sequential predictor feedbacks for systems with multiple delays and controls that are free of distributed terms.

This note is motivated by many engineering applications that are modeled by systems with multiple delays where it may not always be feasible to implement traditional controls. Standard prediction, reduction model, and sequential predictor controls share the common feature that they are expressed using the states of dynamic extensions. However, the standard prediction or reduction model method controls are usually distributed ones, whose formulas involve integrals of past control values or which are only expressed implicitly as solutions of integral equations, instead of closed form control formulas. An alternative is emulation, which achieves input delay compensation under bounds on the allowable delays [15], and so may not be suitable under longer delays.

The sequential predictor feedback design approach consists of two steps. In the first step, one constructs a uniformly globally asymptotically stabilizing feedback control for the system obtained by setting the input delays equal to zero. Then, one interconnects the original system with a collection of dynamic extensions called sequential subpredictors. Each subpredictor consists of a copy of the original system running on a different time scale and additional stabilizing terms. Then, one replaces the state in the feedback from the first step by a new variable, which is a delayed state of the last subpredictor, to eliminate the input delay. This contrasts with standard prediction, which is based on the same two step process except the formula for the new variable in standard prediction is typically defined using distributed terms.

The sequential predictor method is a recent development in a history of works on asymptotic stabilization under long feedback delays, starting with the Smith predictor for linear systems [22], and work during the 1980s and 1990s by Artstein [3], Krstic [11], and others. See [17] and [20] for surveys on input delay compensating control. Other notable works include [10] and [18] on delay systems which can lead to delay compensating controls with bounds on the allowable delays or which are in implicit form. Significant engineering applications of delay compensation include [16] and [21].

Although standard prediction usually yields controls with distributed terms, see [1], [5], and [24] for feedback controls for special cases of time invariant systems where the controls do not contain distributed terms but are based on prediction. See also [7] for chain observers that do not allow different delays in the vector fields and in the input, and which therefore do not cover the problems that we address here. To address more complex systems with any finite number of different constant delays, this work constructs a new class of sequential predictors that include M subsets of subpredictors, where M is the total number of delays in the system. We can view the longest delay as the input delay, and the other delays as delays in the vector fields. Then each of the first $M-1$ subsets of subpredictors compensates for one of the delays in the vector fields, and the Mth subset of subpredictors compensates for the input delay.

In the next section, we state and prove our theorem. Then in Section V, we illustrate our theorem in a benchmark dynamics from identification theory and a two-link manipulator dynamic. In both examples, we allow multiple delays in different input components, without restricting the size of the largest input delay. We end in Section VI by summarizing our contribution and our ideas for future research. This work improves on our conference version [14], which was confined to cases where there are only two different delays and did not include the two-link manipulator application. Moreover, in the special case where there are only two delays, this work

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provides a less conservative lower bound on the required number of sequential predictors than the lower bound from [14], which can reduce the computational burden of the method to make it easier to apply; see Remark 3.

II. Main Result

We study systems of the form

$$\dot{x}(t) = A(t)x(t)+B(t)u(t-k\tau)+\sum_{r=1}^{N} C_r(t)x(t-\omega_r\tau) \quad (1)$$

where x is valued in $\mathbb{R}^n$ for any dimension n, $\tau \geq 0$ is a constant, k and the ω_i's are positive integers such that $\omega_1 < \omega_2 < \ldots \omega_N \leq k$, and the matrix valued functions A, B, and C_r are assumed to be continuous and bounded. Hence, (1) allows $M = N+1$ different constant delays. The feedback u will be specified later, but we write it as a function of only time to simplify the notation. To simplify notation, we also assume throughout that the initial times are always zero, and that the initial functions are constant on $[-k\tau, 0]$. We assume:

Assumption 1: There is a bounded continuous matrix valued function K such that the origin of

$$\dot{x}(t) = A(t)x(t)+B(t)u_s(t,x(t))+\sum_{r=1}^{N} C_r(t)x(t-\omega_r\tau),$$

in closed loop with the control $u_s(t,x) = K(t)x$, is uniformly globally exponentially stable. □

Assumption 1 agrees with the assumptions from [12] and [13] in the special case of time-varying linear systems with constant delays when the C_r's are 0. Systems of the form (1) are important because when linearizing a nonlinear system around a reference trajectory, one obtains linear systems with time-varying coefficients. They can represent a closed loop system with delays $\omega_i\tau$ in the vector fields and a longer input delay $k\tau$ with k as large as desired, or systems with different delays in different components of the feedback.

Assumption 1 is often satisfied. For example, in some cases we can treat the terms

$$\delta(t) = \sum_{r=1}^{N} C_r(t)x(t-\omega_r\tau) \quad (2)$$

as a disturbance and then use high gains to dominate this term with the feedback, in which case no conditions on the sizes of $\omega_i\tau$'s are needed. In other cases, one finds K so that

$$\dot{x}(t) = \left(A(t)+B(t)K(t)+\sum_{r=1}^{N} C_r(t)\right)x(t) \quad (3)$$

is uniformly globally exponentially stable to 0 and then proves that

$$\dot{x}(t) = (A(t)+B(t)K(t))x(t) + \sum_{r=1}^{N} C_r(t)x(t-\omega_r\tau) \quad (4)$$

is uniformly globally exponentially stable to 0, when the $\omega_i\tau$'s are smaller than a constant $\bar{\tau}$, where the bound $\bar{\tau}$ can be found by existing methods, like those of [9] and [15]. On the other hand, since k in (1) can be as large as desired, the emulation approach from [15] would not apply. This motivates our use of sequential predictors that apply without putting upper bounds on the allowable values of $k\tau$; see Section V. Our main result is as follows, where I_n is the identity matrix, $\{a,\ldots,b\}$ for integers a and b is the set of all integers i such that $a \leq i \leq b$, and $|\cdot|_\infty$ is the suprermum:

Theorem 1: Let Assumption 1 hold, let $\ell > 0$ be any positive real constant, and choose any integer

$$p > \tau|A+\ell I_n|_\infty. \quad (5)$$

Set

$$L(t) = -[A(t)+\ell I_n] \text{ and } \phi_i(t) = t+\frac{i\tau}{p} \quad (6)$$

for all $i \in \{1,2,\ldots,kp\}$. Then (1) in closed loop with the control u chosen to be

$$u_{\text{new}}(t) = K(t+k\tau)z_{kp}(t), \quad (7)$$

where z_{kp} is the last n components of the nkp-dimensional system

$$\left\{\begin{aligned} \dot{z}_i(t) &= A(\phi_i(t))z_i(t) \\ &\quad +B(\phi_i(t))u_{\text{new}}(\phi_i(t)-k\tau) \\ &\quad +L(\phi_i(t))\left[z_i\left(t-\tfrac{\tau}{p}\right)-z_{i-1}(t)\right] \\ &\quad +\sum_{r=1}^{N} C_r(\phi_i(t))x\left(t-\omega_r\tau+\tfrac{i\tau}{p}\right) \\ &\quad \text{if } i\in\{1,\ldots,\omega_1 p\} \\ \dot{z}_i(t) &= A(\phi_i(t))z_i(t) \\ &\quad +B(\phi_i(t))u_{\text{new}}(\phi_i(t)-k\tau) \\ &\quad +L(\phi_i(t))\left[z_i\left(t-\tfrac{\tau}{p}\right)-z_{i-1}(t)\right] \\ &\quad +\sum_{r=j+1}^{N} C_r(\phi_i(t))x\left(t-\omega_r\tau+\tfrac{i\tau}{p}\right) \\ &\quad +\sum_{r=1}^{j} C_r(\phi_i(t))z_{p\omega_r}\left(t-2\omega_r\tau+\tfrac{i\tau}{p}\right) \\ &\quad \text{if } i\in\{\omega_j p+1,\ldots,\omega_{j+1}p\} \\ &\quad \text{for some } j\in\{1,\ldots,N-1\} \\ \dot{z}_i(t) &= A(\phi_i(t))z_i(t) \\ &\quad +B(\phi_i(t))u_{\text{new}}(\phi_i(t)-k\tau) \\ &\quad +L(\phi_i(t))\left[z_i\left(t-\tfrac{\tau}{p}\right)-z_{i-1}(t)\right] \\ &\quad +\sum_{r=1}^{N} C_r(\phi_i(t))z_{p\omega_r}\left(t-2\omega_r\tau+\tfrac{i\tau}{p}\right) \\ &\quad \text{if } i\in\{\omega_N p+1,\ldots,kp\} \end{aligned}\right. \quad (8)$$

where $z_0 = x$, is such that the dynamics for $(x(t), z_1(t) - x(t+\tau/p), z_2(t)-z_1(t+\tau/p), \ldots, z_{kp}(t)-z_{kp-1}(t+\tau/p))$ are uniformly globally exponentially stable to 0. □

Remark 1: The z_i subsystem in (8) is called the ith sequential predictor for each i. When $\omega_N < k$, the system (8) can be viewed as a finite sequence of $M = N+1$ subsets of sequential predictors. The number of delayed x terms in each subset of sequential predictors is one less than the number of delayed x terms in the previous subset of sequential predictors in the sequence. Also, the number of delayed $z_{p\omega_r}$ terms in each subset is one more than the number of delayed $z_{p\omega_r}$ terms in the previous subset in the sequence. These increasing and decreasing numbers of delayed terms are essential for producing the cancellations that ensure the exponential stability conclusion of the theorem. When $\omega_N = k$, the last set of sequential predictors in (8) is not present, because in that case, $\{\omega_N p+1,\ldots,kp\}=\emptyset$. □

Remark 2: Condition (9) implies that in the special case where A is the zero function, we can choose any integer $p \geq 1$, by choosing $\ell > 0$ to be small enough, which can allow $2nk$ dimensional dynamic controls that consist of $2k$ sequential predictors. On the other hand, Theorem 1 also applies for nonzero A's, by choosing p large enough to satisfy (9), since the right side of (9) does not depend on p. See Section V for cases with zero and nonzero A. □

Remark 3: Condition (5) is a less restrictive condition on the number of sequential predictors than what was used in in [14] for the two delays case. In [14], the condition was

$$p > \max\left\{2, \tfrac{2\sqrt{2}\tau}{\ell}|A+\ell I_n|_\infty \max\left\{|A|_\infty, |A+\ell I_n|_\infty\right\}\right\}. \quad (9)$$

In Section IV, we show that (5) is a less restrictive condition on p than (9), and also less restrictive than the condition that was required in the work [6] on the LTI case. □

III. Proof of Theorem 1

To prove Theorem 1, we first need a lemma about

$$\dot{\xi}(t) = A(\phi_i(t))\xi(t) + L(\phi_i(t))\xi\left(t-\tfrac{\tau}{p}\right) \quad (10)$$

which uses our condition (9) and the notation $\xi_t(s)=\xi(t+s)$ for all $t \geq 0$ and $s \leq 0$, and which we prove in the appendix:

Lemma 1: Let $A:\mathbb{R}\to\mathbb{R}^{n\times n}$ be a bounded continuous function, let $\ell > 0$ and $\tau > 0$ be constants, and let p be an integer that satisfies condition (9). Let L and the ϕ_i's be defined by (6), and set

$$c = \min\left\{\tfrac{3\ell}{2}, \tfrac{p}{2\tau\ell}\left(\tfrac{p}{\tau} - \tfrac{\tau}{p}|\ell I_n + A|^2_\infty\right)\right\}. \quad (11)$$

Then for each choice of $i \in \{1,\ldots,kp\}$, there is a constant $c_a > 0$ such that the time derivative of the function

$$V^\sharp(\xi_t) = \tfrac{1}{2}|\xi(t)|^2 + v_* \int_{t-\tau/p}^{t}\int_m^t |\dot{\xi}(s)|^2 \mathrm{d}s\mathrm{d}m \quad \text{where } v_* = \tfrac{p}{2\tau\ell} + \tfrac{pc}{8\tau\max\{\ell^2, |\ell I_n + A|^2_\infty\}} \quad (12)$$

along all solutions of (10) satisfies

$$\tfrac{d}{dt}V^\sharp(\xi_t) \leq -c_a V^\sharp(\xi_t) \text{ for all } t \geq 0, \quad (13)$$

so (10) is uniformly globally exponentially stable to 0. □

We also use the fact that for any constant $c_* > 0$ and any continuous function $q:[-c_*,\infty)\to[0,\infty)$, the conditions

$$\tfrac{d}{dt}\int_{t-c_*}^{t}\int_\ell^t q(r)\mathrm{d}r\mathrm{d}\ell = c_* q(t) - \int_{t-c_*}^{t} q(r)\mathrm{d}r \quad (14a)$$

$$\text{and } \int_{t-c_*}^{t}\int_\ell^t q(r)\mathrm{d}r\mathrm{d}\ell \leq c_*\int_{t-c_*}^{t} q(r)\mathrm{d}r \quad (14b)$$

are satisfied for all $t \geq 0$. We also use the error variables

$$\overline{z}_i(t) = z_i(t) - z_{i-1}\left(t+\tfrac{\tau}{p}\right),\ 1 \leq i \leq p \quad (15)$$

where $z_0 = x$. We first show that the dynamics for $\overline{z}_{a1} = (\overline{z}_1,\ldots,\overline{z}_{\omega_1 p})$ are uniformly globally exponentially stable to zero. For all $t \geq 0$, we can combine the closed loop x subsystem from Theorem 1 (with its t arguments replaced by $\phi_1(t)$) with the choice of the z_1 dynamics in (8), and recall that $\overline{z}_1 = z_1 - x(\phi_1)$, to obtain the dynamics

$$\dot{\overline{z}}_1(t) = A(\phi_1(t))\overline{z}_1(t) + L(\phi_1(t))\overline{z}_1\left(t-\tfrac{\tau}{p}\right). \quad (16)$$

The system (16) is uniformly globally exponentially stable to 0, by Lemma 1. By induction, the dynamics

$$\left\{\begin{array}{rcl} \dot{\overline{z}}_1(t) &=& A(\phi_1(t))\overline{z}_1(t) + L(\phi_1(t)\overline{z}_1\left(t-\tfrac{\tau}{p}\right) \\ \dot{\overline{z}}_i(t) &=& A(\phi_i(t))\overline{z}_i(t) + L(\phi_i(t))\overline{z}_i\left(t-\tfrac{\tau}{p}\right) \\ && -L(\phi_i(t))\overline{z}_{i-1}(t),\ i=2,\ldots,\omega_1 p \end{array}\right. \quad (17)$$

for the variable $\overline{z}_{a1} = (\overline{z}_1,\ldots,\overline{z}_{\omega_1 p})$ is uniformly globally exponentially stable to 0. This follows because each system

$$\dot{\overline{z}}_i(t) = A(\phi_i(t))\overline{z}_i(t) + L(\phi_i(t))\overline{z}_i\left(t-\tfrac{\tau}{p}\right) + \delta(t) \quad (18)$$

for all $i \in \{2,\ldots,\omega_1 p\}$ is exponentially input-to-state stable with respect to the disturbance δ, because the function $V^\sharp$ from (12) is also an input-to-state stability Lyapunov-Krasovskii functional for (18) and L is bounded.

We next use the variables $\zeta_j(t) = z_{\omega_j p}(t-\omega_j\tau) - x(t)$ for all j, where we set $\omega_{N+1} = k$ if $\omega_N < k$. Then for each $i \in \{\omega_1 p+1,\ldots,kp\}$, we can rewrite the ith sequential predictor from (8) as

$$\begin{array}{rl} \dot{z}_i(t) =& A(\phi_i(t))z_i(t) \\ & +B(\phi_i(t))u_{\rm new}(\phi_i(t)-k\tau) \\ & +L(\phi_i(t))\left[z_i\left(t-\tfrac{\tau}{p}\right) - z_{i-1}(t)\right] \\ & +\sum_{r=1}^{N} C_r(\phi_i(t))x\left(t-\omega_r\tau+\tfrac{i\tau}{p}\right) \\ & +\sum_{r=1}^{j} C_r(\phi_i(t))\zeta_r\left(t-\omega_r\tau+\tfrac{i\tau}{p}\right) \\ & \text{if } i \in \{\omega_j p+1,\ldots,\omega_{j+1}p\} \\ & \text{for some } j \in \{1,\ldots,N-1\} \end{array} \quad (19)$$

$$\begin{array}{rl} \dot{z}_i(t) =& A(\phi_i(t))z_i(t) \\ & +B(\phi_i(t))u_{\rm new}(\phi_i(t)-k\tau) \\ & +L(\phi_i(t))\left[z_i\left(t-\tfrac{\tau}{p}\right) - z_{i-1}(t)\right] \\ & +\sum_{r=1}^{N} C_r(\phi_i(t))x\left(t-\omega_r\tau+\tfrac{i\tau}{p}\right) \\ & +\sum_{r=1}^{N} C_r(\phi_i(t))\zeta_r\left(t-\omega_r\tau+\tfrac{i\tau}{p}\right) \\ & \text{if } i \in \{\omega_N p+1,\ldots,kp\}. \end{array} \quad (20)$$

Hence, since we set $\omega_{N+1} = k$ if $\omega_N < k$, the dynamics for the ith sequential predictor in (8) can be written as

$$\begin{array}{rl} \dot{z}_i(t) =& A(\phi_i(t))z_i(t) \\ & +B(\phi_i(t))u_{\rm new}(\phi_i(t)-k\tau) \\ & +L(\phi_i(t))\left[z_i\left(t-\tfrac{\tau}{p}\right) - z_{i-1}(t)\right] \\ & +\sum_{r=1}^{N} C_r(\phi_i(t))x\left(t-\omega_r\tau+\tfrac{i\tau}{p}\right) \\ & +\sum_{r=1}^{j} C_r(\phi_i(t))\zeta_r\left(t-\omega_r\tau+\tfrac{i\tau}{p}\right) \\ & \text{if } i \in \{\omega_j p+1,\ldots,\omega_{j+1}p\} \\ & \text{for some } j \in \{1,\ldots,N\}. \end{array} \quad (21)$$

For each $j \in \{1, 2, \ldots, N\}$, let $\mathcal{I}_j$ denote the indicator function of the set $\{\omega_j p+1\}$ for $j = 1, \ldots, N$, so $\mathcal{I}_j(\omega_j p+1) = 1$ and $\mathcal{I}_j(i) = 0$ for all $i \neq \omega_j p+1$. Then the dynamics for the error variables (15) for all $i \in \{\omega_1 p+1, \ldots, kp\}$ are

$$\begin{aligned}\dot{\overline{z}}_i(t) &= A(\phi_i(t))\overline{z}_i(t) - L(\phi_i(t))\overline{z}_{i-1}(t) \\ &+ L(\phi_i(t))\overline{z}_i\left(t - \tfrac{\tau}{p}\right) \\ &+ C_j(\phi_i(t))\zeta_j\left(t - \omega_j\tau + \tfrac{i\tau}{p}\right)\mathcal{I}_j(i)\end{aligned} \tag{22}$$

if $i \in \{p\omega_j + 1, \ldots, p\omega_{j+1}\}$ for some $j \in \{1, \ldots, N\}$. This follows because for each $j \in \{1, 2, \ldots, N\}$, our formulas from (21) for $i \in \{\omega_j p+1, \ldots, \omega_{j+1}p\}$ ensure that there are no terms involving the C_r's in the dynamics for $\overline{z}_i$ for any indices $i \in \{\omega_j p + 2, \ldots, \omega_{j+1}p\}$.

Next note that for all $j = 1, 2, \ldots, N+1$, we have

$$\begin{aligned}&\sum_{s=0}^{p\omega_j-1} \overline{z}_{\omega_j p-s}\left(t - \omega_j\tau + \tfrac{s\tau}{p}\right) \\ &= z_{\omega_j p}(t - \omega_j\tau) - x(t) = \zeta_j(t)\end{aligned} \tag{23}$$

for all $t \geq 0$, which follows from the pairwise cancellation of terms which gives a telescoping sum. Using the uniform global exponential stability of the dynamics (17) for $\overline{z}_{a1} = (\overline{z}_1, \ldots, \overline{z}_{\omega_1 p})$, the telescoping sum (23), and the exponential input-to-state stability of (18) with respect to δ (applied with $\delta(t) = C_1(\phi_i(t))\zeta_1(t - \omega_1\tau + i\tau/p)\mathcal{I}_1(i)$, which exponentially decays to 0 for each i), it follows from the structure of (22) that the dynamics for $(\overline{z}_{a1}, \overline{z}_{a2}) = (\overline{z}_1, \ldots, \overline{z}_{\omega_2 p})$ are uniformly globally exponentially stable to 0. Then, by induction (with the input-to-state stability condition applied with $\delta(t) = C_j(\phi_i(t))\zeta_j(t - \omega_j\tau + i\tau/p)\mathcal{I}_j(i)$ for $j = 2, 3, \ldots, N$), it follows that the dynamics for $(\overline{z}_1, \ldots, \overline{z}_{kp})$ are uniformly globally exponentially stable to 0.

Moreover, (23) implies that $z_{kp}(t) = x(t + k\tau) + \sigma(t)$, where

$$\sigma(t) = \sum_{s=0}^{pk-1} \overline{z}_{pk-s}\left(t + \tfrac{s\tau}{p}\right) \tag{24}$$

for all $t \geq 0$, by setting $j = N+1$ (resp., $j = N$) in (23) if $\omega_N < k$ (resp., $\omega_N = k$). Then the dynamics for

$$\begin{aligned}&\big(x(t), \overline{z}_1(t), \ldots, \overline{z}_{kp}(t)\big) = \big(x(t), z_1(t) - x(t+\tau/p), \\ &z_2(t) - z_1(t+\tau/p), \ldots, z_{kp}(t) - z_{kp-1}(t+\tau/p)\big),\end{aligned} \tag{25}$$

in closed loop with our control (7) are

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u_s(t, x(t)+\sigma(t - k\tau)) \\ &+ \sum_{r=1}^{N} C_r(t)x(t - \omega_r\tau)\end{aligned} \tag{26}$$

interconnected with (17) and (22). Since σ is a linear combination of the error components $\overline{z}_i$, and since the $\overline{z} = (\bar{z}_1, \ldots, z_{kp})$ subsystem of (26) is uniformly globally exponentially stable to 0, we conclude from Assumption 1 and the linearity of u_s in x that (26) is uniformly globally exponentially stable to 0. This proves the theorem.

IV. Comparison of Lower Bounds on p

The work [14] on the two delays case required the lower bound (9) on the value p that is used to compute the number pk of sequential predictors. We next show why (5) is a less restrictive condition than condition (9). To this end, it suffices to note that (9) implies that $p \geq \tau|\ell I_n + A|_\infty \mathcal{D}$ where

$$\mathcal{D} = 2\sqrt{2}\max\left\{\tfrac{|A|_\infty}{\ell}, |I_n + \tfrac{A}{\ell}|_\infty\right\} \tag{27}$$

and then to note that $\mathcal{D} \geq \sqrt{2}$ for all ℓ, A, and n. To check that $\mathcal{D} \geq \sqrt{2}$, note that $\max\{|M|_\infty, |I_n + M|_\infty\} \geq \frac{1}{2}$ for all $n \times n$ matrices M (which follows by separately considering the cases $|M|_\infty \geq \frac{1}{2}$ and $|M|_\infty < \frac{1}{2}$ and noting that in the latter case, the triangle inequality gives $|I_n + M|_\infty \geq |I_n|_\infty - |M|_\infty > \frac{1}{2}$) and then choose $M = A/\ell$.

Our work is new, even in the special case of LTI systems. In the LTI case with constant delays, (5) is less restrictive than the condition in [6, Lemma 2] that was used to provide a lower bound on the number of sequential predictors in [6]. In our notation for the LTI case, [6, Lemma 2] requires

$$\int_0^{\tau/p} |A + \ell I_n| e^{-\ell\theta} \mathrm{d}\theta < 1 \tag{28}$$

and when $\ell I_n + A \neq 0$, (28) can be rewritten as

$$1 - \tfrac{\ell}{|\ell I_n + A|} < e^{-\ell\tau/p}. \tag{29}$$

If $|\ell I_n + A| \leq \ell$, then (29) is satisfied; while if $|\ell I_n + A| > \ell$, then we can rewrite (29) as $p > \tau|\ell I_n + A|\mathcal{S}$ where

$$\mathcal{S} = \frac{-\ell}{\ln\left(1 - \frac{\ell}{|\ell I_n + A|}\right)|\ell I_n + A|} \tag{30}$$

and $\mathcal{S} > 1$. To see why $\mathcal{S} > 1$, it suffices to notice that

$$\ln\left(1 - \tfrac{\ell}{q}\right)q < -\ell \text{ for all } q > \ell; \tag{31}$$

condition (31) holds because its left side is an increasing function of q that converges to $-\infty$ as q converges to ℓ from the right and converges to $-\ell$ as $q \to \infty$. Since $\mathcal{S} > 1$, we conclude that the requirement on p from [6] is more conservative than (5) when there is no $\ell > 0$ such that $|\ell I_n + A| \leq \ell$ (which can occur when A has positive entries). Also, for each choice of A, $\mathcal{S}$ converges to ∞ as $\ell \to \infty$, so our condition (5) is significantly less conservative when ℓ is large, which is important because choosing ℓ large can increase the rate of exponential convergence in our theorem.

V. Illustrations

A. Identification Dynamics

An important dynamics arising in identification theory (from [15], [19] and several references therein) is

$$\dot{x} = -m(t)m^\top(t)u \tag{32}$$

having a control u, where x is valued in $\mathbb{R}^n$ for any dimension n. We use the following assumption from [15]:

Assumption 2: The function $m : \mathbb{R} \to \mathbb{R}^n$ is continuous, there are constants $\alpha' \in (0,1)$, $\beta' > 0$, and $\tilde{c} > 0$ such that

$$\alpha' I_n \leq \int_t^{t+\tilde{c}} m(\tau)m^\top(\tau)\mathrm{d}\tau \leq \beta' I_n \text{ for all } t \in \mathbb{R}, \tag{33}$$

and $|m(t)| = 1$ for all $t \in \mathbb{R}$. □

The following lemma from [15] builds a strict Lyapunov function V for (32) in closed loop with $u_s(t, x) = x$:

Lemma 2: Let m, α', β', and $\tilde{c}$ be such that Assumption 2 is satisfied. Then, with the choices $\kappa = 1 + \frac{\tilde{c}}{2} + \frac{1}{4\alpha'}\tilde{c}^4$ and

$$P(t) = \kappa I_n + \int_{t-\tilde{c}}^{t}\int_{s}^{t} m(l)m^\top(l)\,\mathrm{d}l\,\mathrm{d}s, \qquad (34)$$

the function $V(t,x) = x^\top P(t)x$ satisfies $\dot{V} \le -\alpha'|x|^2/2$ along all trajectories of $\dot{x}(t) = -m(t)m^\top(t)x(t)$ for all $t \ge 0$. Moreover, $|P|_\infty \le \kappa + \tilde{c}^2$. □

Under Assumption 2, [15] computed a bound $\overline{\tau}$ such that $\dot{x}(t) = -m(t)m^\top(t)u_s(t, x_1(t-\tau_1), \dots, x_n(t-\tau_n))$ is uniformly globally asymptotically stable to 0 for all constant delays $\tau_i \in [0, \overline{\tau}]$ for $i = 1, 2, \dots, n$. Moreover, since (32) has no drift term, [15, Remark 4] makes it possible to allow arbitrarily large $\overline{\tau}$ by scaling the control, but this control scaling method would not be viable in practice. Hence, we illustrate Theorem 1 for the input delayed version

$$\dot{x}(t) = -M_a(t)u(t-k\tau) - \sum_{i=1}^{N} M_{bi}(t)x_{bi}(t-\omega_i\tau) \qquad (35)$$

of (32), where $x = (x_a^\top, x_{b1}^\top, \dots, x_{bN}^\top)^\top$ for any dimensions for x_a and for the vectors $x_{bi} \in \mathbb{R}^{n_i}$ respectively that add to n, and M_a (resp., $M_b = (M_{b1}, \dots, M_{bN})$) consists of the first n_a (resp., last n_b) columns of $mm^\top$, where $n = n_a + n_b$. We also let M_{bi*} denote the matrix obtained from M_b by replacing all of its submatrices $M_{bj} \in \mathbb{R}^{n\times n_j}$ for $j \ne i$ by zero matrices, so $M_{bi*} = [M_{i10}\ M_{i20}\ \dots M_{iN0}]$ where $M_{ij0} = 0_{n\times n_j}$ if $j \ne i$ and $M_{ii0} = M_{bi}$ for all i, where $0_{r\times s}$ is the $r \times s$ zero matrix for any r and s. Then (35) can be written as (1) by choosing $C_i = -[0_{n\times n_a}\ M_{bi*}]$ for all i. We use the following lemma:

Lemma 3: Let the assumptions of Lemma 2 hold, and M_a, M_b, and the M_{bi*}'s be defined as above and $\tau > 0$ be a constant such that

$$\tau < \frac{\alpha'}{6\sqrt{2N}\omega_N(N+1)(\kappa+\tilde{c}^2)}. \qquad (36)$$

Then the requirements of Assumption 1 are satisfied with $A = 0_{n\times n}$, $B = -[M_a\ 0_{n\times n_b}]$, $C_i = -[0_{n\times n_a}\ M_{bi*}]$ for $i = 1, \dots, N$, and $u_s(t,x) = x$. □

Proof: By using the bound $|M|_\infty \le 1$ and rewriting

$$\dot{x}(t) = -M_a(t)x_a(t) - \sum_{i=1}^{N} M_{bi}(t)x_{bi}(t-\omega_i\tau) \qquad (37)$$

as $\dot{x}(t) = -m(t)m^\top(t)x(t) + M_b(t)\mathcal{G}_\omega(t)$ with the choice $\mathcal{G}_\omega(t) = (x_{b1}(t) - x_{b1}(t-\omega_1\tau), \dots, x_{bN}(t) - x_{bN}(t-\omega_N\tau))^\top$, we conclude that the time derivative of the V from Lemma 2 along all solutions of (37) for all $t \ge 0$ satisfies

$$\begin{aligned}\dot{V} &\le -\frac{\alpha'|x(t)|^2}{2} + \{|x(t)|\}\left\{2\kappa^\sharp\int_{t-2\omega_N\tau}^{t}|x(\ell)|\mathrm{d}\ell\right\}\\ &\le -\frac{\alpha'|x(t)|^2}{4} + 4(\kappa^\sharp)^2\frac{1}{\alpha'}(2\omega_N\tau)\int_{t-2\omega_N\tau}^{t}|x(\ell)|^2\mathrm{d}\ell\end{aligned} \qquad (38)$$

by the inequality $c_1c_2 \le \frac{\alpha'}{4}c_1^2 + \frac{1}{\alpha'}c_2^2$ where c_1 and c_2 are chosen to be the terms in curly braces in (38) and also using Jensen's inequality, and where $\kappa^\sharp = (N+1)\sqrt{N}(\kappa + \tilde{c}^2)$, and using the fact that for each $i \in \{1, 2, \dots, N\}$, we have

$$\begin{aligned}&\left|x_{bi}(t) - x_{bi}(t-\omega_i\tau)\right| = \left|\int_{t-\omega_i\tau}^{t}\dot{x}_{bi}(\ell)\mathrm{d}\ell\right|\\ &\le (N+1)\int_{t-2\omega_N\tau}^{t}|x(\ell)|\mathrm{d}\ell,\end{aligned} \qquad (39)$$

which implies that $|\mathcal{G}_\omega(t)| \le (N+1)\sqrt{N}\int_{t-2\omega_N\tau}^{t}|x(\ell)|\mathrm{d}\ell$ for all $t \ge 0$. Hence, the time derivative of

$$V^\sharp(x_t) = V(x(t)) + \frac{9\omega_N\tau(\kappa^\sharp)^2}{\alpha'}\int_{t-2\omega_N\tau}^{t}\int_{\ell}^{t}|x(s)|^2\mathrm{d}s\mathrm{d}\ell \qquad (40)$$

along all solutions of (37) for all $t \ge 0$ satisfies

$$\begin{aligned}\frac{d}{dt}V^\sharp(x_t) \;\le\; &-\left\{\frac{\alpha'}{4} - \frac{18\omega_N^2\tau^2}{\alpha'}(\kappa^\sharp)^2\right\}|x(t)|^2\\ &-(\kappa^\sharp)^2\frac{\omega_N\tau}{\alpha'}\int_{t-2\omega_N\tau}^{t}|x(\ell)|^2\mathrm{d}\ell,\end{aligned} \qquad (41)$$

by (14a) with $c_* = 2\omega_N\tau$ and $q(\ell) = |x(\ell)|^2$. By (36), the constant in curly braces in (41) is positive, so we can use (14b) to get an exponential decay estimate on $V^\sharp$ and so also on $|x(t)|$ along all solutions of (35). ■

Since Assumption 1 is satisfied, we can apply Theorem 1, for any $k \in \mathbb{N}$ and any τ satisfying (36). Since Assumption 1 is satisfied with $A = 0$, condition (9) becomes $p > \tau\ell$, so we can choose $p = 2$ by selecting $\ell > 0$ to be small enough.

B. Two-Link Manipulator

Consider the two-link manipulator system from [2], which is a fully actuated system obtained by viewing the robot arm as a segment with mass M and length $\mathcal{L}$. Letting m denote the mass of the hand, r the position of the hand, and θ the angle of the arm, we get the Euler-Lagrange equations

$$\left\{\begin{array}{rcl}\left(mr^2 + M\mathcal{L}^2/3\right)\ddot{\theta} + 2Mr\dot{r}\dot{\theta} &=& T\\ m\ddot{r} - mr\dot{\theta}^2 &=& F\end{array}\right., \qquad (42)$$

with the scalar controls $u_1 = T$ and $u_2 = F$. For simplicity, take $m = M = 1$, $\mathcal{L} = \sqrt{3}$, $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = r$, and $x_4 = \dot{r}$. Then (42) becomes

$$\left\{\begin{array}{rcl}\dot{x}_1(t) &=& x_2(t)\\ \dot{x}_2(t) &=& -\frac{2x_3(t)x_2(t)x_4(t)}{x_3^2(t)+1} + \frac{T(t-\tau)}{x_3^2(t)+1}\\ \dot{x}_3(t) &=& x_4(t)\\ \dot{x}_4(t) &=& x_3(t)x_2^2(t) + F(t-k\tau)\end{array}\right. \qquad (43)$$

if the delay $k\tau$ in F is an integer multiple $k \ge 1$ of the delay τ in T. When $\tau = 0$, the linearization of (43) around the zero vector produces a system of the form $\dot{x}(t) = \mathcal{A}_0x(t) + \mathcal{B}_0u(t)$ where $(\mathcal{A}_0, \mathcal{B}_0)$ is a (constant) controllable pair. Pick a constant matrix K_0 such that $\mathcal{A}_0 + \mathcal{B}_0K_0$ is Hurwitz.

When $\tau = 0$, we can find a constant $\bar{c} > 0$ such that for each pair (x_r, u_r) consisting of a reference trajectory for the original nonlinear system and a corresponding reference input that satisfies $|(x_r, u_r)|_\infty \le \bar{c}$, the linearization $\dot{x}(t) = \mathcal{A}(t)x(t) + \mathcal{B}(t)u$ of (43) around (x_r, u_r) is uniformly globally exponentially stabilized to 0 by $u(t,x) = K_0x$. This $\bar{c}$ exists because $\dot{x}(t) = [\mathcal{A}(t) + \mathcal{B}(t)K_0]x(t)$ will admit the strict Lyapunov function $x^\top P_0x$ when $\bar{c}$ is small enough, where the symmetric positive definite matrix P_0 is chosen to satisfy $H^\top P_0 + P_0H = -I_n$ and $H = \mathcal{A}_0 + \mathcal{B}_0K_0$. Then Theorem 1 applies (with $A = \mathcal{A}$, $B(t) = [0\ \mathcal{B}_R(t)]$, $N = 1$, $C_1(t) = [\mathcal{B}_L(t)\ 0]K_0$, and $u_s(t,x) = K_0x$, where we partitioned $\mathcal{B}$ as $\mathcal{B} = [\mathcal{B}_L\ \mathcal{B}_R]$ in terms of its columns $\mathcal{B}_L$ and $\mathcal{B}_R$), when $\tau \ge 0$ is small enough, by the arguments in Section II. Since k can be as large as desired, we can then use Theorem 1 to compensate for arbitrarily long input delays $k\tau$ in the linearization of (43) around (x_r, u_r).

VI. Conclusion

We derived new uniformly globally exponentially stabilizing sequential predictor feedbacks for time-varying linear systems with multiple delays. Such systems can arise when different components of the feedback control can have different delays. Potential advantage of our approach are that (i) it does not produce the types of distributed terms in the control that would arise from existing methods, and (ii) unlike emulation based controls, it also does not restrict the range of allowable input delays. We hope to extend our work to systems with multiple different time-varying delays, using analogs of our methods for time-varying delays from [12]. We also plan generalizations to nonlinear systems, and to cover measurement delays and sampling. This could extend [13] and [23] that did not allow multiple different delays.

Appendix: Proof of Lemma 1

We use a variant of an argument from [8, Chapter 3]. Set

$$\begin{array}{l} h = \frac{\tau}{p},\ \varrho(t) = \int_{t-h}^{t} \dot{\xi}(m)\mathrm{d}m,\ V_1(\xi) = \frac{1}{2}|\xi|^2, \\ \text{and } V_2(\xi_t) = \frac{1}{2h\ell}\int_{t-h}^{t}\int_{m}^{t} |\dot{\xi}(s)|^2 \mathrm{d}s\mathrm{d}m. \end{array} \tag{A.1}$$

and we rewrite (10) in the form

$$\dot{\xi}(t) = -\ell\xi(t) - L(\phi_i(t))\varrho(t). \tag{A.2}$$

Along all solutions of (A.2) for all $t \geq 0$, we have $\dot{V}_1(t) \leq -2\ell|\xi(t)|^2 - \xi(t)^\top L(\phi_i(t))\varrho(t)$ and

$$\begin{array}{rcl} \dot{V}_2(t) & = & \frac{1}{2h\ell}\left(h|\dot{\xi}(t)|^2 - \int_{t-h}^{t}|\dot{\xi}(s)|^2\mathrm{d}s\right) \\ & = & \frac{1}{2h\ell}\left(h\ell^2|\xi(t)|^2 + 2\ell h\xi^\top(t)L(\phi_i(t))\varrho(t)\right. \\ & & \left. + h|L(\phi_i(t))\varrho(t)|^2 - \int_{t-h}^{t}|\dot{\xi}(s)|^2\mathrm{d}s\right) \\ & \leq & \frac{1}{2h\ell}\left(h\ell^2|\xi(t)|^2 + 2\ell h\xi^\top(t)L(\phi_i(t))\varrho(t)\right. \\ & & \left. + h|L(\phi_i(t))\varrho(t)|^2 - \frac{1}{h}|\varrho(t)|^2\right), \end{array} \tag{A.3}$$

where the last inequality followed from Jensen's inequality.

Next observe that all solutions of (A.2) satisfy

$$\begin{array}{rcl} |\dot{\xi}(t)|^2 & \leq & 2\ell^2|\xi(t)|^2 + 2|\ell I_n + A|_\infty^2\left|\int_{t-h}^{t}\dot{\xi}(m)\mathrm{d}m\right|^2 \\ & \leq & 2m_*\left[|\xi(t)|^2 + \left|\int_{t-h}^{t}\dot{\xi}(m)\mathrm{d}m\right|^2\right], \end{array}$$

where $m_* = \max\{\ell^2, |\ell I_n + A|_\infty^2\}$. Consequently, it follows from our choice of c in (11) that the function $V_3(\xi_t) = V_1(\xi(t)) + V_2(\xi_t)$ satisfies

$$\begin{array}{rcl} \dot{V}_3(t) & \leq & -c|\xi(t)|^2 - c\left|\int_{t-h}^{t}\dot{\xi}(m)\mathrm{d}m\right|^2 \\ & \leq & -\frac{c}{2}|\xi(t)|^2 - \frac{c}{2}\left|\int_{t-h}^{t}\dot{\xi}(m)\mathrm{d}m\right|^2 - \frac{c}{4m_*}|\dot{\xi}(t)|^2. \end{array}$$

Hence, the function $V^\sharp$ defined in (12) satisfies

$$\begin{array}{rcl} \dot{V}^\sharp(t) & \leq & -\frac{c}{2}|\xi(t)|^2 - \frac{c}{2}\left|\int_{t-h}^{t}\dot{\xi}(m)\mathrm{d}m\right|^2 - \frac{c}{8hm_*}|\dot{\xi}(t)|^2 \\ & & -\frac{c}{8hm_*}\int_{t-h}^{t}|\dot{\xi}(m)|^2\mathrm{d}m \end{array}$$

which readily gives the required constant c_a. Then uniform global exponential stability for (10) follows from the quadratic structure of $V^\sharp$.

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